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On monoid algebras: Algebraic and arithmetic properties

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Overview

(I) Monoid algebras

(II) Krull domains/monoids and class groups

(III) Main results

Monoid algebras

Let D be a domain and $(S,+)$ a monoid, i.e. a commutative, cancellative and unitary semigroup.

The monoid algebra $D[S]$ is an overring of D having elements X^s for $s \in S$ such that

- every $f \in D[S]$ has a unique representation $\sum_{i=0}^{n} d_i X^{s_i}$, where $d_i \in D$, $s_i \in S$ and
- $X^s \cdot X^t = X^{s+t}$.

E.g., $D[X] = D[\mathbb{N}_0]$ is the classical polynomial ring.

These objects appear in algebraic geometry, representation theory, invariant theory, in the theory of polytopes, ...

Study $D[S]$

Hilbert's basis theorem: $D[\mathbb{N}_0]$ is noetherian iff D is noetherian.

For arbitrary S this will not hold true anymore (note that $\mathbb{N}_0$ has nice properties!).

Can we generalize such results for general S? $(D[S] \leftrightarrow D, S)$

Yes, and some properties are easy to get.

Known results I

All of the following statements are to be found in Gilmer's book "Commutative semigroup rings".

- $D[S]$ is noetherian iff D is noetherian and S is finitely generated.
- $D[S]$ is integrally closed iff D is integrally closed and S is root closed.
- $D[S]$ is completely integrally closed iff D and S are c.i.c.
- $D[S]$ is a domain iff D is a domain and S is torsion-free.

Other properties are harder to get and we have to make some definitions first.

Krull monoids divisor theoretic

A monoid homomorphism $\varphi : H \to \mathcal{F}(P)$, where $\mathcal{F}(P)$ is a free abelian monoid, is a divisor theory, if

$$a \mid_H b \text{ if and only if } \varphi(a) \mid_{\mathcal{F}(P)} \varphi(b)$$

and $\mathcal{F}(P)$ satisfies a certain minimality condition.

A monoid (domain) H (D) is said to be a Krull monoid if H ($D \setminus \{0\}$) has a divisor theory.

Example: If D is a Dedekind domain, then $D \setminus \{0\} \to \mathcal{I}^*(D)$ is a divisor theory.

In general it turns out, that $H \to \mathcal{I}_v^*(H)$ is a divisor theory for Krull monoids.

Krull monoids ideal theoretic

There are (more than) 2 further approaches to define Krull monoids/domains.

- A monoid (domain) H is Krull if it is c.i.c. and v-noetherian.
- A monoid (domain) H is Krull if
  (1) $H = \bigcap_{P \in \mathfrak{X}(H)} H_P$,
  (2) each H_P is a DVM (DVR) and
  (3) $\{P \in \mathfrak{X}(H) \mid a \in P\}$ is finite for all $a \in H$ (nonzero).

Fact: All of these approaches are equivalent and a domain D is Krull iff $(D \setminus \{0\}, \cdot)$ is Krull.

Class groups

Let H be a Krull monoid, i.e. there exists a divisor theory $\varphi : H \to F = F(P)$.

$\mathcal{C}_v(H) = \mathsf{q}(F)/\mathsf{q}(\varphi(H))$ is independent of the divisor theory and called the (divisor) class group of H. The elements of the set P are called prime divisors.

Example: If $\mathcal{O}_K$ is the ring of integers of a number field K, then $\mathcal{C}_v(\mathcal{O}_K)$ is finite and contains prime divisors in all classes.

Fact: A Krull monoid is uniquely (up to units) determined by its class group and the cardinality of the prime divisors in the classes.

Why class groups?

Let H be a Krull monoid with class group $\mathcal{C}_v(H)$ and let $G_0 \subseteq \mathcal{C}_v(H)$ the set of all classes containing prime divisors.

Then there is a classical object $\mathcal{B}(G_0)$ of additive combinatorics (the monoid of zero-sum sequences) only depending on $G_0 \subseteq \mathcal{C}_v(H)$ that completely reflects lengths of factorizations in D!

