A talk given at Technical Univ. of Graz

Combinatorial Congruences and Infinite Series involving Binomial Coefficients

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Abstract

The central binomial coefficients play important roles in the classical Ramanujan series for $1/\pi$. For integers b, c and $k \geq 0$, the generalized central trinomial coefficient $T_k(b, c)$ denotes the coefficient of x^k in the expansion of $(x^2 + bx + c)^k$. As $T_k(2, 1) = \binom{2k}{k}$, we may view $T_k(b, c)$ as a natural extension of the central binomial coefficient $\binom{2k}{k}$. In 2011 the speaker found new series for $1/\pi$ involving $T_k(b, c)$ of types I-VII (just conjectures), this has stimulated further studies by others on the topic. In 2019 and 2020, the speaker introduced series for $1/\pi$ involving $T_k(b, c)$ of types VIII and IX. In 2026, the speaker found three series for $1/\pi$ involving $T_k(b, c)$ of type X, one of which states that

$$\sum_{k=0}^{\infty} \frac{16k+3}{(-202^2)^k} \binom{2k}{k} T_k(19, -20)\, T_{2k}(9, -5) = \frac{43\sqrt{101}}{75\pi}.$$

These conjectural identities look quite challenging.
In this talk, we review the history of series for $1/\pi$ involving $T_k(b, c)$, and tell how the speaker found them via congruences.

Part I. Ramanujan Series for $\frac{1}{\pi}$ and my Philosophy on Series for $\frac{1}{\pi}$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England
Modular equations and approximations to π,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 Jonathan Borwein and Peter Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

My first impression on Ramanujan-type series

In a year around 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\left(-\frac{27}{512}\right)^k\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 known Ramanujan-type series for $1/\pi$ with $a,b,m\in\mathbb{Z}$. I prefer their forms in terms of binomial coefficients.

What is needed for proving $\sum_{n=0}^{\infty}(6n+1)\binom{2n}{n}^3/256^n = 4/\pi$

The proofs of Ramanujan series involve lots of things such as modulo forms, elliptic integrals, theta functions, hypergeometric series, modular equations and symbolic computation.

$$P(q) := 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} \quad \text{(Eisenstein series)},$$

$$\varphi(q) := \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{(theta function)},$$

$$X = X(q) = q\prod_{j=1}^{\infty} \frac{(1-q^j)^{24}(1-q^{4j})^{24}}{(1-q^{2j})^{48}}.$$

$$\varphi(q)^4 = \sum_{n=0}^{\infty}\binom{2n}{n}X^n, \; P(q^2) = \sqrt{1-64X}\sum_{n=0}^{\infty}(3n+1)\binom{2n}{n}^3 X^n.$$

$$X(e^{-\pi\sqrt{3}}) = \frac{1}{256} \text{ and } P(e^{-2\pi\sqrt{3}}) = \frac{\sqrt{3}}{\pi} + \frac{\sqrt{3}}{4}\varphi(e^{-\pi\sqrt{3}})^4.$$

van Hamme's conjectures

Let p be an odd prime. Similar to Euler's observation $\Gamma(1/2)^2 = \pi$, we have $\Gamma_p(1/2)^2 = -(\frac{-1}{p})$. Motivated by this, in 1997 van Hamme posed p-adic analogues of many series for powers of π.

For the two Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} = \frac{2\sqrt{2}}{\pi} \text{ and } \sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi},$$

in 1997 van Hamme conjectured their following p-adic analogues:

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv p\left(\frac{-2}{p}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) \pmod{p^4}.$$

All the p-adic analogue conjectures of van Hamme were proved before 2017.

A Useful Lemma

Lemma (Z.-W. Sun [Sci. China Math. 54(2011)]) Let p be an odd prime and let $k \in \{0, \ldots, p-1\}$. Then

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Thus,

$$\binom{2(p-k)}{p-k} \equiv \begin{cases} \frac{2p}{k\binom{2k}{k}} \pmod{p} & \text{if } k \in \{\frac{p+1}{2}, \ldots, p-1\}, \\ \frac{-2p}{k\binom{2k}{k}} \pmod{p^2} & \text{if } k \in \{1, \ldots, \frac{p-1}{2}\}. \end{cases}$$

Remark. R. Tauraso [J. Number Theory 130(2010)] realized that

$$\binom{2(p-k)}{p-k} \equiv \frac{2p}{k\binom{2k}{k}} \pmod{p} \text{ for all } k = 1, \ldots, p-1.$$

We have similar lemmas involving $\binom{3k}{k}$ or $\binom{4k}{2k}$.

