On Atomic Domains not Satisfying the ACCP (joint work with Benjamin Li)

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Outline

Atomicity

In the scope of this presentation, a monoid is a cancellative and commutative semigroup with an identity element.

Notation: Let M be a monoid.

- $U(M)$ denotes the group of units (i.e., invertible elements) of M.
- $\mathrm{gp}(M)$ denotes the quotient group of M.
- $\mathcal{A}(M)$ denotes the set of atoms (i.e., irreducible elements) of M.

Definition: M is atomic if every nonunit element of M factors into atoms.

If R is an integral domain, then $R \setminus \{0\}$ is a monoid that we denote by R^*.

Definition (atomicity)

An integral domain R is atomic if the monoid R^* is atomic.

Remarks:

1. Every integral domain satisfying the ACCP is atomic.
2. In particular, Noetherian and Krull domains are atomic.
3. If R satisfies the ACCP, then $R[x]$ satisfies the ACCP.
4. Even if R is atomic, $R[x]$ may **NOT** be atomic (Roitman, 1993).

Strong Atomicity

Definition: Let M be a monoid.

- $d \in M$ is atomic if $d = ua_1 \cdots a_k$ for some $u \in U(M)$ and $a_1, \dots, a_k \in \mathcal{A}(M)$.
- M is strongly atomic provided that for all $x, y \in M$, there exists an atomic element $d \in M$ simultaneously dividing x and y in M such that $\gcd(x/d, y/d) = 1$.

Definition (Strong Atomicity)

An integral domain R is strongly atomic if the monoid R^* is strongly atomic.

Remarks:

1. Every strongly atomic domain is atomic (Anderson-Anderson-Zafrullah, 1990).
2. Every integral domain satisfying the ACCP is strongly atomic (Anderson-Anderson-Zafrullah, 1990).
3. Not every atomic domain is strongly atomic (Roitman, 1993).
4. Not every strongly atomic domain satisfies the ACCP (Roitman, 1993).

The ACCP

Definition: Let M be a monoid.

- $I \subseteq M$ is an ideal if $IM \subseteq I$, and an ideal I is principal if $I = xM$ for some $x \in M$.
- A sequence $(I_n)_{n \in \mathbb{N}}$ of ideals of M is an ascending chain if $I_n \subseteq I_{n+1}$ for every $n \in \mathbb{N}$.
- M satisfies the ascending chain condition on principal ideals (ACCP) if every ascending chain of principal ideals of M stabilizes.

Remark: An integral domain R satisfies the ACCP if and only if the monoid R^* satisfies the ACCP.

Proposition

Every monoid satisfying the ACCP is strongly atomic.

A Strongly Atomic Monoid without the ACCP

Example: Let $(p_n)_{n \ge 0}$ be the strictly increasing sequence whose underlying set is the set of odd primes. Call

$$G = \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle$$

the Grams' monoid.

1. $\mathcal{A}(G) = \left\{ \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\}$, and so G is atomic.
2. The submonoid $V := \left\langle \frac{1}{2^n} \mid n \in \mathbb{N}_0 \right\rangle \subseteq G$ is a valuation monoid (i.e, principal ideals of V form a chain).
3. If $q \in G$, then $q = v + u$, where $v \in V$ and $u \in G$ such that u has a unique factorization into atoms, and such a decomposition is unique.
4. Let $v + u, v' + u' \in G$, where $v, v' \in V$ and u, u' are unique factorization elements. Set $v_m := \min\{v, v'\}$ and $g := \gcd_G(u, u')$. If $d := v_m + g$, then none of the atoms of G divides simultaneously $v + u - d$ and $v' + u' - d$.
5. Hence G is strongly atomic.
6. G does not satisfy the ACCP as $(1/2^n + G)_{n \in \mathbb{N}}$ does not stabilize.

Further Atomic Monoids without the ACCP

Example: Fix $q \in \mathbb{Q} \cap (0,1)$ with $q^{-1} \notin \mathbb{N}$. Set $S_q = \langle q^n \mid n \in \mathbb{N}_0 \rangle$.

