On functions continuous on certain classes of 'thin' sets

by

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Abstract. Continuity properties of functions with continuous restrictions to certain classes of 'thin' sets are considered. By construction of an explicit counterexample it is shown that continuity of the restriction to neither of the following classes of sets is sufficient to guarantee continuity of a real-valued function on the plane: (real) analytic images of compact intervals, zero sets of real analytic functions in two variables and regular $C^{1,\alpha}$-images of compact intervals. However, continuity of a real-valued function in two variables along all regular C^1-curves implies its continuity. In addition it is shown that a function which is continuous on the boundary of some ball of positive radius in a Hilbert space, or even — in the finite dimensional case — on the boundary of a bounded open set must be continuous if it satisfies a Cauchy-type functional equation. In dimension two the same result can be obtained if the set on which continuity is required is connected and contains three noncollinear points.

1. Introduction

In [Z] F. Zorzitto showed that every function $f : \mathbb{C} \to \mathbb{C}$ which satisfies the Cauchy equation

$$f(x+y) = f(x)f(y) \tag{1}$$

and is continuous on circles has necessarily to be continuous globally. He wrote: *"The above proof leads one to speculate, out of mere curiosity, whether any function $f : \mathbb{C} \to \mathbb{C}$, having continuous restrictions to all circles in $\mathbb{C}$, can still possess a point of discontinuity."*

In Section 2 we deal with the problem of constructing several discontinuous functions which have continuous restrictions to certain classes of 'thin' sets, all of them including the class of circles Zorzitto was curious about. More specifically we give an example of a discontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous restrictions $f|_{\gamma(I)}$ ($f|_{N_a}$ resp.) for every analytic image $\gamma(I)$ of a compact interval I and for every zero-set N_a of an arbitrary (non vanishing) real analytic function $a(x,y)$ in two variables. Under the additional regularity assumption that $\dot{\gamma}(t) \neq (0,0)$ for all $t \in I$, we can drop the requirement of analyticity for γ assuming only that γ is taken from some Hölder-class $C^{1,\alpha}$ with $\alpha > 0$.

In contrast to these counterexamples we show in Section 3 that any function $f : \mathbb{R}^2 \to \mathbb{R}$, for which all restrictions to arbitrary regular C^1-images of compact intervals are continuous, has to be continuous itself.

The last Section 4 is devoted to the study of continuity properties of functions satisfying a generalized form of the functional equation (1). Problems of similar structure, dealing with Jensen-convex functions instead of solutions of the Cauchy equation (1), were considered by Ger [G1],[G2] and Kuczma [K].

2. The counterexamples

We consider the well-known function $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x) := \begin{cases} \exp(-x^{-1}) & x > 0 \\ 0 & x \leq 0. \end{cases} \tag{2}$$

This function is of class C^∞ with $\varphi^{(n)}(0) = 0$ for all n. We also note separately the important property that for all $n \in \mathbb{N}_0$

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{\varphi(x)}{x^n} = 0.$$

(This is used to establish the relations $\varphi^{(n)}(0) = 0$ mentioned above.)
Using this we may define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) := 1 - \frac{\varphi(x^2)}{\varphi(x^2) + \varphi(1 - x^2)}. \tag{3}$$

Again, ψ is (well-defined and) of class C^∞. Moreover

$$0 \leq \psi \leq 1, \quad \psi(0) = 1, \quad \text{and} \quad \psi(x) = 0 \qquad \text{outside} \quad [-1, 1].$$

This is used for the definition of the function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) := \begin{cases} 0 & x \leq 0 \\ \psi\left(\frac{2}{\varphi(x)}\left(y - \frac{3}{2}\varphi(x)\right)\right) & x > 0. \end{cases}$$

Remark 1. By the properties of ψ it follows easily that f vanishes outside the closed set A defined by

$$A := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, \varphi(x) \leq y \leq 2\varphi(x)\}. \tag{4}$$

Moreover, $0 \leq f \leq 1$, $f(0, 0) = 0$ and $f(x, \frac{3}{2}\varphi(x)) = 1$ for all $x > 0$.

Lemma 1. *The function f is of class C^∞ on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and discontinuous at $(0, 0)$.*

Proof. Obviously f is C^∞ on the open set $\mathbb{R}^2 \setminus A$ since it is zero there. Moreover it is clear from the definiton that f is C^∞ in the right halfplane. Therefore f is C^∞ on the union of these two open sets, which is $\mathbb{R}^2 \setminus \{(0, 0)\}$.
Finally, f is discontinuous at $(0, 0)$, since $f(0, 0) = 0$ and $\lim_{x \to 0+} f(x, \frac{3}{2}\varphi(x)) = 1$. $\square$

The function f is constructed in such a way that a curve which is either an analytic image of an interval or part of the zero set of an analytic function in two variables and which passes through the origin cannot do so from within A. This property of f is the essential point in the proof of the following two theorems.

Theorem 1. *Let $I \subseteq D \subseteq \mathbb{R}$ with D open and I compact. Let furthermore $\gamma : D \to \mathbb{R}^2$ be (real) analytic and denote by $\gamma(I)$ the image of I under γ. Then the restriction $f|_{\gamma(I)}$ is continuous.*

Proof. We write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, where $\gamma_1, \gamma_2 : D \to \mathbb{R}$ are real analytic. Since f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ the assertion of the theorem is obviously true if $(0,0) \notin \gamma(I)$. Thus we may suppose that $(0,0) \in \gamma(I)$. If, in this case, $\gamma(I) \subseteq \mathbb{R}^2 \setminus A^\circ$ we conclude that $f\big|_{\gamma(I)} = 0$ and thus continuous. Here A° denotes the interior of A. So we are left with the case $\gamma(I) \cap A^\circ \neq \emptyset$.
This means that both γ_1 and γ_2 are different from zero and that there is some $\tilde{t} \in I$ such that $\gamma_1(\tilde{t}) > 0$ and $\varphi(\tilde{t}) < \gamma_2(\tilde{t}) < 2\varphi(\tilde{t})$. In particular γ_1, γ_2 are not constant. Since I is compact and γ_1 is analytic, there are only finitely many zeroes $t_1, t_2, \ldots, t_r$ of γ_1 in I. Now, for $1 \leq i \leq r$ we have representations

$$\gamma_1(t) = (t - t_i)^{\sigma_i} \gamma_{1,i}(t - t_i), \quad \gamma_2(t) = (t - t_i)^{\tau_i} \gamma_{2,i}(t - t_i) \quad (\sigma_i \in \mathbb{N}, \tau_i \in \mathbb{N}_0), \tag{5}$$

where $\gamma_{1,i}(0), \gamma_{2,i}(0) \neq 0$ for $1 \leq i \leq r$.
Now, choose $m \in \mathbb{N}$ such that $m\sigma_i > \tau_i$ for all i. Due to the continuity of the functions $\gamma_{j,i}$ there are positive reals M_1, M_2, δ_0 such that for all $1 \leq i \leq r$ and all $t \in I$ we have that $|t - t_i| < \delta_0$ implies

$$|\gamma_{2,i}(t - t_i)| > M_2 \quad \text{and} \quad |\gamma_{1,i}(t - t_i)| < M_1.$$

