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On the S-class group of formal power series rings

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Notations

Let D be an integral domain with quotient field K and I a fractional ideal of D. We note :

⋆ $I^{-1} = D : I = \{x \in K,\ xI \subseteq D\}$.

⋆ $I_v = D : (D : I) = (I^{-1})^{-1}$.

⋆ $I_t = \bigcup\{J_v;\ J \subseteq I \text{ finitely generated}\}$.

⋆ *I is called divisorial or v-ideal (respectively, t-ideal) if $I_v = I$ (respectively, $I_t = I$.)*

⋆ *I is called v-invertible (respectively, t-invertible) if $(II^{-1})_v = D$ (respectively, $(II^{-1})_t = D$).*

⋆ $\mathcal{F}(D)$: The set of nonzero fractional ideals of D.

⋆ $Prin(D)$: The set of all principal ideals of D.

⋆ $T(D)$: The set of all t-invertible t-ideals of D. ($T(D)$ is a group under the multiplication $I * J = (IJ)_t$).

⋆ $Div(D)$: The set of all fractional divisorial ideals of D.

⋆ [1961, P. Samuel]

$D(D)$: The divisor class group of D

$$\mathrm{D}(D) = Div(D)/Prin(D).$$

⋆ [1982, A. Bouvier]

$\mathrm{Cl}(D)$: The (t)- class group of D.

$$\mathrm{Cl}_t(D) = T(D)/\ \mathrm{Prin}(D).$$

⋆ Theorem : [1982, A. Bouvier]

If D is a Krull domain, then

$$\mathrm{Cl}_t(D) = \mathrm{D}(D).$$

⋆ Theorem : [1988, A. Bouvier and M. Zafrullah]

If D is a Krull domain, then

$$Cl_t(D) = 0 \text{ if and only if } D \text{ is factorial.}$$

⋆ Theorem : [1961, P. Samuel]

Let D be a Krull domain. Then :

$$\begin{array}{cccc} \varphi & : & Cl_t(D) & \longrightarrow & Cl_t(D[\![X]\!]) \\ & & [I] & \longmapsto & [I.D[\![X]\!]] \end{array}$$

is an injective homomorphism.

⋆ Theorem : [1965, L. Claborn]

Let D be a Noetherian regular ring. Then :

$$\begin{array}{cccc} \varphi & : & Cl_t(D) & \longrightarrow & Cl_t(D[\![X]\!]) \\ & & [I] & \longmapsto & [I.D[\![X]\!]] \end{array}$$

is an isomorphism. In particular,

$$Cl_t(D) \simeq Cl_t(D[\![X]\!]).$$

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Proposition: [A. Hamed and S. Hizem]

Let D be an integral domain. Then

$$\begin{array}{rcccl} \varphi & : & Cl_t(D) & \longrightarrow & Cl_t(D[\![X]\!]) \\ & & [I] & \longmapsto & [(I.D[\![X]\!])_t] \end{array}$$

is an injective homomorphism.

Notation

Let D be an integral domain and let I be an integral ideal of $D[\![X]\!]$. We define $I_0 = \{f(0),\ f \in I\}$. Then I_0 is an ideal of D.

The property $(*)$:

Let D be an integral domain with quotient field K, and $(*)$ the following property : For all integral v-invertible v-ideals I and J of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have

$$((IJ)_0)_v = ((IJ)_v)_0.$$

Example

If D is a principal domain, then $D[\![X]\!]$ satisfies the property $(*)$.

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Theorem: [A. Hamed and S. Hizem]

Let D be an integral domain with quotient field K, such that $D[\![X]\!]$ satisfies $(*)$.

For each integral υ-invertible υ-ideal I of $D[\![X]\!]$, there exist a υ-invertible υ-ideal L of D and $h \in qf(D[\![X]\!])$ such that

$$I = hL[\![X]\!].$$

Idea of the demonstration.

Case : $I_0 \neq (0)$.

