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Covers of the Integers by Residue Classes and their Extensions to Groups

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
`zwsun@nju.edu.cn`
http://maths.nju.edu.cn/∼zwsun

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Abstract

A system $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ of k residue classes is called a *cover of* $\mathbb{Z}$ if any integer belongs to one of the k residue classes. This concept was introduced by P. Erdős in the 1950s. Erdős ever conjectured that A is a cover of $\mathbb{Z}$ whenever it covers $1, \ldots, 2^k$.

In this talk we introduce some basic results on covers of $\mathbb{Z}$ as well as their elegant proofs. We will also talk about covers of groups by finitely many cosets, give a proof of the Neumann-Tomkinson theorem, and introduce progress on the Herzog-Schőnheim conjecture and the speaker's disjoint cosets conjecture.

Part I. Covers of $\mathbb{Z}$ by residue classes

Covering systems of residue classes

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let $a(\mathrm{mod}\ n) = a + n\mathbb{Z}$ and

$$[\![x \equiv a \pmod n]\!] = \begin{cases} 1 & \text{if } x \equiv a \pmod n, \\ 0 & \text{otherwise.} \end{cases}$$

For a finite system $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ of residue classes, if $\bigcup_{s=1}^k a_s(\mathrm{mod}\ n_s) = \mathbb{Z}$ then we call A a *covering system* or a *cover of* $\mathbb{Z}$; if A covers each integer exactly once then A is called an *exact cover* of $\mathbb{Z}$. For example, $\{r(\mathrm{mod}\ n)\}_{r=0}^{n-1}$ is an exact cover of $\mathbb{Z}$.

The concept of covering system was introduced by Paul Erdős (in 1950) who gave the following example:

$$\{0(\mathrm{mod}\ 2),\ 0(\mathrm{mod}\ 3),\ 1(\mathrm{mod}\ 4),\ 5(\mathrm{mod}\ 6),\ 7(\mathrm{mod}\ 12)\}.$$

Another Example.

$$A = \{1(\mathrm{mod}\ 2), 2(\mathrm{mod}\ 2^2), \ldots, 2^{k-1}(\mathrm{mod}\ 2^k), 0(\mathrm{mod}\ 2^k)\}$$

is an exact cover of $\mathbb{Z}$. Note that $B = \{2^{s-1}(\mathrm{mod}\ 2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but it does not cover 0.

An application of covers with distinct moduli

P. Erdős: Some residue class $a(\mathrm{mod}\ d)$ with d even and a odd contains no numbers of the form $p+2^n$ with p prime and $n \in \mathbb{N}$.

Proof. Let $A = \{a_1(\mathrm{mod}\ n_1), \ldots, a_6(\mathrm{mod}\ n_6)\}$ be

$\{0(\mathrm{mod}\, 2), 0(\mathrm{mod}\, 3), 1(\mathrm{mod}\, 4), 3(\mathrm{mod}\, 8), 7(\mathrm{mod}\, 12), 23(\mathrm{mod}\, 24)\}$.

This is a cover of $\mathbb{Z}$ with all the moduli distinct. Let $p_1 = 3,\ p_2 = 7,\ p_3 = 5,\ p_4 = 17,\ p_5 = 13,\ p_6 = 241$. Then $p_s \mid 2^{n_s} - 1$ but $p_s \nmid 2^n - 1$ for $0 < n < n_s$. As $2^5 \equiv 1 \pmod{31}$, we have $|\{p_s + 2^n \bmod 31 :\ 1 \leqslant s \leqslant 6,\ n \in \mathbb{N}\}| \leqslant 6 \times 5 < 31$. In fact, $p_s + 2^n \not\equiv 3 \pmod{31}$. Let $a(\mathrm{mod}\ d)$ denote the residue class

$$1(\mathrm{mod}\ 2) \cap 3(\mathrm{mod}\ 31) \cap 2^{a_1}(\mathrm{mod}\ p_1) \cap \ldots \cap 2^{a_6}(\mathrm{mod}\ p_6).$$

(This intersection is nonempty by the Chinese Remainder Theorem.) If $x \equiv a \pmod{d}$ and $x = p + 2^n$ with p prime and $n \in \mathbb{N}$. For some $1 \leqslant s \leqslant 6$, we have $n \equiv a_s \pmod{n_s}$ and hence $2^n \equiv 2^{a_s} \equiv x \pmod{p_s}$. Thus $p_s \mid p$ and $p_s = p$. But $x \not\equiv p_s + 2^n \pmod{31}$, so we get a contradiction.

Erdős' Probelm

Erdős offers 1000 US dollars for the solution to his following problem.

Erdős' Problem. Whether the least modulus in a covering system with distinct moduli can be arbitrarily large?

P. P. Nielsen [J. Number Theory 129(2009)] proved that there exists a cover $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with $n_1 = 40 < n_2 < \ldots < n_k$ and $k > 10^{50}$.

Erdős' Problem was solved negatively by B. Hough [Ann. of Math. 181(2015)]. He used Lovász local lemma and some ideas from the sieve theory to prove that if $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $n_1 < \cdots < n_k$ then we must have $n_1 < 10^{16}$.

Erdős-Selfridge Conjecture

Erdős-Selfridge Conjecture. There is no cover $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with $1 < n_1 < \ldots < n_k$ and $n_1, \ldots, n_k$ all odd.

S. Guo and Z.-W. Sun [Adv. Appl. Math. 35(2005)]: If $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $1 < n_1 < \ldots < n_k$, and all its moduli are odd and squarefree, then $[n_1, \ldots, n_k]$ has at least 22 prime factors.

R. D. Hough and P. P. Nielsen [Duke Math. J. 168 (2019)]: If $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $1 < n_1 < \ldots < n_k$, then one of $n_1, \ldots, n_k$ is divisible by 2 or 3.

P. Balister, B. Bollobás, R. Morros, J. Sahasrabudhe and M. Tiba [Invent. Math. 228 (2022)]: Suppose that $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $1 < n_1 < \ldots < n_k$. Then $n_1 < 616000$, and $n_1, \ldots, n_k$ cannot be all odd and squarefree.

M. Cummings, M. Filaseta and O. Trifonov [Acta Math. Hungar. 175 (2025)]: Suppose that $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $n_1 < \ldots < n_k$. If $n_1, \ldots, n_k$ are all squarefree, then $n_1 \leqslant 118$.

Schinzel's Conjecture

Schinzel's Conjecture: If $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^{k}$ $(k>1)$ is a cover of $\mathbb{Z}$, then $n_i \mid n_j$ for some $1 \leqslant i, j \leqslant k$ with $i \neq j$.

A. Schinzel [Acta Arith. 13(1967)]: The Erdős-Selfridge conjecture is stronger than and Schinzel's conjecture is weaker than the following proposition:

For any polynomial $P(x) \in \mathbb{Z}[x]$ with $P(0) \neq 0$, $P(1) \neq -1$ and $P(x) \neq 1$, there exists an infinite arithmetic progression of positive integers such that $x^n + P(x)$ is irreducible over the rational field $\mathbb{Q}$ for every n in the progression.

P. Balister, B. Bollobás, R. Morros, J. Sahasrabudhe and M. Tiba [Invent. Math. 228 (2022)]: Schinzel's Conjecture holds.