Then, for $0 < \delta < \min(1, M_2/M_1^m, \delta_0)$ and $0 < |t - t_i| < \delta$

$$\begin{aligned} |\gamma_2(t)| = |t - t_i|^{\tau_i} |\gamma_{2,i}(t - t_i)| &\geq |t - t_i|^{m\sigma_i - 1} M_2 > \\ &> |t - t_i|^{m\sigma_i - 1} (\delta M_1^m) > |t - t_i|^{m\sigma_i} M_1^m > \\ &> |t - t_i|^{m\sigma_i} |\gamma_{1,i}(t - t_i)|^m = |\gamma_1(t)|^m . \end{aligned}$$

For $\varepsilon > 0$ put $T_\varepsilon := \{t \in I \,\big|\, |\gamma_1(t)| < \varepsilon\}$. Then T_ε is contained in the union $K_\delta := \bigcup_{i=1}^r]t_i - \delta, t_i + \delta[$ provided that ε is sufficiently small. (For otherwise we could construct a sequence (s_n) of reals contained in the compact set $I \setminus K_\delta$, converging to some $t_0 \in I \setminus K_\delta$, such that $\lim_{n\to\infty} \gamma_1(s_n) = 0 = \gamma_1(t_0)$ contradicting the fact that $t_1, t_2, \ldots, t_r$ is the set of all zeroes of γ_1 in I.)
Taking into account the properties of the function φ we may choose $0 < \varepsilon < 1$ such that

$$T_\varepsilon \subseteq K_\delta \quad \text{and} \quad 2\varphi(|x|) \leq |x|^m \quad (|x| \leq \varepsilon).$$

Thus $t \in I$ and $|\gamma_1(t)| < \varepsilon$ implies $t \in T_\varepsilon$ and

$$|\gamma_2(t)| > |\gamma_1(t)|^m \geq 2\varphi\left(|\gamma_1(t)|\right).$$

But this means that $\gamma(I) \cap (]-\infty, \varepsilon[\times \mathbb{R}) \cap A^\circ = \emptyset$. Accordingly f is continuous on $\gamma(I) \cap (]-\infty, \varepsilon[\times \mathbb{R})$ as it vanishes on this set. But obviously f is also continuous on $\gamma(I) \cap (]0, \infty[\times \mathbb{R})$. So $f\big|_{\gamma(I)}$ is continuous.

$\square$

Now we consider the restrictions of f to locally analytic sets.

Theorem 2. *Let $D \subseteq \mathbb{R}^2$ be an open set and assume that $a : D \to \mathbb{R}$ is a real analytic function, $a \not\equiv 0$. Moreover let*

$$N_a := \{(x, y) \in D \mid a(x, y) = 0\}.$$

Then $f|_{N_a}$ is continuous.

Proof. The assertion is obviously true whenever $(0, 0) \notin N_a$. Thus we suppose that $(0, 0) \in N_a$.
If, in this case, N_a is contained in the complement of A°, we have that f vanishes identically on N_a, implying continuity.
Otherwise we have $N_a \cap A^\circ \neq \emptyset$. It is enough to show continuity of $f\big|_{N_a}$ at $(0, 0)$. We can assume that the power series representation of a at $(0, 0)$ given by

$$a(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j, \tag{6}$$

is absolutely and uniformly convergent on $Q_\delta = [-\delta, \delta] \times [-\delta, \delta]$ for some $\delta > 0$. Without loss of generality we may also assume that a (as given in (6)) is neither divisible by x nor by y. If, for example, $a(x, y)$ is divisible by x, we could write $a(x, y) = x^\ell b(x, y)$ with $\ell \geq 1$ and $b(0, y) \not\equiv 0$. Putting $c(x, y) := x^\ell$ this would imply $N_a \cap Q_\delta = (N_b \cup N_c) \cap Q_\delta$. But $N_c = \{0\} \times \mathbb{R}$ and f is continuous on the y-axes (and, as for the divisibility by y, on the x-axes as well). Thus f is continuous on N_a at $(0, 0)$ iff it is continuous on N_b at $(0, 0)$. This and $a(0, 0) = 0$ implies that

$$a_{0,0} = 0 \quad \text{and} \quad a_{0,\ell},\, a_{n,0} \neq 0 \quad \text{for some } \ell, n \in \mathbb{N}. \tag{7}$$

We will now show that there is some integer $m \geq 1$ and some $\varepsilon > 0$ such that

$$|x| < \varepsilon \quad \text{and} \quad a(x, y) = 0 \quad \text{implies} \quad |y| \geq |x|^m .$$

This can be seen as follows. The properties (7) imply that

$$\sum_{m \geq n} a_{m,0} x^m = - \sum_{(i,j) \in \mathbb{N}_0 \times \mathbb{N}} a_{i,j} x^i y^j . \tag{8}$$

for all $(x, y) \in N_a \cap Q_\delta$, where $n \geq 1$ is minimal with $a_{n,0} \neq 0$. We may rewrite the left-hand side of (8) as

$$x^n (a_{n,0} + a_{n+1,0} x + \cdots) = x^n (a_{n,0} + g(x)),$$

where $a_{n,0} \neq 0$ and $|g(x)| < |a_{n,0}/2|$ for all $|x| < \delta$, if δ is chosen sufficiently small. Hence for $|x| < \delta$ we have

$$|x^n (a_{n,0} + g(x))| \geq |x|^n \left(|a_{n,0}| - |g(x)|\right) \geq |x|^n \left|\frac{a_{n,0}}{2}\right| . \tag{9}$$

Now we consider the right-hand side of (8). This expression is not zero since $a_{0,\ell} \neq 0$ and it does not contain any pure power of x. Thus there is a maximal $k \geq 1$ such that the right-hand side of (8) is divisible by y^k. This yields

$$\left| \sum_{(i,j)\in\mathbb{N}_0\times\mathbb{N}} a_{i,j}x^i y^j \right| = \left| y^k \sum_{\substack{(i,j)\in\mathbb{N}_0\times\mathbb{N} \\ j\geq k}} a_{i,j}x^i y^{j-k} \right| \leq |y|^k \, |h(x,y)| \, , \tag{10}$$

with h continuous on Q_δ. Let $K > 0$ be the maximum of $|h|$ on the compact set Q_δ. Combining (9), (10) and (8) gives

$$|x|^n \left| \frac{a_{n,0}}{2} \right| \leq |x^n(a_{n,0} + g(x))| = \left| y^k h(x,y) \right| \leq |y|^k \, K$$

for all $(x, y) \in N_a \cap Q_\delta$. If we set $M := \left(|a_{n,0}/2| \;\; K^{-1}\right)^{1/k}$ we get

$$|y| \geq M \, |x|^{n/k} \, ,$$

and for $m \in \mathbb{N}$, $m \geq (n/k) + 1$, and $\delta < \min(1, M)$ we finally get

$$|y| \geq \delta \, |x|^{n/k} \geq |x|^{(n/k)+1} \geq |x|^m$$

provided that $(x, y) \in N_a \cap Q_\delta$.
Now the proof can be finished as in Theorem 1.

