CHARACTERIZING VARIOUS CLASSES OF FINITE GROUPS USING SOME INVARIANTS

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JUNE 12, 2025.

Abstract

A few invariants (such as the sum and inverse of element orders, the number of Sylow subgroups, the sum and average of Sylow numbers, etc) have been used to characterise various classes of finite groups. In this talk, we shall discuss some of our results within this area. (This is joint work with Alireza Asboei)

Outline of this presentation

1. Introduction
   - Disproving a 2014 theorem of Tărnăuceanu on characterizing C_2^n
   - Disproving a 1974 conjecture of Street & Whitehead on dihed.
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Disproving a 2014 theorem of Tărnăuceanu

Tărnăuceanu [Archiv der Mathematik 102(1) (2014), 11–14] gave the following theorem: "a finite group G is an elementary abelian 2-group if and only if the set of maximal sum-free sets coincides with the set of complements of the maximal subgroups". The result is wrong.

- If we take 'maximal' in the theorem to mean 'maximal by cardinality', then a counterexample is the cyclic group C_4 of order 4, given by $C_4 = \langle x \mid x^4 = 1\rangle$.
  - Here, there is a unique maximal (by cardinality) sum-free set namely $\{x, x^3\}$, and it is the complement of the unique maximal subgroup. But C_4 is not elementary abelian.
- On the other hand, if we take 'maximal' to mean 'maximal by inclusion', then the theorem will still be wrong since $S = \{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\}$ is a maximal by inclusion sum-free set in $C_2^4 = \langle x_1, x_2, x_3, x_4 \mid x_i^2 = 1, x_ix_j = x_jx_i \text{ for } 1 \leq i, j \leq 4\rangle$, but does not coincide with any complement of a maximal subgroup of C_2^4.

- This theorem of Tărnăuceanu was disproved by A' in late 2016.
- In 2017, Tărnăuceanu's erratum was published.

On filled groups

- We say a product-free subset S of a group G *fills* G if $G^* \subseteq S \sqcup SS$ (where $G^* = G \setminus \{1\}$), and G is called a *filled group* if every locally maximal product-free set in G fills G.
- In 1974, Street and Whitehead investigated filled groups and gave a classification of finite abelian filled groups.
- The finite abelian filled groups are C_3, C_5 and the elementary abelian 2-groups of finite rank n for $n \geq 1$.
- Street and Whitehead verified that all finite dihedral groups of order up to 12 are filled. They asserted that the dihedral group of order $2n$ is not a filled group for $n = 6k + 1$ $(k \geq 1)$, and went further to give the following set

$$S := \{x^{2k+1}, \ldots, x^{4k}, x^{2k+1}y, \ldots, x^{4k}y\}$$

which they believe is a locally maximal product-free set that does not fill D_{2n}.

On filled groups cont'd.

Observation

Let G be a dihedral group of order $2n$ for $n = 6k + 1$ and $k \geq 1$. Then the set

$$S := \{x^{2k+1}, \ldots, x^{4k}, x^{2k+1}y, \ldots, x^{4k}y\}$$

is product-free but not locally maximal in G. In particular,

$$V := \{x^{2k+1}, \ldots, x^{4k}, x^{2k}y, x^{2k+1}y, \ldots, x^{4k}y\},$$

which properly contains S, is product-free in G.

It turns out that D_{14} is a filled group.

- Disproving this conjecture paved way for more classification of filled groups.
- For instance, filled groups of odd orders and filled nilpotent groups are known.

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Finiteness: Nilpotent, Supersolvable and Solvable groups

- Theorem: Every finite nilpotent group is the direct product of its Sylow p-groups.
- We say that G is **supersolvable** if it has a series
$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{k-1} \lhd H_k = G$$
such that $H_j \lhd G$ and H_{j+1}/H_j is cyclic for $j = 0, 1, \ldots, k-1$.
- Thm (Huppert): A finite group is supersolvable IFF all of its maximal subgroups have prime index.
- We say that G is **solvable** if it has a series
$$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{k-1} \lhd H_k = G$$
such that $H_j \lhd G$ and H_{j+1}/H_j is abelian for $j = 0, 1, \ldots, k-1$.
- **Examples and Non-Examples**
  - *Nilpotent groups:* Abelian groups and p-groups.
  - *Non–nilpotent groups:* $D_6, D_{10}, D_{12}, D_{14}, A_n, S_n$ for $n \geq 4$.
  - *Supersolvable groups:* nilpotent groups, D_{2n} for $n \geq 3$.
  - Non–supersolvable groups: A_n, S_n for $n \geq 4$.
  - *Solvable groups:* supersolvable groups, A_4, S_4, groups of odd order (Feit–Thompson thm), groups of order $2^n p^m$ (p prime and $m, n \in \mathbb{N}$).
  - *Non–solvable groups:* A_n, S_n for $n \geq 5$

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On the sum of elements and inverse element orders