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases} (\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod{4}$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy \pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

Another Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

Part II. Series for $\frac{1}{\pi}$ involving $T_n(b, c)$

On central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](x^2+x+1)^n \text{ (the coefficient of } x^n \text{ in } (x^2+x+1)^n)$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Theorem (i) (H. Q. Cao and Sun, 2010). For any prime $p > 3$ we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

(ii) (Z. W. Sun, 2010) For any odd prime p we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

Remark. For any prime $p > 3$ and $k \in \{1, \ldots, p-1\}$, it is easy to show that $T_{p-k} \equiv (\frac{p}{3}) T_{k-1}/(-3)^{k-1} \pmod{p}$.

Central trinomial coefficients

Conjecture (Sun, Jan. 22, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2 T_k$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2 T_k\equiv p\left(20+24\left(\frac{p}{3}\right)(2-3^{p-1})\right) \pmod{p^3}.$$

Two new series for π involving central trinomial coefficients

Conjecture (Sun, 2019) For any prime $p > 3$, we have

$$p^2\sum_{k=1}^{p-1}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} \equiv 11\left(\frac{p}{3}\right)+\frac{p}{2}\left(13-35\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$p^2\sum_{k=1}^{p-1}\frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2(k-1)3^{k-1}} \equiv -\frac{1}{2}\left(\frac{p}{3}\right)-\frac{p}{8}\left(7+\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Two new series for π involving central trinomial coefficients

Conjecture (Sun, 2019) For any prime $p > 3$, we have

$$p^2 \sum_{k=1}^{p-1} \frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} \equiv 11\left(\frac{p}{3}\right) + \frac{p}{2}\left(13 - 35\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$p^2 \sum_{k=1}^{p-1} \frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2 (k-1)3^{k-1}} \equiv -\frac{1}{2}\left(\frac{p}{3}\right) - \frac{p}{8}\left(7 + \left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Conjecture (Sun, Dec. 7, 2019). We have

$$\sum_{k=1}^{\infty} \frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}} = \frac{5\pi}{\sqrt{3}} + 6\log 3,$$

$$\sum_{k=2}^{\infty} \frac{(5k-2)T_{k-1}}{k^2\binom{2k}{k}^2 (k-1)3^{k-1}} = \frac{21 - 2\sqrt{3}\,\pi - 9\log 3}{12}.$$

Remark. As the two series converge very fast, it is easy to check the two identities numerically. I posed the two identities to `MathOverflow` in 2019, nobody has idea to prove it.

Generalized central trinomial coefficients

For real numbers b and c, we define

$$T_n(b,c) := [x^n](x^2+bx+c)^n$$

(the coefficient of x^n in $(x^2+bx+c)^n$)

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

Recursion: $T_0(b,c)=1$, $T_1(b,c)=b$, and

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ (n>0),$$

where $d=b^2-4c$. It is known that if $d \neq 0$ then

$$T_n(b,c) = \sqrt{d}^n P_n\left(\frac{b}{\sqrt{d}}\right)$$

where

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the *Legendre polynomial* of degree n.

Asymptotic Behavior of $T_n(b,c)$

By the Laplace-Heine formula, for $x \notin [-1,1]$ we have

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}} \quad \text{as } n \to +\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c) \sim f_n(b,c) := \frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b,c) = (-1)^n T_n(b,c)$.
Conjecture (Sun, 2011; proved by S. Wagner): For $b,c>0$,

$$T_n(b,c) = f_n(b,c)\left(1+\frac{b-4\sqrt{c}}{16n\sqrt{c}}+O\left(\frac{1}{n^2}\right)\right)$$

as $n \to +\infty$. If $c>0$ and $b=4\sqrt{c}$, then

$$\frac{T_n(b,c)}{\sqrt{c}^n} = \frac{3\times 6^n}{\sqrt{6n\pi}}\left(1+\frac{1}{8n^2}+\frac{15}{64n^3}+\frac{21}{32n^4}+O\left(\frac{1}{n^5}\right)\right).$$

If $c<0$ and $b \in \mathbb{R}$ then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b,c)|} = \sqrt{b^2-4c}.$$

Replace $\binom{2k}{k}$ by $T_k(b,c)$

As $T_k(2,1)=\binom{2k}{k}$, in 2010 I viewed $T_k(b,c)$ as a natural extension of the central binomial coefficients. In contrast with my conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k}$ and $\sum_{k=0}^{p-1}(a+dk)\frac{\binom{2k}{k}^3}{m^k}$ modulo p^2 (with p an odd prime not dividing m), in December 2010 I formulated many conjectures with some $\binom{2k}{k}$ replaced by $T_k(b,c)$. For example, I made the following conjecture.