$\square$

Remark 2.

1) The compactness of the interval I in Theorem 1 is necessary. This can be seen by considering the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \; \gamma(t) = (\cos(t) + \cos(\omega t), \sin(t) + \sin(\omega t))$$

with irrational ω. The image of this curve is dense in the disk of radius 2 centered at the origin (and the origin is in the image). Thus f restricted to $\gamma(\mathbb{R})$ cannot be continuous. Note, however, that although the *restriction* $f|_\gamma(\mathbb{R})$ is discontinuous, the *composition* $f \circ \gamma$ is continuous.

2) Theorems 1 and 2 are not independent. It can be proved that every analytic zero set is *locally* the union of finitely many analytically parametrized curves (c.f. [KP] Theorem 3.2.3, p.84 and Theorem 5.2.1 p.153), thus, in fact, Theorem 2 follows from Theorem 1. However, the proof for this fact requires deep methods from the theory of real analytic functions and is restricted to dimension n=2, so we preferred to give elementary proofs for both cases. It is not known to the authors whether every analytic curve is locally the zero set of an analytic function, so conversely Theorem 2 would imply Theorem 1. A counterexample to an analogous situation is known for 2-dimensional analytically parametrized surfaces in $\mathbb{R}^3$. (c.f. [N] Example 1.4.16, p.28.)

A modified version of the function f can be be shown to have continuous restrictions to all $C^{1,\alpha}$-images of compact intervals. We briefly recall the definition of the *Hölder space* $C^{1,\alpha}(I,\mathbb{R}^2)$: Let I be a compact interval and let $0<\alpha\le 1$. Then $C^{1,\alpha}(I,\mathbb{R}^2)$ consists of all curves $\gamma: I\to\mathbb{R}^2$; $\gamma(t)=(\gamma_1(t),\gamma_2(t))$ for which $\gamma_i\in C^1(I,\mathbb{R})$ and there exists a constant $M=M(\gamma)$ such that

$$|\dot\gamma_i(t)-\dot\gamma_i(s)|\le M|t-s|^\alpha \tag{11}$$

for all $t,s\in I$ and $i=1,2$. An important property of $C^{1,\alpha}$-functions lies in the fact that the remainder in the first-order Taylor expansion is of order $O(1+\alpha)$. Actually we have

$$\gamma_i(t)=\gamma_i(s)+\dot\gamma_i(s)(t-s)+\int_s^t\Big[\dot\gamma_i(\xi)-\dot\gamma_i(s)\Big]\,d\xi, \tag{12}$$

and due to (11),

$$\left|\int_s^t\big[\dot\gamma_i(\xi)-\dot\gamma_i(s)\big]\,d\xi\right|\le M|t-s|^{1+\alpha} \tag{13}$$

for $i=1,2$ and $s,t\in I$.
A curve $\gamma\in C^1(I,\mathbb{R})$ is said to be *regular* if $|\dot\gamma(t)|\neq 0$ for all $t\in I$.
Let ψ be as defined in (3) and let $p:[0,\infty[\to[0,\infty[$ be given by

$$p(x)=-\int_0^x\frac{ds}{\ln s}\qquad\text{for } x\in\Big[0,\frac{1}{2}\Big],$$

and suppose that p is continued outside $[0,\frac{1}{2}]$ in such a way that p is monotone and C^∞-smooth on $[0,\infty[$. It follows easily by de l' Hospitals rule that

$$\lim_{x\to 0^+}\frac{p(x)}{x^{1+\alpha}}=\infty \tag{14}$$

for all $\alpha>0$ and

$$\lim_{x\to 0^+}\frac{p(x)}{x}=0. \tag{15}$$

We now define the function $\tilde f:\mathbb{R}^2\to\mathbb{R}$ in analogy to f as

$$\tilde f(x,y):=\begin{cases}0 & x\le 0\\ \psi\left(\frac{2}{p(x)}\left(y-\frac{3}{2}p(x)\right)\right) & x>0.\end{cases} \tag{16}$$

Note that, by definition, $\tilde f(x,y)=0$ if $x\le 0$ or if $y\notin]p(x),2p(x)[$.

Theorem 3. *For any $\alpha\in]0,1]$ the function $\tilde f$ has continuous restrictions to all regular $C^{1,\alpha}$-images of compact intervals.*

Proof. Let $\gamma: I\to\mathbb{R}^2$ be a regular curve of class $C^{1,\alpha}$. Obviously $\tilde f|_\gamma(I)$ is continuous at every point $(x,y)\in\gamma(I)$ which is not the origin. We therefore assume that there exists $t_0\in I$ such that $\gamma(t_0)=(0,0)$ and, as γ is regular, $\dot\gamma(t_0)=(v_1,v_2)\neq(0,0)$.

Let us first suppose that $v_2 \neq 0$. If we choose some neighborhood V_{t_0} of t_0 small enough that $|t-t_0|^\alpha \leq \min\{1, \frac{|v_2|}{2M}\}$ is satisfied for all $t \in V_{t_0}$, we can conclude from (12) and (13) that

$$|\gamma_2(t)| \geq |v_2|\,|t-t_0| - M|t-t_0|^{1+\alpha} \geq \frac{|v_2|}{2}|t-t_0|,$$

and

$$|\gamma_1(t)| \leq (|v_1| + M)|t-t_0|$$

holds for all $t \in V_{t_0}$. Consequently we obtain

$$|\gamma_1(t)| \geq \frac{|v_2|}{2(|v_1|+M)}|\gamma_1(t)| \tag{17}$$

on V_{t_0}. Now since

$$\frac{p(|\gamma_1(t)|)}{|\gamma_1(t)|} \to 0 \qquad \text{for } t \to t_0,$$

due to (11), we have

$$p(|\gamma_1(t)|) \leq \frac{|v_2|}{4(|v_1|+M)}|\gamma_1(t)| \tag{18}$$

on some neighborhood $W_{t_0} \subset V_{t_0}$ of t_0. Combining (17) and (18) we find

$$|\gamma_2(t)| \geq 2p(|\gamma_1(t)|)$$

and therefore $\tilde{f}(\gamma_1(t), \gamma_2(t)) = 0$ on W_{t_0}. Hence $f \circ \gamma$ is continuous in this case.
Now assume that $v_2 = 0$. Then $v_1 \neq 0$ since γ is regular. Let the neighborhood V_{t_0} be choosen in such a way that $|t-t_0|^\alpha \leq \frac{|v_1|}{2M}$ on V_{t_0} Then, using again (12) and (13) we obtain

$$|\gamma_2(t)| \leq M|t-t_0|^{1+\alpha}$$

and

$$|\gamma_1(t)| \leq M\Big(\frac{2}{|v_1|}\Big)^{1+\alpha}|\gamma_1(t)|^{1+\alpha}.$$

