Inverse Sumset Results mod p at High Density

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Sumsets

Let G be an abelian group.

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Theorem (Folklore)

For finite, nonempty A, $B \subseteq \mathbb{Z}$, we have

$$|A + B| \geq |A| + |B| - 1.$$

If equality holds, then A and B are arithmetic progressions with common difference (or $|A| = 1$ or $|B| = 1$).

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Let $A, B \subseteq \mathbb{Z}$ be finite and nonempty with $|A| \geq |B|$ and

$$|A + B| = |A| + |B| + r \leq |A| + 2|B| - 3 - \delta,$$

where

$$\delta = \begin{cases} 1 & \textit{if} \quad A = (\min A - \min B) + B, \\ 0 & \textit{otherwise.} \end{cases}$$

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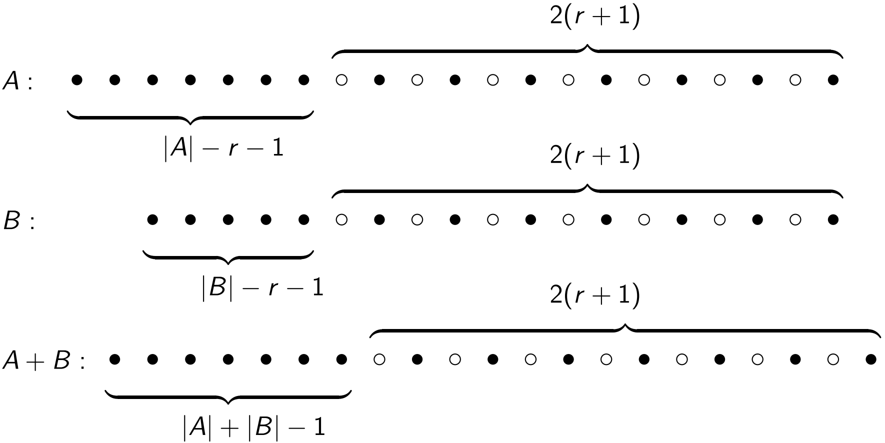
Then there are arithmetic progressions P_A, P_B, $P_{A+B} \subseteq \mathbb{Z}$ having common difference such that

$$X \subseteq P_X \quad \text{and} \quad |P_X| \leq |A| + r + 1 \qquad \text{for all } X \in \{A, B\},$$

$$P_{A+B} \subseteq A + B \quad \text{and} \quad |P_{A+B}| \geq |A| + |B| - 1.$$

Freiman (1962); Lev and Smeliansky (1995); Freiman (2009); Bardaji and G (2010); G (2013)

Some Examples: r is tight



Extension modulo p

Definition (General Setup)

$G = \mathbb{Z}/p\mathbb{Z}$ with $p \geq 2$ prime, $A, B \subseteq G$ nonempty, $A + B \neq G$,
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Assume **General Setup**. *If*

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- Upshot: $3k - 4$ Theorem should in $\mathbb{Z}/p\mathbb{Z}$ too so long as $A + B$ isn't too large
- Much partial Progress. General Idea: Impose additional small doubling and density constraints to obtain **Target Conclusions**.

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- $C \subseteq P_C \iff P_{A+B} = -(P_C)^c \subseteq -C^c = A+B$
- $|P_C| \leq |C| + r + 1 \iff |P_{A+B}| \geq |A| + |B| - 1$

$$|P_{A+B}| = p - |P_C| \geq p - |C| - r - 1 = |A+B| - r - 1 = |A| + |B| - 1$$

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- Suppose **Target Conclusions** holds if $|A + B| \leq p - r - 3$ and $|A + B| \leq (|A| + |B|) + \alpha|B| - 3$

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- Thus $r+3 \le \alpha'|B| < \alpha'\frac{p}{2}$, so its true for $\alpha' \le 2\epsilon$
- Summary:

$$|A+B| \le (|A|+|B|)+2\epsilon|B|-3 \quad \text{and} \quad |A+B| \le (1-\epsilon)p$$

ensure A, B and C contained in small length arithmetic progressions (for small $\epsilon < \frac{1}{2}\alpha$.)

Partial Progress: Low Density

$$|A + B| = |A| + |B| + r \leq (|A| + |B|) + \alpha|B| - 3, \qquad \text{where } \alpha \in (0, 1]$$

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- Low density implies "isomorphic" to a sumset $A' + B' \subseteq \mathbb{Z}$.
- $|A \cup B| \leq \log_4 p \quad \longrightarrow \quad$ Bilu, Lev, Ruzsa (1998)
- $|A \cup B| \leq \lceil \log_2 p \rceil \quad \longrightarrow \quad$ Lev (2008), + technical issue G. (2013)
- $A = B$ and $|A| \leq cp$ with $c = (1/96)^{108} \quad \longrightarrow \quad$ Green, Ruzsa (2006)

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  - $|A+A| \leq 2|A| + (0.59)|A| - 3$ and $101 \leq |A| < (0.0045)p$
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- In many cases, "flexible" versions are available: density restriction grows stronger as small doubling constraint gets weaker.
- $(0.001)|A|^{2/3} \leq |B| \leq |A|, \quad |A+B| \leq (|A| + |B|) + (0.03)|B|$ and $|A| < (0.0045)p \quad \longrightarrow \quad$ Huichochea (2022)

Partial Progress: High/Ideal Density

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- This gives no density restriction for $|A+B| \leq |A| + (1+2c)|B| - 3$.