Conjecture (Sun, 2010-12-25). Let p be any odd prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and } p=x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e.,}\ p\equiv 3,5,6 \pmod 7).\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2}.$$

Story happened around Jan 1, 2011

$$T_n(1,16) \sim \frac{(1+2\sqrt{16})^{n+1/2}}{2\sqrt[4]{16}\sqrt{n\pi}} = \frac{9^{n+1/2}}{4\sqrt{n\pi}} = \frac{9^n}{12\sqrt{n\pi}}.$$

This is very similar to the fact that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}$. On Dec. 18, 2010, I conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2},$$

which is very similar to Ramanujan-type congruences.

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which is very similar to Ramanujan-type congruences.

Conjecture (Z. W. Sun, Jan. 2, 2011). We have

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 T_k(1,16) = \frac{24}{\pi},$$

```
T[n_]:=If[n>0,Coefficient[(x^2+x+16)^n,x^n],1]
S[n_]:=Sum[(30k+7)Binomial[2k,k]^2*T[k]/(-256)^k,{k,0,n}]
Print[N[S[200]Pi,20]]
```

Output: 24.000000000000000000

New series for $1/\pi$ involving $T_k(b,c)$

For $b, c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a, b, c, d, m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

Type I. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In October 2011, I found 10 conjectural series for $1/\pi$ of two new types:

Type VI. $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

Type VII. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k} T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, W. Zudilin.

My conjectural series of type I

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi}, \tag{I1}$$

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2T_k(34,1)=\frac{12}{\pi}, \tag{I2}$$

$$\sum_{k=0}^{\infty}\frac{30k-1}{4096^k}\binom{2k}{k}^2T_k(194,1)=\frac{80}{\pi}, \tag{I3}$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{4096^k}\binom{2k}{k}^2T_k(62,1)=\frac{16\sqrt{3}}{\pi}, \tag{I4}$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{256^k}\binom{2k}{k}^2T_k(8,-2)=\frac{2}{\pi}\sqrt{8+6\sqrt{2}}. \tag{I5}$$

Remark. (I1)-(I4) were found in 2011, and (I5) was found in 2019.

Some variants of series of type I

Conjecture (Z.-W. Sun [Electron. Res. Arch. 28(2020)]).

$$\sum_{k=0}^{\infty} \frac{50k+1}{(-256)^k} \binom{2k}{k} \binom{2k}{k+1} T_k(1,16) = \frac{8}{3\pi}, \qquad (\mathrm{I1}')$$

$$\sum_{k=0}^{\infty} \frac{30k+23}{(-1024)^k} \binom{2k}{k} \binom{2k}{k+1} T_k(34,1) = \frac{20}{3\pi}, \qquad (\mathrm{I2}')$$

$$\sum_{k=0}^{\infty} \frac{110k+103}{4096^k} \binom{2k}{k} \binom{2k}{k+1} T_k(194,1) = \frac{304}{\pi}, \qquad (\mathrm{I3}')$$

$$\sum_{k=0}^{\infty} \frac{238k+263}{4096^k} \binom{2k}{k} \binom{2k}{k+1} T_k(62,1) = \frac{112\sqrt{3}}{3\pi}, \qquad (\mathrm{I4}')$$

$$\sum_{k=0}^{\infty} \frac{2k+3}{256^k} \binom{2k}{k} \binom{2k}{k+1} T_k(8,-2) = \frac{6\sqrt{8+6\sqrt{2}} - 16\sqrt[4]{2}}{3\pi}. \qquad (\mathrm{I5}')$$

Remark. (I1′)-(I5′) were found in 2019.

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

Two unsolved conjectural series of type VII

Conjecture (Sun, 2011). We have

$$\sum_{k=0}^{\infty}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2=\frac{10406669}{2\sqrt{6}\,\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}).$$

Remark. The last identity was motivated by my following conjecture: For any odd prime $p\neq 7$, we have

$$\sum_{k=0}^{p-1}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2\equiv p\left(\frac{-6}{p}\right)\left(4+\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

My 2019 conjectural series of type VIII

In November 2019, I introduced a new type series for $1/\pi$.

Type VIII. $\sum_{k=0}^{\infty}(a+dk)T_k(b_1,c_1)T_k(b_2,c_2)^2/m^k = C/\pi$.