By (14) and since $\gamma_1(t) \to 0$ as $t \to t_0$, we can choose a neighborhood W_{t_0} of t_0 so that

$$p(|\gamma_1(t))|) \geq M\Big(\frac{2}{|v_1|}\Big)^{1+\alpha}|\gamma_1(t)|^{1+\alpha}.$$

for all $t \in W_{t_0}$. We thus have

$$|\gamma_2(t)| \leq p(|\gamma_1(t)|)$$

and accordingly $\tilde{f}(\gamma_1(t), \gamma_2(t)) = 0$ if $t \in W_{t_0}$. Hence also in this case we obtain that $\tilde{f} \circ \gamma$ is continuous at t_0.
So far we have shown that $\tilde{f} \circ \gamma : I \to \mathbb{R}$ is continuous. It is still to prove that $\tilde{f}$ restricted to $\gamma(I)$ is a continuous function. To this aim suppose that $(x_n, y_n) \to (x, y)$ with $(x_n, y_n) = \gamma(t_n)$ and $(x, y) = \gamma(t)$ and assume that

$$|\tilde{f}(\gamma_1(t_n)) - \tilde{f}(\gamma(t))| \geq \epsilon \qquad \text{for all } n \in \mathbb{N}. \tag{19}$$

By the compactness of I, we can find a convergent subsequence $t_{n_k} \to s$ of $\{t_n\}$. The continuity of $\tilde{f} \circ \gamma$ implies that

$$\tilde{f}(\gamma(t_{n_k})) \to \tilde{f}(\gamma(s)) \qquad (20).$$

Since γ is continuous we also have $\gamma(t_{n_k}) \to \gamma(s)$. Thus $\gamma(t) = \gamma(s)$ in contradiction to (19) and (20). $\square$

Remark 3. By means of the functions f and $\tilde{f}$ we can easily construct functions F and $\tilde{F}$ respectively which have discontinuities of the above discussed types at all points of an arbitrary countable set $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$. In fact, we set $f_n(x) := f(x - x_n)$ and we define

$$F(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x). \tag{21}$$

The series (21) is uniformly convergent on $\mathbb{R}^2$, and every function f_n is continuous on $\mathbb{R}^2 \setminus \{x_n\}$, hence the limit-function F is continuous at every point $x \notin \{x_n\}$. Let G be either an analytic image of a compact interval, or the zero-set of a real analytic function. Then every restriction $f_n|_G$ is continuous and therefore F is again a continuous function on G. Finally F cannot be continuous at $x_n \in \mathbb{R}^2$ since it is the sum of the continuous function $\sum_{k\neq n} \frac{1}{2^k} f_k(x)$ and the discontinuous f_n.
The construction of $\tilde{F}$ is completely analoguous.

3. Functions continuous on regular C^1-curves are continuous

In contrast to the counterexamples given in Theorem 1 and Theorem 2 we shall now show that the continuity of f along regular C^1-curves implies its continuity.

Theorem 4. *Suppose $D \subseteq \mathbb{R}^2$ is open and $f : D \to \mathbb{R}$. Moreover assume that $f \circ \gamma : I \to \mathbb{R}$ is continuous for every regular curve $\gamma : I \to D$, $\gamma \in C^1(I, \mathbb{R}^2)$ and $I \subset \mathbb{R}$ compact. Then f is continuous.*

Proof. Suppose that f is discontinuous at $z_0 \in D$. Then there is a sequence $z_n \in D$, $z_n \neq z_0$, $z_n \to z_0$, and $\varepsilon > 0$ such that

$$|f(z_n) - f(z_0)| \geq \varepsilon \qquad (n \in \mathbb{N}). \tag{22}$$

Our aim is to find an appropriate subsequence of $\{z_n\}$ which can be interpolated by a C^1-curve γ. Since f is continuous along γ this would contradict (22).
Before we proceed any further, some technical comments have to be made. In the following we shall frequently choose subsequences of $\{z_n\}$ which we shall always denote by the same expression $\{z_n\}$. Furthermore let $t \in C^1([-1,1], \mathbb{R})$ be given, fulfilling the requirements $t(1) = 1$, $t(-1) = -1$, $t'(1) = t'(-1) = 0$. For instance we could choose $t(x) = \frac{1}{2}(3x - x^3)$. We set

$$M = \|t\|_\infty = \sup\{\, |t(x)| \;\big|\; -1 \leq x \leq 1\}. \tag{23}$$

Using polar coordinates centered at $z_0 = (x_0, y_0)$ we can write $z_n = (x_n, y_n) = (x_0 + r_n \cos\varphi_n, y_0 + r_n \sin\varphi_n)$ with $r_n > 0$ and $\varphi_n \in \mathbf{T} = \mathbb{R}/2\pi\mathbb{Z}$. From the compactness of the torus group $\mathbf{T}$ we conclude that there exists a convergent subsequence $\varphi_n \to \varphi$. Without loss of generality we assume $\varphi = 0$, for otherwise we could rotate the coordinate system and use $\{(\cos\varphi, \sin\varphi), (-\sin\varphi, \cos\varphi)\}$ as a new basis on $\mathbb{R}^2$. Since $\varphi_n \to 0$ it follows that $x_n - x_0 > 0$ for sufficiently large n. By taking a subsequence we can assume that this property holds for all n.

We shall now list several properties which can be supposed to hold for some subsequence of the current (sub)sequence $z_n = (x_n, y_n)$ and which remain valid if we take further subsequences.