Covering function

For $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$, its *covering function* $w_A : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$w_A(x) = |\{1 \leqslant s \leqslant k : \ x \equiv a_s \pmod{n_s}\}|.$$

Clearly w_A is periodic modulo $N_A = [n_1, \ldots, n_k]$. Observe that

$$\begin{aligned}\frac{1}{N_A}\sum_{x=0}^{N_A-1} w_A(x) =& \frac{1}{N_A}\sum_{x=0}^{N_A-1}\sum_{s=1}^{k}[\![x \equiv a_s \pmod{n_s}]\!]\\ =& \sum_{s=1}^{k}\frac{1}{N_A}|\{0 \leqslant x < N_A : \ x \equiv a_s \pmod{n_s}\}|\\ =& \sum_{s=1}^{k}\frac{1}{N_A}\cdot\frac{N_A}{n_s} = \sum_{s=1}^{k}\frac{1}{n_s}.\end{aligned}$$

If A covers each integer at least m times, then we call A an *m-cover* (of $\mathbb{Z}$) and note that $\sum_{s=1}^k \frac{1}{n_s} \geqslant m$. If A covers each integer exactly m times, then we call A an *exact m-cover* (of $\mathbb{Z}$) and note that $\sum_{s=1}^k \frac{1}{n_s} = m$ in this case.

Davenport-Mirsky-Newman-Rado Result

In the 1960s Paul Erdős made the following conjecture: *If* $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k>1)$ *is a disjoint system with the moduli* $n_1,\ldots,n_k$ *distinct, then it cannot be a cover of* $\mathbb{Z}$.

H. Davenport, L. Mirsky, D. Newman and R. Radó (1960s): If $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k>1)$ is a disjoint cover of $\mathbb{Z}$ with $1<n_1\leqslant n_2\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$, then we must have $n_{k-1}=n_k$.

Proof. Without loss of generality we let $0\leqslant a_s<n_s$ $(1\leqslant s\leqslant k)$. For $|z|<1$ we have

$$\sum_{s=1}^k\frac{z^{a_s}}{1-z^{n_s}}=\sum_{s=1}^k\sum_{q=0}^\infty z^{a_s+qn_s}=\sum_{n=0}^\infty z^n=\frac{1}{1-z}.$$

If $n_{k-1}<n_k$ then

$$\infty=\lim_{\substack{z\to e^{2\pi i/n_k}\\|z|<1}}\frac{z^{a_k}}{1-z^{n_k}}=\lim_{\substack{z\to e^{2\pi i/n_k}\\|z|<1}}\left(\frac{1}{1-z}-\sum_{s=1}^{k-1}\frac{z^{a_s}}{1-z^{n_s}}\right)<\infty,$$

a contradiction!

On covering equivalence

Fundamental Theorem on Covering Equivalence (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly, 1987]). Let M be a left R-module where R is a ring with identity. Let F be a map to M with $\mathrm{Dom}(F) \subseteq \mathbb{C} \times \mathbb{C}$ such that

$$\left\{\left\langle \frac{x+r}{n}, ny \right\rangle : r = 0, 1, \ldots, n-1\right\} \subseteq \mathrm{Dom}(F)$$

for any $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$. Then the following two statements are equivalent:

(a) Whenever $\{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^l$, i.e.,

$$\sum_{\substack{1\leqslant s\leqslant k\\ x\equiv a_s \pmod{n_s}}} \lambda_s = \sum_{\substack{1\leqslant t\leqslant l\\ x\equiv b_t \pmod{m_t}}} \mu_t \quad \text{for all } x \in \mathbb{Z}$$

(with $\lambda_s, \mu_t \in R$, $0 \leqslant a_s < n_s$ and $0 \leqslant b_t < m_t$), we have

$$\sum_{s=1}^{k} \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \text{ for all } \langle x, y\rangle \in \mathrm{Dom}(F).$$

On covering equivalence (continued)

(b) For any $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$, we have

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y).$$

Remark. I call a function F satisfying (b) a uniform function. There are many examples of uniform functions. For example,

$$F(x,y) = \lfloor x\rfloor,\ \frac{\cot \pi x}{y},\ y^{m-1}B_m(x),\ \log\left(\Gamma(x)\frac{y^{x-1/2}}{\sqrt{2\pi y}}\right)$$

are uniform functions, where $B_m(x)$ is the Bernoulli polynomial of degree m, and Γ is the well known Gamma function.

Burshtein's conjecture

Let $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ be a disjoint cover of $\mathbb{Z}$ with each modulus occurring at most M times. Write $[n_1,\ldots,n_k]=\prod_{t=1}^r p_t^{\alpha_t}$, where $p_1<\cdots<p_r$ are distinct primes and $\alpha_1,\ldots,\alpha_r$ are positive integers. N. Burshtein [Discrete Math. 14(1976)] conjectured that

$$p_r\leqslant M\prod_{p\leqslant p_r}\frac{p}{p-1}.$$

R. J. Simpson [Discrete Math. 59(1986)] proved further that

$$p_r\leqslant M\prod_{t=1}^{r-1}\frac{p_t}{p_t-1}.$$

The last inequality implies that $M\geqslant p_1>1$; in fact, if $r\geqslant 2$ then

$$M\geqslant p_r\prod_{t=1}^{r-1}\frac{p_t-1}{p_t}\geqslant p_{r-1}\prod_{t=1}^{r-2}\frac{p_t-1}{p_t}\geqslant\cdots\geqslant p_2\frac{p_1-1}{p_1}>p_1-1.$$

This gives a combinatorial approach to the Erdös conjecture.

A conjecture of Stein

A Conjecture of S.K. Stein [Math. Ann. 134(1958)]: If $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(1 < n_1 < \ldots < n_k)$ is a disjoint system, then one of $1, \ldots, 2^k$ is not covered by A.

Note that if $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(1 < n_1 < \ldots < n_k)$ is a disjoint system then it is not a cover of $\mathbb{Z}$ by the Davenport-Mirsky-Newman-Rado result.

P. Erdős [Mat. Lapok 13(1962)] proved a weaker version of Stein's Conjecture with 2^k replaced by $k2^k$.

Erdős' Conjecture

In 1965, P. Erdős offered $25 prize for a proof of his following conjecture which is a refinement of Stein's conjecture.

Erdős' Conjecture (1962). $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^{k}$ is a covering system if it covers all those integers from 1 to 2^k.

Remark. The 2^k in Erdős' conjecture is best possible because $\{2^{s-1}(\mathrm{mod}\ 2^s)\}_{s=1}^{k}$ covers $1, \ldots, 2^k - 1$ but does not cover any multiple of 2^k.

In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [Bull. Amer. Math. Soc. 1969; Proc. Amer. Math. Soc. 1970] supplied a long and awkward proof of the Erdős conjecture which involves some deep results concerning the distribution of primes. They actually proved that if the conjecture failed then there would be a counterexample with $k < 20$, and then claimed that the case $k < 20$ "*may be checked by more special arguments*".

A local-global theorem

As usual, the fractional part of a real number x is denoted by $\{x\}$.

For real numbers α and $\beta > 0$, we define

$$\alpha + \beta\mathbb{Z} := \{\alpha + \beta x : \ x \in \mathbb{Z}\}.$$

The First Local-Global Theorem (Z.-W. Sun [Acta Arith. 72(1995)]). Let $\alpha_1, \ldots, \alpha_k$ be real numbers and $\beta_1, \ldots, \beta_k$ be positive real numbers. Then $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ covers all the integers at least m times if it covers $|S|$ consecutive integers at least m times, where

$$S = \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : \ I \subseteq \{1, \ldots, k\} \right\}.$$

Remark. For $1 \leqslant m \leqslant k$, clearly an integer x is covered by $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ at least m times if and only if it is covered by $\{\alpha_s + \beta_s\mathbb{Z}\}_{\substack{s=1 \\ s \notin J}}^k$ for all $J \subseteq \{1, \ldots, k\}$ with $|J| = m - 1$. So the theorem is reduced to the case $m = 1$.

