IDENTIFICATION OF THE LOAD OF A PARTIALLY BREAKING BEAM FROM INCLINATION MEASUREMENTS

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ABSTRACT. The problem considered here is to identify an unknown load applied to a steel-concrete composite beam by using measurements of the inclination along the axis of the beam. This problem arises in an industrial setting, where it is required to stabilize steep mountain slopes by reinforced concrete piles. A nonlinear, non-smooth constitutive relation is used to model the partial breaking of the pile at points where the bending moment exceeds a critical value. A two-step approach for the inverse problem is considered. In the first step the broken and unbroken parts of the beam are determined from the solution of a regularized least squares problem, where a total variation-type regularization term is used. In the second step a linearly constrained least squares problem is solved. Existence, stability, and convergence results are presented. Numerical experiments are carried out and compared.

1. INTRODUCTION

We investigate the problem of identifying the load applied to a steel-concrete composite beam by using measurements of the inclination of the beam along its axis. In the same solution procedure we also identify the bending moment generated by the load. This problem arises in an industrial setting where it is required to protect against landslides on steep mountain slopes by stabilizing them with steel-concrete piles. Very often, the exact load acting on the pile is not known in advance and even difficult to estimate due to the inhomogeneity of the ground. Also, for existing piles, it is difficult to measure the distribution of the load directly. On the other hand, it is comparatively easy to get measurements of the *deformation* of the pile. Measurements of the inclination of the pile with respect to the vertical direction can be obtained by means of a sensor (an inclinometer probe) which slides within a tube inside the pile. It is our goal to reconstruct the load and the bending moment from these measurements.

Another important feature of steel-concrete composite beams is that the concrete part breaks on the tensile side of the beam if the bending moment exceeds a certain critical value. In this case, the structure is maintained

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only by the steel rods in the beam. We will model this phenomenon in a non-smooth constitutive relation.

In Section 2 we shall discuss the model equation and derive an existence and uniqueness result for the direct problem. For the solution of the inverse problem, we first determine the broken and unbroken part of the beam and resolve the state dependence of the material in this way. In order to determine the broken and unbroken part, a regularized inverse problem is solved, where we use a regularization term of total variation type. What remains to be solved in a second step is a linearly constrained inverse problem. The setup of this two-step procedure is given in Section 3. In Section 4 we prove solvability, stability with respect to data, and convergence if the noise in the data goes to 0. Finally, Section 5 contains numerical examples for the two-step procedure derived and analyzed in Section 3.

Throughout the paper we use the following notation. All function spaces considered are defined over the interval $(0,1)$ and we write L^2 for $L^2(0,1)$ and H^k for $H^k(0,1)$ with $k \in \mathbb{R}$. Moreover, if not specified otherwise, integrals are supposed to be defined over the interval $(0,1)$.

2. The direct problem

We consider a beam of length 1 which is exposed to some load $p \in L^2(0,1)$. Let $m(x)$ denote the bending moment and let $u(x)$ be the displacement at the point $x \in (0,1)$. Then, using the simplest model for an elastic beam, (cf. Strang [16, p.172ff]) we have the following relations:

$$m'' + p = 0 \tag{2.1}$$

$$EIu'' + m = 0 \text{ on } (0,1), \tag{2.2}$$

where E is the modulus of elasticity of the material and I is the moment of inertia of the cross section of the beam. We want to apply the model to a steel-concrete composite material for which the tensile strength is much smaller than the compressive strength. If the bending moment exceeds a certain critical value m_c the concrete will break at the tensile side of the beam. This partial breaking of the beam is modeled by a discontinuous dependence of the moment of inertia I on the bending moment m. We suppose that $0 < I_2 < I_1$ and we set

$$I = I(m) = \begin{cases} I_1 & \text{if } |m| < m_c, \\ [I_2, I_1] & \text{if } |m| = m_c, \\ I_2 & \text{if } |m| > m_c. \end{cases} \tag{2.3}$$

Note that we defined I as a set-valued mapping, and consequently we have to replace (2.2) by

$$0 \in EI(m)u'' + m. \tag{2.4}$$

The nonlinearity introduced in (2.3) describes the dominant effects for the deformation of the beam. A possible (nonlinear) dependence of the

