SEQUENTIAL PREDICTOR-CORRECTOR REGULARIZATION METHODS AND THEIR LIMITATIONS

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ABSTRACT. A well-known sequential predictor-corrector regularization method for the stable solution of ill-posed Volterra problems is analyzed. Under a certain stability assumption well-posedness and convergence results are obtained. Under the additional assumption of smoothness for the exact solution, convergence rates are proved. It is also proved that the stability condition cannot be satisfied for Volterra problems with ν-smoothing kernels, where $\nu \geq 5$. Numerical examples are presented showing the stable and the unstable situation and verifying numerically the convergence rate result.

1. INTRODUCTION

The general aim of this paper is the analysis of a class of sequential (structure preserving) regularization methods for the solution of ill-posed Volterra Problems. A discrete sequential regularization method for ill-posed Volterra equations of the first kind was introduced and investigated by J. V. Beck already in the early 1960's under the name of "sequential (constant) function specification method". (See [3], [1], [2] and the monograph [4].) The idea behind Beck's method and its variants is to choose a constant value u_i for the discrete solution on the time-interval $[t_{i-1}, t_i]$ in such a way that a constant continuation with this value into the interval $[t_i, t_{i+r}]$ fits the given data not only for the current time t_i but also, in a least-squares sense, at a few time steps t_{i+j}, $j = 1, \ldots, r$ *in the future*. In the infinite dimensional limit Beck's method leads to a Volterra integral equation of the second kind. (See [10] for a derivation of the limiting second-kind equation.)

More recently, P. K. Lamm has made considerable contributions to the development and understanding of sequential regularization. Specifically the equivalence of Beck's sequential constant function specification method with the discretization of a properly defined Volterra equation of the second kind is due to Lamm (see [11], [12], and [10]). In [12] it could be shown that the second-kind equation is solvable in a stable way. Also it has been shown that, as the data noise goes to zero, the solution of the approximating second-kind Volterra equation converges to the exact solution for finitely smoothing kernels under a certain stability condition (see [12]). For 1-smoothing kernels, a convergence rate of order $O(\delta^{\frac{1}{2}})$ was derived in [10]. A regularization theory for nonconvolution kernels was developed in [14] and several generalizations were considered in [15] and [7]. We also refer to the

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survey article [13] and the numerous references cited therein for an overview of the literature on stable numerical methods for the sequential solution of ill-posed Volterra equations. We should also mention that the class of regularization methods under consideration occurs under various names in the literature such as "local regularization method", "predictor-corrector regularization", "future-sequential regularization", "Beck's method", or combinations of these notions.

However, the analysis of sequential regularization methods cannot be considered as complete. Specifically convergence rate results for ν-smoothing kernels with $\nu > 1$ are still lacking. Moreover, the frequently raised question, whether or not sequential methods are suitable for the numerical solution of severely ill-posed Volterra problems, such as the inverse heat conduction problem, is, up to this point, not satisfactorily answered. We shall fill part of this gap in the subsequent investigations. We shall provide a regularization theory for finitely smoothing convolution kernels including convergence rates. We shall, moreover, prove that a certain stability condition, which is crucial for the analysis and, as numerical experiments indicate, also for the numerical stability of the method, cannot be satisfied for ν-smoothing kernels with $\nu \geq 5$, thus limiting the applicability of the method to mildly ill-posed Volterra problems.

Section 2 is devoted to the convergence analysis of the (infinite dimensional) regularization method derived in [10]. We obtain stability estimates and a convergence result in the case of exact data. For the class of finitely smoothing kernels we can also prove a convergence rate result. As usual in inverse problems the rate of convergence is determined by smoothness properties of the exact solution (cf. Theorem 3). In spirit, our analysis is similar to the considerations in [12]. Apart from the convergence rate result, which is completely new, we could avoid to impose strong smoothness conditions on the exact solution to get convergence, as it is necessary in the analysis in [12], and we get a better stability estimate (see Theorem 1). In section 3 we investigate limitations of the method. We show that we cannot construct a sequential regularization scheme for which the stability condition (A_2) is satisfied if the convolution kernel is ν-smoothing with $\nu \geq 5$. The last section 4 contains numerical experiments showing different situations where the stability condition (A_2) is or is not satisfied and we verify numerically the validity of the convergence rate result in Theorem 3.

2. Convergence Analysis

We consider the linear Volterra equation of the first kind

$$\int_0^t k(t-s)\,u(s)\,ds = f(t) \quad \text{for } t \in [0,1] \tag{2.1}$$

with kernel $k \in \mathcal{C}^{(\nu)}[0,1]$ and data $f \in \mathcal{C}[0,1]$. It follows from the Arzela-Ascoli Theorem that the Volterra operator

$$K : \mathcal{C}[0,1] \to \mathcal{C}[0,1]; \quad Ku(\cdot) = \int_0^{\cdot} k(\cdot - s)\,u(s)\,ds$$

is compact. Thus, a solution to (2.1), if it exists, does not depend continuously on the right hand side f. Moreover, it is known ([12], [16]) that

the degree of ill-posedness depends on local properties of k at $t = 0$. More precisely we say that the kernel k is *ν-smoothing* if:

(2.2)
$$k \in \mathcal{C}^{(\nu)}[0,1] \text{ and } k^{(l)}(0) = 0 \text{ for } l = 0, \dots, \nu - 2, \text{ and } k^{(\nu-1)}(0) \neq 0,$$

where $k^{(l)}$ denotes the l'th derivative of k. The numerical solution of (2.1) becomes more and more ill-conditioned the larger ν is. **We assume for the rest of the paper that k is ν-smoothing with $\nu \geq 1$.** In [10] the following family of approximating Volterra equations of the second kind for the stable solution of (2.1) was defined. Let $R > 0$ be given. For every $\rho \in (0, R)$ suppose that $\omega = \omega_\rho$ is a positive, regular Borel measure on $[0, \rho]$. We set:

$$\kappa(\tau) = \int_0^\tau k(s)\, ds, \tag{2.3}$$

$$\tilde{k}(t) = \int_0^\rho k(t+\tau)\, d\omega(\tau), \tag{2.4}$$

$$\tilde{f}(t) = \int_0^\rho f(t+\tau)\, d\omega(\tau), \tag{2.5}$$

and

$$a = a_\rho = \int_0^\rho \kappa(\tau)\, d\omega(\tau) \tag{2.6}$$

and we consider the second-kind equation

$$a\, u(t) + \int_0^t \tilde{k}(t-s)\, u(s)\, ds = \tilde{f}(t) \text{ for } t \in [0,1]. \tag{2.7}$$