Proof of the Local-Global Theorem with $m = 1$

For any integer x, clearly

$$\begin{aligned}
& x \text{ is covered by } A \\
\iff & e^{2\pi i(\alpha_s - x)/\beta_s} = 1 \text{ for some } s = 1, \ldots, k \\
\iff & \prod_{s=1}^{k} \left(1 - e^{2\pi i(\alpha_s - x)/\beta_s}\right) = 0 \\
\iff & \sum_{I \subseteq \{1,\ldots,k\}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s} \cdot e^{-2\pi i x \sum_{s\in I} 1/\beta_s} = 0 \\
\iff & \sum_{\theta \in S} e^{-2\pi i x\theta} z_\theta = 0,
\end{aligned}$$

where

$$z_\theta = \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\sum_{s\in I} 1/\beta_s\} = \theta}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s}.$$

Proof of the Local-Global Theorem with $m = 1$

Suppose that A covers $|S|$ consecutive integers

$$a, a+1, \ldots, a+|S|-1$$

where $a \in \mathbb{Z}$. By the above,

$$\sum_{\theta \in S} (e^{-2\pi i\theta})^r (e^{-2\pi i a\theta} z_\theta) = 0$$

for $r = 0, 1, \ldots, |S|-1$. As the determinant

$$\left| (e^{-2\pi i\theta})^r \right|_{0 \leqslant r < |S|,\, \theta \in S}$$

is of Vandermonde's type and hence nonzero, by Cramer's rule we have $z_\theta = 0$ for all $\theta \in S$. Therefore

$$\sum_{\theta \in S} e^{-2\pi i x\theta} z_\theta = 0$$

for all $x \in \mathbb{Z}$, i.e., any $x \in \mathbb{Z}$ is covered by A.

A corollary

Corollary. Let $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ and $M=\max_{n\in\mathbb{Z}^+}|\{1\leqslant s\leqslant k:\ n_s=n\}|$. If A covers $2^{k-M}(M+1)$ consecutive integers at least m times then A is an m-cover.

Proof. Choose $n\in\mathbb{Z}^+$ with $J=\{1\leqslant s\leqslant k:\ n_s=n\}$ of cardinality M. Then

$$\begin{aligned}&\left|\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\ldots,k\}\right\}\right|\\ \leqslant&\left|\left\{\sum_{s\in I\cap J}\frac{1}{n_s}+\sum_{s\in I\setminus J}\frac{1}{n_s}:\ I\subseteq\{1,\ldots,k\}\right\}\right|\\ \leqslant&\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq J\right\}\right|\times\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\ldots,k\}\setminus J\right\}\right|\\ \leqslant&\left|\left\{\frac{|I|}{n}:\ I\subseteq J\right\}\right|\times|\{I:\ I\subseteq\{1,\ldots,k\}\setminus J\}|\\ =&(|J|+1)2^{k-|J|}=(M+1)2^{k-M}.\end{aligned}$$

Crittenden-Vanden Eynden Conjecture

Example: Let $1 \leqslant \ell \leqslant k$. Then the residue classes

$$2^{i-1} (\mathrm{mod}\ 2^i) \ \ (i = 1, \ldots, k - \ell + 1)$$

are disjoint and their union is $\mathbb{Z} \setminus 0 (\mathrm{mod}\ 2^{k-\ell+1})$. Thus the k residue classes

$$1 (\mathrm{mod}\ \ell), \ldots, \ell - 1 (\mathrm{mod}\ \ell), \ell (\mathrm{mod}\ 2\ell), \ldots, 2^{k-\ell} \ell (\mathrm{mod}\ 2^{k-\ell+1} \ell)$$

are disjoint and their union is $\mathbb{Z} \setminus 0 (\mathrm{mod}\ 2^{k-\ell+1} \ell)$. So, the system A of these k residue classes covers $1, \ldots, 2^{k-\ell+1} \ell - 1$ but it is not a cover of $\mathbb{Z}$. Note that each modulus in A occurs at most $\ell - 1$ times and also every modulus of A is at least ℓ.

Crittenden-Vanden Eynden Conjecture [Amer. Math. Monthly 79(1972)]. Let $A = \{a_s (\mathrm{mod}\ n_s)\}_{s=1}^{k}$ with each modulus at least ℓ, where $1 \leqslant \ell \leqslant k$. A is a cover of $\mathbb{Z}$ if it covers $1, \ldots, 2^{k-\ell+1} \ell$.

Remark. When $\ell = 1, 2$ this reduces to Erdős' conjecture. The above conjecture in the case $\ell = 3$ was proved by R.J. Simpson [J. Austral Math. Soc. 63 (1997)].

An application of the First Local-Global Theorem

Theorem. Let $m_1, \ldots, m_{n-1}$ $(n > 1)$ be integers. If there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid s m_{\sigma(s)}$ for all $s = 1, \ldots, n-1$, then the set

$$\left\{ \sum_{i \in I} m_i : \ I \subseteq \{1, \ldots, n-1\} \right\}$$

contains a complete system of residues modulo n.

Proof. $A = \{s + (n/m_{\sigma(s)})\mathbb{Z}\}_{s=1}^{n-1}$ covers $1, \ldots, n-1$ but it does not cover 0. By the First Local-Global Theorem, the fractional parts

$$\left\{ \sum_{s \in I} \frac{1}{n/m_{\sigma(s)}} \right\} \quad (I \subseteq \{1, \ldots, n-1\})$$

must have more than $n-1$ distinct values. Thus, the set

$$\left\{ \sum_{i \in I} m_i : \ I \subseteq \{1, \ldots, n-1\} \right\} = \left\{ \sum_{s \in I} m_{\sigma(s)} : \ I \subseteq \{1, \ldots, n-1\} \right\}$$

contains a complete system of residues modulo n.

Ge and Sun's result for finite abelian groups

For a finite multiplicative group G, its exponent $\exp(G)$ is defined to be the least positive integer such that $x^n = e$ for all $x \in G$, where e is the identity of G. For a finite abelian group G, $\exp(G)$ is known to be $\max\{o(x) : \ x \in G\}$, where $o(x)$ denotes the order of x. If G is an additive group, then for $k \in \mathbb{Z}^+$ and $a \in G$ we write ka for the sum $a_1 + \ldots + a_k$ with $a_1 = \cdots = a_k = a$.

Theorem (F. Ge and Z.-W. Sun [Electron. J. Combin. 24(2017)]). Let G be a finite additive abelian group with exponent $n > 1$. For any $a_1, \ldots, a_{n-1} \in G$, there is a permutation $\sigma \in S_{n-1}$ such that all the elements $sa_{\sigma(s)}$ $(s = 1, \ldots, n-1)$ are nonzero if and only if

$$\left|\left\{1 \leqslant s < n : \ \frac{n}{d} a_s \neq 0\right\}\right| \geqslant d - 1 \ \text{ for all } d \in D(n).$$

where $D(n)$ denotes the set of all positive divisors of n.

Remark. This theorem for the cyclic group $\mathbb{Z}/n\mathbb{Z}$ was conjectured by Sun in 2004.

Ming-Zhi Zhang's Result

For convenience, we simply write $a(n)$ to stand for the residue class $a(\mathrm{mod}\ n) = a + n\mathbb{Z}$.

In 1989, by using the Riemann zeta function, M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] showed the following surprising result.

Zhang's Result (1989): If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ then $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq [1,k] = \{1,\ldots,k\}$.

The starting point of Zhang is that A forms a cover of $\mathbb{Z}$ if and only if

$$\prod_{s=1}^{k}\left(1 - e^{2\pi i(n+a_s)/n_s}\right) = 0 \ \text{ for all } n = 1,2,3,\ldots.$$

The crucial trick in Zhang's proof is that for a real number c the series $\sum_{n=1}^{+\infty} \frac{e^{2\pi icn}}{n}$ diverges if and only if c is an integer.

