On the Additive Monoid of Simple Extension Semirings

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Part II: Atomicity and Factorization Invariants

—*Timothy Chen*

Overview

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Theorem (Correa-Morris and Gotti, 2022)

For positive algebraic α, M_α is atomic if and only if not antimatter.

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Antimatter decompositions are important.

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  - etc.
- Formally, an antimatter decomposition is some polynomial $f(x) \in x\mathbb{N}_0[x] - 1$ that is a multiple of $m_\alpha(x)$.

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If $\alpha \in (0, 1)$ is algebraic, then M_α is antimatter if and only if α^{-1}

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- is at least each of its conjugates by norm.

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- $M_{\sqrt{2}/2} \cong (M_{1/2})^2$ is antimatter.
- Not a valuation monoid since $-\sqrt{2}/2$ has the same norm as $\sqrt{2}/2$.
- We simply drop the simplified requirement.

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  - Then, $h(x) - 1 \in x\mathbb{N}_0[x]$ is an antimatter decomposition.

Sketch of Sufficiency Proof (cont.)

- Let $w_\alpha(x)$ be the polynomial obtained by clearing the denominators of $m_\alpha(x)$ (the unique primitive multiple with positive leading coefficient). Let $r_\alpha(x)$ be the reciprocal polynomial of $-w_\alpha(x)$.

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- By a theorem of Handelman, $(x+1)^N r_\alpha(x)$ has a single sign change for sufficiently large N.
- Let $r(x)$ denote the resulting polynomial and $u(x)$ its negative reciprocal polynomial. We claim that we can construct a multiple of $u(x)$ that lies in $x\mathbb{N}_0[x] - 1$, thus an antimatter decomposition.

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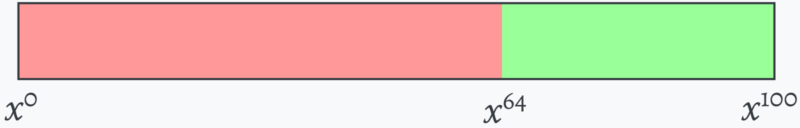

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- Set $d \coloneqq \deg m_\alpha(x)$ and $D \coloneqq \deg r(x) = \deg u(x) = d + N$.

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- Let the coefficients a_D and onward satisfy $\sum_{i=0}^{D} c_{D-i} a_{n-i} = 0$.
- Since $c_D = -1$, we have $a_n = \sum_{i=1}^{D} c_{D-i} a_{n-i} \in \mathbb{Z}$.

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Coefficients are getting bigger and bigger.

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- α^{-1} being an algebraic integer and having no positive conjugate aside from itself follow exactly as before our valuation proof.
- Previously, M_α had to be simplified or else it would be a nontrivial product of monoids, which is never a valuation monoid.

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- Hence, we can relax the strict inequality. ∎

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  - In fact, not M_q for any rational $0 < q < 1$.
- Example — $M_{2-\sqrt{2}}$, as $M_{2-\sqrt{2}} \cong M_{2+\sqrt{2}}$. Since $2 + \sqrt{2} > 1$, then $M_{2+\sqrt{2}}$ can be listed increasingly and must satisfy the ACCP.

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- Backward direction follows similarly to our proof for $M_{2-\sqrt{2}}$.

Proof Sketch of Forward Direction

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  - $M = N + 2d$, where $d = \deg m_\alpha(x)$.

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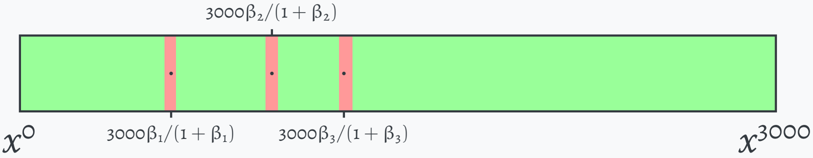

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- We then claim that we can express $(x+1)^N m_\alpha(x)^2$ as $(x^k-1)f(x)+g(x)$ for some $f(x), g(x) \in \mathbb{N}_0[x]$ and $k \in \mathbb{N}$.

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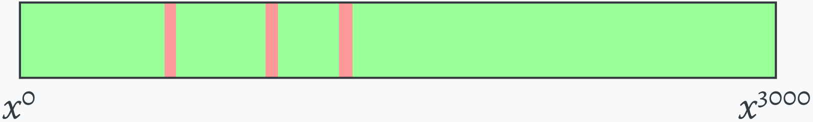
Let k slightly exceed the length of the smallest interval. Since the intervals are so small, k is still less than the distance between the two closest intervals.