1. $\mathcal{A}(S_q) = \{q^n \mid n \in \mathbb{N}_0\}$, and so S_q is atomic.
2. S_q is indeed strongly atomic (it is not obvious).
3. S_q does not satisfy the ACCP. If $q = a/b$ with $a, b \in \mathbb{N}$, then $bq^n = (b-a)q^n + bq^{n+1}$ for every $n \in \mathbb{N}$.
   Hence $(bq^n + S_q)_{n \in \mathbb{N}}$ does not stabilize.

Proposition (Correa-Morris-G., 2022)

For any $d \in \mathbb{N}$, there exists a positive algebraic α of degree d such that the monoid $S_\alpha = \langle \alpha^n \mid n \in \mathbb{N}_0 \rangle$ is atomic but does not satisfy the ACCP.

Proposition (Tirador-G., 2021)

There exists a submonoid of $(\mathbb{Z}^2, +)$ that is atomic but does not satisfy the ACCP.

Atomic Domains without the ACCP (Historical Outline)

Back in 1968, P. Cohn wrongly asserted (without proof) that every atomic domain satisfies the ACCP. Here are (some of?) the known constructions of atomic integral domains without the ACCP.

- 1974: A. Grams constructed the first class of atomic domains without the ACCP.
- 1982: A. Zaks proved that certain quotient of a polynomial ring (over a field) in countably many variables is atomic but does not satisfy the ACCP.
- 1993: M. Roitman constructed a class of atomic domains to disprove the believed conjecture that atomicity should transfer from a ring R to its ring of polynomials $R[x]$. In particular, rings in this class are atomic but not ACCP (as the ACCP property ascends from R to $R[x]$).
- 2019: J. Boynton and J. Coykendall used a pullback construction to provide another class of atomic domains without the ACCP.
- 2021: G. and B. Li constructed the first class of atomic monoid domains without the ACCP (using infinite-rank monoids).
- 2022: G. and B. Li constructed another class of atomic monoid domains without the ACCP (using rank-two monoids).

Atomic Domains without ACCP: Grams' Construction

Grams' Construction: Let $G = \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N}_0 \right\rangle$ be the Grams' monoid, and consider the monoid algebra $R := \mathbb{Q}[G]$.

- G is atomic but does not satisfy the ACCP.
- $S := \{ f \in R \mid f(0) \neq 0 \}$ is a multiplicative set of R.

Theorem (Grams, 1974)

The localization $S^{-1}R$ of R at S is atomic but does not satisfy the ACCP.

Questions:

1. What is the essential property of the Grams' monoid G that makes her construction work?
2. Can we generalize such a property to obtain new atomic domains not satisfying the ACCP?

The Weak ACCP Property

Definition: Let M be a monoid.

- $x \in M$ satisfies the ACCP if every ascending chain of principal ideals starting at xM stabilizes.
- M is weak ACCP if every nonempty finite subset S of M has an atomic common divisor $d \in M$ such that s/d satisfies the ACCP for some $s \in S$.

Definition (Weak ACCP)

An integral domain R is weak ACCP provided that the monoid R^* is weak ACCP.

Proposition (Li-G., 2022)

The following statements hold.

1. *Every domain/monoid satisfying the ACCP is weak ACCP.*
2. *Every weak ACCP domain/monoid is strongly atomic.*
3. *ACCP $\Rightarrow$ weak ACCP $\Rightarrow$ strong atomicity $\Rightarrow$ atomicity.*

Examples of Weak ACCP Monoids

Remark: None of the implications in the following chain is reversible.

ACCP $\Rightarrow$ **weak ACCP** $\Rightarrow$ **strong atomicity** $\Rightarrow$ **atomicity**

Proposition (Li-G., 2021-2022)

There exists a submonoid of $(\mathbb{Q}_{\geq 0}, +)$ that is atomic (resp., strongly atomic) but not strongly atomic (resp., weak ACCP).

