IDEMPOTENT FACTORIZATION OF MATRICES OVER INTEGRAL DOMAINS II

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Recall:

An integral domain $R$ satisfies the property:

1. **$ID_m$:** if every $m \times m$ singular matrix over $R$ is a product of idempotent matrices.

2. **$GE_m$:** if every $m \times m$ invertible matrix over $R$ is a product of elementary matrices.

$R$ admits the idempotent (elementary) matrix factorization 1(E)MF if it satisfies $ID_m$ ($GE_m$) for an integer $m \geq 2$.

$ID_m(R) = \{ M \in M_m(R) : M \text{ product of idempotent singular matrices} \}$

$GE_m(R) = \{ M \in GL_m(R) : M \text{ product of elem. matrices} \}$
Over a Bézout domain

\[1D_2 \iff IMF \iff EMF \iff GE_2\]

What happens outside Bézout domains?
THEOREM Bhaskara Rao 2009:

Let $R$ be a projective-free domain (i.e., every finitely-generated projective $R$-module is free).

If $R$ satisfies $1D_2$, then $R$ is a Bézout domain.

This result and those by Laffey, Ruitenburg ... led to

CONJECTURE (Salce, Zanardo 2014)

(c): $R$ integral domain

$R$ satisfies $1D_2 \implies R$ is a Bézout domain
(C) : $1D_2 \Rightarrow \text{Bézout}$

**NOTE:**

- If (C) is true $1D_2 \Rightarrow 1D_{m \forall m \geq 2} (\equiv \text{IMF}) \iff GE_{m \forall m \geq 2} (\equiv \text{EMF})$

- $GE_2 \not\Rightarrow \text{Bézout}$ : local non-valuation domains are non Bézout domains satisfying $GE_2$

**EXAMPLES (in support of (C)):**

- $R$ proj.-free
- $R$ UFD
- $R$ local
- $R$ PRINC (domains introduced by Salce, Zanardo 2014)

+ $1D_2 \Rightarrow \text{Bézout}$
THEOREM 1:

Let $R$ be an integral domain.

If $R$ satisfies $1D_2 \Rightarrow R$ is a Prüfer domain

($=$ every f.g. ideal of $R$ is invertible)

(C): $1D_2 \Rightarrow$ Bézout
TH1: $1D_2 \Rightarrow$ Prüfer

$(C) \iff (C')$: Prüfer non- Bézout $\Rightarrow$ non- $1D_2$

EQUIVALENT FORMULATION OF THE CONJECTURE
PROOF $1D_2 \Rightarrow $ Prüfer: $R$ domain

We first prove that

$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in 1D_2 (R) \Rightarrow (x, y)$ invertible

If either $x=0$ or $y=0$, or if either $x | y$ or $y | x$, trivial.

Let $x, y \neq 0$, $x | y$ and $y | x$, and let

$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x' & y' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$

$\Rightarrow \begin{cases} b (nx = nx'a + y'c) \\ a (y' = nx'b + y'(1-a)) \end{cases} \Rightarrow bnx = nx'ab + y' a(1-a) = ay$
Thus: \( nb = ya \)

and analogously \( nc(1-a) = yc \)

\[
(n, y)(1-a, b) = (n(1-a), nb, y(1-a), yb) = \\
= (yc, ya, y(1-a), yb) = \\
= y(c, a, 1-a, b) = (y)
\]

\( \implies (n, y) \) invertible

Now,

\[
R \text{ ID}_2 \implies \forall n, y \in R \begin{pmatrix} n & y \\ 0 & 0 \end{pmatrix} \in \text{ID}_2(R)
\]

\( \implies \forall n, y \in R \ (n, y) \) is invertible, i.e.,

\( R \) is a Prüfer domain. \( \qedsymbol \)
THEOREM 2

Let $R$ be (any) integral domain.

$R$ satisfies $1D_2 \Rightarrow R$ satisfies $GE_2$.