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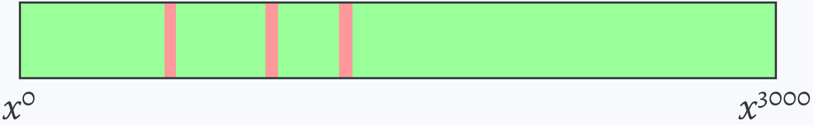

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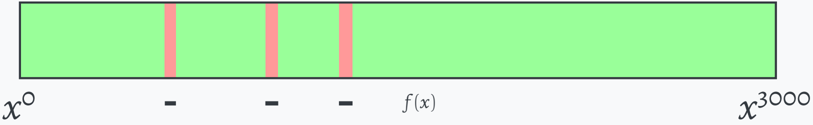

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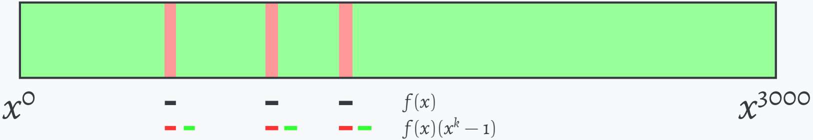

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- $(x+1)^2(3x-1)^2 - (x^2-1)f(x) = 7x^4 + 8x^3 + 1 = g(x) \in \mathbb{N}_0[x]$.

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In general, atomic $\supsetneq$ strongly atomic $\supsetneq$ almost ACCP $\supsetneq$ ACCP.

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    - CSDs greatly simplify working with Puiseux monoids.

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- But when q is not a unit fraction, M_q satisfies the almost ACCP.
- The proof involves a canonical sum decomposition (CSD).
  - A CSD is able to represent each element as the sum of generators uniquely.
  - This representation is typically constrained with bounds on coefficients.
  - CSDs greatly simplify working with Puiseux monoids.
  - Recently, CSDs have been discovered with certain classes of M_α.

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- An element satisfies the ACCP iff all its coefficients are less than 3.
- We can manipulate these CSDs via the identity $3q^k = 5q^{k+1}$ in order to find s and d and show that M_q satisfies the almost ACCP.

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More generally, if M_α satisfies the almost ACCP but not the ACCP itself, then $m_\alpha(x)$ is simplified. As before, this is due to the fact that the product of two almost ACCP monoids need not be almost ACCP.

Strongly Atomic and Atomic

It is unknown whether there are atomic additive monoids of simple semiring extensions that are not strongly atomic.

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  - Let $\mathcal{A}(M)$ denote the set of atoms of M.

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  - What about algebraic integers?
- We extend this result to a certain subclass of algebraic integers.

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Theorem (Chen, Gotti, Lu, and Yao, 2025)

If $\alpha > 1$ and M_α is almost antimatter, then M_α is densely elastic.

Almost Antimatter Proof Sketch

- It suffices to show that if L and ℓ are respectively the lengths of the longest and shortest factorizations of some $\beta \in M_\alpha$, then we can exhibit an element β' whose longest and shortest have lengths $L+1$ and $\ell+1$, respectively.

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- **Idea** — If $\beta' = \beta + \alpha^k$ for some large enough k, every new factorization of β' should be x^k plus a factorization of β.

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- It remains an open question to determine $\rho(M_\alpha)$ for all algebraic integers $\alpha > 1$, since it is not always ∞.
- Also, the structure of $R(M_\alpha)$ remains mostly unknown.

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- $\omega \colon \mathcal{A}(M_\alpha) \to \mathbb{N} \cup \{\infty\}$, with $\omega(a) = 1$ iff a is prime.
- Recent interest has seen studies on algorithms, closed-form formulae, and asymptotic results in various classes of monoids, most notably numerical, Krull, and block monoids.

ω-Primality Definition

- $\omega(a)$ is the smallest positive integer with the property that whenever a divides some sum $q_1 + q_2 + \cdots + q_r$ for $r > \omega(a)$, then there exists some subset T of $\{1, \ldots, r\}$ of size at most $\omega(a)$ such that a divides the subsum whose indices are elements of T.

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- One can show that $\omega(a)$ is the supremum of the lengths of a.

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ω-Primality Theorem

We determine the ω-primality for all atomic M_α, which resolves several conjectures left by Jiang et al. in their 2022 paper.

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- However, they left open the case when $\alpha > 1$.
- We split into two cases based on the number of atoms of M_α.

Infinitely Many Atoms

Suppose $|\mathcal{A}(M_\alpha)| = \infty$.

- Interestingly enough, this proof involves M_β, where $\beta = \alpha^{-1}$.

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  - This is equivalent to $m_\beta(x)$ having no multiple in $x\mathbb{N}_0[x] - 1$, which naturally leads to M_β being atomic. ∎

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- Multiplying any of these bullets by α^N for sufficiently large N will bring them into M_α, which establishes that $\omega(M_\alpha) = \infty$ because bullets are preserved under this multiplication.
- It is not hard to show that multiplication preserves the property of being a bullet (so long as we also shift the atom).

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- Jiang et al. conjectured that $\omega(M_\alpha) < \infty$ iff $\alpha \in \mathbb{N}$.
- Instead if M_α has finitely many atoms, then $\omega(M_\alpha)$ is finite. Remarkably, this holds in the general case.

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- Consider the infinite grid $\mathbb{N}_0^N$ consisting of the formal sums of other atoms. Let S denote the subset of the grid that a divides.
- The bullets of a correspond to the minimal elements of S under the pointwise partial order given by $(m_1, m_2, \ldots, m_N) \leqslant (n_1, n_2, \ldots, n_N)$ if and only if $m_i \leqslant n_i$ for every i.

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