Example 1. Let $G = \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle$ be the Grams' monoid.

1. If $q \in G$, then $q = v + u$, where $v \in V := \left\langle \frac{1}{2^n} \mid n \in \mathbb{N}_0 \right\rangle \subseteq G$ and $u \in G$ such that u has a unique factorization.
2. Let $v_1 + u_1, \dots, v_k + u_k \in G$ with $v_1, \dots, v_k \in V$ and unique factorization elements $u_1, \dots, u_k$. If $v_j = \min\{v_1, \dots, v_k\}$, then v_j is a common divisor of $v_1 + u_1, \dots, v_k + u_k$ in G and $u_j = (v_j + u_j) - v_j$ satisfies the ACCP.
3. Hence G is a weak ACCP monoid that does not satisfy the ACCP.

Example 2. The monoid $S_q = \langle q^n \mid n \in \mathbb{N}_0 \rangle$, where $q \in \mathbb{Q} \cap (0,1)$ with $q^{-1} \notin \mathbb{N}$ is also a weak ACCP monoid that does not satisfy the ACCP.

Nested Atomic Classes of Monoids Weaker than ACCP

Remark: Even in the special class of integral domains, none of the implications in the following chain is reversible.

ACCP $\Rightarrow$ **weak ACCP** $\Rightarrow$ **strong atomicity** $\Rightarrow$ **atomicity**

Theorem (Roitman, 1993)

There is an atomic domain that is not strongly atomic.

Theorem (Li-G., 2021-2022 via Roitman's example)

There is a strongly atomic domain that is not weak ACCP.

Theorem (Grams 1974 and Li-G., 2021-2022)

Grams' counterexample of an atomic domain without the ACCP is indeed a weak ACCP domain.

Weak ACCP and Grams' Construction

Definition: A positive monoid is an additive submonoid of the nonnegative cone of an ordered field.

Theorem (Li-G., 2022)

Let F be a field, and let M be a positive monoid. If M is weak ACCP, then so is the localization of $F[M]$ at the multiplicative subset $\{f \in F[M] \mid f(0) \neq 0\}$.

Corollary: Let F be a field, and take $M \in \{G, S_q\}$, where G is the Grams' monoid and $q \in \mathbb{Q} \cap (0,1)$ with $q^{-1} \notin \mathbb{N}$. Set $R := F[M]$ and $S := \{f \in R \mid f(0) \neq 0\}$. Then

- $S^{-1}R$ is a weak ACCP domain, and so
- $S^{-1}R$ is an atomic domain without the ACCP.

Boynton-Coykendall Pullback Construction

Boynton-Coykendall Construction: Let x and y be two distinct indeterminates over a field F, and consider the multiplicative set

$$S := \{s \in F[x, y] \mid s \notin (x) \text{ and } s \notin (y)\}.$$

1. Set $D := S^{-1}F[x, y]$, and let K be the quotient field of D.
2. Consider the overrings $D_1 := D[\frac{y}{x^2}]$ and $D_2 := D[\frac{x}{y^2}]$ of D.
3. Set $f(X) = X(X-1)$ and $I := K[X]f(X)$.

The canonically defined homomorphisms

$$\eta\colon K[X] \to K[X]/I \cong K \times K \quad \text{and} \quad \iota\colon D_1 \times D_2 \hookrightarrow K \times K$$

are surjective and injective, respectively. Then the pullback of η and ι is

$$R := \big\{g(X) \in K[X] \mid g(0) \in D_1 \text{ and } g(1) \in D_2\big\}.$$

Theorem

- *R is atomic but does not satisfy the ACCP [Boynton-Coykendall, 2019].*
- *R is a weak ACCP domain [Li-G., 2021-2022].*

The Weak ACCP Under Polynomial-Like Extensions

Definition: Let M be an (additive) monoid.