Conjecture (Sun, Nov. 2019). We have

$$\sum_{k=0}^{\infty}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2=\frac{55\sqrt{15}}{9\pi}, \qquad \text{(VIII1)}$$

$$\sum_{k=0}^{\infty}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2=\frac{1452\sqrt{5}}{\pi}, \qquad \text{(VIII2)}$$

$$\sum_{k=0}^{\infty}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2=\frac{189\sqrt{15}}{2\pi}, \qquad \text{(VIII3)}$$

$$\sum_{k=0}^{\infty}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2=\frac{6795\sqrt{5}}{\pi}. \qquad \text{(VIII4)}$$

A congruence related to the identity (VIII4)

Conjecture (Sun, 2019). Let $p>5$ be a prime with $p\neq 11$. Then

$$\sum_{k=0}^{p-1}\frac{T_k(14,1)T_k(11,-11)^2}{(-29700)^k}$$

$$\equiv\begin{cases}4x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=1,\ p=x^2+165y^2\\ 2x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=x^2+165y^2,\\ 12x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,(\frac{p}{3})=(\frac{p}{11})=1,\ p=3x^2+55y^2,\\ 6x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,(\frac{p}{3})=(\frac{p}{11})=-1,\ 2p=3x^2+55y^2,\\ 2p-20x^2 & \text{if } (\frac{-1}{p})=(\frac{p}{11})=1,(\frac{p}{3})=(\frac{p}{5})=-1,\ p=5x^2+33y^2,\\ 2p-10x^2 & \text{if } (\frac{-1}{p})=(\frac{p}{11})=-1,(\frac{p}{3})=(\frac{p}{5})=1,\ 2p=5x^2+33y^2,\\ 44x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,(\frac{p}{5})=(\frac{p}{11})=1,\ p=11x^2+15y^2,\\ 22x^2-2p & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,(\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=11x^2+15y^2,\\ 0 & \text{if } (\frac{-165}{p})=-1,\end{cases}$$

modulo p^2, where x and y are integers.

Remark. The quadratic field $\mathbb{Q}(\sqrt{-165})$ has class number 8.

My 2020 conjectural series of type IX

In August 2019, I introduced a new type series for $1/\pi$.
`Type IX.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k(b_1,c_1)T_k(b_2,c_2)/m^k = C/\pi$.

Conjecture (Sun, August 2020). We have

$$\sum_{k=0}^{\infty}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)=\frac{5390}{\pi} \qquad \text{(IX1)}$$

and

$$\sum_{k=0}^{\infty}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)=\frac{98}{3\pi}(10+7\sqrt{5}). \quad \text{(IX2)}$$

Remark. All of my such conjectural series for $1/\pi$ came from **combinations of philosophy, intuition, inspiration, experience and computation!**

Congruences related to (IX2)

Conjecture (Sun, August 2020). (i) For any prime $p>7$, we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(2,81)T_k(14,81)$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+15y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) Let p be an odd prime with $p\neq 7$. Then

$$\sum_{k=0}^{p-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$

$$\equiv\frac{p}{3}\left(270\left(\frac{-1}{p}\right)-104\left(\frac{-2}{p}\right)+245\left(\frac{-5}{p}\right)\right)\pmod{p^2}.$$

My 2026 conjectural series of type X

In March 2026, I introduced a new type of series for $1/\pi$.

`Type X.`

$$\sum_{k=0}^{\infty}\frac{a+dk}{m^k}\binom{2k}{k}T_k(b_1,c_1)T_{2k}(b_2,c_2)=\frac{C}{\pi},$$

where $a\in\mathbb{Z}$, $b_1,b_2,c_1,c_2,d,m\in\mathbb{Z}\setminus\{0\}$, $b_1^2\ne 4c_1$, $b_2^2\ne 4c_2$, and C is an algebraic number.

Conjecture (Z.-W. Sun, arXiv:2603.29973). We have

$$\sum_{k=0}^{\infty}\frac{24k+5}{76^{2k}}\binom{2k}{k}T_k(1,-12)T_{2k}(8,-3)=\frac{\sqrt{38(120+73\sqrt{3})}}{6\pi},\tag{X1}$$

$$\sum_{k=0}^{\infty}\frac{585k+136}{(-85^2)^k}\binom{2k}{k}T_k(8,-9)T_{2k}(7,-9)=\frac{85\sqrt{255}}{6\pi},\tag{X2}$$

$$\sum_{k=0}^{\infty}\frac{16k+3}{(-202^2)^k}\binom{2k}{k}T_k(19,-20)T_{2k}(9,-5)=\frac{43\sqrt{101}}{75\pi}.\tag{X3}$$

Part III. Zeilberger-type Series and Fastest Series for K and $\zeta(5)$

Zeilberger's series for $\zeta(2)$

In 1993, D. Zeilberger used the Wilf-Zeilberger method to obtain the new identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