- Since $x_n \to x_0$ and $x_n > 0$ we may choose a subsequence such that

$$x_{n+1} - x_0 < \frac{x_n - x_0}{2} \qquad (n \in \mathbb{N}). \tag{24}$$

- As $y_n \to y_0$ there is a subsequence such that

$$|y_{n+1} - y_0| \le |y_n - y_0| \qquad (n \in \mathbb{N}). \tag{25}$$

- Since $\varphi_n \to 0$ we also have $\tan \varphi_n \to 0$ as $n \to \infty$. Therefore $\frac{y_n - y_0}{x_n - x_0} \to 0$ and we find a subsequence for which

$$\frac{|y_n - y_0|}{x_n - x_0} \le \frac{1}{n} \qquad (n \in \mathbb{N}). \tag{26}$$

For (x_n, y_n) satisfying properties (24), (25) and (26) we now construct a regular C^1-curve $\gamma = (\gamma_1(x), \gamma_2(x)) : [x_0, x_1] \to \mathbb{R}^2$ such that $\gamma_2(x_0) = y_0$ and $\gamma_2(x_n) = y_n$ for all $n \in \mathbb{N}$. We define

$$\ell_n(x) := \frac{2(x - x_{n+1})}{x_n - x_{n+1}} - 1 \qquad (x \in [x_{n+1}, x_n], n \in \mathbb{N}) \tag{27}$$

and

$$\gamma(x) := \begin{cases} (x,\, \frac{1}{2}(y_n - y_{n+1}) t(\ell_n(x)) + \frac{1}{2}(y_{n+1} + y_n)) & \text{if } x \in]x_{n+1}, x_n], \\ (x_0, y_0) & \text{if } x = x_0. \end{cases} \tag{28}$$

Note that $\ell_n : [x_{n+1}, x_n] \to [-1, 1]$ is linear, $\ell_n(x_{n+1}) = -1$ and $\ell_n(x_n) = 1$. The only delicate points with respect to continuity of γ and $\dot\gamma$ are the points $x = x_n$ for $n \in \mathbb{N}$ and the point $x = x_0$. It is easy to see that γ is continuous at $x = x_n$. For the derivative we find

$$\dot\gamma(x) = (1,\, (y_n - y_{n+1}) t'(\ell_n(x)) \frac{1}{x_n - x_{n+1}}) \qquad (x \in]x_{n+1}, x_n]). \tag{29}$$

Again we easily find that the left sided and the right sided derivatives of γ at $x = x_n$ exist and that they both equal $(1, 0)$. Thus $\dot\gamma(x_n)$ exists and $\dot\gamma$ is continuous at $x = x_n$. From (25) we obtain

$$|y_n - y_{n+1}| \le 2\, |y_n - y_0| \qquad (n \in \mathbb{N}) \tag{30}$$

and (24) implies

$$x_n - x_{n+1} = (x_n - x_0) - (x_{n+1} - x_0) \ge \frac{x_n - x_0}{2}. \tag{31}$$

Consequently

$$\frac{|y_n - y_{n+1}|}{x_n - x_{n+1}} \le 4\frac{|y_n - y_0|}{x_n - x_0} \le \frac{4}{n} \qquad (n \in \mathbb{N}). \tag{32}$$

Due to (26) (using also (23) and (32)) we have

$$\left| \frac{y_n - y_{n+1}}{x_n - x_{n+1}} t'(\ell_n(x)) \right| \le \frac{4}{n} M \qquad \text{for } x_{n+1} < x \le x_n, \tag{33}$$

hence (recall (29)) $\lim_{x \to x_0} \dot\gamma(x) = (1,0)$. It remains to prove that γ is differentiable at $x = x_0$ and that $\dot\gamma(x_0) = (1,0)$. Obviously only the second component γ_2 of γ has to be considered. For $x \in]x_{n+1}, x_n]$ we have

$$\begin{aligned}
&\frac{|\gamma_2(x) - \gamma_2(x_0)|}{x - x_0} \le \frac{|\gamma_2(x) - \gamma_2(x_{n+1})| + |\gamma_2(x_{n+1}) - \gamma_2(x_0)|}{x - x_0} \le \\
&\le \frac{|\gamma_2(x) - \gamma_2(x_{n+1})|}{x - x_{n+1}} + \frac{|y_{n+1} - y_0|}{x_{n+1} - x_0} \le \sup_{\xi \in [x_{n+1}, x_n]} |\gamma_2'(\xi)| + \frac{1}{n+1} \le \\
&\le \frac{4M}{n} + \frac{1}{n}
\end{aligned}$$

by (26), (29) and (32). But this implies the desired properties of $\dot\gamma$.
Now, for x sufficiently close to x_0 we may assume that $\gamma(x) \in D$. Thus for some $n_0 \in \mathbb{N}$ the function $f \circ \gamma\big|_{[x_0, x_{n_0}]}$ is continuous. This implies $f(z_n) \to f(z_0)$ as $n \to \infty$ since $z_n = \gamma(x_n)$ in contradiction to (22).

□

Remark 4.

1) Once the existence of an appropriate subsequence fulfilling (24)–(26) is established, we can conclude the existence of an interpolating C^1-curve by applying WHITNEY's extension theorem (see [N, p.31]). But since the character of this paper is *constructive* we preferred however to give an explicit construction of γ.
2) We stress the fact that the velocity of the parametrization of the curves along which f is assumed to be continuous is bounded away from zero. In fact $|\dot\gamma(x)| \ge 1$ would be a sufficient requirement. By this assumption 'trivial' interpolating curves, such as C^∞-parametrizations of broken lines with vanishing velocity at the vertices, are excluded. By this argument one can easily see that if we drop the assumtion of regularity, then it is suffitient to claim that f is continuous on all C^∞ images of compact intervals in order to obtain continuity of f.
3) The restriction to the two-dimensional case $D \subseteq \mathbb{R}^2$ is not essential. With some minor modifications the proof works for $D \subseteq \mathbb{R}^n$, $n \ge 2$. However it is not clear to the authors what happens in the infinite dimensional case, since we do not have compactness of the unit sphere in this situation.

4. Functions with a continuous addition law and continuity on 'thin' sets

Here we present a generalization of ZORZITTO's result. We start with a lemma.