(A) On the sum of element orders of G and average order of G, we have:

$$\eta(G) := \sum_{g \in G} \circ(g) \text{ and } \Lambda(G) := \frac{\eta(G)}{|G|}.$$

- In 2021, Khukhro, Moretó and Zarrin conjectured that if a finite group G satisfies $\Lambda(G) < \frac{211}{60} = \Lambda(A_5)$, then G is solvable.
- This conjecture was proved by Herzog, Longobardi and Maj in 2022.
- Herzog, Longobardi and Maj also proved that if G is a finite nonsolvable group, then $\Lambda(G) = \frac{211}{60}$ if and only if $G \cong A_5$.
- In 2022, Tărnăuceanu proved that for a finite group G, if $\Lambda(G) < \frac{31}{12}$, then G is supersolvable. Moreover, $\Lambda(G) = \frac{31}{12}$ if and only if $G \cong A_4$.

(B) On the sum of the inverses of the element orders, we have

$$\beta(G) := \sum_{g \in G} \frac{1}{\circ(g)}.$$

Theorem (Azad et al. 2023)

Let G be a finite group. (a) If $\beta(G) < \frac{599}{30} = \beta(A_5)$, then G is solvable. (b) If $\beta(G) < \frac{31}{6} = \beta(A_4)$, then G is supersolvable.

Some invariants used to characterize finite groups

- The sum of element orders in a finite group
- The sum of inverse of element orders in a finite group
- The sum/average of Sylow numbers of a finite group
- The number of Sylow subgroups of a finite group

We say a little about the last part here.

In 2020, Robati conjectured that "if G is a finite group and $n_p(G) \leq p^2 - p + 1$ for each odd prime p, then G is solvable".

Later that year, we proved a stronger version of Robati's conjecture.

Theorem (A, Moretó, Zarrin 2020)

Let G be a group. If $n_3(G) \leq 7$ and $n_5(G) \leq 1456$, then G is solvable.

Definition

For distinct primes p and q (with $p < q$), we say finite groups are '(p, q)–**recognizable**' if there exist two natural numbers a and b such that the following two conditions are satisfied:
(i) if G is a finite group such that $n_p(G) < a$ and $n_q(G) < b$, then G is solvable;
(ii) there are nonsolvable groups K and L such that $n_p(K) = a$ and $n_q(L) = b$.

Theorem (A, Asboei 2025)

If finite groups are (p, q)–recognizable, then $(p, q) \in \{(3, 5)\} \cup \{(2, q) : q \geq 3\}$.

Order polynomials and some properties

Definition

Let G be a finite group. We denote the *Order polynomial* of G by $\mathcal{P}(G, x)$, and define it as

$$\mathcal{P}(G, x) := \sum_{g \in G} x^{\circ(g)}. \tag{1}$$

- It is clear that $\mathcal{P}(G, 1) = |G|$.
- Differentiating (1) and substituting 1 for x, we get $\mathcal{P}'(G, 1) = \sum_{g \in G} \circ(g)$.
- Let G_1 and G_2 be two finite groups. Suppose $\pi_e(G_1) = \{a_0, a_1, ..., a_k\}$ and $\pi_e(G_2) = \{b_0, b_1, ..., b_l\}$, where $a_0 = 1 = b_0$. Then

$$\mathcal{P}(G_1, x) = m_{a_0} x^{a_0} + m_{a_1} x^{a_1} + ... + m_{a_k} x^{a_k} = \sum_{i=0}^{k} m_{a_i} x^{a_i},$$

where m_{a_i} is the number of elements of order a_i in G_1. Similarly,

$$\mathcal{P}(G_2, x) = m_{b_0} x^{b_0} + m_{b_1} x^{b_1} + ... + m_{b_l} x^{b_l} = \sum_{j=0}^{l} m_{b_j} x^{b_j},$$

where m_{b_j} is the number of elements of order b_j in G_2.

Multiplication

We define $\mathcal{P}(G_1, x) \times \mathcal{P}(G_2, x)$ as follows:

$$\mathcal{P}(G_1, x) \times \mathcal{P}(G_2, x) = \sum_{i=0}^{k} \sum_{j=0}^{l} m_{a_i} m_{b_j} x^{[a_i, b_j]},$$

where $[a_i, b_j]$ is the least common multiple of a_i and b_j.

Example

We know that $\pi_e(C_6) = \{1, 2, 3, 6\}$, with $m_1 = 1, m_2 = 1, m_3 = 2$ and $m_6 = 2$. One can easily use GAP[35] to obtain that $\pi_e(A_5) = \{1, 2, 3, 5\}$, with $m_1 = 1, m_2 = 15, m_3 = 20$ and $m_5 = 24$. Therefore, $\mathcal{P}(A_5, x) \times \mathcal{P}(C_6, x) = x + 15x^2 + 20x^3 + 24x^5 + x^2 + 15x^2 + 20x^6 + 24x^{10} + 2x^3 + 30x^6 + 40x^3 + 48x^{15} + 2x^6 + 30x^6 + 40x^6 + 48x^{30} = x + 31x^2 + 62x^3 + 24x^5 + 122x^6 + 24x^{10} + 48x^{15} + 48x^{30}$.