Here we indicated ρ-depending expressions by a corresponding subscript. In the following we often omit the subscript for the sake of compact notation. From (2.5) and (2.7) we see that we need to have knowledge of the data f on the *future* time-interval $[t, t+\rho]$ to solve (2.7) at the current time t. Thus, in order to solve (2.7) on the whole interval $[0,1]$, it is necessary to have data $f(t)$ on the extended interval $[0, 1+\rho]$. In [12] it is proved that the solution to (2.7) converges to the solution to (2.1) (in the case of exact data) if the regularization parameter ρ goes to zero. To prove this convergence result the following assumptions on the measure ω are made.

(A_1) There exist an integer $s \geq 0$ and positive numbers C_j independent of ρ such that
$$\int_0^\rho \tau^j\, d\omega_\rho(\tau) = \rho^{s+j}\big(C_j + o(\rho)\big) \text{ for } j = 0, \dots, \nu + 1.$$

(A_2) All roots of the polynomial
$$p(x) = p_{\nu,\omega}(x) = \sum_{j=0}^{\nu} \frac{C_j}{j!}\, x^j$$
have negative real parts.

In the following we shall improve the stability estimate in [12] and we shall prove a result on convergence rates for ν-smoothing kernels. We set

$$k_\nu = k^{(\nu-1)}(0) \neq 0.$$

Without loss of generality we assume that $k_\nu > 0$. This assumption implies that $k(s) > 0$ for s small enough. We therefore usually omit absolute-value bars on k if we integrate over small intervals of the form $[0, \rho]$. With (A_1) holding we get the following asymptotic estimates:

$$k^{(l)}(s) = s^{\nu-l-1}\left(\frac{k_\nu}{(\nu-l-1)!} + o(s)\right) \text{ for } l = 0, \dots, \nu-1, \tag{2.8}$$

$$\kappa(\tau) = \tau^\nu\left(\frac{k_\nu}{\nu!} + o(\tau)\right), \tag{2.9}$$

$$a = \rho^{s+\nu}\left(C_\nu\,\frac{k_\nu}{\nu!} + o(\rho)\right). \tag{2.10}$$

From definition (2.4) it easily follows that $\tilde{k} \in \mathcal{C}^\nu[0,1]$. Using (2.8) and (A_1), we obtain:

$$\tilde{k}^{(l)}(0) = \rho^{s+\nu-l-1}\left(C_{\nu-l-1}\,\frac{k_\nu}{(\nu-l-1)!} + o(\rho)\right) \text{ for } l = 0, \dots, \nu-1. \tag{2.11}$$

We use the transformation $y(t) = y_\rho(t) = u(\rho t)$ to obtain from (2.7) the Volterra equation

$$a\,y(t) + \rho\int_0^t \tilde{k}(\rho(t-s))\,y(s)\,ds = \tilde{f}(\rho t) \text{ for } t \in \left[0, \frac{1}{\rho}\right]. \tag{2.12}$$

We can express the solution to (2.12) with help of the resolvent $r = r_\rho$, i.e., the solution to

$$a\,r(t) + \rho\int_0^t \tilde{k}(\rho(t-s))\,r(s)\,ds = \rho\,\tilde{k}(\rho t) \text{ for } t \in \left[0, \frac{1}{\rho}\right]. \tag{2.13}$$

With r given we find for the solution to (2.12)

$$y(t) = \frac{1}{a}\left(\tilde{f}(\rho t) - \int_0^t r(t-s)\,\tilde{f}(\rho s)\,ds\right) \text{ for } t \in \left[0, \frac{1}{\rho}\right]. \tag{2.14}$$

(See Gripenberg, Londen and Staffans [9]). The advantage of using the resolvent r for the estimation of y is that the right-hand side in (2.13) is ν-times continuously differentiable. Due to this property we can transform (2.13) into a system of integro-differential equations. If we make this transformation for equation (2.12) (as in [12]), we have to assume additional smoothness for f. In the transformed situation we can use Lyapunov stability theory to obtain estimates for the resolvent r. These estimates can then be used in combination with (2.14) to get bounds for y. The idea to introduce and estimate the resolvent is the main technical difference to [12].

Lemma 1. *Assume that conditions (A_1) and (A_2) hold. Then there exist constants $\tilde{C} > 0$ (depending on ω) and $m > 0$ (depending on k and ω) such that*

$$\|r\|_{L^1(0,\frac{1}{\rho})} \leq m \tag{2.15}$$

independent of ρ if

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}.$$

Here r denotes the resolvent defined in (2.13).

Proof. Since $\tilde{k}$ is ν times continuously differentiable we can differentiate (2.13) with respect to t:

$$r^{(j)}(t) = -\frac{1}{a}\sum_{l=0}^{j-1} \rho^{l+1}\, \tilde{k}^{(l)}(0)\, r^{(j-l-1)}(t) - \frac{\rho^{j+1}}{a}\int_0^t \tilde{k}^{(j)}(\rho(t-s))\, r(s)\, ds + \frac{\rho^{j+1}}{a}\tilde{k}^{(j)}(\rho t) \tag{2.16}$$

for $j = 0, \dots, \nu$ and $t \in [0, \frac{1}{\rho}]$. We set

$$\gamma_l = \gamma_l(\nu) = \frac{C_l}{C_\nu}\frac{\nu!}{l!} \text{ for } l = 0, \cdots, \nu-1. \tag{2.17}$$

Using (2.16), (2.11), (2.10), and (2.17) we find for $j = \nu$ that

$$r^{(\nu)}(t) = -\sum_{l=0}^{\nu-1}\Big(\gamma_{\nu-l-1} + o(\rho)\Big)\, r^{(\nu-l-1)}(t) - \frac{\rho^{\nu+1}}{a}\int_0^t \tilde{k}^{(\nu)}\big(\rho(t-s)\big)\, r(s)\, ds + \frac{\rho^{\nu+1}}{a}\tilde{k}^{(\nu)}(\rho t). \tag{2.18}$$