On exact m-covers

If $A=\{a_s(n_s)\}_{s=1}^k$ is an exact m-cover, then $\sum_{s=1}^k\frac{1}{n_s}=m$.

In 1976 Š. Porubský asked whether every exact m-cover is a union of m disjoint covers. Choi supplied the following exact 2-cover

$$\{1(2);0(3);2(6);0,4,6,8(10);1,2,4,7,10,13(15);5,11,12,22,23,29(30)\},$$

which is not a union of two exact covers.

Ming-Zhi Zhang [J. Sichuan Univ. (Nat. Sci. Ed.), 1991]: For each $m=2,3,\ldots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be a union of an n-cover and an $(m-n)$-cover with $0<n<m$. (By a *graph-theoretic approach*.)

Hao Pan and Li-Lu Zhao [Adv. in Appl. Math. 43(2009)]: For each $m=2,3,\ldots$ there is an exact m-cover of $\mathbb{Z}$ which is not a union of two covers of $\mathbb{Z}$.

On exact m-covers

For any exact m-cover $\{a_s(n_s)\}_{s=1}^k$, $\sum_{s=1}^k 1/n_s = m \in \mathbb{Z}^+$. So Zhang's result for exact m-covers is trivial.

Z.-W. Sun [Israel J. Math. 77(1992)]. Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover of $\mathbb{Z}$. Then, for any $n = 0 \ldots, m$ we have

$$\left|\left\{I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{1}{n_s} = n\right\}\right| \geqslant \binom{m}{n}.$$

Remark. Considering the exact m-cover of $\mathbb{Z}$ consisting m copies of $0(1)$, we see that the lower bound $\binom{m}{n}$ is best possible.

Lemma. $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$ if and only if we have the identity

$$\prod_{s=1}^k \left(1 - x^{N/n_s} e^{2\pi i a_s/n_s}\right) = (1 - x^N)^m,$$

where N is the least common multiple of $n_1, \ldots, n_k$.

Proofs

Proof of the Lemma. It suffices to note that

$$1 - x^{N/n_s} e^{2\pi i a_s/n_s} = 0 \Longrightarrow x^N = 1$$

and

$$1 - (e^{-2\pi i r/N})^{N/n_s} e^{2\pi i a_s/n_s} = 0 \iff r \equiv a_s \pmod{n_s}.$$

Proof of the Theorem. Comparing the coefficients of x^{nN} in both sides of the identity

$$\prod_{s=1}^{k} \left(1 - x^{N/n_s} e^{2\pi i a_s/n_s}\right) = (1 - x^N)^m,$$

we get

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \sum_{s\in I} \frac{1}{n_s} = n}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s/n_s} = (-1)^n \binom{m}{n}.$$

So there are at least $\binom{m}{n}$ subsets I of $\{1, \ldots, k\}$ with $\sum_{s\in I} \frac{1}{n_s} = n$.

Further extensions

Z.-W. Sun [Acta Arith. 81(1997)]: Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover. Then, for any $t = 1, \ldots, k$ and $a = 0, 1, 2, \ldots$ we have

$$\left|\left\{I \subseteq [1,k] \setminus \{t\} :\ \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_t}\right\}\right| \geqslant \binom{m-1}{\lfloor a/n_t \rfloor}.$$

Z.-W. Sun [Bull. Austral. Math. Soc. 81(2010)]: If $\{a_s(n_s)\}_{s=0}^k$ covers each integer more than $m = \lfloor \sum_{s=1}^k \frac{1}{n_s} \rfloor$ times, then

$$\left|\left\{I \subseteq [1,k] :\ \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_0}\right\}\right| \geqslant \binom{m}{\lfloor a/n_0 \rfloor}$$

for all $a \in \mathbb{N}$. In particular, if $A = \{a_s(n_s)\}_{s=1}^k$ has covering multiplicity $m(A) = \lfloor \sum_{s=1}^k \frac{1}{n_s} \rfloor$, then for any $n \in \mathbb{N}$ we have

$$\left|\left\{I \subseteq [1,k] :\ \sum_{s \in I} \frac{1}{n_s} = n\right\}\right| \geqslant \binom{m(A)}{n}.$$

Characterizing *m*-covers

As usual, the fractional part of a real number x is denoted by $\{x\}$.

For real numbers α and $\beta > 0$, we define

$$\alpha + \beta\mathbb{Z} := \{\alpha + \beta x : \ x \in \mathbb{Z}\}.$$

Z.-W. Sun [Acta Arith. 72(1995)]: Let $\alpha_1, \ldots, \alpha_k$ be real numbers and $\beta_1, \ldots, \beta_k$ be positive real numbers. Then $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^{k}$ covers all the integers at least m times if and only if

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\sum_{s\in I} \frac{1}{\beta_s}\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s\in I} 1/\beta_s \rfloor}{n} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s} = 0$$

for all $n = 0, \ldots, m-1$ and $0 \leqslant \theta < 1$.

Remark. The starting point is that $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^{k}$ covers x at least m times if and only if

$$\prod_{s=1}^{k} \left(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s}\right) = o((1-r)^{m-1}) \ \ (r \to 1).$$

m-covers and unit fractions

Using the above characterization of m-covers, we deduced the following properties of m-covers related to unit fractions.

Let $\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$ and let $m_1, \ldots, m_k$ be positive integers.

Z.-W. Sun [Trans. Amer. Math. Soc. 348(1996)]: There are at least m positive integers in the form $\sum_{s\in I} m_s/n_s$ with $I \subseteq [1,k]$.

Z.-W. Sun [Proc. Amer. Math. Soc. 127(1999)]: For any $J \subseteq \{1,\ldots,k\}$, there are at least m subsets I of $\{1,\ldots,k\}$ with $I \neq J$ such that $\{\sum_{s\in I} \frac{m_s}{n_s}\} = \{\sum_{s\in J} \frac{m_s}{n_s}\}$.

H. Pan and Z.-W. Sun [Proc. Amer. Math. Soc. 135(2007)]: For any $0 \leqslant \theta < 1$, if the set

$$\left\{I \subseteq \{1,\ldots,k\} :\ \left\{\sum_{s\in I} \frac{m_s}{n_s}\right\} = \theta\right\}$$

is nonempty, then its cardinality is at least 2^m.

Classical results on zero-sums

EGZ Theorem (P. Erdős, A. Ginzburg and A. Ziv [Bull. Research Council. Israel, 1961]): For any $c_1, \ldots, c_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [1, 2n-1]$ with $|I| = n$ such that $\sum_{s \in I} c_s \equiv 0 \pmod{n}$. In other words, given $2n-1$ (not necessarily distinct) elements of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, we can select n of them with the sum vanishing.

For a finite abelian group G (written additively), the **Davenport constant** $D(G)$ is defined as the smallest positive integer k such that any sequence $(c_1, \ldots, c_k)$ (repetition allowed) of elements of G has a subsequence $c_{i_1}, \ldots, c_{i_l}$ $(i_1 < \ldots < i_l)$ with zero-sum (i.e. $c_{i_1} + \ldots + c_{i_l} = 0$). It is easy to see that $D(G) \leqslant |G|$.

We may explain $D(\mathbb{Z}_n) = n$ via covers of $\mathbb{Z}$. As $A = \{r(n)\}_{r=1}^{n}$ is a cover of $\mathbb{Z}$, for any $m_1, \ldots, m_n \in \mathbb{Z}^+$ we have $\sum_{s \in I} \frac{m_s}{n} \in \mathbb{Z}^+$ (i.e., $\sum_{s \in I} m_s \equiv 0 \pmod{n}$) for some $\emptyset \neq I \subseteq \{1, \ldots, k\}$.