Theorem (Kainrath, 1999)

Let H be a Krull monoid with infinite class group and prime divisors in all classes. Then $\mathcal{L}(H)$ is full, i.e. for each finite set of integers $2 \le \ell_1 < \ldots < \ell_k$ there exists an element of H whose lengths of factorizations are exactly $\ell_1, \ldots, \ell_k$.

E.g. there is an element of H that factors as a product of 3 atoms, 8 atoms and 100000 atoms but in no other number of atoms.

Back to monoid algebras: Known results II

Theorem (Chouinard, 1981, Gilmer)

Let D be an integral domain and let S be a torsion-free monoid.

(a) *$D[S]$ is Krull iff D is Krull and S is Krull (and $S^\times$ has ACCC).*

(b) *Let $D[S]$ be Krull. Then $\mathcal{C}_v(D[S]) = \mathcal{C}_v(D) \oplus \mathcal{C}_v(S)$.*

Theorem (Chang, 2009)

Let D be a domain and S a torsion-free monoid such that $\mathsf{q}(S)$ satisfies A.C.C.C. Then $D[S]$ is weakly factorial iff D is a weakly factorial GCD-domain and S is a weakly factorial GCD-semigroup.

2 Questions

We have a characterization of when a monoid algebra is Krull and we know something about the class group.

Two questions arise:

(I) Can we characterize other ring-theoretic properties?

(II) What about the distribution of prime divisors in the class groups resp. factorization theory?

Weakly Krull domains

Let D be a domain. Then D is said to be a weakly Krull domain if

- $D = \bigcap_{P \in \mathfrak{X}(D)} D_P$
- $\{P \in \mathfrak{X}(D) \mid a \in P\}$ is finite for all $a \in D \setminus \{0\}$.

D is a weakly Krull domain iff $D \setminus \{0\}$ is a weakly Krull monoid.

Weakly Krull domains: D.D. Anderson, Mott, Zafrullah, 1992
Weakly Krull monoids: Halter-Koch, 1995

Examples

- All orders in number fields are weakly Krull, but the non-principal ones are never Krull.
- Every 1-dim. noetherian domain is weakly Krull (v-noetherian suffices).
- Cohen-Macaulay domains are weakly Krull.

Question I for weakly Krull

When is $D[S]$ weakly Krull? (Chang, 2009)

A domain D is said to be a UMT-domain if the **U**ppers to zero in $D[X]$ are **M**aximal **t**-ideals.

Proposition (D.D Anderson, Houston, Zafrullah, 1993)

Let A be a domain. Then $A[X]$ is weakly Krull iff A is a weakly Krull UMT-domain.

Main result

Theorem (Chang, F., Windisch)

Let D be a domain with char$(D) = 0$ (resp. $= p$) and S a torsion-free monoid with quotient group G. Then $D[S]$ is weakly Krull iff D is a weakly Krull UMT-domain, S is a weakly Krull UMT-monoid, and G satisfies A.C.C.C. (resp. is of type $(0, 0, 0, \ldots)$ except p).

With not too much effort, we reobtain the two results by Chouinard and Chang with our main result.

Question II: Prime divisors and factorizations

What follows are results on prime divisors for

- Krull monoid algebras and
- affine monoid algebras.

Finally, our characterization of weakly Krull monoid algebras and the results for affine monoid algebras will lead to factorization theoretic statements for weakly Krull affine monoid algebras.

Prime divisors in the Krull case

There were attempts to show: Every class of $D[S]$ contains a prime divisor.
But the proofs have gaps.

Theorem (F., Windisch)

Let $D[S]$ be a (nontrivial) Krull monoid algebra. Then every class of $\mathcal{C}_v(D[S])$ contains infinitely many prime divisors.

So lengths of factorizations in $D[S]$ are completely determined by $\mathcal{C}_v(D[S])$.

Corollary

If $\mathcal{C}_v(D)$ or $\mathcal{C}_v(S)$ is infinite, then $\mathcal{L}(D[S])$ is full, i.e. for each finite set of integers $2 \leq \ell_1 < \ldots < \ell_k$ there exists an element of $D[S]$ whose lengths of factorizations are exactly $\ell_1, \ldots, \ell_k$.