We write the ν'th-order equation (2.18) as a system of integro-differential equations in the usual way, setting

$$\mathbf{r} = (r^{(0)}, \dots, r^{(\nu-1)})^T.$$

With this we obtain the system

$$\mathbf{r}'(t) = A\,\mathbf{r}(t) + M_\rho \mathbf{r}(t) + \int_0^t D_\rho(t-s)\,\mathbf{r}(s)\, ds + \mathbf{g}_\rho(t) \tag{2.19}$$

where:

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ -\gamma_0 & -\gamma_1 & \dots & -\gamma_{\nu-1} \end{pmatrix}, \tag{2.20}$$

$$\|M_\rho\| \to 0 \text{ as } \rho \to 0, \tag{2.21}$$

$$D_\rho(t) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & & \vdots \\ -\frac{\rho^{\nu+1}}{a}\tilde{k}^{(\nu)}(\rho t) & 0 \quad \dots & 0 \end{pmatrix}, \tag{2.22}$$

and

$$\mathbf{g}_\rho(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\rho^{\nu+1}}{a}\tilde{k}^{(\nu)}(\rho t) \end{pmatrix}. \tag{2.23}$$

Let $Z(t)$ be the $\nu \times \nu$-matrix whose columns are solutions to the homogeneous part of (2.19), i.e. for (2.19) with $\mathbf{g}_\rho = 0$. Let us assume further that Z satisfies the initial condition $Z(0) = I$ (the $\nu \times \nu$ identity matrix). Later on we shall use Z in the variation of constants formula to express and estimate $\mathbf{r}$. Our next step is to derive (ρ-independent) estimates for the columns of Z.

The eigenvalues of the matrix A are given by the roots of the polynomial,

$$q(x) = (-1)^\nu \left(\sum_{l=0}^{\nu-1} \gamma_l \, x^l + x^\nu \right).$$

Upon multiplying q by $\frac{C_\nu}{\nu!}$ and using (2.17) we see that the roots of q coincide with the roots of the polynomial $p(x)$ defined in (A_2). By assumption (A_2) we also know that all roots of p have negative real parts. Then, for sufficiently small ρ, all real parts of eigenvalues of $A_\rho = A + M_\rho$ are negative and bounded away from zero. With the notation $A_0 = A$ we define

$$B_\rho = \int_0^\infty (\exp(A_\rho t))^T \, \exp(A_\rho) \, dt$$

for $\rho \geq 0$ sufficiently small. The matrix B_ρ is well defined, symmetric, positive definite, and satisfies

$$A_\rho^T \, B_\rho + B_\rho \, A_\rho = -I$$

(cf. [6, p. 124]). Note that B_ρ cannot be defined if A_ρ was not a stable matrix. Using Lebesgue's dominated convergence theorem we find $B_\rho \to B_0$ as $\rho \to 0$. It follows that there exist positive constants L, K, and R independent of ρ such that:

$$|\mathbf{x}| \geq 2L \, (\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{2.24}$$

$$|B_\rho \mathbf{x}| \leq K \, (\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{2.25}$$

$$|\mathbf{x}| \leq R \, (\mathbf{x}^T B_\rho \mathbf{x})^{\frac{1}{2}} \tag{2.26}$$

for all $\mathbf{x} \in \mathbb{R}^\nu$ and all ρ sufficiently small. We define the Lyapunov functional

$$V_\rho\big(t, \mathbf{x}(\cdot)\big) = \big(\mathbf{x}^T(t) B_\rho \mathbf{x}(t)\big)^{\frac{1}{2}} + \overline{K} \int_0^t \int_t^{\frac{1}{\rho}} \|D_\rho(\xi - s)\| \, d\xi \, |\mathbf{x}(s)| \, ds \tag{2.27}$$

with some $\overline{K} > 0$ chosen independently of ρ. Let $\mathbf{z}_j$, $j = 1, \ldots, \nu$ denote the j'th column of Z. Following the calculations in [6, p. 38] we find for the

derivative of V_ρ along $\mathbf{z}_j$ that

$$\frac{d}{dt}\left[V_\rho\big(t,\mathbf{z}_j(t)\big)\right] \le -\left(L-\overline{K}\int_t^{\frac{1}{\rho}}\|D_\rho(\xi-t)\|\,d\xi\right)|\mathbf{z}_j(t)| - (\overline{K}-K)\int_0^t \|D_\rho(t-s)\|\,|\mathbf{z}_j(s)|\,ds. \tag{2.28}$$

Using (2.9) we have:

$$\begin{aligned}\int_t^{\frac{1}{\rho}}\|D_\rho(\xi-t)\|\,d\xi &\le \frac{\rho^{\nu+1}}{a}\int_t^{\frac{1}{\rho}}|\tilde{k}^{(\nu)}(\rho(\xi-t))|\,d\xi\\ &= \frac{\rho^\nu}{a}\int_0^{1-\rho t}|\tilde{k}^{(\nu)}(s)|\,ds\\ &\le \frac{\rho^\nu}{a}\int_0^\rho\int_0^{1-\rho t}|k^{(\nu)}(s+\tau)|\,ds\,d\omega(\tau)\\ &\le \|k^{(\nu)}\|_{L^1(0,1+\rho)}\left(\frac{\nu!}{k_\nu}\frac{C_0}{C_\nu}+o(\rho)\right). \end{aligned}\tag{2.29}$$

We want to choose $\overline{K}$ in such a way that $\overline{K}-K\ge 0$ and that there exists $\overline{L}>0$ satisfying

$$\left(L-\overline{K}\int_t^{\frac{1}{\rho}}\|D_\rho(\xi-t)\|\,d\xi\right)\ge\overline{L} \tag{2.30}$$

for all ρ sufficiently small and for all $t\in[1,\frac{1}{\rho}]$. Using the estimate (2.29) we see that $\overline{K}$ can be chosen satisfying the above inequalities if

$$K\le\overline{K}\le\frac{(L-\overline{L})\,k_\nu\,C_\nu}{\nu!\,\|k^{(\nu)}\|_{L^1(0,1+\rho)}\,C_0}+o(\rho).$$

It is now easily seen that we can find appropriate $\overline{L}$ and $\overline{K}$ if we have