Lemma 2. *Let $(X, \langle , \rangle)$ be a real inner product space of dimension greater than 1. Let $e \in X$ with $\|e\| = 1$, and put $B := B(e,1) := \{x \in X \mid \|x - e\| < 1\}$. Then there exist continuous functions $u, v : B \to \partial B$, where ∂B denotes the boundary of B, such that $u + v = \mathrm{id}\big|_B$.*

Proof. We first derive some necessary conditions for the function $u : B \to \partial B$ Suppose, we have $u = u(z) \in \partial B$, $v = v(z) \in \partial B$ and $z = u + v$. Since u and v have the same distance from e, we get that

$$0 = \|u-e\|^2 - \|v-e\|^2 = \langle u, u\rangle - \langle v, v\rangle - \langle u-v, 2e\rangle = \langle u-v, u+v-2e\rangle = \langle 2u-z, z-2e\rangle.$$

Upon dividing by 4 in the above inequality it follows that $\langle u - \frac{z}{2}, \frac{z}{2} - e\rangle = 0$. On the other hand, if we can find a continuous function $w : B \to X \setminus \{0\}$ which satisfies the orthogonality condition

$$\langle w(z), \frac{z}{2} - e\rangle = 0, \tag{34}$$

then the construction

$$u(z) := \frac{z}{2} + \sqrt{1 - \|\frac{z}{2} - e\|^2}\,\frac{w(z)}{\|w(z)\|} \quad \text{and} \quad v(z) := z - u(z)$$

will do the job. Note that $\|\frac{z}{2} - e\| \le \frac{1}{2}(\|z - e\| + \|e\|) < 1$, hence the square root in the definition of u makes sense.

Now let $e_0 \in X$ be given such that $\|e_0\| = 1$, $\langle e_0, e\rangle = 0$, and let $P(z) = z - \langle z, e_0\rangle e_0$ denote the projection onto the orthogonal complement of the subspace generated by e_0. We have $P(z) \neq 0$ for all $z \in B$, for otherwise we would have $z = \lambda e_0$ and consequently $1 > \|z - e\|^2 = \lambda^2 + 1$ due to the orthogonality of e_0 and e.

To determine the function w, we put

$$w(z) = \alpha(z)e_0 + \beta(z)P(z).$$

The orthogonality condition (34) implies

$$\alpha(z)\langle e_0, \frac{z}{2}\rangle + \beta(z)\langle P(z), \frac{z}{2} - e\rangle$$

which allows the determination of the coefficients

$$\alpha(z) = -\langle P(z), \frac{z}{2} - e\rangle \quad \text{and} \quad \beta(z) = \langle e_0, \frac{z}{2}\rangle.$$

To prove continuity of w, we still have to ensure that $\alpha(z)^2 + \beta(z)^2 \neq 0$ for $z \in B$. However $\beta(z) = \alpha(z) = 0$ would imply $z \perp e_0$ and

$$0 = \langle z - \langle z, e_0\rangle e_0, \frac{z}{2} - e\rangle = \frac{1}{2}\|z\|^2 - 2\langle z, e\rangle,$$

leading to the contradiction

$$1 > \|z - e\|^2 = \|z\|^2 - 2\langle z, e\rangle + \|e\|^2 = 1.$$

Thus we have constructed a function w which has all the properties we want.

$\square$

We have the following generalization of ZORZITTO's result.

Theorem 5. *Let $(X, \langle , \rangle)$ be a real inner product space of dimension greater than 1, let Y be a topological space, $g : Y \times Y \to Y$ continuous. Let furthermore $f : X \to Y$ fulfill the functional equation*

$$f(x+y) = g(f(x), f(y)) \qquad (x, y \in X). \tag{35}$$

Then f is continuous on X provided that its restriction to the boundary of some ball of positive radius is continuous.

Proof. At first we suppose that $f|_{\partial B(e,1)}$ is continuous, where $\|e\| = 1$. By Lemma 2, we can write $z = u(z) + v(z)$ where $z \in B(e,1)$, $u(z), v(z) \in \partial B(e,1)$, and u, v are continuous. Then from (35) it follows that

$$f(z) = f(u(z) + v(z)) = g(f(u(z)), f(v(z)))$$

which implies continuity of f on $B(e,1)$.
But the continuity of any solution f of (35) in *some* point implies global continuity. To see this consider $f(x_1 + h) = f((x_0 + h) + (x_1 - x_0)) = g(f(x_0 + h), f(x_1 - x_0))$ and $f(x_1) = g(f(x_0), f(x_1 - x_0))$. The same arguments show that continuity of $f|_{\partial B(x_0,r)}$ for fixed r implies the continuity of $f|_{\partial B(x_1,r)}$ for all x_1. So if we suppose that $f|_{\partial B(x_0,r)}$ is continuous, we can conclude that $f|_{\partial B(0,r)}$ must be continuous. Hence also the function f_r defined by $f_r(x) := f(rx)$ has continuous restriction onto $\partial B(0,1)$. But f_r also satisfies (35). Thus $f_r|_{\partial B(e,1)}$ is continuous, and, by the first part of the proof, we have continuity on X. Accordingly also f is continuous on X since $f(x) = f_r(r^{-1}x)$.
$\square$

It is clear that this result generalizes ZORZITTO's result ([Z]) when we consider $\mathbb{C}$ as a (2-dimensional real) Hilbert space. But in the finite dimensional case an even stronger result may be derived. In the sequel the inner product space X is replaced by $\mathbb{R}^n$ equipped with the usual euclidean norm.

Theorem 6. *Let $X = \mathbb{R}^n$ with $n \geq 2$, Y, f, g be given as in Theorem 5. Suppose moreover that there exist a bounded, open and connected set $O \subset \mathbb{R}^n$ and a compact, connected set $C \subset \mathbb{R}^n$ containing the boundary ∂O of O such that the restriction $f|_C$ is continuous. Then f is continuous on $\mathbb{R}^n$.*

Before we start proving Theorem 6, we formulate a topological lemma which we shall need henceforth.

Lemma 3. *Let $O \subset \mathbb{R}^n$ be a bounded, open and connected set and assume that $C \subset \mathbb{R}^n$ is compact and connected satisfying $\partial O \subset C$. Then*

$$2O \subset C + C \quad \textit{and therefore} \quad (C+C)^\circ \neq \emptyset.$$

Proof. It has to be proved that, for every $x \in O$, we can find $u, v \in C$ with $u + v = 2x$. Let $x \in O$. We translate $\mathbb{R}^n$ in such a way that x is shifted into the origin and consider the sets $C^x = -x + C$ and $O^x = -x + O$