An example explaining the order polynomials of the two groups of size 6

- $C_6 = \langle a \mid a^6 = 1 \rangle = \{1, a, a^2, a^3, a^4, a^5\}$.
  - $\pi_e(C_6) = \{1, 2, 3, 6\}$, with $m_1 = 1, m_2 = 1, m_3 = 2$ and $m_6 = 2$.
  - $\mathcal{P}(C_6, x) = x + x^2 + 2x^3 + 2x^6$.
- $D_6 = \langle a, b \mid a^3 = 1 = b^2, ab = ba^2 \rangle = \{1, a, a^2, b, ab, ab^2\}$.
  - $\pi_e(D_6) = \{1, 2, 3\}$, with $m_1 = 1,\ m_2 = 3$ and $m_3 = 2$.
  - $\mathcal{P}(D_6, x) = x + 3x^2 + 2x^3$.
- Let us look at $\mathcal{P}(C_2, x) \times \mathcal{P}(C_3, x)$.
  - $\pi_e(C_2) = \{1, 2\}$, with $m_1 = 1$ and $m_2 = 1$. So $\mathcal{P}(C_2, x) = x + x^2$.
  - $\pi_e(C_3) = \{1, 3\}$, with $m_1 = 1$ and $m_3 = 2$. So $\mathcal{P}(C_3, x) = x + 2x^3$.
  - Now, $\mathcal{P}(C_2, x) \times \mathcal{P}(C_3, x) = x + 2x^3 + x^2 + 2x^6$.
- It is clear that $\mathcal{P}(C_6, x) = \mathcal{P}(C_2, x) \times \mathcal{P}(C_3, x) \neq \mathcal{P}(D_6, x)$.

Thompson's problem as the motivation for introducing order polynomials

- All groups here are finite. Let G be a finite group.
- The set of prime divisors of $|G|$ is denoted by $\pi(G)$.
- The set of all element orders of G is denoted by $\pi_e(G)$.
- Given $i \in \pi_e(G)$, we write $m_i(G) := |\{g \in G : \circ(g) = i\}|$.

• Two groups G_1 and G_2 are said to be of the *same order type* IFF $\pi_e(G_1) = \pi_e(G_2)$ and $m_t(G_1) = m_t(G_2)$ for all $t \in \pi_e(G_1)$.

• G_1 and G_2 being of the same order type $\Rightarrow |G_1| = |G_2|$.

• In 2009, Vasil'ev et al. proved that if G is any finite group and S is a FSG such that $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$, then $G \cong S$.

• The well–known 1987 Thompson's problem (given as Problem 12.37 of the Kourovka Notebook) asks whether for two finite groups G_1 and G_2 of the same order type, is G_2 necessarily solvable if G_1 is solvable?

What we did for the order polynomials

- Let Property X be a certain property of some finite groups; for instance, nilpotent, supersolvable, solvable et cetera.
- The Thompson-like problem asks whether for two finite groups G_1 and G_2 of the same order type, does G_2 always satisfy Property X if G_1 satisfies Property X?
- We introduced a new kind of polynomials, which we called 'the order polynomials' and used it to propose a way of solving the Thompson-like problem when Property X is nilpotent.
- Furthermore, we showed that the answer to Thompson-like problem is in the negative when Property X is supersolvable.
- Piwek (2024) answered the Thompson-like problem when Property X is solvable. Also answered for simple groups (A' 2019).
- $\mathcal{P}^{'}(G,1) = \sum\limits_{g \in G} \circ(g)$ is well-studied by Amiri et al. (from 2008).
- We worked on the definite integral part.

Decomposable

Definition

For a finite group G, we say $\mathcal{P}(G,x)$ is *decomposable* if there exist groups A and B such that $\mathcal{P}(G,x) = \mathcal{P}(A,x) \times \mathcal{P}(B,x)$. If $\mathcal{P}(G,x)$ is not decomposable, then $\mathcal{P}(G,x)$ is *indecomposable*.

Example

There are subgroups H_1 and H_2 of $A_5 \times C_6$ such that $H_1 \cong A_5$ and $H_2 \cong C_6$. From the earlier example, we know that $\mathcal{P}(A_5,x) = x + 15x^2 + 20x^3 + 24x^5$ and $\mathcal{P}(C_6,x) = x + x^2 + 2x^3 + 2x^6$. Using Magma[17] or GAP[35], we see that $\pi_e(A_5 \times C_6) = \{1,2,3,5,6,10,15,30\}$, with $m_1 = 1, m_2 = 31, m_3 = 62$, $m_5 = 24$, $m_6 = 122$, $m_{10} = 24$, $m_{15} = 48$ and $m_{30} = 48$. So, $\mathcal{P}(A_5 \times C_6, x) = x + 31x^2 + 62x^3 + 24x^5 + 122x^6 + 24x^{10} + 48x^{15} + 48x^{30}$. Since $\mathcal{P}(A_5 \times C_6, x)$ gotten here is the same as $\mathcal{P}(A_5,x) \times \mathcal{P}(C_6,x)$ gotten from the earlier example, we conclude that $\mathcal{P}(A_5 \times C_6, x)$ is decomposable.