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu}<\frac{L\,C_\nu}{K\,\nu!\,C_0}=:\tilde{C}.$$

This is the constant $\tilde{C}$ referred to in the formulation of the Lemma. Note that $\tilde{C}$ depends exclusively on the C_l's and hence only on the choice of ω. With (2.30) and $\overline{K}-K\ge 0$ we conclude from (2.28) that

$$\frac{d}{dt}\left[V_\rho\big(t,\mathbf{z}_j(t)\big)\right]\le-\overline{L}|\mathbf{z}_j(t)|\text{ for all }j.$$

Integrating this relation yields

$$V_\rho\big(t,\mathbf{z}_j(t)\big)\le V_\rho\big(0,\mathbf{z}_j(0)\big)-\overline{L}\int_0^t|\mathbf{z}_j(s)|\,ds.$$

With (2.27), (2.24), and (2.26) we obtain

$$\frac{1}{R}|\mathbf{z}_j(t)|\le\frac{1}{2L}|\mathbf{z}_j(0)|-\overline{L}\int_0^t|\mathbf{z}_j(s)|\,ds.$$

We have $|\mathbf{z}_j(0)|=1$. Using Gronwall's inequality we find

$$|\mathbf{z}_j(t)|\le\frac{R}{2L}e^{-R\overline{L}t}$$

and

$$\|\mathbf{z}_j\|_{L^1(0,\frac{1}{\rho})} \leq \frac{1}{2L\overline{L}}$$

for $j = 1, \dots, \nu$. For the fundamental matrix $Z(t)$ we therefore obtain

$$\|Z(t)\| \leq \frac{(\nu+1)R}{2L} e^{-R\overline{L}t} \tag{2.31}$$

and

$$\|Z\|_{L^1(0,\frac{1}{\rho})} \leq \frac{\nu+1}{2L\overline{L}}. \tag{2.32}$$

Using the variation of constants formula (see e.g. [6]), we find

$$\mathbf{r}(t) = Z(t)\,\mathbf{r}(0) + \int_0^t Z(t-s)\,\mathbf{g}_\rho(s)\,ds \tag{2.33}$$

for $t \in [0, \frac{1}{\rho}]$. From (2.10), (2.11), and (2.16) it is easily seen by induction that there exists a constant $d > 0$ independent of ρ such that

$$|\mathbf{r}(0)| \leq d \tag{2.34}$$

for all ρ sufficiently small. For $\mathbf{g}_\rho$ we have the following estimate:

$$\begin{aligned}
|\mathbf{g}_\rho(t)| &\leq \frac{\rho^{\nu+1}}{a} |\tilde{k}^{(\nu)}(\rho t)| \leq \frac{\rho^{\nu+1}}{a} \int_0^\rho |k^{(\nu)}(\rho t + \tau)|\, d\omega(\tau) \\
&\leq \frac{\rho^{s+\nu+1}\, \|k^{(\nu)}\|_{\mathcal{C}[0,1+\rho]} (C_0 + o(\rho))}{a} \\
&= \rho \|k^{(\nu)}\|_{\mathcal{C}[0,1+\rho]} \left(\frac{C_0}{C_\nu} \frac{\nu!}{k_\nu} + o(\rho) \right) \\
&\leq \rho\, \tilde{m} \qquad (2.35)
\end{aligned}$$

with some constant $\tilde{m}$ independent of ρ. From (2.33) (using (2.31), (2.32), (2.34), and (2.35)) it follows that

$$|\mathbf{r}(t)| \leq \frac{(\nu+1)\,R\,d}{2L} e^{-R\overline{L}t} + \rho \frac{(\nu+1)\tilde{m}}{2L\overline{L}}.$$

Consequently we have

$$\begin{aligned}
\|\mathbf{r}\|_{L^1(0,\frac{1}{\rho})} &\leq \int_0^{\frac{1}{\rho}} \left(\frac{(\nu+1)\,R\,d}{2L} e^{-R\overline{L}t} + \rho \frac{(\nu+1)\tilde{m}}{2L\overline{L}} \right) dt \\
&\leq \frac{(\nu+1)\,(d+\tilde{m})}{2L\overline{L}} =: m. \qquad (2.36)
\end{aligned}$$

Obviously estimate (2.36) also holds for the first component r of $\mathbf{r}$. This completes the proof. □

With Lemma 1 we get the following stability estimate.

Theorem 1. *Let f be continuous on $[0, 1+R)$ for some $R > 0$ and let k be ν-smoothing with $\nu > 0$. Let a be given by (2.6) and assume that $\tilde{f}$ and $\tilde{k}$ are defined by (2.5) and (2.4), respectively. Assume moreover that (A_1)*

and (A_2) *hold and let* $\tilde{C} > 0$ *be given as in Lemma 1. With* u *denoting the solution to* (2.7) *we have*

$$|u(t)| \leq \frac{\overline{m}}{\rho^\nu} \|f\|_{\mathcal{C}[0,1+\rho]} \text{ for all } t \in [0,1] \tag{2.37}$$

for some constant $\overline{m} > 0$ *independent of* ρ *provided that*

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}$$

holds.

Proof. From (2.14) and Lemma 1 we conclude that

$$|y(t)| \leq \frac{1}{a} \|\tilde{f}\|_{\mathcal{C}[0,1]} (1+m),$$

where m is the bound for $\|r\|_{L^1(0,\frac{1}{\rho})}$ derived in Lemma 1. With (2.5), (2.10), and (A_1) we find

$$\frac{|\tilde{f}(t)|}{a} \leq \frac{1}{\rho^\nu} \left(\frac{\|f\|_{\mathcal{C}[0,1+\rho]} C_0 \, \nu!}{C_\nu \, k_\nu} + o(\rho) \right) \text{ for all } t \in [0,1].$$

Hence, there exists a constant $\overline{m} > 0$ (depending on ω and k) such that

$$|y(t)| \leq \frac{\overline{m}}{\rho^\nu} \|f\|_{\mathcal{C}[0,1+\rho]} \text{ for all } t \in [0, \frac{1}{\rho}].$$

Since $u(t) = y(\frac{t}{\rho})$ the above estimate also holds for u. □

Suppose now that we are given functions f_0 and u_0 which are continuous on $[0, 1+R)$ and which satisfy

$$\int_0^t k(t-s) \, u_0(s) \, ds = f_0(t) \text{ on } [0, 1+R). \tag{2.38}$$

We refer to u_0 as the exact solution to (2.38) with attainable data f_0. We have the following convergence result for the approximating regularization scheme (2.7).