We shall prove that $C^x \cap (-C^x) \neq \emptyset$. We choose w from the compact set C^x such that $\|w\| = \sup\{\|y\| \mid y \in C^x\}$. Since $\partial O^x \subset C^x$, we have

$$\|w\| \geq \sup\{\|y\| \mid y \in \partial O^x\} = \sup\{\|y\| \mid y \in O^x\},$$

implying that $-w \notin O^x$, for otherwise we could find some $v \in O^x$ with $\|v\| > \|w\|$ contradicting the above inequality. We can also suppose that $w \notin \partial O^x$, for otherwise we would have $C^x \cap (-C^x) \neq \emptyset$ and the assertion is proved. Now we choose $\rho \in \mathbb{R}$ in such a way that $n := \rho\, w \in C^x$ and

$$|\rho| = \inf\{|\sigma| \mid \sigma w \in C^x, \sigma \in \mathbb{R}\}.$$

Again we assume that $-n \notin C^x$ for otherwise we would conclude that $C^x \cap (-C^x) \neq \emptyset$. We consider $[\![-n, n[\![$, the straight segment joining $-n$ and n with n excluded. Obviously $0 \in [\![-n, n[\![$ and also $0 \in O^x$. Moreover $[\![-n, n[\![$ does not intersect C^x by the minimality property of ρ, hence it also does not intersect ∂O^x. Since $[\![-n, n[\![$ is connected, has one point in O^x and does not intersect the boundary of O^x, it must lie completely within O^x. (c.f. [R, p.141, Prop. 14.5].) Therefore $-n \in O^x$. Thus we have $-n, -w \in -C^x$, $-n \in O^x$ and $-w \notin \overline{O^x}$. Employing again [R, p.141, Prop. 14.5] and the fact that $-C^x$ is connected, we conclude that $-C^x \cap \partial O^x \neq \emptyset$ and, since $\partial O^x \in C^x$, we obtain $C^x \cap (-C^x) \neq \emptyset$.
Now we choose $u' \in C^x \cap (-C^x)$, i.e. $u' \in C^x$ and $-u' \in C^x$. By definition of C^x there exist $u, v \in C$ such that $u' = -x + u$ and $-u' = -x + v$ and therefore $2x = u + v$, which completes the proof. □

Proof of Theorem 6. Let $x \in O$. We want to show that f is continuous at $2x$. Assume that it is not. Then there exists a sequence $x_n \in O$, $x_n \to x$, such that $f(2x_n)$ does not converge to $f(2x)$. This implies the existence of some neighbourhood V of $f(2x)$ and of a subsequence x'_n of the original one such that

$$f(2x'_n) \notin V \qquad (n \in \mathbb{N}). \tag{36}$$

But every point $2x'_n$ may be written as

$$2x'_n = u'_n + v'_n \qquad (u'_n, v'_n \in C)$$

and we get

$$f(2x'_n) = g(f(u'_n), f(v'_n)).$$

Using the compactness of C we may (by choosing suitable subsequences, if necessary) assume that $u'_n \to u \in C$ and $v'_n \to v \in C$. We also have $2x = u + v$. Passing to the limit and using the continuity of f on C we derive

$$f(2x'_n) = g(f(u'_n), f(v'_n)) \to g(f(u), f(v)) = f(u + v) = f(2x)$$

which contradicts (36).
Hence f is continuous at $2x$. Using the arguments given in the proof of Theorem 4 this implies the continuity of f on X.

□

If $O \in \mathbb{R}^n$ is an open, bounded and connected set which has a connected boundary ∂O, then it is possible to choose $C = \partial O$ in Lemma 3 and we obtain

$$2O \subset \partial O + \partial O.$$

Furthermore we conclude by Theorem 6 that f (satisfying the functional equation (35)) must be continuous on $\mathbb{R}^n$ if its restriction to ∂O is continuous. Thus Theorem 6 is in fact a generalization of Theorem 5 in the finite dimensional case.
It is however possible to derive an even more general result.

Lemma 3′. *Let $O \subset \mathbb{R}^n$ be a bounded, open and connected set. Then*

$$2O \subset \partial O + \partial O \quad \textit{and therefore} \quad (\partial O + \partial O)^\circ \neq \emptyset.$$

Corollary 1. *Let $X = \mathbb{R}^n$ with $n \geq 2$, Y, f, g be given as in Theorem 5. Suppose moreover that there exist a bounded, open and connected set $O \subset \mathbb{R}^n$ such that the restriction $f|_{\partial O}$ is contiuous. Then f is continuous on $\mathbb{R}^n$.*

Once the assertion in Lemma 3′ is established, Corollary 1 follows from Lemma 3′ in exactly the same way as Theorem 6 follows from Lemma 3. Therefore we focus on the proof of Lemma 3′.

Proof of Lemma 3′. Let $O \subset \mathbb{R}^n$ be open, bounded and connected. The complement $\mathbb{R}^n \setminus \overline{O}$ contains exactly one unbounded, open, connected component which we denote by O_∞. Let $C = \partial O_\infty$. Then C is compact and $C \subset \partial O$. (c.f. [D, p.356, Th. 1.2] for these facts.) C is *separating* the space $\mathbb{R}^n$ in the sense that $\mathbb{R}^n \setminus C$ is not connected. Especially O and O_∞ lie in different components of $\mathbb{R}^n \setminus C$ (Suppose they lie in the same component, then there exists a path γ in this component which connects some point in O with some other point in O_∞. This path however must intersect $\partial O_\infty = C$ contradicting the fact that γ lies entirely in $\mathbb{R}^n \setminus C$.)
It can be proved by means of Zorn's Lemma that there exists a minimal compact set (with respect to inclusion) $C^* \subset C$ such that O and O_∞ lie in different components of $\mathbb{R}^n \setminus C^*$.
The set C^* is connected. This is proved as follows. Assume that $C^* = C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$, $C_i \neq \emptyset$ and C_i closed for $i = 1, 2$ and let $p \in O$ and $q \in O_\infty$ be given. Suppose that p and q belong to the same component of $\mathbb{R}^n \setminus C_1$ and that they also belong to the same component of $\mathbb{R}^n \setminus C_2$. Let $\beta_p : \mathbb{R}^n \setminus \{p\} \to S^{n-1}$ be defined as $\beta_p(x) = \dfrac{x - p}{|x - p|}$ and $\beta_q : \mathbb{R}^n \setminus \{q\} \to S^{n-1}$ be analoguosly given by $\beta_q(x) = \dfrac{x - q}{|x - q|}$. It is known [D, p.359, Prop. 4.1] that two points p and q belong to the same component of $\mathbb{R}^n \setminus A$, where A is some compact set, if and only if the restrictions $\beta_p|_A$ and $\beta_q|_A$ are homotopic. Hence there exist homotopies $\Phi_i : [0,1] \times C_i \to S^{n-1}$ with $\Phi_i(0, x) = \beta_p(x)$ and $\Phi_i(1, x) = \beta_q(x)$ for $x \in C_i$ and $i = 1, 2$. But now it is easy to construct a homotopy Φ between β_p and β_q on $C^* = C_1 \cup C_2$. We define $\Phi : [0,1] \times C^* \to S^{n-1}$ by

$$\Phi(t, x) := \begin{cases} \Phi_1(t, x) & \text{if } x \in C_1 \\ \Phi_2(t, x) & \text{if } x \in C_2. \end{cases}$$