TH. 1 + TH. 2:

$1D_2 \iff$ Pr"ufer + $GE_2$

NOTE: (C'): Pr"ufer non-Bézout $\Rightarrow$ not $1D_2$

TH. 2

$1D_2 \Rightarrow GE_2 \iff$ non-$GE_2 \Rightarrow$ non-$1D_2$

Thus EVERY PR"UFER non-Bézout domain NOT SATISFYING $GE_2$ VERIFIES (C'), SINCE IT DOES NOT EVEN SATISFY $1D_2$.
We proved that the following Prüfer non-Bézout domains do not satisfy GE2, thus verifying (C'):

- the affine coordinate ring $R$ of a smooth plane curve of degree $\geq 2$ and conjugate points at infinity (not GE2 even if $R$ is a PID)

  \[ R = \frac{\mathbb{R}[x,y]}{(x^4 + y^4 + 1)} \text{ is a Dedekind - not PID} \]
  \[ (x^2 + y^2 - 1)(x^2 + y^2 + 1) = 2(xy - 1)(xy + 1) \text{ non-unique factorization over } R \]

- the ring of \textbf{integer-valued polynomials} $\text{Int}(\mathbb{Z}) = \{ f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z} \}$

Prüfer, non-Bézout domain
We proved the above results by properly applying Cohen's characterizations of GEz(R) when R is a:

- \textit{k-ring with degree function}, i.e., a ring R with a degree function \( d : R \to \mathbb{N}_0 \cup \{-\infty\} \) and s.t. \( R^x = k \) field, such as our coordinate rings

- \textit{discretely ordered ring}, i.e., a totally ordered ring such that \( \forall r \in R, r > 0 \Rightarrow r \geq 1 \), such as \( \text{Int}(\mathbb{Z}) \)
PRODUCTS OF IDEMPOTENT MATRICES OVER SPECIAL PRÜFER DOMAINS:

- Minimal Prüfer-Dress rings

C.-Zamardo, "Minimal Prüfer-Dress rings and products of idempotent matrices", 2019

- Rings of integers in quadratic number fields

C.-Zamardo, "Idempotent factorizations of singular 2x2 matrices over quadratic integer rings", 2020
MINIMAL PRÜFER-DRess RINGS

DEF: Given a field $K$, not containing square roots of $-1$, the MINIMAL (PRÜFER) DRESS RING $D_k$ of $K$ is the SMALLEST SUBRING of $K$ containing every element of the form

$$\frac{1}{1+nx^2}, \ n \in K$$

WHY "DRESS"? These rings were introduced by A. Dress in 1965

WHY "PRÜFER"? Dress proved that if $a, b \in D_k$ $(a, b)^2 = (a^2 + b^2)$, then every 2-generated ideal over $a$ $D_k$ is invertible
Examples (propositions):

- \( k = \text{ordered field such that every positive element is a square} \) (ex. \( \mathbb{R} \))
  \[ \Rightarrow \quad D_k = k \]

- \( k = \mathbb{Q} \implies D_k = \mathbb{Z} S, \) with \( S = \langle \{ p \text{ prime} : p = 1 \mod 4 \} \rangle \subset \mathbb{Z} \)

The minimal Dress ring \( D \) of \( \mathbb{R}(X) \): complete characterization

- \( D = \left\{ \frac{f}{\gamma} : f, \gamma \in \mathbb{R}[X], \gamma > 0, \deg f \leq \deg \gamma \right\} \)

- \( D \) is a Dedekind domain (= noetherian Prüfer domain)

\( D \) is not a PID: \( \left( \frac{1}{1+X^2}, \frac{X}{1+X^2} \right) \) is not principal
CONJECTURE (C'): Prüfer non-Bézout \( \Rightarrow \) not-ID2

EXPECTATION: D does not satisfy ID2. \( \leftarrow \) STILL OPEN PROBLEM

On the other hand, against expectations:

**THEOREM:**

Let \( p, q \in \mathbb{D} \) satisfy one of the following conditions:

1. \( \deg p \geq \deg q \), \( q(z) > 0 \) \( (\leq 0) \) \( \forall \) root of \( p \)
2. \( \deg q \geq \deg p \), \( p(z) > 0 \) \( (\leq 0) \) \( \forall \) root of \( q \)
3. \( p \) and \( q \) have no common roots and either \( p \geq 0 \) \( (\leq 0) \) or \( q \geq 0 \) \( (\leq 0) \)
4. \( p \) and \( q \) have no common roots, \( \deg p \) is odd and \( q \) has a unique root

new results
Then \((p, q) \in \text{ID}_2(D)\).