- M is reduced if $U(M) = \{0\}$.
- M is torsion-free if whenever $nx = ny$ for some $n \in \mathbb{N}$ and $x, y \in M$, the equality $x = y$ holds.

Theorem (Li-G., 2021-2022)

Let R be an integral domain, and let M be a reduced torsion-free monoid satisfying the ACCP.

- *R is weak ACCP if and only if $R[M]$ is weak ACCP.*
- *R is weak ACCP if and only if $R[x, x^{-1}]$ is weak ACCP.*

Strongly Atomic Domains not Weak ACCP

Taking $M = (\mathbb{N}_0, +)$ in the previous theorem, we obtain that being a the weak ACCP property ascends to polynomial extensions.

Corollary: Let R be an integral domain. Then R is weak ACCP if and only if $R[x]$ is weak ACCP.

Definition: We say that an integral domain R is a Roitman domain if R is atomic but $R[x]$ is not atomic. As proved by Roitman, the class of Roitman domains contains strongly atomic domains.

Theorem (Roitman, 1993)

The class of Roitman domains contains strongly atomic domains.

Corollary: There are strongly atomic domains that are not weak ACCP.

The Weak ACCP Under the D+M Construction

Both atomicity and the ACCP behave well under the $D + M$ construction.

Proposition (Anderson-Anderson-Zafrullah, 1990)

Let T be an integral domain, and let K and M be a subfield of T and a nonzero maximal ideal of T, respectively, such that $T = K + M$. For a subring D of K, set $R = D + M$. Then

- *R is atomic if and only if T is atomic and D is a field, and*
- *R satisfies the ACCP if and only if T satisfies the ACCP and D is a field.*

The same statement still hold when we replace ACCP by weak ACCP.

Proposition (Li-G., 2021-2022)

Let $T = K + M$ and $R = D + M$ be as in the $D + M$ construction. Then

- *R is weak ACCP if and only if T is weak ACCP and D is a field.*

The Weak ACCP Property Under Localization

Definition: An extension $R \subseteq S$ of integral domains is called inert if for all $x, y \in S^*$ with $xy \in R$ there exists $u \in S^\times$ such that $ux, u^{-1}y \in R$.

Proposition (Li-G., 2021-2022)

Let R be an integral domain, and let S be a multiplicative subset of R such that the extension $R \subseteq R_S$ is inert. If R is weak ACCP, then R_S is weak ACCP.

Definition: A multiplicative subset S of an integral domain R is called splitting if every $r \in R$ can be written as $r = as$ for some $a \in R$ and $s \in S$ such that $Ra \cap Rs' = Ras'$ for all $s' \in S$.

Proposition (Li-G., 2021-2022)

Let R be an integral domain, and let S be a splitting multiplicative subset of R generated by primes. If R_S is weak ACCP, then R is weak ACCP.

Remark: The statements obtained from the previous two propositions after replacing the weak ACCP property by either being atomic or satisfying the ACCP also hold (Anderson-Anderson-Zafrullah, 1992).

The Weak ACCP Under Directed Unions

Definition:

1. A poset $(\Gamma, \preceq)$ is directed if for all $\alpha, \beta \in \Gamma$, there exists $\theta \in \Gamma$ such that $\alpha \preceq \theta$ and $\beta \preceq \theta$.
2. A family $(R_\gamma)_{\gamma \in \Gamma}$ of integral domains indexed by a directed poset Γ is a directed family if for all $\alpha, \beta \in \Gamma$ with $\alpha \preceq \beta$ the integral domain R_α is a subring of R_β.

Theorem (Anderson-Anderson-Zafrullah, 1992)

Let $(R_\gamma)_{\gamma \in \Gamma}$ be a directed family of integral domains such that the extension $R_\alpha \subseteq R_\beta$ is inert whenever $\alpha \preceq \beta$.