Ideal Density

Theorem (Serra and Zémor 2009)

Assume **General Setup**. *If* $|A| \geq 4$, $p > 2^{94}$,

$$|A+A| \leq 2|A| + (0.0001)|A| \quad \textit{and} \quad |A+A| \leq p - r - 3,$$

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Ideas for the Proof: Additive Trios

Definition (Boothbay, DeVos, Montejano 2015)

Let G be a finite abelian group; let $A, B, C \subseteq G$ be nonempty sets. Then (A, B, C) is and **additive trio** if $A + B + C \neq G$ with $\mathrm{r}(A, B, C) := |G| - |A| - |B| - |C|$.

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Example

$A, B \subseteq G$, $C := -(A + B)^{\mathrm{c}}$, $|A + B| = |A| + |B| + r$. Then (A, B, C) is an additive trio, since

$$0 \notin (A + B) - (A + B)^{\mathrm{c}} = A + B + C,$$

with $\mathrm{r}(A, B, C) = |G| - |A| - |B| - (|G| - |A| - |B| - r) = r$

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- $|A+B| = |A| + |B| + r \leq (|A| + |B|) + \alpha|B| - 3$ and $|A+B| \leq p - \alpha(r+3)$ equivalent to $|A|, |B|, |C| \geq \alpha^{-1}(r+3)$.

Ideas for the Proof: Huicochea's Reduction

Definition

For $A \subseteq G$ and $d \in G$, let $\ell_d(A)$ be the minimal length of an arithmetic progression with difference d containing A.

Theorem (Huicochea 2017; modified by G. 2024)

Let $G = \mathbb{Z}/p\mathbb{Z}$ with $p \geq 2$ prime, let (A, B, C) be an additive trio from G, and let $r,\ h \in \mathbb{Z}$ be integers.

1. *If $\ell_d(A) \leq |A| + h$ for some $d \in G \setminus \{0\}$, $r(A, B, C) \leq r$, $|A| \geq r + 3 + h$ and $|B| \geq r + 3 + 2h$ with strict inequality in one of these estimates, and $|C| \geq r + 3$, then*

   $$\ell_d(A) \leq |A| + r + 1,\ \ell_d(B) \leq |B| + r + 1, \quad \text{and} \quad \ell_d(C) \leq |C| + r + 1.$$

2. *If $\ell_d(A) \leq |A| + h$ for some $d \in G \setminus \{0\}$, $r(A, B, C) \leq h - 1$, $h + 2 \leq |A| \leq \max\{|B|, |C|\}$, $|B|, |C| \geq h + 3$ and $35h + 10 \leq p$, then*

   $$\ell_d(B) \leq |B| + h \quad \text{and} \quad \ell_d(C) \leq |C| + h.$$

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- Problem 1: Simple extremal combinatorial argument for showing $A = A'$ fails at higher densities $|A + A| > \frac{p+1}{2}$.
- Problem 2: Issues with Fourier calculation estimates when $A \neq B$ and adapting the combinatorial argument for showing $A = A'$.

Ideas for the Proof: Combinatorial Reduction

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- These arguments needed to be adapted from $A + A$ to $A + B$ (G. 2024)

Theorem (G. 2024)

Let p be prime, let A, $B \subseteq \mathbb{Z}/p\mathbb{Z}$ be nonempty subsets with

$$|A + B| = |A| + |B| + r \leq \frac{3}{4}(p + 1) \quad \text{and} \quad p \geq 4r + 9,$$

and set $C = -(A + B)^c$. Suppose there exist subsets $A' \subseteq A$ and $B' \subseteq B$ or $A' \subseteq B$ and $B' \subseteq A$ such that $A' + B'$ rectifies, $|B'| \leq |A'|$ and

$$|A'| + 2|B'| - 4 \geq |A + B|. \tag{1}$$

Then **Target Conclusions** *hold.*

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Lemma

If $z_1, \ldots, z_N \in \mathbb{C}$ is a sequence of points lying on the complex unit circle such that every open half-arc contains at most n of the terms z_i for $i \in [1, N]$, then $|\sum_{i=1}^{N} z_i| \leq 2n - N$.

The “Base Case”

Theorem (G. 2024)

Assume **General Setup**. *Let* $\beta \in [0.731, 1]$ *and* $\alpha \in (0, 0.212]$ *be real numbers. Suppose*

$$|A + B| \le |A| + (1 + \alpha)|B| - 3 \quad \textit{and} \quad \beta|A| \le |B| \le |A| \le f(\alpha, \beta)\, p.$$

Then **Target Conclusions hold**.

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- Used Huichochea reduction argument transfers desired structure back to A, B and C.

Near Optimal Density

Theorem (G. 2024)

Assume **General Setup**. *If*

$$|A+B| \leq (|A|+|B|) + \frac{1}{9}|B| - 3 \quad \textit{and} \quad |A+B| \leq p - 29(r+3),$$

then **Target Conclusions** *hold.*

Moving from near-optimal to optimal density

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Let G be an abelian group, let $A, B \subseteq G$ be finite, nonempty subsets and let $A' \subseteq A$ be a nonempty subset attaining the minimum

$$\alpha := \min\left\{\frac{|A'+B|}{|A'|} : \emptyset \neq A' \subseteq A\right\} \leq \frac{|A+B|}{|A|}.$$

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$$|A'+nX| > |A'| + \frac{10}{9}|nX| - 3$$

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- Repeat for $4X = 2X+2X$ to get improved lower bound for $|4X|$.
- Compare with upper bound, and obtain a contradiction.

Thanks!