Affine monoid algebras

Let K be a field and let S be an affine monoid, that is, a finitely generated monoid (not necessarily Krull!) with quotient group $(\mathbb{Z}^n, +)$ for some $n > 0$.

If $n = 1$, then S can be viewed as submonoid of $\mathbb{N}_0$ with $\mathbb{N}_0 \setminus S$ finite (or $S \cong \mathbb{Z}$).

$K[S]$ is an important object in the theory of polytopes.

Class groups of affine monoid algebras

Let K be a field and S be a numerical monoid with Frobenius number $\mathrm{f}(S) = \max(\mathbb{N}_0 \setminus S)$. We define

$$\mathcal{C} = \{1 + \sum_{i=1}^{\mathrm{f}(S)} k_i X^i \mid k_i \in K\}.$$

One can endow $\mathcal{C}$ with a group structure.

Proposition (F., Windisch)

Let K be a field, $m \in \mathbb{N}$ and S be numerical monoid with $\mathbb{N} \setminus S = [1, m]$. Then there is an isomorphism $\mathcal{C}_v(K[S]) \cong \mathcal{C}$.

Proposition (F., Windisch)

Let K be a field and $S \subseteq T$ numerical monoids. Then there exists an epimorphism

$$\mathcal{C}_v(K[S]) \to \mathcal{C}_v(K[T]).$$

Prime divisors in affine monoid algebras

Theorem (F., Windisch)

(1) *Let* $n = 1$ *and* $\mathsf{f}(S) = \max(\mathbb{N}_0 \setminus S)$*. Then* $K[S]$ *has at least one prime divisor (resp. infinitely many) in all classes if for all* $a_0, \ldots, a_{\mathsf{f}(S)} \in K$ *with* $a_0 \neq 0$ *there exists at least one irreducible polynomial (resp. infinitely many) in* $K[X]$ *whose coefficient at the monomial* X^i *equals* a_i *for all* $i \in [0, \mathsf{f}(S)]$*.*

(2) *Let* $n > 1$ *and* $\widehat{S}$ *be factorial. Then* $K[S]$ *has infinitely many prime divisors in each divisor class.*

Factorizations in the weakly Krull affine case

Now we know something about weakly Krull monoid algebras and prime divisors in the affine case. So one can combine these results.

Most important ingredient:

- The factorization theory of Krull monoids boils down to the investigation of a combinatorial object ($\mathcal{B}(G_0)$) depending on the class group and the distribution of the prime divisors of the Krull monoid.
- For weakly Krull monoids there is a weak analogon to this with a more complicated combinatorial object and more involved control over the factorizations.

1. Theorem

Theorem (Chang, F., Windisch)

Let D be a weakly Krull UMT-domain with non-zero conductor $\mathfrak{f}_D = (D : \widehat{D}) \neq (0)$ and with infinite pseudo-Hilbertian quotient field K. Let $\Gamma \neq \mathbb{N}_0$ be a numerical monoid and suppose that $D[\Gamma]$ is v-noetherian (e.g. this is the case if D is noetherian). Then $\mathcal{L}(D[\Gamma])$ is full.

Corollary

Let $\Gamma \neq \mathbb{N}_0$ be a numerical monoid.

1. *If D is an order in an algebraic number field, then $\mathcal{L}(D[\Gamma])$ is full.*
2. *If D is a noetherian weakly Krull UMT-domain with non-zero conductor, then $\mathcal{L}(D[X][\Gamma])$ is full.*

2. Theorem

Theorem (Chang, F., Windisch)

Let K be a field and let Γ be a weakly Krull affine monoid not being a numerical monoid such that the root closure $\widetilde{\Gamma}$ is factorial.

1. *If $\mathcal{C}_v(K[\Gamma])$ is infinite, then $\mathcal{L}(K[\Gamma])$ is full.*
2. *If $\mathcal{C}_v(K[\Gamma])$ is finite, then $K[\Gamma]$ satisfies the Structure Theorem for Sets of Lengths.*
3. *If $\mathcal{C}_v(K[\Gamma])$ is finite and Γ is seminormal, then the set of distances $\Delta(K[\Gamma])$ and the unions $\mathcal{U}_k(K[\Gamma])$ are finite intervals for all $k \geq 2$.*

Publications

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Thank you for your attention.