A proposition

Proposition

Let G be a finite group and $G \neq 1$. If $G = A \times B$, then $\mathcal{P}(G,x) = \mathcal{P}(A,x) \times \mathcal{P}(B,x)$.

Proof.

Let $\pi_e(A) = \{a_0 = 1, a_1, \ldots, a_n\}$, $\pi_e(B) = \{b_0 = 1, b_1, \ldots, b_r\}$. Then

$$\pi_e(A \times B) = \{1, a_1, \ldots, a_n, b_1, \ldots, b_r, [a_1, b_1], \ldots, [a_1, b_r], \ldots, [a_n, b_1], \ldots, [a_n, b_r]\}.$$

The rest of the proof follows from $m_{c_k} x^{c_k} = \sum_{[a_i, b_j] = c_k} m_{a_i} x^{a_i} m_{b_j} x^{b_j}$. □

Remark. The converse of the above proposition is not necessarily true.

- Let $G := \langle a, b \mid a^4 = b^2 = (aba)^2 = (ba^{-1})^4 = (baba^{-1})^2 = 1\rangle \cong (C_4 \times C_2) \rtimes C_2 \cong$ SmallGroup(16, 3). Here, $\pi_e(G) = \{1, 2, 4\}$, with $m_1 = 1$, $m_2 = 7$ and $m_4 = 8$.
- Let $H := \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb\rangle \cong (C_4 \times C_2) \times C_2 \cong$ SmallGroup(16, 10). Here, $\pi_e(H) = \{1, 2, 4\}$, with $m_1 = 1$, $m_2 = 7$ and $m_4 = 8$.
- By the proposition, we conclude that $\mathcal{P}(H, x) = \mathcal{P}(C_4 \times C_2, x) \times \mathcal{P}(C_2, x) = \mathcal{P}(G, x)$.
- But $G \not\cong (C_4 \times C_2) \times C_2$.

Connection with earlier work

Observe that

$$\int_0^1 \mathcal{P}(G,x)\ dx = \sum_{g\in G} \frac{1}{\circ(g)+1}$$

and

$$\int_0^1 \frac{1}{x}\mathcal{P}(G,x)\ dx = \sum_{g\in G} \frac{1}{\circ(g)}.$$

The latter " $\sum_{g\in G} \frac{1}{\circ(g)}$ " has been studied by some authors; for instance, see the 2023 paper of Azad, Khosravi and Rashidi entitled **"On the sum of the inverses of the element orders in finite groups": Communications in Algebra 51(2)(2023), 694–698.**

Nilpotency on Thompson-like problem via Order polynomials

Theorem

Let G be a nilpotent group of order $n = p_1^{k_1} \cdots p_t^{k_t}$, where each p_i is a prime number for $i \in \{1, \ldots, t\}$. Then $\mathcal{P}(G, x) = \mathcal{P}(\mathrm{Syl}_{p_1}(G), x) \times \cdots \times \mathcal{P}(\mathrm{Syl}_{p_t}(G), x)$.

If G is a nilpotent group which is not a p–group, then $\mathcal{P}(G, x)$ is decomposable.

Theorem

Let G and H be two groups having the same order type. If G is a nilpotent group, then H is nilpotent too.

Supersolvability counterexamples on Thompson-like problem

• One cannot answer Thompson's question affirmatively if asked specifically for just two supersolvable groups.

• Let

$$G_1 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1, (ab)^4 = (bc)^2 = (bd)^2 = 1, ac = ca, ad = da, dc = cd \rangle$$

and

$$G_2 = \langle x, y, z \mid x^2 = y^2 = z^3 = 1, (xy)^4 = (xz)^2 = (xyz^2)^4 = 1, yxyzyxy = z, (yz^2)^2(yz)^2 = 1 \rangle .$$

• Note that $G_1 \cong$ SmallGroup(72,35) and $G_2 \cong$ SmallGroup(72,40).

• One can use GAP[35] to see that G_1 and G_2 have the same order type; in particular, each has twenty-one involutions, eight elements of order 3, eighteen elements of order 4 and twenty-four elements of order 6. But G_1 is supersolvable while G_2 is not supersolvable.

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Some results on an associated invariant via order polynomials

$$\beta(G) := \int_0^1 \frac{1}{x}\mathcal{P}(G,x)\, dx = \sum_{g\in G} \frac{1}{\circ(g)}.$$

Theorem (Azad et al. 2023)

Let G be a finite group. If $\beta(G) < \frac{599}{30}$, then G is solvable.