It is easily seen that Φ is continuous and satisfies $\Phi(0,x) = \beta_p(x)$ and $\Phi(1,x) = \beta_q(x)$ for all $x \in C^*$. Using again [D, p.359, Prop. 4.1], we conclude that p and q belong to the same component of $\mathbb{R}^n \setminus C^*$. This however is a contradiction to the fact that C^* separates O from O_∞. Therefore p and q are separated by at least one of the C_i-s say by C_1. Since O and O_∞ are connected and $O \cap C_1 = O_\infty \cap C_1 = \emptyset$, it follows that O and O_∞ are subsets of different components of $\mathbb{R}^n \setminus C_1$, which cannot be true since C^* is a minimal set fulfilling this requirement. Thus C^* is connected.
By O^* we denote the component of $\mathbb{R}^n \setminus C^*$ which includes O as a subset. Then we have $\partial O \subset C^*$ [D, p356, Th. 1.2 (3)] and O^* is bounded since it is separated from the (unique) unbounded component of $\mathbb{R}^n \setminus C^*$ (which includes O_∞) by C^*. Therefore Lemma 3 can be applied to O^* and C^* respectively and we finally get

$$2O \subset 2O^* \subset C^* + C^* \subset \partial O + \partial O$$

which completes the proof. $\square$

In dimension $n = 2$, this result can be generalized.

Lemma 4. *Let $C \subset \mathbb{R}^2$ be a compact, connected set, containing three noncollinear points a, b and c. Then*

$$(C + C)^\circ \neq \emptyset.$$

Proof.
1. case: $C^\circ \neq \emptyset$, then obviously $(C + C)^\circ \neq \emptyset$.
2. case: If $\mathbb{R}^2 \setminus C$ has a bounded component O, then $\partial O \subset C$ and Lemma 3 implies that $(C + C)^\circ \neq \emptyset$.
3. case: Assume that $C^\circ = \emptyset$ and $\mathbb{R}^2 \setminus C$ is connected. Under these assumptions C cannot contain all edges of the triangle, whose vertices are a, b and c. Without loss of generality we suppose that there exists a point $x \in [\![a, b]\!] \setminus C$. Here $[\![a, b]\!]$ denotes the closed line-segment joining a and b. We denote by L the straight line passing through x, perpendicular to $[\![a, b]\!]$. Then there exist two points $x_1, x_2 \in L \setminus \{x\}$, such that $[\![x_1, x_2]\!] \cap C = \emptyset$ and $[\![x_1, x_2]\!] \cap [\![a, b]\!] = \{x\}$, since $\mathbb{R}^2 \setminus C$ is open.
We set $\tilde{C} := C \cup [\![a, b]\!]$ and we shall show now that $\mathbb{R}^2 \setminus \tilde{C}$ has at least one bounded component. Assume to the contrary that $\mathbb{R}^2 \setminus \tilde{C}$ is connected. Since $x_1, x_2 \in \mathbb{R}^2 \setminus \tilde{C}$, there exists a broken line $P_{x_1,x_2} \subset \mathbb{R}^2 \setminus \tilde{C}$ with endpoints x_1 and x_2, which does not cross itself. Here we use the fact that the *open*, connected set $\mathbb{R} \setminus \tilde{C}$ is also polygonaly connected (in the sense described above, see [R, p150, Thm. 14.29]). We also may assume that $P_{x_1,x_2} \cap [\![x_1, x_2]\!] = \{x_1, x_2\}$. Hence $J := P_{x_1,x_2} \cup [\![x_1, x_2]\!]$ is a closed polygonal Jordan curve. Moreover a and b lie in different components of $\mathbb{R}^2 \setminus J$, since the line connecting a and b intersects the Jordan curve at exactly one point x. On the other hand, a and b are elements of the connected set C, and therefore C must have a nonempty intersection with the boundary of the bounded component of $\mathbb{R}^2 \setminus J$, which is J. However we assumed that $[\![x_1, x_2]\!] \cap C = \emptyset$ and $P_{x_1,x_2} \cap C = \emptyset$, in contradiction to $J \cap C \neq \emptyset$.
Let O be one of the (open) bounded components of $\mathbb{R}^2 \setminus \tilde{C}$ and let $w \in \partial O$ be a point with maximal distance $\delta > 0$ from $\overline{a, b}$, the line through a, b. Moreover let L_* be the line parallel to $\overline{a, b}$, with distance $\delta/2$ from $\overline{a, b}$ which lies between $\overline{a, b}$ an w. We now consider the nonempty and open intersection of the open halfplane H_*, defined by

$\partial H_* = L_*$ and $w \in H_*$, with O. We put $O_* := H_* \cap O$ and take any $z \in O_*$. Then, by Lemma 3, there exist points $u, v \in \partial O \subset C \cup [\![a, b]\!]$ such that $z = \frac{u+v}{2}$. It is easy to see that $u, v \notin [\![a, b]\!]$, because $u \in [\![a, b]\!] \subset \overline{a, b}$ would imply that the distance between $v \in \partial O$ and $\overline{a, b}$ is twice the distance between z and $\overline{a, b}$, which is greater that δ, in contradiction to the maximality of δ. Therefore we have $u, v \in C$ and we can write $2O_* \subset C + C$, i.e. $(C + C)^\circ \neq \emptyset$.

□

With Lemma 4, we can generalize a result by Kuczma [K, p.219 (§5 plane curves) Thm. 2].

Theorem 7. *Let $D \subset \mathbb{R}^2$ be open and convex and let $f : D \to \mathbb{R}$ be a Jensen convex function, i.e.*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \qquad \text{for all } x, y \in D.$$

Furthermore let $C \subset D$ be a connected compact set containing three noncollinear points, such that f is bounded from above on C. Then f is continuous on D.

Proof. Let $f(z) \leq M$ for all $z \in C$, then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \leq \frac{M + M}{2} = M$$

for all $x, y \in C$, or, what is the same, $f(u) \leq M$ for all $u \in \frac{1}{2}(C + C)$. By Lemma 4, we have $(\frac{1}{2}(C + C))^\circ \neq \emptyset$, and therefore, by the Theorem of Bernstein-Doetsch, we get continuity of f on D.

By the same method we get the following generalization of Theorem 6 in the case of dimension 2.

Theorem 6′. *Let $f : \mathbb{R}^2 \to Y$ fulfill the equation*

$$f(x + y) = g(f(x), f(y))$$

with Y a topological space and $g : Y \times Y \to Y$ a continuous function. If $C \subset \mathbb{R}^2$ is a compact connected set, containing three noncollinear points, and if $f|_C$ is continuous, then f is continuous on $\mathbb{R}^2$.

Proof. Imitate the proof of Theorem 6 by using Lemma 4.

Conjecture: Let $n \geq 2$ and let $C \subset \mathbb{R}^n$ be a compact connected set which contains $n+1$ affinely independent points, then the n-fold sum $C + \cdots + C$ has nonempty interior.

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