$\star$ We put $J = aI(I_0)^{-1}[\![X]\!]$, where $0 \neq a \in I_0$.
Then $(J_0)_v = a(I_0(I_0)^{-1})_v = aD$. Thus by the property $(*)$, $J_v = fD[\![X]\!] + XJ_v$ for some $f \in J_v$ such that $f(0) = a$.

$\star$ We show that $J_v = fD[\![X]\!]$.
Let $h \in J_v$, by induction on n, we prove that, for each $n \in \mathbb{N}$, $h = fs_n + X^{n+1}L_n$ for some $s_n \in D[X]$ and $L_n \in D[\![X]\!]$. So the sequence $(fs_n)_{n\in\mathbb{N}}$ converges to h in $D[\![X]\!]$ for the $XD[\![X]\!]$-adic topology.

Idea of the demonstration.

⋆ We note $s = \lim_{n \to +\infty} s_n = \sum_{i=0}^{+\infty} a_i X^i$.
Thus $h = fs \in fD[\![X]\!]$ and $J_v = fD[\![X]\!]$.

⋆ $I = hI_0[\![X]\!]$ where $h = \frac{1}{a} f$.

Case : $I_0 = (0)$.

⋆ Let $n \in \mathbb{N}$ such that $I \subseteq (X^n)$ and $I \not\subseteq (X^{n+1})$. We put $I' = X^{-n} I$. Then I' is an integral v-invertible v-ideal of $D[\![X]\!]$ and $(I')_0 \neq (0)$. So by the first case $I' = h(I')_0[\![X]\!]$. Then $I = hX^n (I')_0[\![X]\!]$ with $h = \frac{1}{a} f$.

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Theorem: [A. Hamed and S. Hizem]

Let D be an integral domain. Assume that :

1. $D[\![X]\!]$ *satisfies* $(*)$.
2. *Each* v*-invertible* v*-ideal of* D *is* v*-finite type.*

Then :

$$Cl_t(D) \simeq Cl_t(D[\![X]\!]).$$

Recall that D is a TV-domain, if the v- and the t-operation on D are the same.

Corollary

Let D be a TV-domain such that $D[\![X]\!]$ satisfies the property $(*)$, then :

$$Cl_t(D) \simeq Cl_t(D[\![X]\!]).$$

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Theorem: [A. Hamed and S. Hizem]

Let D be an integral domain with quotient field K such that $D[\![X]\!]$ is a TV-domain. Then $D[\![X]\!]$ satisfies the property $(*)$ if and only if the homomorphism

$$\varphi : Cl_t(D) \longrightarrow Cl_t(D[\![X]\!])$$

is an isomorphism.

Proposition

Let D be an integral domain. If D is a regular ring, then $D[\![X]\!]$ satisfies the property $(*)$.

Example

Let $D = \mathbb{Z}[i\sqrt{5}]$, then D is a regular integral domain. Thus $\mathbb{Z}[i\sqrt{5}][\![X]\!]$ satisfies $(*)$.
Note that $\mathbb{Z}[i\sqrt{5}]$ is not a UFD. Hence

$$(0) \neq Cl_t(\mathbb{Z}[i\sqrt{5}]) \simeq Cl_t(\mathbb{Z}[i\sqrt{5}][\![X]\!]).$$

The S-class group of an integral domain

Definition

The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w = \{x \in K,\ xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is a star operation on D.

Definition: [2014, H. Kim, O. Kim, J. Lim]

Let D be an integral domain and S a multiplicative subset of D. We say that a nonzero ideal I of D is S-w-principal if there exist an $s \in S$ and $a \in A$ such that $sI \subseteq aD \subseteq I_w$.
We also define D to be an S-factorial domain if each nonzero ideal of D is S-w-principal.

⋆ Theorem : [1988, A. Bouvier and M. Zafrullah]

If D is a Krull domain, then

$$Cl_t(D) = 0 \text{ if and only if } D \text{ is factorial.}$$

Definition

Let D be an integral domain, S be a multiplicative subset of D and I a nonzero fractional ideal of D. We say that I is S-principal if there exist an $s \in S$ and $a \in I$ such that $sI \subseteq aD \subseteq I$.