Define

$$F(n,k) = \frac{1}{\binom{2n}{n}(n+1)^2\binom{2n+k+1}{n+1}^2}$$

and

$$G(n,k) = \frac{n!^4(n+k)!^2}{2(2n+1)!(2n+k+2)!^2}P(n,k),$$

where $P(n,k)$ denotes

$$(n+1)^2(21n+13) + 2k^3 + k^2(13n+11) + k(28n^2+48n+20).$$

Then $\langle F, G\rangle$ is a **WZ pair** in the sense that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Zeilberger's proof

$$\sum_{k=0}^{N-1}(F(n+1,k)-F(n,k)) = \sum_{k=0}^{N-1}(G(n,k+1)-G(n,k)) = G(n,N)-G(n,0).$$

$$\sum_{k=0}^{N-1}\sum_{n=0}^{N}(F(n+1,k)-F(n,k)) = \sum_{n=0}^{N}G(n,N) - \sum_{n=0}^{N}G(n,0).$$

$$\sum_{k=0}^{N-1}F(N+1,k) - \sum_{k=0}^{N-1}F(0,k) = \sum_{n=0}^{N}(G(n,N)-G(n,0)).$$

$$F(0,k) = \frac{1}{(k+1)^2},\ \ G(n,0) = \frac{21(n+1)-8}{(n+1)^3\binom{2n+2}{n+1}^3}.$$

$$\sum_{k=0}^{N-1}F(N+1,k) - \sum_{n=1}^{N}\frac{1}{n^2} = \sum_{n=0}^{N}G(n,N) - \sum_{n=1}^{N+1}\frac{21n-8}{n^3\binom{2n}{n}^3}$$

and hence $\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3} = \zeta(2) = \frac{\pi^2}{6}$ since $\sum_{k=0}^{N-1}F(N+1,k) \to 0$ and $\sum_{n=0}^{N}G(n,N) \to 0$.

Other Zeilberger-type series

J. Guillera [Ramanujan J. 15(2008)] used the WZ method to give three new Zeilberger-type series:

$$\sum_{k=1}^{\infty}\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}=-16G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G,\quad \sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2},$$

where G is the Catalan constant $L(2,(\frac{-4}{\cdot}))=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$.

Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [Proc. Amer. Math. Soc. 147(2019)] provided a q-analogue of the last identity with $|q|<1$:

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3}=(1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4},$$

where $(a;q)_n:=\prod_{0\leqslant i<n}(1-aq^i)$ and $(a;q)_\infty:=\prod_{i=0}^{\infty}(1-aq^i)$.

Similar series conjectured by the speaker

Conjecture (Z.-W. Sun, 2010; Sci. China Math. 54(2011)).

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K:=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}$$ (confirmed by Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012),

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2$$ (confirmed by J. Guillera in 2013),

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},\quad \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

Remark. The last three conjectural identities were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97 (2014)].

The fastest series for $K = L(2, (\frac{-3}{\cdot}))$

In 2024 F. Calegari, V. Dimitrov and Yunqing Tang released a paper over 100 pages in arXiv to prove the irrationality of

$$K := \sum_{k=1}^{\infty} \frac{(\frac{k}{3})}{k^2} = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right).$$

Conjecture (Z.-W. Sun, Oct. 2023). We have

$$\sum_{k=1}^{\infty} \frac{(4k-1)3^k}{(2k-1)k^2\binom{2k}{k}\binom{3k}{k}} = 2K$$

and

$$\sum_{k=1}^{\infty} \frac{10k-3}{(2k-1)k^2 3^k\binom{2k}{k}\binom{3k}{k}} = \frac{K}{2}. \tag{0.1}$$

Remark. The converging rate of the last series is $1/81$. As pointed out by Jorge Zuniga, the last identity provides the **fastest algorithm** to compute the important constant K. In June 2025, Lorenz Milla used the formula to compute the first 10^{12} decimal digits of K.

The fastest series for $\zeta(5)$

W. Chu and W. Zhang [Math. Comp. 2014]:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(7k-2)}{(2k-1)k^2\binom{2k}{k}\binom{3k}{k}} = \frac{\pi^2}{12},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(56k^2-32k+5)}{(2k-1)^2k^3\binom{2k}{k}\binom{3k}{k}} = 4\zeta(3).$$

Z.-W. Sun (conjectured in 2023):

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(28k^2-18k+3)}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}} = \frac{\pi^4}{45}.$$

Z.-W. Sun (conjectured on Jan. 18, 2025):

$$\sum_{k=1}^{\infty} \frac{(-1)^{k}(560k^4-640k^3+408k^2-136k+17)}{(2k-1)^4k^5\binom{2k}{k}\binom{3k}{k}} = 180\zeta(5) - \frac{56}{3}\pi^2\zeta(3).$$

The last formula was confirmed by K. C. Au via the WZ-seed method. Au wrote "*A peculiarity of this example is that, we need to combine two transformation formulas to complete the proof. As far as I know, infinite series requiring this step has never appeared in literature before.*"

My posting in MathOverflow

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A curious hypergeometric series related to Riemann's zeta function

Asked 1 month ago Modified 1 month ago Viewed 1k times

11

Last week I found the following curious hypergeometric identity:

$$\sum_{k=1}^{\infty}\frac{(-1)^k(560k^4-640k^3+408k^2-136k+17)}{(2k-1)^4k^5\binom{2k}{k}\binom{3k}{k}}=180\zeta(5)-\frac{56}{3}\pi^2\zeta(3). \qquad (1)$$

As the series has converging rate $-1/27$, it is easy to check the identity numerically via Mathematica. I conjecture that the identity (1) does hold.