**Proof (Sketch):** based on analytic methods.

i)-- iii) we find a suitable \(Te \text{M}_2(D)\) idempotent and \(u \in D^x\) s.t.

\[
(p, q) = (u, 0)^T = 
\begin{pmatrix}
1 & -1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
\]

iv) \(\exists \epsilon \in \mathbb{R}\) s.t. \((p, rp+q) \in \text{ID}_2(D)\) and we conclude since \((p, q) \sim (p, rp+q)\).
NOTE: the couples $p, q \in D$ of the previous theorem can generate both principal and non-principal ideals.

**EASY COROLLARY:**

If $p = \frac{\alpha}{\gamma}, \quad q = \frac{\gamma^2}{\gamma} \in D$ and $\deg \alpha, \deg \gamma \leq 1$,

then $(p \quad q) \in \text{ID}_2(D)$.

**STILL UNANSWERED QUESTION:** Does $D$ satisfy (or not) $\text{ID}_2$?
QUADRATIC INTEGER RINGS

d = square-free integer

\[ R = \begin{cases} \mathbb{Z}[\sqrt{d}] = \{ a + \sqrt{d} b : a, b \in \mathbb{Z} \} & \text{if } d \equiv 2, 3 \mod 4 \\ \mathbb{Z}\left[\frac{1 + \sqrt{d}}{2}\right] = \{ a + \frac{\sqrt{d} b}{2} : a, b \in \mathbb{Z}, a \equiv b \mod 2 \} & \text{if } d \equiv 1 \mod 4 \end{cases} \]

R is a Dedekind domain.

EXPECTATION: Whenever R is not a PID, then it doesn't satisfy 1D2 (in view of (C'))

we couldn't prove this fact
CASE $d < 0$

**THEOREM** Cohn ’66

Let $R$ be the ring of integers in $\mathbb{Q}[^{\sqrt{d}}]$, with $d < 0$ square-free integer.

$R$ satisfies $GE_2 \iff R$ is Euclidean ($d = -1, -2, -3, -7, -11$)

**CONSEQUENCE:**

$R$ non-Euclidean (PID or not) $\Rightarrow$ not-$GE_2 \Rightarrow$ not-$ID_2$
CASE $d > 0$

$R =$ ring of integers in $\mathbb{Q}[\sqrt{d}]$

$d =$ square-free positive integer

THEOREM Vaserstein '72

The special linear group $SL_2(R)$ of $R$ is generated by transvections ($\iff R$ satisfies $GE_2$)

THEOREM

If $x, y \in R$ generate a principal ideal of $R$, then $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in ID_2(R)$
PROOF (SKETCH): We first reduce to the case $x, y$ comaximal (not restrictive). Say $xz + yt = 1$, $z, t \in R$.

Then we use the following chain of implications:

\[
\begin{pmatrix}
  x & y \\
  -t & z
\end{pmatrix}
\text{ product of transvections (by Vaseršteīm) } \Rightarrow
\]

$\Rightarrow x, y$ admit a weak Euclidean algorithm $\Rightarrow$

$\Rightarrow \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in \text{ID}_2(R)$

proved last week
**THEOREM**

For every \( a, y \in \mathbb{R} \), \((a, y) \in \text{ID}_2(\mathbb{R})\).

**STRATEGY OF THE PROOF:**

If \((a, y)\) is not principal, we find a suitable \( T \in \text{M}_2(\mathbb{R}) \) idempotent, such that

\[
\begin{pmatrix}
  a \\
  y
\end{pmatrix}
= \begin{pmatrix}
  a' \\
  y'
\end{pmatrix}
\]

with \( a', y' \in \mathbb{R} \) s.t. \((a', y')\) is principal.
RECALL: \( M \in M_2(\mathbb{R}) \) is column-row if \( \exists x,y,a,b \in \mathbb{R} \) s.t.

\[
M = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}
\]

**Corollary**

Every column-row matrix over \( \mathbb{R} \) is a product of idempotent matrices.

- Column-row matrices form a large class among 2x2 singular matrices over \( \mathbb{R} \). E.g., every 2x2 singular matrix with elements of a row or column that generate a principal ideal is column row.

- We found several explicit factorizations into idempotent factors of non-column-row matrices over \( \mathbb{Z}[\sqrt{7}] \).
OPEN QUESTIONS

• Do real quadratic integer rings satisfy \(1D_2\)?

• Is the conjecture \((C') \equiv (C)\) true?

• \(1D_2 \iff GE_2\) over Prüfer domains?

NOTE: local non-valuation domains, that verify \(GE_2\) but not \(1D_2\), are not Prüfer domains.