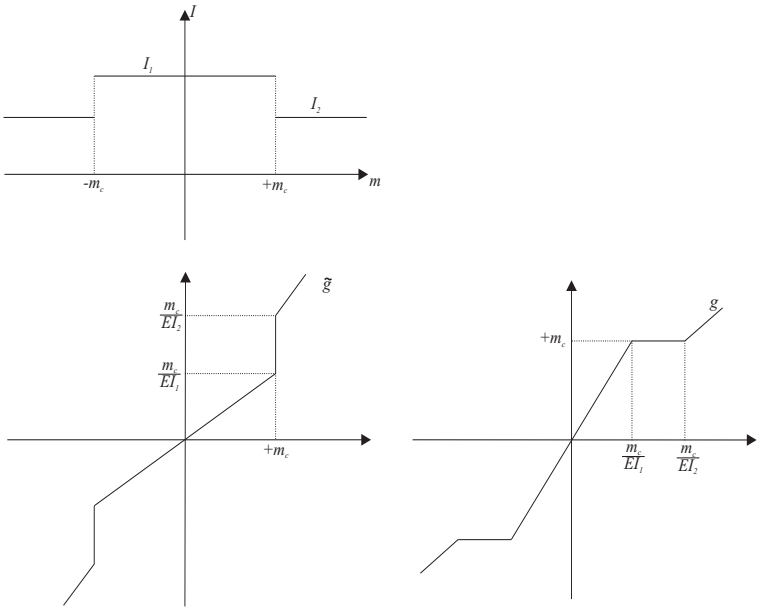

FIGURE 1. The function I and the monotone graphs g and $\tilde{g}$

modulus of elasticity E on the moment m is neglected. In fact, for the sake of compact notation, we set $E = 1$ in the following.

We define the piecewise affine function (cf. Figure 1)

$$g(x) = \begin{cases} I_1 x & \text{if } |x| < \frac{m_c}{I_1}, \\ m_c & \text{if } \frac{m_c}{I_1} \le |x| < \frac{m_c}{I_2}, \\ I_2 x & \text{if } |x| \ge \frac{m_c}{I_2}. \end{cases} \tag{2.5}$$

From (2.4) (recall that we set $E = 1$) we conclude

$$u'' \in -\frac{m}{I(m)}. \tag{2.6}$$

The graph $\tilde{g}(x) = \frac{x}{I(x)}$ is strictly maximal monotone on $\mathbb{R}$. A short calculation shows that its inverse is the function g defined in (2.5). (cf. Figure 1) We can therefore write (2.6) in the equivalent form

$$g(u'') + m = 0. \tag{2.7}$$

We consider the following set of boundary conditions:

$$m(0) = m(1) = 0 \ \ldots \text{ no bending moment in the endpoints,} \tag{2.8}$$

$$m'(0) = m'(1) = 0 \ \ldots \text{ no normal force in the endpoints.} \tag{2.9}$$

The boundary conditions reflect the hypothesis that we do not fix the beam at the endpoints in any way. Suppose $m \in H^2$ and $p \in L^2$ satisfy (2.1) and

m satisfies also the boundary conditions (2.8) and (2.9). Then it follows that

$$\int p\,dx = -(m'(1) - m'(0)) = 0 \tag{2.10}$$

and

$$\int x\,p\,dx = -m'(1) + \int m'\,dx = m(1) - m(0) = 0. \tag{2.11}$$

We therefore assume that the solvability conditions

$$\int p\,dx = \int x\,p\,dx = 0 \tag{S}$$

hold for p. On the other hand, if $p \in L^2$ is given and (S) is satisfied, it is well known that a unique solution $m \in H^2 \cap H_0^1$ to (2.1) exists and we can read (2.11) from right to left and obtain $m'(1) = 0$. Then we can use (2.10) to conclude that also $m'(0) = 0$.

Remark. The conditions (S) have a physical meaning. They say that the resulting normal force and the resulting moment of the external forces are zero. Hence the beam is held in a static equilibrium by the external forces and no normal forces or moments need to be absorbed at the endpoint of the beam. This matches the boundary conditions (2.8) and (2.9).

If $m \in H^2$ is given, we consider $\omega \in g^{-1}(m) \subset L^2$ and we can solve

$$u'' = -\omega \tag{2.12}$$

for $u \in H^2$. Note that $\omega \in g^{-1}(m)$ is not necessarily unique. However, if we have $\text{meas}\{x \in (0,1) \,:\, |m(x)| = m_c\} = 0$, then $\omega \in L^2$ is uniquely determined by m. We have also non-uniqueness for the solution of (2.12) due to the lack of boundary conditions on u. To obtain a unique solution for (2.12) we suppose that

$$\int u\,dx = \int u'\,dx = 0 \tag{U}$$

and we set

$$\mathcal{H} = \{f \in H^2 \,:\, \int f\,dx = \int f'\,dx = 0\}. \tag{2.13}$$

We summarize the existence and uniqueness results in the following Proposition

Proposition 1. *Suppose $p \in L^2$ is given and p satisfies the solvability conditions* (S)*. Then there exist a unique function $m \in H^2 \cap H_0^1$ and a function $u \in \mathcal{H}$ (possibly not unique) such that*

$$m'' + p = 0$$
$$g(u'') + m = 0$$

and

$$m'(0) = m'(1) = 0.$$

If $\text{meas}\{x \in (0,1) \,:\, |m(x)| = m_c\} = 0$, *then* $u \in \mathcal{H}$ *is unique.*