Theorem (A', Asboei 2025)

Let G be a finite nonabelian simple group. Then $\beta(G) = \frac{599}{30}$ if and only if $G \cong A_5$.

Theorem (A', Asboei 2025)

Let G be a finite nonsolvable group. Then $\beta(G) = \frac{599}{30}$ IFF $G \cong A_5$.

Theorem (A', Asboei 2025)

If G is a finite solvable group, then $\beta(G) \neq \frac{599}{30}$.

Definitions

We write $\mathcal{SP}(G,x)$ for the *Sylow polynomial* of G, and define it as follows:

$$\mathcal{SP}(G,x) := \sum_{P \in \mathrm{Sylow}(G)} x^{|P|}.$$

Let $\pi(G) = \{p_1, p_2, ..., p_n\}$ such that $p_1 < p_2 < ... < p_n$. Then

$$\mathcal{SP}(G,x) = n_{p_1}(G)x^{|\mathrm{Syl}_{p_1}(G)|} + n_{p_2}(G)x^{|\mathrm{Syl}_{p_2}(G)|} + \cdots + n_{p_n}(G)x^{|\mathrm{Syl}_{p_n}(G)|}. \quad (2)$$

Note the following:

- $\mathcal{SP}(G,1) = n_{p_1}(G) + n_{p_2}(G) + \cdots + n_{p_n}(G) = \sum_{p \in \pi(G)} n_p(G) = \delta(G)$;
- $\left[\frac{d}{dx}\mathcal{SP}(G,x)\right]\Big|_{x=1} = \mathcal{SP}'(G,1) = \sum_{p \in \pi(G)} n_p(G)|\mathrm{Syl}_p(G)|$;
- $\left[\frac{d}{dx}\mathcal{SP}'(G,x)\right]\Big|_{x=1} = \mathcal{SP}''(G,1) = \sum_{p \in \pi(G)} n_p(G)|\mathrm{Syl}_p(G)|\left(|\mathrm{Syl}_p(G)|-1\right)$.

The first two invariants here are $\mathcal{SP}'(G,1)$ and $\mathcal{SP}''(G,1)$, which we denote by $\upsilon(G)$ and $\mu(G)$ respectively.

Recall: Some preliminary results

Theorem (A, Moreto, Zarrin 2020)

Let G be a finite group.
If $n_3(G) \leq 7$ and $n_5(G) \leq 1456$, then G is solvable.

Lemma

Let G be a group and N be a normal subgroup of G. Then $n_p(N)n_p(G/N)$ divides $n_p(G)$ for every prime p.

A non–cyclic finite simple group is called a K_n–group if its order is divisible by exactly n different prime numbers. There is no K_n–group for $n \leq 2$. Result for $n = 3$ and $n = 4$ are given in the next two Lemmas.

Lemma (Herzog 1968)

If G is a K_3–group, then G is isomorphic to one of the following eight groups: A_5, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,2)$ *and* $\mathrm{PSL}(2,17)$.

A classification of K_4–groups

Lemma (Shi 1991)

Let G be a K_4–group. Then G is isomorphic to one of the following groups: (1) A_7, A_8, A_9, A_{10}, $\mathrm{PSL}(2,11)$, M_{11}, M_{12}, J_2, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,25)$, $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(2,81)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(3,5)$, $\mathrm{PSL}(3,7)$, $\mathrm{PSL}(3,8)$, $\mathrm{PSL}(3,17)$, $\mathrm{PSL}(4,3)$, $S_4(4)$, $S_4(5)$, $\mathrm{S}_4(7)$, $S_4(9)$, $\mathrm{S}_6(2)$, $O_8^+(2)$, $G_2(3)$, $\mathrm{PSU}(3,4)$, $\mathrm{PSU}(3,5)$, $\mathrm{PSU}(3,7)$, $\mathrm{PSU}(3,8)$, $\mathrm{PSU}(3,9)$, $\mathrm{PSU}(4,3)$, $\mathrm{PSU}(5,2)$, $S_z(8)$, $S_z(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $\mathrm{PSL}(2,\mathrm{r})$, *where r is an odd prime, $17 \neq r \geq 11$, $r^2-1 = 2^a3^bv^c$, $a,b,c \geq 1$, and a prime $v > 3$;*

(3) $\mathrm{PSL}(2,2^m)$, *where $m \geq 5$ satisfies*

$$u = 2^m - 1 \text{ and } t^b = (2^m+1)/3, \text{ with } u \text{ and } t \text{ primes, } t > 3 \text{ and } b \geq 1;$$

(4) $\mathrm{PSL}(2,3^n)$, *where $n \geq 3$ satisfies*

$$t^b = (3^n+1)/4 \text{ and } u = (3^n-1)/2, \text{ with } u \text{ and } t \text{ odd primes and } b \geq 1$$

or

$$t = (3^n+1)/4 \text{ and } u^c = (3^n-1)/2, \text{ with } u \text{ and } t \text{ odd primes and } c \geq 1.$$

A characterization of A_5 using the first invariant

Recall: $\mathcal{SP}'(G,1) = \sum_{p\in\pi(G)} n_p(G)|\mathrm{Syl}_p(G)| = \upsilon(G)$.