Olson's Theorem [J. Number Theory 1(1969)]. Let p be any prime. For an abelian p-group $G \cong \mathbb{Z}_{p^{h_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{h_l}}$ we have $D(G) = 1 + \sum_{t=1}^{l}(p^{h_t} - 1)$.

Kemnitz's Conjecture

For a finite abelian group G, define $s(G)$ to be the least positive integer k such that any sequence $(a_1, \ldots, a_k)$ of elements of G has a zero-sum subsequence of length $\exp(G)$, where the exponent $\exp(G)$ of G is the least $n \in \mathbb{Z}^+$ with $nx = 0$ for all $x \in G$.

By the EGZ theorem, $s(\mathbb{Z}_n) = 2n - 1$ for any positive integer n.

What is the smallest integer $l = s(\mathbb{Z}_n^2)$ such that every sequence of l elements in $\mathbb{Z}_n^2 = \mathbb{Z}_n \oplus \mathbb{Z}_n$ contains a zero-sum subsequence of length n?

In 1983 Kemnitz [Ars Combin.] conjectured that $s(\mathbb{Z}_n^2) = 4n - 3$, and the conjecture can be reduced to the case with n prime. This was finally confirmed by C. Reiher [Ramanujan J. 13 (2007)].

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

Alon-Dubiner Lemma. Let q be a prime power, and let $c_1, \ldots, c_{3q}$ be elements of $\mathbb{Z}_q^2$ with $c_1 + \ldots + c_{3q} = 0$. Then there is an $I \subseteq [1, 3q]$ with $|I| = q$ such that $\sum_{i \in I} c_i = 0$.

Connections between covers of $\mathbb{Z}$ and zero-sum problems

In 2003 Z.-W. Sun established connections between covers of $\mathbb{Z}$ and some classical theorems on zero-sums such as $D(\mathbb{Z}_m) = m$, the EGZ theorem, the Alon-Dubiner lemma and Olson's theorem.

The results were first announced in Electron. Res. Announc. Amer. Math. Soc. 9(2003). Full proofs of them were published in Israel J. Math. 170(2009).

Note that the speaker's discovery is **quite different from** Gao and Geroldinger's work in the paper

W. Gao and A. Geroldinger, *Zero-sum problems and coverings by proper cosets*, European J. Combin. **24** (2003), 531–549.

Connections between covers of $\mathbb{Z}$ and zero-sum problems

The following theorem in the case $n_1 = \cdots = n_k = 1$ reduces to known results on zero-sums.

Theorem (Z.-W. Sun [Israel J. Math. 170(2009)]). Let $A = \{a_s(n_s)\}_{s=1}^k$ and let G be an abelian group with $|G| = q$ a prime power.

(i) If A forms a q-cover of $\mathbb{Z}$, then for any $m_1, \ldots, m_k \in \mathbb{Z}$ there exists a nonempty $I \subseteq [1, k]$ such that $\sum_{s\in I} m_s/n_s \in q\mathbb{Z}$.

(ii) If $\{w_A(x) : x \in \mathbb{Z}\} \subseteq \{D(G) + q - 1, \ldots, 2q\}$, then for any $c_1, \ldots, c_k \in G$ there exists an $I \subseteq [1, k]$ such that $\sum_{s\in I} 1/n_s = q$ and $\sum_{s\in I} c_s = 0$.

(iii) If A is an exact $3q$-cover of $\mathbb{Z}$, then for any $c_1, \ldots, c_k \in G \oplus G$ with $c_1 + \ldots + c_k = 0$, there exists an $I \subseteq [1, k]$ such that $\sum_{s\in I} 1/n_s = q$ and $\sum_{s\in I} c_s = 0$.

(iv) Suppose that A is a $D(G)$-cover of $\mathbb{Z}$. Then, for any $m_1, \ldots, m_k \in \mathbb{Z}$ and $c_1, \ldots, c_k \in G$, there is a nonempty $I \subseteq [1, k]$ such that $\sum_{s\in I} c_s = 0$ and $\sum_{s\in I} m_s/n_s \in \mathbb{Z}$.

Conjecture

Conjecture (Z.-W. Sun [Israel J. Math. 170(2009)]) The above theorem remains valid if the prime power q is replaced by any positive integer. In particular, if $\{a_s(n_s)\}_{s=1}^k$ is an m-cover of $\mathbb{Z}$, then $\sum_{s\in I}\frac{1}{n_s}\in m\mathbb{Z}$ for some $\emptyset\neq I\subseteq\{1,\ldots,k\}$.

Why can I prove the theorem with q a prime power? The following lemma is of technical importance.

Lemma (Z.-W. Sun, 2003). Let p be a prime, and let $a\in\mathbb{N}$ and $m\in\mathbb{Z}$. Then we have the following congruence

$$\binom{m-1}{p^a-1}\equiv\begin{cases}1\pmod p & \text{if } p^a\mid m,\\ 0\pmod p & \text{otherwise.}\end{cases}.$$

Remark. Let m be an integer and let p be a prime. Fermat's little theorem tells that we can characterize whether p divides m as follows:

$$1-m^{p-1}\equiv\begin{cases}1\pmod p & \text{if } p\mid m,\\ 0\pmod p & \text{if } p\nmid m.\end{cases}$$

Main Theorem

Actually the theorem follows from the following Main Theorem.

Main Theorem (Z.-W. Sun [Israel J. Math. 170(2009)]). Let G be an additive abelian p-group where p is a prime. Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is a $(D(G) + p^h - 1)$-cover of $\mathbb{Z}$ with $h \in \mathbb{N} = \{0, 1, \ldots\}$. Let $c_1, \ldots, c_k \in G$ and $m_1, \ldots, m_k \in \mathbb{Z}$. Then

$$\left|\left\{ I \subseteq [1,k] : \sum_{s\in I} c_s = c \text{ and } \sum_{s\in I} \frac{m_s}{n_s} \in \alpha + p^h\mathbb{Z} \right\}\right| \neq 1$$

for any $c \in G$ and rational number α. In particular, $(c_1, \ldots, c_k)$ has a zero-sum subsequence $(c_s)_{s\in I}$ with $\emptyset \neq I \subseteq [1,k]$ satisfying the restriction $\sum_{s\in I} m_s/n_s \in p^h\mathbb{Z}$.

The Second Local-Global Theorem

The Second Local-Global Theorem (Z.-W. Sun [J. Algebra, 293(2005)]). Let G be any abelian group written additively, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to G with periods $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. Set $\psi = \psi_1 + \cdots + \psi_k$ and

$$S(n_1, \ldots, n_k) = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : \ r = 0, \ldots, n_s - 1 \right\}.$$

(i) There are periodic maps $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1} : \mathbb{Z} \to \mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$ such that $\psi(x) = \sum_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\psi(r)$ for all $x \in \mathbb{Z}$. In particular, values of ψ are completely determined by the set $S(n_1, \ldots, n_k)$ and the initial values $\psi(0), \ldots, \psi(|S(n_1, \ldots, n_k)| - 1)$.

(ii) ψ is constant if $\psi(x)$ equals a constant for $|S(n_1, \ldots, n_k)|$ $(\leqslant n_1 + \cdots + n_k - k + 1)$ consecutive integers x.

Remarks on $|S(n_1,\ldots,n_k)|$

Let $D=\{d\in\mathbb{Z}^+:\ d\mid n_s \text{ for some } s=1,\ldots,k\}$. Then

$$|S(n_1,\ldots,n_k)|=\left|\bigcup_{d\in D}\left\{\frac{c}{d}:\ 0\leqslant c<d\ \&\ (c,d)=1\right\}\right|=\sum_{d\in D}\varphi(d),$$

where φ is the well-known Euler function.