Question. Any idea towards proving (1)? Can one prove it via the WZ method or transformations of hypergeometric series?

Edits. Y. Zhao [Question 281009] proposed the following hypergeometric identity:

$$\sum_{k=1}^{\infty}\frac{Q(k)(-2^{10}/5^5)^k(1)_k^9}{k^9(1/2)_k^5(1/5)_k(2/5)_k(3/5)_k(4/5)_k}=-380928\zeta(5), \qquad (2)$$

where $Q(k)=5532k^4-5600k^3+2275k^2-425k+30$. This identity was proved by K. C. Au in the paper arXiv:2312.14051 by the WZ method. In Remark 4.2 of his paper, Au mentioned that (2) seems to be the only currently known (conjectural or proven) hypergeometric series for $\zeta(5)$ with geometric converging rate. In 2024, D. Broadhurst [arXiv:2401.08997] reported that he had used the identity (2) to obtain the numerical value of $\zeta(5)$ to 200 billion decimal digits.

As pointed out by J. Zuniga in his comments, (1) seems to be the most efficient formula to calculate numerical values of $\zeta(5)$. Note that the series in (1) converges much faster than the series in (2).

nt.number-theory sequences-and-series riemann-zeta-function hypergeometric-functions combinatorial-identities Edit tags

Comments on MathOverflow

11 You just happened to write down this identity, and it checked out? – Dave Benson Jan 21 at 9:11

Is it correct, that the right side is rational, actually -34.81109130749055680098535958677530288696289 0625 – Dieter Kadelka Jan 21 at 10:18

Numerically: The left side seems to converge to -34.8110913074905784984277818840679138229263987555907252185774130074434376628890040418 592905421519554773428985471154729079064369892428036908399319349490949 86 – Dieter Kadelka Jan 21 at 10:21

You may calculate (the sum of the first 2000 terms)/RHS. – Zhi-Wei Sun Jan 21 at 10:41

@Dave Benson MathOverflow is a place to ask and answer questions. If one is interested in how I found those series identities, he/she may visit my homepage and look at some talks of mine to learn how I was led to the discoveries. – Zhi-Wei Sun Jan 21 at 10:45

@Zhi-Wei Sun My calculation of the right side is wrong. With sage I got the sum. – Dieter Kadelka Jan 21 at 10:48

Mathematica Program: P[k_]:=P[k]=560k^4-640k^3+408k^2-136k+17; N[Sum[(-1)^k*P[k]/((2k-1)^4*k^5*Binomial[2k,k]Binomial[3k,k]),{k,1,3000}]/(180Zeta[5]-56/3*Pi^2*Zeta[3]),100] – Zhi-Wei Sun Jan 21 at 10:54

6 @Zhi-WeiSun, You can check if this identity results proven under Au's technique on WZ seeds. (arXiv:2312.14051v4 [math.CO] 10 Oct 2024). Several similar identities are proven in this work. Formula is very interesting for numerics. It seems to be the most efficient series to calculate $\zeta(5)$ by binary splitting (to million digits and beyond) since $\zeta(3)$ (my formula) and π (Chudnovsky) are computed very fast. – Jorge Zuniga Jan 21 at 16:17

@Jorge Zuniga Au's method does not work for some conjectural identities proposed by me. If the identity gives the most efficient series for calculating $\zeta(5)$, it will be of particular value and interest. – Zhi-Wei Sun Jan 21 at 21:46

Comments on MathOverflow

2 @Dave Benson and Steven Clark As you can check, I have proposed many conjectural series identities. At my homepage there are several talks and papers on such identities with some backgrounds and the story behind the conjectures. Some inspirations I got can hardly be explained. You may ask similar questions about Ramanujan's formulae and doubt their values if you wish. – Zhi-Wei Sun Jan 21 at 21:54

To discuss how one found a certain result is the task of mathematical history, not the main goal of MathOverflow. By the way, I have added certain edits after the original posting. You can judge yourself whether the proposed identity is interesting. – Zhi-Wei Sun Jan 22 at 9:49