- It is an easy exercise to show that if G is a finite group such that $\upsilon(G) < 80$, then G is solvable.
- The converse of this result is not necessarily true. The dihedral group of size 54 is an example of a solvable group which does not satisfy $\upsilon(G) < 80$, since $n_2(D_{54}) = 27$, $|\mathrm{Syl}_2(D_{54})| = 2$, $n_3(D_{54}) = 1$ and $|\mathrm{Syl}_3(D_{54})| = 27$; so $\upsilon(D_{54}) = \sum_{p\in\{2,3\}} n_p(D_{54})|\mathrm{Syl}_p(D_{54})| = (27\times 2) + (1\times 27) = 81$.
- It is pertinent to note that $\pi(A_5) = \{2,3,5\}$, $n_2(A_5) = 5$, $|\mathrm{Syl}_2(A_5)| = 4$, $n_3(A_5) = 10$, $|\mathrm{Syl}_3(A_5)| = 3$, $n_5(A_5) = 6$ and $|\mathrm{Syl}_5(A_5)| = 5$. Therefore $\upsilon(A_5) = (5\times 4) + (10\times 3) + (6\times 5) = 80$.

Theorem (A, Asboei 2024)

Let G be a non–solvable finite group. Then $\upsilon(G) = 80$ if and only if $G \cong A_5$.

Proof

- We know that $\upsilon(A_5) = 80$. On the other hand, as G is non–solvable, it has at least three prime divisors. Also, $\exists$ prime p with $n_p(G) \neq 1$.
- We shall show that the maximum such prime p for which $n_p(G) > 1$ is at most 7. Suppose $p > 7$. Then $p \geq 11$ and $n_p(G) \geq 12$; so $\upsilon(G) \geq 12 \times 11 = 132$, a contradiction.
- Now, we shall show that the maximum such prime p for which $n_p(G) > 1$ is at most 5. We first prove that if $n_7(G) > 1$, then the only possibility is that $n_7(G) = 8$. Assume $n_7(G) > 8$. Then $n_7(G) \geq 15$. So $\upsilon(G) \geq 15 \times 7 = 105$; a contradiction. Suppose $n_7(G) = 8$. We need to look for $n_2(G) \in \{1, 3, 5, 7\}$, $n_3(G) \in \{0, 1, 4\}$, $n_5(G) \in \{0, 1\}$, $n_{11}(G) \in \{0, 1\}$, $n_{13}(G) \in \{0, 1\}$, $a \in \{1, 2, 3, 4\}$, $b \in \{1, 2\}$ such that

$$56 + (2^a \times n_2) + (3^b \times n_3) + (5 \times n_5) + (11 \times n_{11}) + (13 \times n_{13}) = 80.$$

In any such possibility, we have that $n_3(G) \leq 4$ and $n_5(G) \leq 1$. Hence G is solvable; a contradiction. Therefore the maximum such prime p for which $n_p(G) > 1$ is at most 5.
- Since G is a non–solvable group, it has a non–abelian composition factor S. On the other hand, $n_p(S) \mid n_p(G)$ for every prime divisor p of $|G|$.
- From $\upsilon(G) = 80$, we deduce that $S \cong A_5$. Therefore $G \cong A_5$.

A characterization of A_5 using the second invariant

Recall: $\mathcal{SP}^{''}(G,1) = \sum_{p\in\pi(G)} n_p(G)|\mathrm{Syl}_p(G)|\big(|\mathrm{Syl}_p(G)|-1\big) = \mu(G)$.

- If G is a p-group of size p^n for some natural number n, then $\mu(G) = p^n(p^n-1)$. From this, we see immediately that if G is a group of size 16, then $\mu(G) = 16\times 15 = 240$.
- There are also solvable groups G whose $\mu(G)$ is less than 240 $\big($for instance, $\mu(C_6)=8$, $\mu(S_3)=12$, $\mu(A_4)=36$, $\mu(C_{12})=18$, $\mu(D_{12})=42$ and $\mu(S_4)=192\big)$ or greater than 240 $\big($for instance, $\mu(C_{17})=272$ and $\mu(D_{54})=756\big)$.

We give a characterization of A_5 using this invariant as follows:

Theorem (A, Asboei 2024)

Let G be a non–solvable finite group. Then $\mu(G)=240$ if and only if $G\cong A_5$.

The proof of this result is similar to the first one, but shorter.