Theorem 2. *Let* u_0 *and* f_0 *be given satisfying* (2.38). *Moreover let*

$$\mu(u_0, \rho) = \max\{ \, |u_0(s) - u_0(t)| \, : \, s, t \in [0, 1+\rho], \, |s-t| \leq \rho \}.$$

Furthermore let u_ρ *denote the solution to* (2.7) *with data given by* $\tilde{f}_0$ *(constructed from* f_0*) and assume that* (A_1) *and* (A_2) *hold. With* $\tilde{C} > 0$ *denoting the constant in Lemma 1, there exists a constant* $\overline{M} > 0$ *(depending on* k *and* ω*) such that*

$$|u_\rho(t) - u_0(t)| \leq \overline{M} \, \mu(u_0, \rho) \text{ for all } t \in [0,1] \tag{2.39}$$

whenever

$$\frac{\|k^{(\nu)}\|_{L^1(0,1)}}{k_\nu} < \tilde{C}. \tag{2.40}$$

Proof. Evaluating (2.38) at the point $t+\tau$ and integrating from 0 to ρ with respect to the measure ω yields:

$$\int_0^\rho \int_0^{t+\tau} k(t-s+\tau)\, u_0(s)\, ds\, d\omega(\tau) = \int_0^\rho f_0(t+\tau)\, d\omega(\tau),$$

and hence

$$\int_0^t \int_0^\rho k(t-s+\tau)\, d\omega(\tau)\; u_0(s)\, ds + \int_0^\rho \int_t^{t+\tau} k(t-s+\tau)\, u_0(s)\, ds\, d\omega(\tau) = \int_0^\rho f_0(t+\tau)\, d\omega(\tau).$$

With (2.4) and (2.5) we obtain

$$\int_0^t \tilde{k}(t-s)\, u_0(s)\, ds + \int_0^\rho \int_0^\tau k(s)\, u_0(t-s+\tau)\, ds\, d\omega(\tau) = \tilde{f}_0(t).$$

We set $v_\rho = u_\rho - u_0$. Since u_ρ is solution to (2.7) with right-hand side given by $\tilde{f}_0$ we obtain, using (2.6):

$$a\, v_\rho(t) + \int_0^t \tilde{k}(t-s)\, v_\rho(s)\, ds = \int_0^\rho \int_0^\tau k(\tau-s) \Big(u_0(t+s) - u_0(t)\Big)\, ds\, d\omega(\tau). \tag{2.41}$$

Using the transformation $w_\rho(t) = v_\rho(\rho t)$ we get:

$$a\, w_\rho(t) + \rho \int_0^t \tilde{k}(\rho(t-s))\, w_\rho(s)\, ds = \int_0^\rho \int_0^\tau k(\tau-s) \Big(u_0(\rho t+s) - u_0(\rho t)\Big)\, ds\, d\omega(\tau)$$

for $t \in [0, \frac{1}{\rho}]$. We set

$$F_\rho(t) = \frac{1}{a} \int_0^\rho \int_0^\tau k(\tau-s) \Big(u_0(\rho t+s) - u_0(\rho t)\Big)\, ds\, d\omega(\tau).$$

We have the following estimate for F_ρ:

$$\begin{aligned} |F_\rho(t)| &\le \max_{\xi\in[\rho t, \rho t+\rho]} |u_0(\xi) - u_0(\rho t)|\, \frac{1}{a} \int_0^\rho \int_0^\tau k(s)\, ds\, d\omega(\tau) \\ &= \max_{\xi\in[\rho t, \rho t+\rho]} |u_0(\xi) - u_0(\rho t)| \le \mu(u_0, \rho) \end{aligned} \tag{2.42}$$

for all $t \in [0, \frac{1}{\rho}]$.

We use again the resolvent r (defined by (2.13)) to estimate the solution w_ρ. We have

$$w_\rho(t) = \left(F_\rho(t) + \int_0^t r(t-s)\, F_\rho(s)\, ds\right).$$

Consequently, using (2.42) and Lemma 1 we find

$$|w_\rho(t)| \le \mu(u_0, \rho) + m\, \mu(u_0, \rho).$$

With $\overline{m} = 1 + m$ and $|v_\rho(t)| = |y_\rho(\frac{t}{\rho})|$ the proof is complete. □

Remark 1. Since u_0 is uniformly continuous on $[0, 1+\rho]$, it is clear that $\mu(u_0, \rho) \to 0$ as $\rho \to 0$. If the exact solution u_0 is Hölder continuous of order $0 < \alpha \leq 1$, then we have $\mu(u_0, \rho) = O(\rho^\alpha)$ and we get the estimate

$$|u_\rho(t) - u_0(t)| \leq m\, \rho^\alpha \tag{2.43}$$

for all $t \in [0, 1]$ provided that the assumptions of Theorem 2 hold. We mention that μ is closely related to the modulus of continuity of the function u.

We now come to the announced convergence rate result in the case of noisy data. Note that, as usual in inverse problems, a convergence rate can only be guaranteed if the data satisfy certain smoothness conditions.

Theorem 3. *Suppose the notation and conditions of Theorems 1 and 2 hold. Let a family of functions $\{f_\delta\}_{\delta>0} \subset \mathcal{C}[0, 1+\rho_0]$ be given satisfying*

$$|f_\delta(t) - f_0(t)| \leq \delta$$

for all $t \in [0, 1+\rho_0]$ with some $\rho_0 > 0$. Moreover denote by u_ρ^δ the solution to (2.7) with data $\tilde{f}_\delta$ defined by means of $f = f_\delta$ in (2.5). Then there exists $\rho = \rho(\delta)$ such that

$$|u^\delta_{\rho(\delta)}(t) - u_0(t)| \to 0 \quad \text{as } \delta \to 0$$

uniformly in $t \in [0, 1]$.