3. Two-step formulation of the inverse problem

We now address the inverse problem of recovering p and m from measurements of

$$v = u'.$$

A standard approach would be, to write the inverse problem as a regularized least-squares problem for the variables (v, m, p) and to treat the relations between the variables (i.e. (2.7),(2.10),(2.11),(S) and (U)) as explicit contraints. It turns out, however, that the non-convex, non-smooth equality constraint (2.7) is difficult to handle from the numerical point of view. An implementation of an augmented Lagrangian SQP algorithm (cf. Kunisch and Ito [10]) where we replaced (2.7) by some smooth approximation, was convergent only if the startup value was chosen close to the exact solution and we did not put too much noise on the data (less that 1%).

We shall avoid dealing with the piecewise linear constraint (2.7) in the least-squares problem by the following two-step procedure. In the first step, we determine the broken and the unbroken part of the beam. In a second step, we solve a (linearly) constrained, regularized inverse problem where the constraint (2.7) is replaced by $I_1 v'(x) + m(x) = 0$ and $I_2 v'(x) + m(x) = 0$ for x in the broken and unbroken part respectively. This new constraint is linear in the optimization variables v and m.

Let us discuss the first step. It is our aim to identify the function

$$\tilde{I}(x) = I(m(x))$$

from given data $v_d \in L^2$. Suppose we have functions $(v, m) \in H^1 \times H_0^1$ satisfying

$$g(v') + m = 0. \tag{3.1}$$

We set $w = v'$ and we write the constraint (3.1) in the form

$$w(x) \in -\frac{m(x)}{I(m(x))} \tag{3.2}$$

as originally proposed in (2.6). Given any point $x \in (0,1)$, we want to decide whether the beam is broken at this point or not. In the unbroken case, we have $\tilde{I}(x) = I(m(x)) = I_1$ and $|m(x)| < m_c$. Hence, by (3.2), we get

$$|w(x)| < \frac{m_c}{I_1}.$$

In the case where the beam is broken at $x \in (0,1)$, we have $\tilde{I}(x) = I_2$ and $|m(x)| > m_c$ and therefore

$$|w(x)| > \frac{m_c}{I_2}.$$

Note that $w(x)$ has a jump discontinuity at every point $x \in (0,1)$ for which $|m(x)| = m_c$ and m is strictly monotone on some neighborhood of x. The height of the jump is given by (cf. Figure 2)

$$|w_+(x) - w_-(x)| = m_c \left(\frac{1}{I_2} - \frac{1}{I_1} \right).$$

We set

$$w_l = \frac{m_c}{I_1}, \quad w_u = \frac{m_c}{I_2}, \quad w_m = \frac{1}{2}(w_l + w_m), \tag{3.3}$$

and we have

$$\tilde{J}_u = \{x \in (0,1) \,:\, |w(x)| < w_l\}$$

and

$$\tilde{J}_b = \{x \in (0,1) \,:\, |w(x)| > w_u\}$$

for the sets of broken and unbroken points respectively.

In general we do not have $(0,1) = \tilde{J}_u \cup \tilde{J}_b$. The residual set is given by

$$\tilde{J}_r = \{x \in (0,1) \,:\, w_l \le |w(x)| \le w_u\} = \{x \in (0,1) \,:\, |m(x)| = m_c\}.$$

We assume that

$$\operatorname{meas}(\tilde{J}_r) = 0, \tag{A}$$

where $\operatorname{meas}(E)$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}$. Under this assumption we have

$$\tilde{I}(x) = I(m(x)) = \begin{cases} I_1 & \text{for } x \in J_u \\ I_2 & \text{for } x \in J_b \end{cases} \tag{3.4}$$

where

$$J_u = J_u(w) = \{x \in (0,1) \,:\, |w(x)| \le w_m\} \tag{3.5}$$

$$J_b = J_b(w) = \{x \in (0,1) \,:\, |w(x)| > w_m\}. \tag{3.6}$$

This is correct since we had to assign a different function value to $I(m(x))$ at most on a set of measure 0. Note also that the element sign in (3.2) can be replaced by equality almost everywhere if assumption (A) holds.