Conjecture

$$\gamma(G) = \int_0^1 \mathcal{SP}(G,x)\ dx = \sum_{p\in\pi(G)} \frac{n_p(G)}{|\mathrm{Syl}_p(G)|+1}.$$

Conjecture (A, Asboei 2024)

Let G be a nonsolvable finite group. Then $\gamma(G) = \frac{9}{2}$ if and only if $G \cong A_5$.

Gammas of some solvable groups:
$\gamma(D_6) = \frac{5}{4} < \frac{9}{2}$; $\gamma(A_4) = \frac{6}{5} < \frac{9}{2}$; $\gamma(D_{54}) = 9\frac{1}{28} > \frac{9}{2}$.

Gammas of some non–solvable groups:
$\boxed{\gamma(A_5) = \frac{9}{2}}$; $\gamma(GL(2,4)) = 3 < \frac{9}{2}$; $\gamma(A_6) = 12 > \frac{9}{2}$.

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A third characterization of A_5

Theorem

Let G be a noncyclic simple group. Then $\gamma(G) = \frac{9}{2}$ if and only if $G \cong \mathrm{A}_5$.

A sketch of the proof.

- It is known that $\gamma(\mathrm{A}_5) = \frac{9}{2}$. For the reverse case, let G be a finite noncyclic simple group such that $\gamma(G) = \frac{9}{2}$.
- Lemma: If G is a noncyclic simple group and $p \in \pi(G)$, then $n_p(G) \geq |\mathrm{Syl}_p(G)| + 1$.
- Since $\gamma(G) = \frac{9}{2}$, we have $|\pi(G)| = 3$ or 4.
- Use results on the classification of K_3 and K_4 groups to finish up!

∎

Note that $n_2(A_5) = 5 = |\mathrm{Syl}_2(A_5)| + 1$. The converse of the mentioned Lemma is not necessarily true. For instance, take $G = S_5$. It is easy to see that $\pi(G) = \{2, 3, 5\}$, $n_2(G) = 15 > 9 = |\mathrm{Syl}_2(G)| + 1$, $n_3(G) = 10 > 4 = |\mathrm{Syl}_3(G)| + 1$ and $n_5(G) = 6 = |\mathrm{Syl}_5(G)| + 1$.

Characterizing each finite simple group by its Sylow sum and average

- The set of all prime divisors of $|G|$ and the number of Sylow p-subgroups of G are denoted by $\pi(G)$ and $n_p(G)$ respectively.
- Let

$$\delta(G) := \sum_{p \in \pi(G)} p(G),$$

$$S(G) := \{p \in \pi(G) |\ n_p(G) > 1\},$$

and

$$\delta_0(G) := \sum_{p \in S(G)} n_p(G).$$

- Note that $\pi(G) = S(G)$ if and only if G is a nonabelian simple group; in this case, $\delta(G) = \delta_0(G)$.
- For a finite nonabelian simple group G, the sum and average of the Sylow numbers of G are what we denote by $\delta_0(G)$ and $\alpha(G)$ respectively, where

$$\alpha(G) := \frac{\delta_0(G)}{|S(G)|}.$$

- These invariants are well studied; for instance, see [14], [15] and [24].

Recall: Some preliminary results

Theorem (Asboei, Darafsheh 2018)

Let G be a nonabelian simple group. Then there exists $p \in \pi(G)$ such that $n_p(G)^2 > |G|$.

Theorem (A, Moreto, Zarrin 2020)

Let G be a finite group. If $n_3(G) \leq 7$ and $n_5(G) \leq 1455$, then G is solvable.

Lemma (A, Asboei 2024)

If G is a non–cyclic simple group and $p \in \pi(G)$, then $n_p(G) \geq |Syl_p(G)| + 1$, where $Syl_p(G)$ is a Sylow p-subgroup of G.

Theorem (A 2025)

Let G be a finite group. If $n_2(G) < 5$, then G is solvable.

Proposition (A 2025)

Let G be a finite nonsolvable group. If $\delta_0(G) \leq 1464$, then $n_3(G) \geq 10$.

The bound "$n_3(G) \geq 10$" is tight since $n_3(A_5) = 10$.

A characterization of PSL(2,7) by its Sylow sum and average

Theorem

Let G be a finite nonabelian simple group. Then $G \cong PSL(2,7)$ if and only if $\delta_0(G) = 57$. On the other hand, $G \cong PSL(2,7)$ if and only if $\alpha(G) = 19$.

Proof.

- Let $G = PSL(2,7)$. We know that $|G| = 168 = 2^3 \times 3 \times 7$. A simple calculation gives $n_2(G) = 21$, $n_3(G) = 28$ and $n_7(G) = 8$. Therefore $\delta_0(PSL(2,7)) = 57$.
- Suppose G is a nonabelian simple group such that $\delta_0(G) = 57$. Let $p \in S(G)$ such that $n_p(G)^2 > |G|$. Then $n_p(G) \leq 57 - 5 - 10 = 42$. Hence, $|G| \leq 42^2 = 1764$.
- Using GAP [35], we observe that all nonabelian simple groups of order less than 1764 are A_5, $PSL(2,7)$, A_6, $PSL(2,8)$, $PSL(2,11)$ and $PSL(2,13)$.
- On the other hand, $\delta_0(A_5) = 21$, $\delta_0(PSL(2,7)) = 57$, $\delta_0(A_6) = 91$, $\delta_0(PSL(2,8)) = 73$, $\delta_0(PSL(2,11)) = 188$ and $\delta_0(PSL(2,13)) = 274$. So $G \cong PSL(2,7)$.