As $|\bigcap_{s\in I}\{r/n_s:\ r=0,\ldots,n_s-1\}|=\gcd(n_s:\ s\in I)$ for all $\emptyset\neq I\subseteq\{1,\ldots,k\}$, by the inclusion-exclusion principle, we have

$$|S(n_1,\ldots,n_k)|=\sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}}(-1)^{|I|-1}\gcd(n_s:\ s\in I).$$

Two corollaries

As $|S(m,n)| = m+n-\gcd(m,n)$, we have the following consequence.

Corollary 1. Let g and h be maps from $\mathbb{Z}$ to an additive abelian group G with positive periods m and n respectively. Then $\{g(x)-h(x):\ x\in\mathbb{Z}\}$ is contained in the subgroup of G generated by those $g(r)-h(r)$ with $0\leqslant r<m+n-\gcd(m,n)$; in particular, g and h are identical if $g(r)=h(r)$ for all $r=0,\ldots,m+n-\gcd(m,n)-1$.

Fine-Wilf Theorem (N.J. Fine and H.S. Wilf [Proc. Amer. Math. Soc. 16(1965)]). Let g and h be maps from $\mathbb{Z}$ to the real field $\mathbb{R}$ with positive periods m and n respectively. If $g(r)=h(r)$ for all $r=0,\ldots,m+n-\gcd(m,n)-1$, then we have $g=h$.

Corollary 2. $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is an exact m-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^k\{r/n_s:\ \ r=0,\ldots,n_s-1\}|$ $(\leqslant\sum_{s=1}^k n_s-k+1)$ consecutive integers exactly m times.

Part II. Covers of Groups by Cosets

Covers of groups by cosets

The addition group $\mathbb{Z}$ is an infinite cyclic group. Its subgroups have the form $n\mathbb{Z}$ $(n = 0, 1, 2, \ldots)$.

For $n \in \mathbb{Z}^+$, $n\mathbb{Z}$ is a normal subgroup of the additive group $\mathbb{Z}$ and the quotient group $\mathbb{Z}/n\mathbb{Z}$ is of order n. A residue class $a + n\mathbb{Z}$ is a coset of $n\mathbb{Z}$ in $\mathbb{Z}$. Note also that n is the index of the subgroup $n\mathbb{Z}$ in $\mathbb{Z}$.

Instead of covers of $\mathbb{Z}$ by finitely many residue classes $a_s + n_s\mathbb{Z}$ $(s = 1, \ldots, k)$, we may also study covers of a group G by finitely many left cosets $a_1G_1, \ldots, a_kG_k$. Instead of the moduli n_s in the residue class $a_s + n_s\mathbb{Z}$, we may investigate the indices $n_s = [G : G_s]$.

Let H be a subgroup of a group G. Note that a right coset Ha of H is also a left coset $a(a^{-1}Ha)$ of $a^{-1}Ha$.

Disjoint covers of a group by left or right cosets

Let H be a subgroup of a group G with $[G : H] = k < \infty$. Then we can partition G into k left cosets $g_1H, \ldots, g_kH$, and $\{g_iH\}_{i=1}^k$ forms a disjoint cover of G by left cosets.

Let $\{Ha_i\}_{i=1}^k$ be a right coset decomposition of G. Then $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of G where $G_i = a_i^{-1}Ha_i$. Observe that

$$\bigcap_{i=1}^{k} G_i = \bigcap_{i=1}^{k} \bigcap_{h\in H} a_i^{-1}h^{-1}Hha_i = \bigcap_{g\in G} g^{-1}Hg$$

is the normal core H_G of H in G which is the largest normal subgroup of G contained in H.

In group theory, it is known that G/H_G can be embedded into the symmetric group $S_{[G:H]} = S_k$ and thus

$$\left[G : \bigcap_{i=1}^{k} G_i\right] = |G/H_G| \leqslant k!.$$

A basic theorem on covers of groups

An Example of M. J. Tomkinson. Let $k > 1$ be a positive integer, and let G be the symmetric group S_k and H be the stabilizer of 1. Then $G_i = (1i)^{-1}H(1i)$ is the stabilizer of i for each $i = 1, \ldots, k$. Clearly,

$$\{G_1, (12)G_2, \ldots, (1k)G_k\} = \{H, H(12), \ldots, H(1k)\}$$

forms a disjoint cover of G with $\bigcap_{i=1}^{k} G_i = H_G = \{e\}$. Note that $[G : \bigcap_{i=1}^{k} G_i] = |G| = k!$.

A Basic Theorem on Covers of Groups. Let $\mathcal{A} = \{a_i G_i\}_{i=1}^{k}$ be a finite system of left cosets in a group G where $G_1, \ldots, G_k$ are subgroups of G. Suppose that $\mathcal{A}$ forms a minimal cover of G (i.e. $\mathcal{A}$ covers all the elements of G but none of its proper systems does).

(i) (B. H. Neumann, 1954) There is a constant c_k depending only on k such that $[G : G_i] \leqslant c_k$ for all $i = 1, \ldots, k$.
(ii) (M. J. Tomkinson, 1987) We have $[G : \bigcap_{i=1}^{k} G_i] \leqslant k!$, where the upper bound $k!$ is best possible.

Tomkinson's proof of the second part

We show by induction that

$$\left[\bigcap_{i\in I} G_i : \bigcap_{i=1}^{k} G_i\right] \leqslant (k-|I|)! \qquad (*_I)$$

for all $I \subseteq \{1,\ldots,k\}$, where $\bigcap_{i\in\emptyset} G_i$ is regarded as G.

Clearly $(*_I)$ holds for $I=\{1,\ldots,k\}$. Now let $I\subset\{1,\ldots,k\}$ and assume $(*_J)$ for all $J\subseteq\{1,\ldots,k\}$ with $|J|>|I|$. Since $\{a_iG_i\}_{i\in I}$ is not a cover of G, there is an $a\in G$ not covered by $\{a_iG_i\}_{i\in I}$. Clearly $a(\bigcap_{i\in I}G_i)$ is disjoint from the union $\bigcup_{i\in I}a_iG_i$ and hence contained in $\bigcup_{j\notin I}a_jG_j$. Thus

$$a\left(\bigcap_{i\in I} G_i\right) = \bigcup_{\substack{j\notin I\\ a_jG_j\cap a(\bigcap_{i\in I}G_i)\neq\emptyset}} \left(a_jG_j \cap a\left(\bigcap_{i\in I} G_i\right)\right),$$

$$\left[\bigcap_{i\in I} G_i : H\right] \leqslant \sum_{j\notin I}\left[G_j\cap\bigcap_{i\in I} G_i : H\right] \leqslant \sum_{j\notin I}(k-(|I|+1))! = (k-|I|)!$$

where $H=\bigcap_{i=1}^{k} G_i$. This concludes the induction proof.

Herzog-Schöheim Conjecture

Herzog-Schönheim Conjecture [Canad. Math. Bull. 17(1974)]: Let $\{a_i G_i\}_{i=1}^{k}$ ($k > 1$) be a partition of a group G into left cosets of subgroups $G_1, \ldots, G_k$. Then the (finite) indices $n_1 = [G : G_1], \ldots, n_k = [G : G_k]$ cannot be distinct.

This is an extension of the confirmed Erdős conjecture to covers of groups.

Berger, Felzenbaum and Fraenkel [Canad. Bull. Math. 1986]: The Herzog-Schöheim Conjecture holds for finite nilpotent groups.

L. Margolis and O. Schnabel [Beitr. Algebra Geom. 2019]: The Herzog-Schöheim Conjecture holds for all groups G with $|G| < 1440$.

Subnormal subgroups

A subgroup H of a group G is said to be *subnormal* in G if there are a finite chain of subgroups

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

such that H_i is normal in H_{i+1} for all $i = 0, \ldots, n-1$.