If additionally $u_0 \in \mathcal{C}^\alpha[0, 1+\rho_0]$ for some $0 < \alpha \leq 1$ then there exists a constant $M > 0$ such that, for the choice

$$\rho(\delta) = \hat{m}\, \delta^{\frac{1}{\alpha+\nu}}$$

with some constant $\hat{m} > 0$, we have

$$|u^\delta_{\rho(\delta)}(t) - u_0(t)| \leq M\, \delta^{\frac{\alpha}{\alpha+\nu}}$$

for all $t \in [0, 1]$.

Proof. Using Theorem 1 and Theorem 2 we get

$$\begin{aligned} |u_\rho^\delta(t) - u_0(t)| &\leq |u_\rho^\delta(t) - u_\rho(t)| + |u_\rho(t) - u_0(t)| \\ &\leq \frac{\overline{m}}{\rho^\nu}\, \delta + \overline{M}\, \mu(u_0, \rho). \end{aligned}$$

For the choice $\rho(\delta) = \delta^{\frac{1}{2\nu}}$ we obtain $\mu(u_0, \rho(\delta)) \to 0$ and $\rho(\delta)^{-\nu}\, \delta \to 0$ as $\delta \to 0$. This proves the first assertion.

If u_0 is Hölder continuous with exponent $\alpha > 0$ we have $\mu(u_0, \rho) \leq m\, \rho^\alpha$ with some constant $m > 0$. Thus, we have

$$|u_\rho^\delta(t) - u_0(t)| \leq \frac{\overline{m}}{\rho^\nu}\, \delta + \overline{M}\, m\, \rho^\alpha \leq \tilde{M} \left(\frac{\delta}{\rho^\nu} + \rho^\alpha \right)$$

where $\tilde{M} = \max(\overline{m}, \overline{M} m)$. For the choice $\rho(\delta) = \hat{m}\, \delta^{\frac{1}{\alpha+\nu}}$ we get

$$|u^\delta_{\rho(\delta)}(t) - u_0(t)| \leq \tilde{M} \left(\left(\frac{1}{\hat{m}} \right)^\nu + \hat{m}^\alpha \right) \delta^{\frac{\alpha}{\alpha+\nu}}$$

for all $t \in [0, 1]$. Setting $M = \tilde{M} \left((\frac{1}{\hat{m}})^\nu + \hat{m}^\alpha \right)$ completes the proof. ∎

3. Limitations of the Method

The crucial condition for stability and convergence of the sequential predictor-corrector method is the stability criterion (A_2). It has been shown (see [12]) that the uniform (Lebesgue-) measure $\omega(A) = \int_A dx$ satisfies (A_2) for $\nu = 1, 2, 3$. Also in [12] a sophisticated construction of a stabilizing Borel-Stieljes measure for $\nu = 4$ is given. In this section we shall prove that it is impossible to find a positive measure ω which satisfies (A_2) for $\nu \geq 5$.

We start our considerations by studying structural properties of the polynomial

$$p_{\nu,\omega}(x) = \sum_{j=0}^{\nu} \frac{C_j}{j!} x^j.$$

defined in (A_2). For the coefficients C_l we find the following relation:

$$\begin{aligned} C_l^2 &= \lim_{\rho\to 0} \left(\frac{1}{\rho^{2(s+l)}} \left(\int_0^\rho \tau^l d\omega_\rho(\tau) \right)^2 \right) \\ &\leq \lim_{\rho\to 0} \left(\frac{1}{\rho^{2(s+l)}} \int_0^\rho \left(\tau^{\frac{l-1}{2}} \right)^2 d\omega_\rho(\tau) \int_0^\rho \left(\tau^{\frac{l+1}{2}} \right)^2 d\omega_\rho(\tau) \right) \\ &= \lim_{\rho\to 0} \left(\frac{1}{\rho^{s+l-1}} \int_0^\rho \tau^{l-1} d\omega_\rho(\tau) \right) \lim_{\rho\to 0} \left(\frac{1}{\rho^{s+l+1}} \int_0^\rho \tau^{l+1} d\omega_\rho(\tau) \right) \\ &= C_{l-1}\, C_{l+1} \end{aligned} \tag{3.1}$$

for all $l \geq 1$ by the Cauchy-Schwarz inequality. Moreover, (3.1) implies that

$$C_l^4 \leq C_{l-1}^2\, C_{l+1}^2 \leq C_{l-2}\, C_l^2\, C_{l+2}$$

and hence

$$C_l^2 \leq C_{l-2}\, C_{l+2}, \tag{3.2}$$

for all $l \geq 2$.

We define the following sets of polynomials

$$\begin{aligned} \mathcal{P}_\nu = \Big\{ q \in \mathbb{R}[x] \,:\, q(x) = \sum_{j=0}^{\nu} \frac{D_j}{j!} x^j \\ \text{with } D_l^2 \leq D_{l-1}\, D_{l+1} \text{ for } l \geq 1;\ D_k > 0 \text{ for } k \geq 0 \Big\} \end{aligned} \tag{3.3}$$

It follows by the preceding remarks that $p_{\nu,\omega} \in \mathcal{P}_\nu$. It is also easily seen that

$$q \in \mathcal{P}_\nu \Rightarrow q' \in \mathcal{P}_{\nu-1}. \tag{3.4}$$

We shall prove in a sequence of two Lemmas that all polynomials in $\mathcal{P}_\nu$ have at least one root with non-negative real part if $\nu \geq 5$. Thus (A_2) cannot be satisfied for $\nu \geq 5$.

Lemma 2. *Every $q \in \mathcal{P}_5$ has at least one root with non-negative real part.*

