Arithmetics of Flatness for Monoids

Gerhard Angermüller

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Flatness for Monoids

Acts

In this talk, a *monoid* H is a multiplicatively written commutative and cancellative semigroup with unit element 1.

- A non-empty set A is called an *H-act*, if there is a map $H \times A \to A$, $(s, a) \mapsto sa$ such that $1a = a$ and $(st)a = s(ta)$ for all $s, t \in H$ and $a \in A$.
- A map $\varphi : A \to B$ with H-acts A, B is a *morphism* of H-acts, if $\varphi(sa) = s\varphi(a)$ for all $s \in H$ and $a \in A$.

Let R be a domain and M an R-module.
Then $R^\bullet := R \setminus \{0\}$ is a monoid and M is an $R^\bullet$-act.
If M is torsion-free and $\neq \{0\}$, then $M^\bullet := M \setminus \{0\}$ is an $R^\bullet$-act too.

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Flatness for Monoids

Tensor Product

Let A, B be H-acts.

- A map $\rho : A \times B \to X$ to a set X is called *H-balanced*, if $\rho(sa, b) = \rho(a, sb)$ for all $s \in H$, $a \in A$ and $b \in B$.
- An H-act T together with an H-balanced map $\tau : A \times B \to T$ is called (the) *tensor product of A and B (over H)* if for every set X every H-balanced map $\rho : A \times B \to X$ factors uniquely through τ; it is denoted by $A \otimes B$.

FACTS:

- The tensor product of A and B exists and is unique up to isomorphism.
- For all $a, a' \in A$, $b, b' \in B$: $a \otimes b = a' \otimes b'$ if and only if there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ and $s_1, \ldots, s_{n+1}, t_1, \ldots, t_n \in H$ such that $a = s_1 a_1$, $s_1 b = t_1 b_1$, $t_i a_i = s_{i+1} a_{i+1}$, $s_{i+1} b_i = t_{i+1} b_{i+1}$ for $i = 1, \ldots, n-1$, and $t_n a_n = s_{n+1} a'$, $s_{n+1} b_n = b'$.

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Flatness for Monoids

Definition

Any H-act A defines a covariant functor $A \otimes -$ from the category of H-acts to the category of sets; A is called

- *flat* if $A \otimes -$ preserves monomorphisms,
- *weakly flat* if $A \otimes -$ preserves all embeddings of ideals into H,
- *principally weakly flat* if $A \otimes -$ preserves all embeddings of principal ideals into H.

Further, A is said to be *torsion-free* if for all $s \in H$ and $a, b \in A$ the equality $sa = sb$ implies $a = b$.

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Theorem

Let H be a monoid and A an H-Act.
Then the following conditions are equivalent:
(1) A is flat,
(2) A is weakly flat,
(3) A is principally weakly flat and for all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u \in Hs \cap Ht$ such that $sa = tb = uc$,
(4) A is torsion-free and for all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u \in Hs \cap Ht$ such that $sa = tb = uc$,
(5) A is torsion-free and for all ideals I and J of H: $(I \cap J)A = IA \cap JA$.
(6) For all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u, v \in H$ such that $a = uc$, $b = vc$ and $us = vt$.

Flatness for Monoids

Properties

Corollary

Let T be a submonoid of a monoid H. Then $T^{-1}H$ is a flat H-act.

Corollary

Let $\varphi : H \to D$ be a morphism of monoids making D a flat H-act. Then for all $u, v \in H$ such that $\varphi(u)|_D \varphi(v)$ there are $w \in \varphi^{-1}(D^\times)$ such that $u|_H vw$.

In particular, if $q(\varphi) : q(H) \to q(D)$ denotes the canonical morphism induced in the quotient monoids, then $q(\varphi)^{-1}(D) = \varphi^{-1}(D^\times)^{-1}H$.

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Corollary

Let H, D be monoids such that $H \subseteq D \subseteq q(H)$. The following conditions are equivalent:
(1) *D is a flat H-act,*
(2) $D = (H \cap D^{\times})^{-1}H$,
(3) *for every $x \in D$ there are $u \in H \cap D^{\times}$ such that $ux \in H$,*
(4) $(H :_H x)D = D$ *for every $x \in D$.*

Corollary

For every monoid H the following conditions are equivalent:
(1) *H is a valuation monoid,*
(2) *every torsion free H-act is flat,*
(3) *every overmonoid of H is a flat H-act.*

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An H-act A is called *strongly faithful* if for all $s, t \in H$ the equality $sa = ta$ for some $a \in A$ implies $s = t$.

Let R be a domain and M a torsion-free R-module $\neq \{0\}$.
Then $M^\bullet$ is a strongly faithful torsion-free $R^\bullet$-act.

Corollary

Let H be a monoid and A a strongly faithful H-Act. Then the following conditions are equivalent:
(1) A is flat,
(2) for all $a, b \in A$ and $s, t \in H$ such that $sa = tb$ there exist $c \in A$ and $u, v \in H$ such that $a = uc$ and $b = vc$.

An H-act A is called *locally cyclic* if for any $a, b \in A$ there is some $c \in A$ such that $Ha, Hb \subseteq Hc$.

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Corollary

Let H be a monoid and A a strongly faithful torsion-free H-Act. Then the following conditions are equivalent:
(1) *A is flat,*
(2) *$q(H)x \cap A$ is locally cyclic for each $x \in A$.*

Corollary

Let H, D be monoids such that $H \subseteq D \subseteq q(H)$. Then the following conditions are equivalent:
(1) *D is a flat H-act,*
(2) *$D = T^{-1}H$ for some submonoid T of H,*
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Let R be a Dedekind domain whose class group is not a torsion group. Then there is a flat overdomain S of R, such that $S \neq T^{-1}R$ for every multiplicatively closed subset T of R, i.e. S is a flat R-module, but $S^{\bullet}$ is not a flat $R^{\bullet}$-act.