- *If R_γ is atomic for each $\gamma \in \Gamma$, then $\bigcup_{\gamma \in \Gamma} R_\gamma$ is atomic.*
- *If R_γ satisfies the ACCP for each $\gamma \in \Gamma$, then $\bigcup_{\gamma \in \Gamma} R_\gamma$ satisfies the ACCP.*

Proposition (Li-G., 2022)

Let $(R_\gamma)_{\gamma \in \Gamma}$ be a directed family of integral domains such that the extension $R_\alpha \subseteq R_\beta$ is inert whenever $\alpha \preceq \beta$.

- *If R_γ is weak ACCP for each $\gamma \in \Gamma$, then $\bigcup_{\gamma \in \Gamma} R_\gamma$ is weak ACCP.*

An Atomic Monoid Domain Without the ACCP

Let $(p_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of primes such that $\sum_{n=1}^{\infty} \frac{1}{p_n} < \frac{1}{3}$, and consider the following set:

$$A := \left\{ \frac{1}{p_{j_k}} + \sum_{i=1}^{\ell} \frac{1}{p_{j_i}} \;\middle|\; k, \ell \in \mathbb{N} \text{ and } j_1 < j_2 < \cdots < j_\ell \right\}.$$

In addition, let β be an irrational number such that $\beta > 1$, and consider the following subset of $\mathbb{N}_0 \beta + \mathbb{Q}$:

$$B := \{\beta\} \bigcup \left\{ \beta_\ell := \beta - \sum_{i=1}^{\ell} \frac{1}{p_i} \;\middle|\; \ell \in \mathbb{N} \right\}.$$

Theorem (Li-G., 2022)

Set $M = \langle A \cup B \rangle$, and let F be a field. Then the following statements hold.

1. *M is atomic with $\mathcal{A}(M) = A \cup B$.*
2. *$F[M]$ is weak ACCP and, therefore, atomic.*
3. *M does not satisfy the ACCP, and so $F[M]$ does not satisfy the ACCP.*

An Atomic Monoid Domain without the ACCP

Definition: The rank of a monoid M is the rank of $\text{gp}(M)$ as a $\mathbb{Z}$-module or, equivalently, the dimension of the $\mathbb{Q}$-space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$.

Remark: The monoid domain $F[M]$ exhibited in the previous slide seems to be the only atomic monoid domain without the ACCP known so far. However, the rank of M is 2.

Open Question: Is there a rank-one monoid M and a field F such that $F[M]$ is atomic but does not satisfy the ACCP?

A Potential Answer: The monoid S_q, where $q \in \mathbb{Q} \cap (0,1)$ such that $q^{-1} \notin \mathbb{N}$, satisfies the following conditions.

- S_q is an atomic rank-one monoid.
- S_q does not satisfy the ACCP.

Conjecture

With S_q as above, the monoid domain $\mathbb{Q}[S_q]$ is atomic but does not satisfy the ACCP.

References

- D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990) 1–19.
- D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorization in integral domains, II*, J. Algebra **152** (1992) 78–93.
- J. G. Boynton and J. Coykendall: *An example of an atomic pullback without the ACCP*, J. Pure Appl. Algebra **223** (2019) 619–625.
- P. M. Cohn: *Bezout rings and their subrings*, Proc. Cambridge Phil. Soc. **64** (1968) 251–264.
- J. Correa-Morris and F. Gotti: *On the additive structure of algebraic valuations of polynomial semirings*, J. Pure Appl. Algebra **226** (2022) 107104
- F. Gotti and B. Li: *Atomic semigroup rings and the ascending chain condition on principal ideals*, Proc. Amer. Math. Soc. (to appear).

References

A. Grams: *Atomic rings and the ascending chain condition for principal ideals*, Proc. Cambridge Philos. Soc., **75** (1974) 321–329.

M. Roitman: *Polynomial extensions of atomic domains*, J. Pure Appl. Algebra **87** (1993) 187–199.

A. Zaks: *Atomic rings without a.c.c. on principal ideals*, J. Algebra **80** (1982) 223–231.

THANK YOU!