Examples :

1. Every principal ideal is S-principal.
2. An S-principal ideal is not necessarily a principal ideal. Indeed, let $A = \mathbb{Z} + X\mathbb{Z}[i][X]$ and consider the ideal $I = 2\mathbb{Z} + (1+i)X\mathbb{Z}[i][X]$. We put $S = \{2^n,\ n \in \mathbb{N}\}$. then I is an S-principal ideal but is not a principal ideal of D.

Notation

Let D be an integral domain with quotient field K. We note,

S-$P(D)$ the set of fractional S-principal t-invertible t-ideals of D.

Proposition

Let D be an integral domain with quotient field K and S a multiplicative subset of D. Then S-$P(D)$ is a subgroup of $T(D)$ under the t-multiplication : $I \star J = (IJ)_t$.

Definition

Let D be an integral domain and S a multiplicative subset of D. The quotient group $S\text{-}Cl_t(D) = T(D)/S\text{-}P(D)$ is called the S-class group of D.

Remark

If S consists of units of D, then $S\text{-}Cl_t(D) = Cl_t(D)$.

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Theorem: [A. Hamed and S. Hizem]

Let D be a Krull domain. Then :

$$S\text{-}Cl_t(D) = 0 \textit{ if and only if } D \textit{ is an } S\textit{-factorial domain.}$$

Corollary

If D is a Krull domain, then

$$Cl_t(D) = 0 \textit{ if and only if } D \textit{ is factorial.}$$

Definition

et D be an integral domain and S a multiplicative subset of D. We say that a nonzero ideal I of D is S-υ-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_\upsilon$.
We also define D to be an S-GCD-domain if each finitely generated nonzero ideal of D is S-υ-principal.

Remark

If $S = \{1\}$, then D is an S-GCD-domain if and only if D is a GCD-domain.

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Theorem: [A. Hamed and S. Hizem]

Let D be a $P\upsilon MD$. Then

$$S\text{-}Cl_t(D) = 0 \text{ if and only if } D \text{ is an } S\text{-GCD-domain.}$$

Corollary

If D is a $P\upsilon MD$, then

$$Cl_t(D) = 0 \text{ if and only if } D \text{ is a GCD-domain.}$$

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Theorem: [A. Hamed and S. Hizem]

Let $A \subseteq B$ be an extension of integral domains such that B is a flat A-module and S a multiplicative subset of A. Then the canonical mapping $\varphi : S\text{-}Cl_t(A) \longrightarrow S\text{-}Cl_t(B)$, $[I] \mapsto [IB]$ is well-defined and it is a homomorphism.

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Proposition: [A. Hamed and S. Hizem]

Let $A \subseteq B$ be an extension of integral domains such that B is a flat A-module and S be a multiplicative subset of A. If B is integrally closed and $B \subseteq frac(A)$, then $S\text{-}Cl_t(A) \simeq S\text{-}Cl_t(A + XB[X])$.

Corollary: [D. F. Anderson, S. E. Baghdadi and S. E. Kabbaj]

Let $A \subseteq B$ be an extension of integral domains such that B is a flat A-module. If B is integrally closed and $B \subseteq frac(A)$, then $Cl_t(A) \simeq Cl_t(A + XB[X])$.

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Theorem: [A. Hamed and S. Hizem]

Let D be a TV-domain such that $D[\![X]\!]$ satisfies the property $(*)$ and S be a multiplicative subset of D. Then

$$S\text{-}Cl_t(D) \simeq S\text{-}Cl_t(D[\![X]\!]).$$

Corollary

Let D be a Krull domain, such that $D[\![X]\!]$ satisfies $(*)$ and S a multiplicative subset of D. Then

D is an S-factorial domain if and only if $D[[X]]$ is an S-factorial domain.

Thank you for your attention