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Flatness for Monoids

Factorable Acts

Let H be a monoid, A be an H-act and $a \in A$..

- An element $h \in H$ (resp. $b \in A$) is called an *H-divisor* of a (resp. an *A-divisor of a*), if $a = hc$ for some $c \in A$ (resp. $a = sb$ for some $s \in H$); h (resp. b) is called a *greatest H-divisor* (resp. a *smallest A-divisor*) of a, if any H-divisor of a divides h (resp. b is an A-divisor of any A-divisor of a).
- a is called *irreducible* if any H-divisor of a is a unit of H.
- a is called *primitive* if a is a smallest A-divisor of any element of Ha.
- A is called *atomic*, if any element of A has an irreducible A-divisor.
- A is called *factorable*, if any element of A has a smallest A-divisor.

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- An element $h \in H$ (resp. $b \in A$) is called an *H-divisor* of a (resp. an *A-divisor of a*), if $a = hc$ for some $c \in A$ (resp. $a = sb$ for some $s \in H$); h (resp. b) is called a *greatest H-divisor* (resp. a *smallest A-divisor*) of a, if any H-divisor of a divides h (resp. b is an A-divisor of any A-divisor of a).

- a is called *irreducible* if any H-divisor of a is a unit of H.

- a is called *primitive* if a is a smallest A-divisor of any element of Ha.

- A is called *atomic*, if any element of A has an irreducible A-divisor.

- A is called *factorable*, if any element of A has a smallest A-divisor.

Flatness for Monoids

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Flatness for Monoids

Factorable Acts

FACTS: Let H be a monoid, A be an H-act and $a \in A$.

- Any greatest H-divisor of a is unique up to units of H.
- Any smallest A-divisor of a is unique up to units of H, if A is strongly faithful.

Theorem

Let H be a monoid and A be a strongly faithful torsion-free H-act. The following conditions are equivalent:
(1) A is factorable,
(2) every element of A has a greatest H-divisor,
(3) every $a \in A$ has a representation $a = hb$ with $h \in H$, b an irreducible element of A and this representation is unique up to a unit of H,
(4) A is atomic and every irreducible element of A is primitive,
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Corollary

Let H be a monoid.
H is a GCD-monoid if and only if $H \times H$ is a factorable H-act.

Theorem

Let H be a monoid and A be a strongly faithful torsion-free H-act. The following conditions are equivalent:
(1) *A is factorable,*
(2) *A is flat and atomic,*
(3) *$q(H)x \cap A$ is cyclic for every $x \in A$.*

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Flatness for Monoids

Domains

Let R be a domain and M a torsion-free R-module $\neq \{0\}$. Then:
M is a factorable R-module if and only if $M^\bullet$ is a factorable $R^\bullet$-act.
M is an atomic R-module if and only if $M^\bullet$ is an atomic $R^\bullet$-act.

Corollary

Let R be a domain with quotient field K and M a torsion-free R-module $\neq \{0\}$. The following conditions are equivalent:
(1) M is a factorable R-module,
(2) $M^\bullet$ is a flat $R^\bullet$-act and M is an atomic R-module,
(3) $Kx \cap M$ is a cyclic R-module for every $x \in M$.

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Flatness for Monoids

Pre-Schreier Acts

Let H be a monoid and A be a strongly faithful torsion-free H-act.

- A is a *pre-Schreier H-act* if for every equality $ua = vb$ with $u, v \in H$ and $a, b \in A$ there are $r, s, t \in H$ and $c \in A$ such that $u = rt$, $v = rs$, $a = sc$ and $b = tc$; (r, s, t, c) is called a *refinement of $ua = vb$*.
- H is a *pre-Schreier monoid* if H is a pre-Schreier H-act.

Example

Every GCD-monoid is pre-Schreier.

Theorem

Let H be a monoid and A be a strongly faithful torsion-free H-act.
a) If A is pre-Schreier, then A is flat.
b) If H is pre-Schreier and A is flat, then A is pre-Schreier.

Corollary

Let H be a pre-Schreier monoid and A be an H-act.
A is pre-Schreier if and only if A is strongly faithful and flat.

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Flatness for Monoids

Pre-Schreier Acts

Theorem

Let H be an atomic monoid and T be the submonoid of H generated by the nonprime atoms of H.
a) $T^{-1}H$ is a factorial monoid.
b) Every pre-Schreier H-act is a pre-Schreier $T^{-1}H$-act.
c) Every pre-Schreier $T^{-1}H$-act is a pre-Schreier H-act.
d) The pre-Schreier H-acts are exactly the strongly faithful flat $T^{-1}H$-acts.

Theorem

Let H be a monoid and I a finite set.
H is pre-Schreier if and only if H^I is pre-Schreier.

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Corollary

Let R be a domain and M a torsion-free R-module $\neq \{0\}$.
a) If M is a pre-Schreier R-module, then $M^\bullet$ is a flat $R^\bullet$-act.
b) If R is a pre-Schreier domain and $M^\bullet$ is a flat $R^\bullet$-act, then M is a pre-Schreier R-module.

There are examples of factorial domains R and pre-Schreier R-modules M which are not flat; by a), $M^\bullet$ is a flat $R^\bullet$-act.
If R is a pre-Schreier domain and M a flat R-module, then M is a pre-Schreier R-module, i.e. $M^\bullet$ is a flat $R^\bullet$-act.

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Let R be a domain and M a torsion-free R-module $\neq \{0\}$.
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Thank you.