Proof. Suppose otherwise we are given a polynomial $q = \sum_{j=0}^{5} \frac{D_j}{j!} x^j \in \mathcal{P}_5$ for which all roots have negative real parts. From the Routh-Hurwitz Criterion [8, p.545] we then conclude that

$$\det \begin{pmatrix} 5\,D_4 & 60\,D_2 & 120\,D_0 & 0 \\ D_5 & 20\,D_3 & 120\,D_1 & 0 \\ 0 & 5\,D_4 & 60\,D_2 & 120\,D_0 \\ 0 & D_5 & 20\,D_3 & 120\,D_1 \end{pmatrix} > 0,$$

and hence

$$\begin{aligned} (3.5)\quad & 150\,D_4\,D_3\,D_2\,D_1 + 30\,D_5\,D_4\,D_1\,D_0 + 30\,D_5\,D_3\,D_2\,D_0 \\ & - 50\,D_4\,D_3^2\,D_0 - 75\,D_4^2\,D_1^2 - 90\,D_5\,D_2^2\,D_1 - 3\,D_5^2\,D_0^2 > 0 \end{aligned}$$

must hold. From (3.2) it follows that

$$D_3^2 \le D_1\,D_5. \tag{3.6}$$

We now claim that (3.5) and (3.6) cannot be simultaneously satisfied. Suppose otherwise that

$$\begin{aligned} (3.7)\quad & 150\,D_4\,D_3\,D_2\,D_1 + 30\,D_5\,D_4\,D_1\,D_0 + 30\,D_5\,D_3\,D_2\,D_0 \\ & - 50\,D_4\,D_3^2\,D_0 - 75\,D_4^2\,D_1^2 - 90\,D_5\,D_2^2\,D_1 - 3\,D_5^2\,D_0^2 - \kappa = 0 \end{aligned}$$

with some $\kappa > 0$, and assume that (3.6) holds. Equation (3.7) is of the form

$$A\,D_4^2 + B\,D_4 + C = 0,$$

with

$$\begin{aligned} A =& -75\,D_1^2, \\ B =& 150\,D_1\,D_2\,D_3 - 50\,D_0\,D_3^2 + 30\,D_0\,D_1\,D_5, \\ C =& -90\,D_1\,D_2^2\,D_5 + 30\,D_0\,D_2\,D_3\,D_5 - 3\,D_0^2\,D_5^2 - \kappa. \end{aligned}$$

Hence, we have $D_4 = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC})$. We shall prove that $d = B^2 - 4AC < 0$, thus D_4 cannot be a positive real number. We have

$$\begin{aligned} d =& (150\,D_1\,D_2\,D_3 - 50\,D_0\,D_3^2 + 30\,D_0\,D_1\,D_5)^2 \\ & + 300\,D_1^2\,(-90\,D_1\,D_2^2\,D_5 + 30\,D_0\,D_2\,D_3\,D_5 - 3\,D_0^2\,D_5^2 - \kappa) \\ =& 100\Big(-3\,\kappa\,D_1^2 + 5(-D_1\,D_2 + D_0\,D_3)^2(5\,D_3^2 - 6\,D_1\,D_5)\Big). \end{aligned}$$

Obviously $-3\,\kappa\,D_1^2 < 0$, $(-D_1\,D_2 + D_0\,D_3)^2 \ge 0$ and

$$5\,D_3^2 - 6\,D_1\,D_5 < 5(D_3^2 - D_1\,D_5) \le 0$$

due to (3.6). Thus, we have $d < 0$ and $D_4 \notin \mathbb{R}$ contradicting our assumption. □

Lemma 3. *Every $q \in \mathcal{P}_\nu$ for $\nu \ge 5$ has at least one root with non-negative real part.*

Proof. We prove the assertion by induction. The case $\nu = 5$ is already proved in Lemma 2. Suppose that the assertion holds for $\nu - 1 \ge 5$ and assume that all roots of $\tilde{q} \in \mathcal{P}_\nu$ have negative real parts. It follows from Lucas' Theorem (cf. Borwein and Erdélyi [5, Lemma 1.3.2.]) that all roots of $\tilde{q}'$ must have negative real parts. But we also have $\tilde{q}' \in \mathcal{P}_{\nu-1}$, contradicting

our assumption. Hence, $\tilde{q}$ must have at least one root with non-negative real part. □

From Lemma 3 we can now immediately conclude the following Theorem.

Theorem 4. *There exists no family of positive Borel measures ω_ρ for $\rho > 0$ for which (A_2) is satisfied if $\nu \geq 5$.*

Theorem 4 describes the limitations of sequential regularization methods defined on the basis of (2.3)–(2.7). We shall see in the next section that these limitations are not only of theoretical nature, but are also relevant in concrete numerical experiments. It could be of interest to investigate whether it is possible to define a sequential regularization scheme for ill-posed Volterra problems which is stabilizing for all finitely smoothing kernels, or maybe even for infinitely smoothing kernels as the heat kernel.

4. Numerical Examples

In all our examples we use the uniform measure $\omega(A) = \int_A dx$. The plots in Figure 1 show results for the sequential predictor-corrector method for different noise-levels in the data. We chose $k(t) = t^2$, i.e., k is 3-smoothing. The data are given by $f_0(t) = t^3/3 - t^5/30$. To the data we added artificial 0-1 distributed Gaussian noise scaled with a factor $\sigma\|f\|_\infty$. The plots in Figure 1 show the results for $\sigma = 0$, $\sigma = 0.005$, $\sigma = 0.01$, and $\sigma = 0.02$. The lengths of the future intervals are $\rho = 0.05$, $\rho = 0.4$, $\rho = 0.5$, and $\rho = 0.55$, respectively. We used $N = 600$ time steps on the time interval $[0, 3]$. The plots show the exact solution as dashed line and the reconstructed solutions from noisy data as solid lines. It is seen that the sequential regularization method gives stable results in the case of noisy data.

In Table 4 the convergence rate result in Theorem 3 is numerically checked in a series of calculations with decreasing noise-level δ in the first column of the Table. For this example we use again $k(t) = t^2$ ($\nu = 3$). The data are given by $f_0(t) = -\frac{1}{32}\sin(4t) + \frac{1}{8}t$. The exact solution for unperturbed data is given by $u_0 = \cos 4t$. Obviously $u_0 \in \mathcal{C}^{1,1}$, hence we have $\alpha = 1$ in Theorem 4. For the first experiment, (the first row in the table) the length of the future interval ρ was chosen experimentally to give 'best looking' results. For the further experiments (the following rows) $\rho = \rho(\delta)$ was chosen such that the ratio $\rho/\delta^{\frac{1}{\nu+\alpha}}$ was kept (close to) constant. The two last columns of the table show the absolute reconstruction-error $\|u_\rho^\delta - u_0\|_\infty$ and the ratio $\|u_\rho^\delta - u_0\|_\infty/\delta^{\frac{\alpha}{\alpha+\nu}}$. In good accordance with the convergence rate result in Theorem 4 it is seen that the last column of the Table is bounded over a large range of noise levels ($\delta = 10^{-2}$ to $\delta = 10^{-6}$).