I never ask any problem solver at MathOverflow how he/she found the solution or proof. – Zhi-Wei Sun Jan 22 at 10:20

7 @Steven Clark For the identity in this posting, I have no derivation. Last week, when I visited the China Dinosaur Park at Changzhou, I suddenly realized that there should exist such formulas for $\zeta(5)$, and then I returned to the hotel to find the identity (1) via Mathematica. Of course, I'm not Ramanjuan! I'm a Chinese, and I don't believe in any gods. – Zhi-Wei Sun Jan 22 at 23:18

Ok, thanks. It seems despite a lack of derivation there is some interest in your formula and its speed of convergence. – Steven Clark Jan 22 at 23:26

3 "i visited the dinasour park and realised there should be rapidly converging hypergeometric series to $\zeta(5)$" is one of the favourite things i've read here:)) – tomos Jan 23 at 14:39

8 In fact, in the day time of Jan. 18, 2025, I gave six-hour lessons on combinatorics. After the supper I felt relaxed and then walked to the China Dinosaur Park. Seeing a huge dinosaur stimulated my inspiration: I realized that I should not restrict myself to hypergeometric series for π, π^2, $\zeta(3)$, π^4, I should consider hypergeometric series for the challenging $\zeta(5)$ as well. In my eyes, π is a child dinosaur while $\zeta(5)$ is a huge dinosaur whose irrationality is even open. – Zhi-Wei Sun Jan 23 at 15:16

J. Zuniga's Comments

12

This is not an answer, it is just a complement of my comment above.

Hypergeometric series are particularly well suited to computation by the binary splitting algorithm yielding some of the most efficient methods for obtaining the numerical value of many mathematical constants, up to billions of digits in some cases.

For $\zeta(5)$ there are currently some BBP type series and 3 linearly convergent hypergeometric series known, Zhao's Eq.(2) above and Eqs.(III.1 - III.2) here, but they all have in common poor efficiency.

The current decimal places record for $\zeta(5)$ was obtained using a BBP-type formula by Oliver Kruse. See here that is the fastest known algorithm to compute this constant.

I made some test-bench in y-cruncher to verify the efficiency of Eq.(1) following my comments above. This output from an old notebook displays 50 million digits for $\zeta(5)$ in less than 3 minutes that is compared to Kruse algorithm which took over 6 minutes under the same conditions.

```
Begin Computation:

Series CommonP2B3...  34,931,875 terms  (Expansion Factor = 46.540)
Time:    143.748 seconds  ( 2.396 minutes )
Large Division...
Time:    0.347 seconds  ( 0.006 minutes )

Zeta(3) - Zuniga (2023-vi):
Series CommonP2B3...  4,217,363 terms  (Expansion Factor = 8.700)
Time:    25.015 seconds  ( 0.417 minutes )
Large Division...
Time:    0.524 seconds  ( 0.009 minutes )

Pi - Chudnovsky (1988):
Series CommonP2B3...  3,525,695 terms  (Expansion Factor = 2.424)
Time:    6.203 seconds  ( 0.103 minutes )
Large Division...
```

J. Zuniga's Comments

```
Zeta(5):  177.104 seconds  ( 2.952 minutes )

Processing Hexadecimal Digits:
Time:    0.013 seconds  ( 0.000 minutes )
Base Converting:
Time:    0.570 seconds  ( 0.009 minutes )
Processing Decimal Digits:
Time:    0.015 seconds  ( 0.000 minutes )

Verifying Base Conversion:
Time:    0.245 seconds  ( 0.004 minutes )
Verifying Binary Output:
Time:    0.001 seconds  ( 0.000 minutes )

Start Time: Wed Jan 22 09:43:36 2025
End Time:   Wed Jan 22 09:46:34 2025

Total Computation Time:     177.675 seconds  ( 2.961 minutes )
Start-to-End Wall Time:     178.087 seconds  ( 2.968 minutes )

CPU Utilization:          197.12 %  +  19.80 % kernel overhead
Multi-core Efficiency:     49.28 %  +   4.95 % kernel overhead

Last Decimal Digits:  Zeta(5)
7129651407 4652758567 6528303467 6882217154 5010337299  :  49,999,950
4936970076 8617385739 4483376770 3845318050 1570246376  :  50,000,000

Spot Check: Good through 50,000,000
```

Therefore, from this limited quick testing, there is some evidence that Eq.(1) is about twice faster than high performance Kruse formula for $\zeta(5)$, yielding a new, much more efficient algorithm.