The proof continues

- For the converse, let G be a nonabelian simple group such that $\alpha(G) = 19$.
- We prove that $|S(G)| \leq 12$. Suppose for contradiction that $|S(G)| \geq 13$. Using the preliminary results, we have that

$$\alpha(G) = \frac{\delta_0(G)}{|S(G)|} \geq \frac{5+10+6+8+12+14+18+20+24+30+32+38+42}{13} = \frac{259}{13} > 19;$$

  a contradiction to the hypothesis that $\alpha(G) = 19$. So $|S(G)| \leq 12$.
- As $\alpha(G) = \dfrac{\delta_0(G)}{|S(G)|} = 19$, we have that $\delta_0(G) = 19|S(G)|$.
- Using $|S(G)| \leq 12$, we deduce that $\delta_0(G) \leq 19 \times 12 = 228$.
- Let $p \in S(G)$ such that $n_p(G)^2 > |G|$. Then $n_p(G) \leq 228 - 5 - 10 = 213$.
  So $|G| < 213^2 = 45369$.
- Using GAP [35], we observe that there are 26 nonabelian simple groups of order less than 45369.
- For these 26 simple groups, we have that $\alpha(A_5) = 7$, $\alpha(PSL(2,7)) = 19$, $\alpha(A_6) = \frac{91}{3}$, $\alpha(PSL(2,8)) = \frac{73}{3}$, $\alpha(PSL(2,11)) = 47$, $\alpha(PSL(2,13)) = \frac{137}{2}$, $\alpha(PSL(2,17)) = \frac{307}{3}$, $\alpha(A_7) = \frac{631}{4}$, $\alpha(PSL(2,19)) = \frac{333}{2}$, $\alpha(PSL(2,16)) = \frac{409}{4}$, $\alpha(PSL(3,3)) = \frac{547}{3}$, $\alpha(PSU(3,3)) = \frac{505}{3}$, $\alpha(PSL(2,23)) = 328$, $\alpha(PSL(2,25)) = \frac{813}{2}$, $\alpha(M_{11}) = \frac{545}{2}$, $\alpha(PSL(2,27)) = 394$, $\alpha(PSL(2,29)) = \frac{2292}{5}$, $\alpha(PSL(2,31)) = \frac{1489}{4}$, $\alpha(A_8) = \frac{1891}{4}$, $\alpha(PSL(3,4)) = \frac{3361}{4}$, $\alpha(PSL(2,37)) = 879$, $\alpha(PSp(4,3)) = \frac{1591}{3}$, $\alpha(Sz(8)) = \frac{4161}{4}$, $\alpha(PSL(2,32)) = \frac{1553}{4}$, $\alpha(PSL(2,41)) = \frac{6845}{5}$ and $\alpha(PSL(2,43)) = 1230$.
- Therefore $G \cong PSL(2,7)$. □

Concluding remarks and Conjecture

- The invariants δ_0 and α can be used to characterize many finite nonabelian simple groups; an example was just given with $PSL(2,7)$.
- If S_1 and S_2 are finite nonabelian simple groups such that $|S_1| < |S_2|$, then it is not always true that one of $\delta_0(S_1) < \delta_0(S_2)$ and $\alpha(S_1) < \alpha(S_2)$ must be true. For instance, $|A_6| < |PSL(2,8)|$, but $\delta_0(A_6) = 91 > 73 = \delta_0(PSL(2,8))$ and $\alpha(A_6) = \frac{91}{3} > \frac{73}{3} = \alpha(PSL(2,8))$.
- On another point, $|S_1| = |S_2|$ for two finite nonabelian simple groups S_1 and S_2 does not guarantee that $\delta_0(S_1) = \delta_0(S_2)$ or $\alpha(S_1) = \alpha(S_2)$. For instance, $|A_8| = 20160 = |PSL(3,4)|$, but $\delta_0(A_8) = 1891 < 3361 = \delta_0(PSL(3,4))$ and $\alpha(A_8) = \frac{1891}{4} < \frac{3361}{4} = \alpha(PSL(3,4))$.
- We conclude this discussion with the conjecture below. Through GAP [35], we proved its validity for any simple group G s.t. $|G| < 10^{12}$.

Conjecture

A finite nonabelian simple group can be distinguished from another nonabelian simple group using the sum of its Sylow numbers. A similar result holds for the average of its Sylow numbers.

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