In Figure 2 the stars ($*$) and circles (o) show the locations of the roots of the polynomial

$$p_{\nu,\omega}(x) = \sum_{l=0}^{\nu} \frac{C_l}{l!}\, x^l$$

for $\nu = 3$ and $\nu = 4$, respectively, in the complex plane. It is seen that for $\nu = 3$, all roots have negative real parts. For $\nu = 4$, one pair of roots has crossed the imaginary axis, implying that assumption (A_2) is not satisfied in

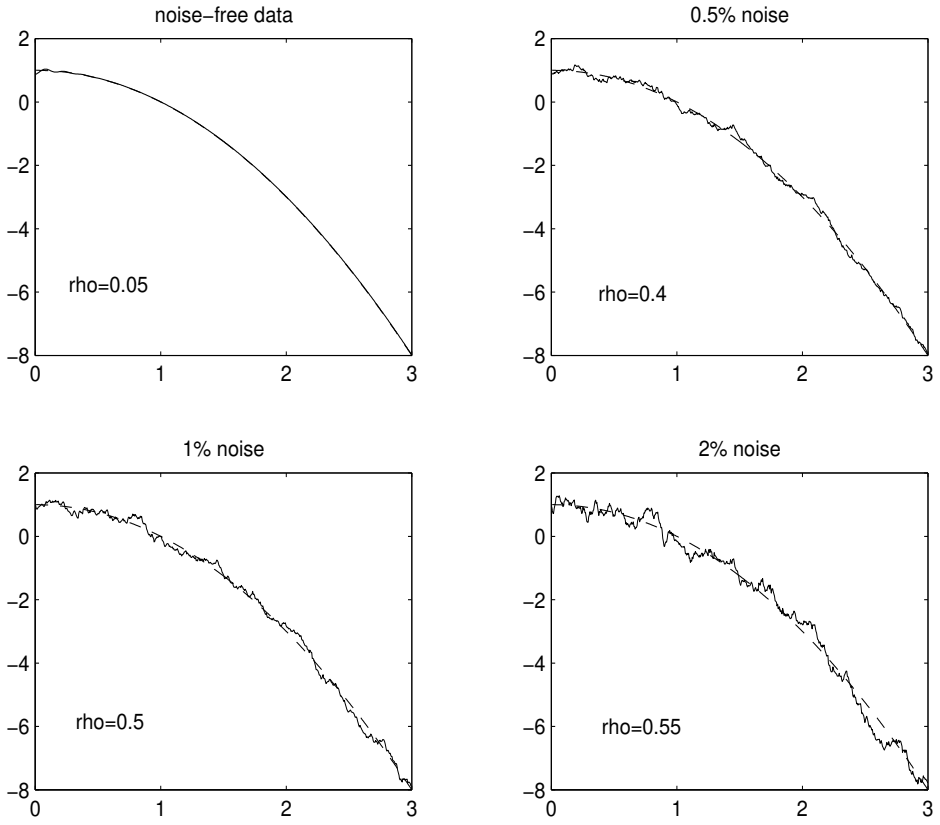

FIGURE 1. Results using sequential regularization for different noise-levels.

δ	ρ	$\rho/\delta^{\frac{1}{\alpha+\nu}}$	$\|u_\rho^\delta - u_0\|_\infty$	$\|u_\rho^\delta - u_0\|_\infty/\delta^{\frac{\alpha}{\alpha+\nu}}$
0.0258	0.5	1.2482	0.3606	0.9002
0.0046	0.325	1.2502	0.1752	0.6740
$8.8145 \cdot 10^{-4}$	0.215	1.2478	0.1336	0.7752
$1.8263 \cdot 10^{-4}$	0.145	1.2473	0.1006	0.8656
$3.3662 \cdot 10^{-5}$	0.095	1.2472	0.0748	0.9819
$7.3502 \cdot 10^{-6}$	0.065	1.2484	0.0426	0.8173

this case. The instability effect of condition (A_2) not being satisfied is also indicated in Figure 2. Here we chose $k(t) = t^3$ (i.e. $\nu = 4$) on the interval $[0, 8]$ with zero noise on the data. We set $\rho = 0.8$ for the length of the future interval. The true solution is again given by $u_0 = \cos 4t$. It is seen that the solution obtained by the sequential predictor-corrector method behaves like an unstable oscillator.

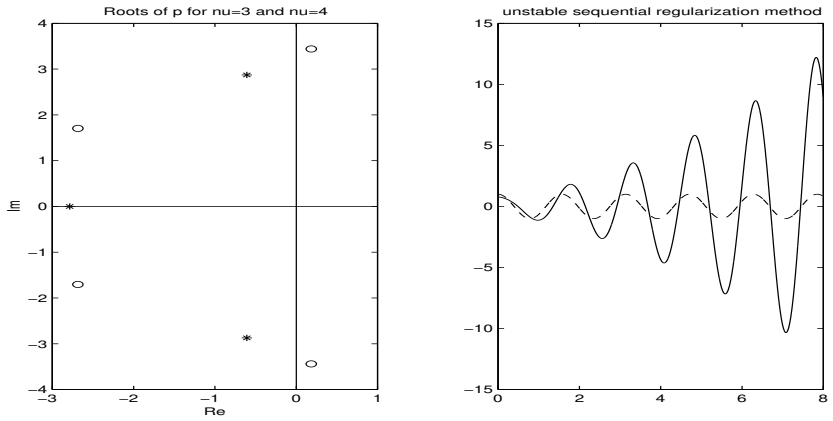

FIGURE 2. Location of roots of the polynomial p_ν for $\nu = 3$ ($*$) and $\nu = 4$ (o). Solution for $\nu = 4$ in case of exact data.

5. Conclusions

We investigated a well-known sequential regularization method for the stable solution of ill-posed Volterra equations of the first kind. Using resolvent- and Lyapunov function techniques we obtained improved stability estimates (Theorem 1) and convergence rates (Theorem 3) which were previously not known in the literature. We could also prove that a crucial stability condition can not be satisfied for ν-smoothing kernels if $\nu > 4$ (Theorem 4). Numerical experiments indicate that it is not advisable to use sequential predictor-corrector regularization methods for the inversion of highly smoothing Volterra operators (Figure 2). The convergence rate result was numerically checked in Table 4.

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