It is known that all the subgroups of a nilpotent group G are subnormal in G.

My result on the Herzog-Schöheim Conjecture

Z.-W. Sun [J. Algebra 273(2004)]. Let $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group G with not all the G_i equal to G. Suppose that $\mathcal{A}$ covers all the elements of G the same number of times, and that among the (finite) indices

$$n_1 = [G : G_1] \leqslant \ldots \leqslant n_k = [G : G_k].$$

each occurs at most $M \in \mathbb{Z}^+$ times. If all the G_i are subnormal in G, then $M > 1$ and

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log\log M).$$

A lemma on subnormal subgroups

One of the key lemmas is the following one which is the main reason why covers involving subnormal subgroups are better behaved than general covers.

A Lemma on Indices of Subnormal Subgroups (Z. W. Sun). Let G be a group, and let $P(n)$ denote the set of prime divisors of a positive integer n.

(i) [European J. Combin. 2001] If $G_1, \ldots, G_k$ are subnormal subgroups of G with finite index, then

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \Bigm| \prod_{i=1}^{k} [G : G_i] \text{ and } P\left(\left[G : \bigcap_{i=1}^{k} G_i\right]\right) = \bigcup_{i=1}^{k} P([G : G_i]).$$

(ii) [J. Algebra, 2004] Let H be a subnormal subgroup of G with finite index. Then

$$P(|G/H_G|) = P([G : H]).$$

A lemma on unions of cosets

Here is another useful lemma of combinatorial nature.

A Lemma on Unions of Cosets (Z. W. Sun [J. Algebra, 2004]). Let G be a group and H its subgroup with finite index N. Let $a_1, \ldots, a_k \in G$, and let $G_1, \ldots, G_k$ be subnormal subgroups of G containing H. Then the union $\bigcup_{i=1}^{k} a_i G_i$ contains at least $|\bigcup_{i=1}^{k} 0(\mathrm{mod}\ n_i) \cap \{0, 1, \ldots, N-1\}|$ left cosets of H, where $n_i = [G : G_i]$.

This lemma implies the following result of Z. W. Sun [Internat. J. Math. 2006]: *If $G_1, \ldots, G_k$ are normal Hall subgroups of a finite group G, then*

$$\left|\bigcup_{i=1}^{k} a_i G_i\right| \geqslant \left|\bigcup_{i=1}^{k} G_i\right|.$$

(A subgroup H of a finite group G is called a *Hall subgroup* of G if $|H|$ is relatively prime to $[G : H]$.)

Tools from analytic number theory

We also need the following theorems in analytic number theory.

The Prime Number Theorem with Error Terms For $x \geqslant 2$ we have

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where $\pi(x) = \sum_{p \leqslant x} 1$ is the number of primes not exceeding x.

Mertens' Theorem. For $x \geqslant 2$ we have

$$\prod_{p \leqslant x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

Using the above two theorems we can deduce the following lemma.

An Analytic Lemma (Z. W. Sun [J. Algebra, 2004]). For $M \geqslant 2$, if $q > 1$ is an integer with $q < M\prod_{p \leqslant q} p/(p-1)$ then

$q < e^{\gamma} M \log M + O(M \log\log M)$ and $\pi(q) \leqslant e^{\gamma} M + O(M/\log M)$,

where the O-constants are absolute.

Huhn-Megyesi's problems on disjoint residue classes

A finite sequence $\{n_s\}_{s=1}^k$ of positive integers is called *harmonic* if $n_1, \ldots, n_k$ are the moduli of pairwise disjoint residue classes.

In 1982 A. P. Huhn and L. Megyesi [Discrete Math. 41(1982)] posed two open problems on harmonic sequences. Both were solved by Sun [Chinese Ann. Math. 13A(1992)] negatively.

A Problem of Huhn-Megyesi: Let $n_1, \ldots, n_k$ be positive integers with $\sum_{s=1}^k 1/n_s \leqslant 1$. Is the following a necessary and sufficient condition for $\{n_s\}_{s=1}^k$ to be harmonic?

$$\max_{\substack{i,j\in I\\ i\neq j}}(n_i, n_j) \geqslant |I| \ \text{ for all } I \subseteq \{1, \ldots, k\} \text{ with } |I| \geqslant 2. \qquad (*)$$

Sun [Chinese Ann. Math. 13A(1992)] showed that for $k \geqslant 5$ there are positive integers $n_1, \ldots, n_k$ satisfying $\sum_{s=1}^k 1/n_s \leqslant 1$ and $(*)$ but $\{n_s\}_{s=1}^k$ is *not* harmonic.

A conjecture on harmonic sequences

Z.-W. Sun [Discrete Math. 104(1992)]: Let $n_1, \ldots, n_k$ be positive integers. If

$$|\{\{i,j\}:\ 1 \leqslant i < j \leqslant k \text{ and } \gcd(n_i, n_j) = d\}| < \sqrt{\frac{d+7}{8}}$$

for all $d = 1, 2, 3, \ldots$, then $\{n_s\}_{s=1}^{k}$ is harmonic.

Conjecture (Z.-W. Sun [Discrete Math. 104(1992)]). Let $n_1, \ldots, n_k$ be positive integers. If

$$|\{\{i,j\}:\ 1 \leqslant i < j \leqslant k \text{ and } \gcd(n_i, n_j) = d\}| < 2d - 1$$

for all $d = 1, 2, 3, \ldots$, then $\{n_s\}_{s=1}^{k}$ is harmonic.

Y.-G. Chen [Discrete Math. 162(1996)]: Let $n_1, \ldots, n_k$ $(k > 1)$ be positive integers. If

$$|\{\{i,j\}:\ 1 \leqslant i < j \leqslant k \text{ and } \gcd(n_i, n_j) = d\}| \leqslant \frac{kd}{4(k-1)}$$

for all $d = 1, \ldots, 2k-3$, then $\{n_s\}_{s=1}^{k}$ is harmonic.

A challenging conjecture on disjoint cosets

Finally we mention a challenging conjecture arising from the speaker's study of Huhn-Megyesi problems and covers of groups.

A Conjecture on Disjoint Cosets (Z.-W. Sun, [Internat. J. Math., 2006]). Let G be a group, and $a_1G_1, \ldots, a_kG_k$ $(k > 1)$ be pairwise disjoint left cosets of G with all the indices $[G : G_i]$ finite. Then, for some $1 \leqslant i < j \leqslant k$ we have $\gcd([G : G_i], [G : G_j]) \geqslant k$.

Z.-W. Sun [Internat. J. Math. 2006] noted that this conjecture holds for p-groups as well as the special case $k = 2$. If G_1 and G_2 are subgroups of G with $[G : G_1]$ and $[G : G_2]$ finite and relatively prime, then $G_1G_2 = G$ and $a_1G_1 \cap a_2G_2 \neq \emptyset$ for all $a_1, a_2 \in G$.

W.-J. Zhu [Int. J. Mod. Math. 3(2008)] proved the conjecture for $k = 3, 4$ via several sophisticated lemmas. In 2009, Z. Gong confirmed the conjecture in the case $k = 5$. K. O'Bryant [Integers 2007] confirmed the conjecture for $G = \mathbb{Z}$ in the case $k \leqslant 20$.

Reduce the Disjoint Cosets Conjecture to finite groups

Suppose that $a_1G_1, \ldots, a_kG_k$ $(k > 1)$ be pairwise disjoint left cosets of a group G with all the indices $[G : G_i]$ finite. By Poincare's theorem, the intersection $F = \bigcap_{i=1}^{k} G_i$ has a finite index in G and hence $H = F_G$ also has a finite index in G.