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answered Jan 22 at 13:28
Jorge Zuniga

Comments from the World of Numbers

Report from the website http://www.numberworld.org/y-cruncher/news/2025.html#2025_1_25

`New Fastest Formula for Zeta(5): (January 25, 2025) - permalink`

`As new fastest formula for Zeta(5) has been found by Zhi-Wei Sun which Jorge Zuniga has turned into custom formula file for use in y-cruncher!`

`The new formula is nearly twice as fast as the previous fastest (Kruse 2019). You can read more about this here.`

New record for computing $\zeta(5)$

https://www.numberworld.org/y-cruncher/records.html

Record Size Computations:

Blue: Current World Record
Green: Former World Record
Red: Unverified computation.*

*To protect against computation errors from software bugs and/or hardware instability, a computation must be verified before it counts as a world record. This is usually done by repeating the computation using a different and independent formula.

Date Announced	Date Completed:	Source:	Who:	Constant:	Decimal Digits:	Time:	Computer:
December 11, 2025	November 19, 2025	Source	Kevin O'Brien Divyansh Jain Brian Beeler (Storage Review)	Pi	314,000,000,000,000	Compute: 110 days Verify: 4.37 hours	2 x AMD Epyc 9965 1.5 TB
November 15, 2025	October 23, 2025		Dmitriy Grigoryev	Zeta(5)	600,000,000,000	Compute: 105 hours Verify: 190 hours	Intel Xeon W7-3465X 2 TB
November 15, 2025	October 5, 2025		Dmitriy Grigoryev	Gamma(1/5)	270,000,000,000	Compute: 41.8 hours Verify: 43.9 hours	Intel Xeon W7-3465X 2 TB
November 15, 2025	October 2, 2025		Mamdouh Barakat	Log(2)	3,100,000,000,000	Compute: 167 hours Verify: 144 hours	AMD Ryzen TR 5965WX 512 TB
August 9, 2025	August 6, 2025		Mamdouh Barakat	Gamma(1/3)	1,300,000,000,000	Compute: 115 hours Verify: 90.7 hours	AMD Ryzen TR 5965WX 512 TB
June 20, 2025	June 13, 2025		Dmitriy Grigoryev	Gamma(1/4)	1,200,000,000,000	Compute: 1.23 days Verify: 1.22 days	Intel Xeon W9-3595X 2 TB
June 20, 2025	June 9, 2025		Dmitriy Grigoryev	Gamma(1/3)	1,200,000,000,000	Compute: 33.6 hours Verify: 32.3 hours	Intel Xeon W9-3595X 2 TB

Algorithm based on my formula

```
Program:              y-cruncher v0.8.6 Build 9545-icx
Binary:               18-CNL ~ Shinoa

User:                 Dmitriy Grigoryev

Operating System:     Windows 10.0 Build 26100

Processor(s):
    Name:             Intel(R) Xeon(R) w9-3595X
    Logical Cores:    60
    Physical Cores:   60
    Sockets:          1
    NUMA Nodes:       1
    Base Frequency:   1,992,115,696 Hz
    Est. Frequency:   1,987,974,415 Hz

Motherboard:
    Manufacturer:     Dell Inc.
    Model:            006CX9
    Version:          A02
    Serial Number:    Suppressed - Personally identifiable information is opt-in only.

Memory:
    Usable Memory:    2,197,949,710,336 (2.00 TiB)
    Total Memory:     2,199,023,255,552 (2.00 TiB)
    DIMM:              128 GiB - 00AD069D00AD - HMCT04MEERA137N
    DIMM:              128 GiB - 00AD069D00AD - HMCT04MEERA137N
    DIMM:              128 GiB - 00AD069D00AD - HMCT04MEERA137N
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    DIMM:              128 GiB - 00AD069D00AD - HMCT04MEERA137N

Constant:             Custom - Zeta(5)
Algorithm:            Zhi-Wei Sun (2025)
Decimal Digits:       600,000,000,000
```

Main References about my conjectures:

1. Z.-W. Sun, *Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$*, in: Number Theory and Related Area (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Adv. Lect. Math. 27, Higher Education Press and Internat. Press, Beijing-Boston, 2013, pp. 149–197.

2. Z.-W. Sun, *On sums related to central binomial and trinomial coefficients*, in: M. B. Nathanson (ed.), Combinatorial and Additive Number Theory: CANT 2011 and 2012, Springer Proc. Math. Stat., Vol. 101, Springer, New York, 2014, pp. 257-312.

3. Z.-W. Sun, *New series for powers of π and related congruences*, Electron. Res. Arch. **28** (2020), 1273–1342.

4. Z.-W. Sun, *Some new series for $1/\pi$ motivated by congruences*, Colloq. Math. **173** (2023), no. 1, 89–109.

5. Z.-W. Sun, *Various conjectural series identities*, arXiv:2603.29973.

Thank you!

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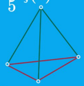
$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}=\frac{2}{5}\zeta(3)$$



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