Let $\bar{G} = G/H$, $\overline{a_i} = a_iH$ and $\overline{G_i} = G_i/H$ for all $i = 1, \ldots, k$. Then

$$\overline{a_i}\,\overline{G_i} \quad (i = 1, \ldots, k)$$

are pairwise disjoint left cosets in the finite group $\bar{G} = G/H$. Note that

$$[\bar{G} : \overline{G_i}] = [G : G_i] \quad \text{for all } i = 1, \ldots, k.$$

By the above, the Disjoint Cosets Conjecture can be reduced to finite groups.

m-covers of groups by cosets

Let m be a positive integer, and let $A = \{a_i G_i\}_{i=1}^{k}$ be a finite system of left cosets in a group G. If each element of G is covered by A at least (resp., exactly) m times, then we call A an *m-cover* (resp., *exact m-cover*) of G. If A is an m-cover of G but none of its proper subsystems does, then A is said to be a *minimal m-cover* of G.

The Neumann-Tomkinson theorem can be extended to minimal m-covers of groups (cf. Corollary 1 of Z. W. Sun [Fund. Math. 134(1990)]); it also has applications in Galois theory, groups rings, Banach spaces, projective geometry and Riemann surfaces as pointed out by T. Soundararajan and K. Venkatachaliengar [Acta Math. Vietnam 19(1994)].

Extremal problems for exact m-covers

Let $A = \{a_i G_i\}_{i=1}^k$ be an exact m-cover of a group G with $\bigcap_{i=1}^k G_i = H$. By the Neumann-Tomkinson theorem, $[G : H] \leqslant k!$. How to provide a sharp lower bound of k in terms of G and H?

An Example of Š. Znám. Let $n > 1$ be an integer with the factorization $\prod_{t=1}^r p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then $0 (\mathrm{mod}\ n)$ and the following $f(n) = \sum_{s=1}^r \alpha_s (p_s - 1)$ residue classes

$$jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1} (\mathrm{mod}\ p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha})$$
$$(\alpha = 1, \ldots, \alpha_s;\ j = 1, \ldots, p_s - 1;\ s = 1, \ldots, r)$$

form a disjoint cover of $\mathbb{Z}$ whose moduli have the least common multiple n. As a convention we define $f(1) = 0$. The function f is called the **Mycielski function**.

An example on exact m-covers of groups

An Example of Z. W. Sun. Let H be a subnormal subgroup of a group G with finite index. Let

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G. For each $i = 0, \ldots, n-1$, write

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i.$$

Then the following $d(G,H) = \sum_{i=0}^{n-1}([H_{i+1} : H_i] - 1)$ left cosets

$$b_j^{(i)} H_i \ (0 \leqslant i < n;\ 1 \leqslant j < [H_{i+1} : H_i]),$$

together with H and $m-1$ copies of G, form an exact m-cover of G by $m + d(G,H)$ left cosets of subgroups whose intersection is H. (In the case $H = G$ we define $d(G,H) = 0$.)

Mycielski's Conjecture and its refinements

Relation between the Mycielski Function f and $d(G,H)$ (Z. W. Sun, Fund. Math. 1990; European J. Combin. 2001). Let H be any subnormal subgroup of G with finite index. Then

$$d(G,H) \geqslant f([G:H]) \geqslant \log_2[G:H].$$

Also, $d(G,H) = f([G:H])$ if and only if G/H_G is solvable.

Mycielski's Conjecture (J. Mycielski, 1966). If $\{a_i G_i\}_{i=1}^k$ is a disjoint cover of an abelian group G, then $k \geqslant 1 + f([G:G_i])$ for all $i = 1, \ldots, k$.

Related Results on Exact m-covers. Let $A = \{a_i G_i\}_{i=1}^k$ be an exact m-cover of a group G with $\bigcap_{i=1}^k G_i = H$.

(i) (I. Korec [Fund. Math., 1974]) If $m = 1$ and $G_1, \ldots, G_k$ are normal in G, then $k \geqslant 1 + f([G:H])$.

(ii) (Z. W. Sun [European J. Combin., 2001]) If $G_1, \ldots, G_k$ are subnormal in G, then $k \geqslant m + d(G,H)$, with the lower bound best possible.

On minimal *m*-covers of abelian groups

Korec's and Sun's results on exact m-covers can be extended to minimal m-covers of $\mathbb{Z}$, see R. J. Simpson [Acta Arith., 1985] for the case $m = 1$ and Z. W. Sun [Internat. J. Math. 17(2006)] for general $m \geqslant 1$. However, they cannot be extended to minimal m-covers of abelian groups as illustrated by the following example.

An Example of G. Lettl and Z. W. Sun. Let G be the abelian group $C_p \times C_p$ where p is a prime and C_p is the cyclic group of order p. Then any element $a \neq e$ of G has order p. Let $G_1, \ldots, G_k$ be all the distinct subgroups of G with order p. If $1 \leqslant i < j \leqslant k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal cover of G with $\bigcap_{s=1}^k G_s = \{e\}$. Since $1 + k(p-1) = |\bigcup_{s=1}^k G_s| = |G| = p^2$, we have $k = p + 1 \geqslant 1 + f([G : G_s]) = 1 + f(p) = p$. However,

$$k = p + 1 \leqslant 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\left(G, \bigcap_{s=1}^{k} G_s\right),$$

and the last inequality becomes strict when $p > 2$.

On minimal m-covers of abelian groups

G. Lettl and Z.-W. Sun [Acta Arith. 131(2008)]: Let $A = \{a_sG_s\}_{s=1}^k$ be a minimal m-cover of an abelian group G by left cosets. Then

$$k \geqslant m + f([G : G_t]) \qquad \text{for any} \quad t = 1, \ldots, k.$$

This theorem (proved via characters of finite abelian groups) implies the following conjecture of W. D. Gao and A. Geroldinger [European J. Combin. 2003] who proved it for elementary abelian p-groups.

Gao-Geroldinger Conjecture (2003). Let G be a finite abelian group with identity e. If $G \setminus \{e\}$ is a union of k cosets $a_1G_1, \ldots, a_kG_k$, then we have $k \geqslant f(|G|)$.

In fact, if we set $a_0 = e$ and $G_0 = \{e\}$ then $\{a_sG_s\}_{s=0}^k$ forms a cover of G with a_0G_0 irredundant and hence

$$k + 1 \geqslant 1 + f([G : G_0]) = 1 + f(|G|).$$

Some further conjectures

A Conjecture of Z. W. Sun. (i) (2008) Whenever $A=\{a_iG_i\}_{i=1}^k$ forms an m-cover of a group G by left cosets with a_tG_t irredundant, we have the inequality $k\geqslant m+f([G:G_t])$.

(ii) (2004) If $A=\{a_iG_i\}_{i=1}^k$ forms a minimal m-cover of an abelian group G by left cosets or an exact m-cover of a solvable group G by left cosets, then we have $k\geqslant m+f(N)$, where N is the least common multiple of the indices $[G:G_1],\ldots,[G:G_k]$.

When $\{a_iG_i\}_{i=1}^k$ forms an exact m-cover of a solvable group G, the inequality $k\geqslant m+f([G:G_t])$ was shown by Berger, Felzenbaum and Fraenkel [Colloq. Math. 1988] in the case $m=1$ and proved by the speaker [European J. Combin. 2003] for general m.

A Conjecture of S. Guo and Z. W. Sun (2004). If $\{G_i\}_{i=1}^k$ forms a minimal m-cover of an abelian group G with $[G:\bigcap_{i=1}^k G_i]=\prod_{t=1}^r p_t^{\alpha_t}$, where $p_1,\ldots,p_r$ are distinct primes and $\alpha_1,\ldots,\alpha_r$ are positive integers. Then we have the inequality $k>m+\sum_{t=1}^r(\alpha_t-1)(p_t-1)$.

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Thank you!