Once $\tilde{I}(x)$ is identified, we can substitute if for $I(m)$ in (3.2) and obtain a *linear* constraint for the variables v and m. The nonlinearity is already resolved in the assignment of the sets of broken and unbroken points in (3.4)–(3.6).

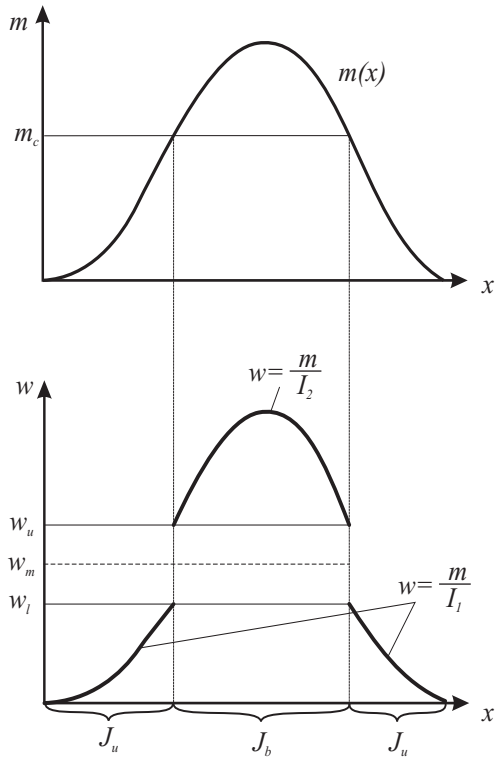

FIGURE 2. Functions $m(x)$ and $w(x)$ with discontinuities at the transition from unbroken to broken parts.

If we proceed in this way, it is crucial how we identify $w = v'$ from the measurement data v_d, i.e., in which way we (approximately) solve the problem

$$\min_w \|Kw - v_d\|_{L^2}^2 \tag{3.7}$$

where

$$Kw(x) = \int_0^x w(\xi)\,d\xi - \int_0^1 (1-\xi)w(\xi)\,d\xi. \tag{3.8}$$

The operator $K : L^2 \to L^2$ is compact and constructed in such a way that, for $v = Kw$, we have $v' = w$ and $\int_0^1 v\,dx = 0$. The operator K is well defined on L^1, but we leave it open for the moment to specify or restrict the space of feasible functions w in the minimization problem (3.7).

We have to deal with several difficulties. On the one hand, numerical differentiation of noisy data is an ill-posed problem which has to be stabilized. Thus, oscillations which would be amplified by some ad hoc numerical differentiation process have to be damped in some way. On the other hand, we want to resolve the location of the jump discontinuities in w (i.e. the transition points between broken and unbroken parts) as well as possible

and therefore we must not smear out the jumps in the reconstruction. A Tikhonov-like regularization method which damps oscillations but does not penalize jump discontinuities is a regularization based on the total variation functional. (cf. Rudin, Osher and Fatemi [13], Chavent and Kunisch [4], Casas, Kunisch and Pola [3]). We set

$$BV = BV(0,1) = \{f \in L^1 \,:\, f' \in \mathcal{M}(0,1)\}, \tag{3.9}$$

where f' denotes the distributional derivative of f and $\mathcal{M}(0,1)$ stands for the Banach space of all signed regular Borel measures on $(0,1)$, i.e. $\mathcal{M}(0,1) = \mathcal{C}_0(0,1)^*$ due to the Riesz representation theorem (Rudin [14, Thm.2.14,p.40]). BV is a Banach space with norm

$$\|f\|_{BV} = \|f\|_{L^1} + \int |f'|,$$

where $\int |f'|$ denotes the total variation of the measure f'. We shall use the total variation functional as regularization term for the identification of w. More precisely we proceed in the following way.

Step 1: Solve

$$\min \|Kw - v_d\|_{L^2}^2 + \beta \int |w'| \text{ over } w \in BV \tag{3.10}$$

for some (suitably chosen) regularization parameter $\beta > 0$. With w^{β,v_d} denoting the solution to (3.10), define

$$J_u(w^{\beta,v_d}) = \{x \in (0,1) \,:\, |w^\beta(x)| \le w_m\} \tag{3.11}$$

$$J_b(w^{\beta,v_d}) = \{x \in (0,1) \,:\, |w^\beta(x)| > w_m\} \tag{3.12}$$

and

$$\tilde{I}^{\beta,v_d}(x) = \begin{cases} I_1 & \text{for } x \in J_u(w^{\beta,v_d}), \\ I_2 & \text{for } x \in J_b(w^{\beta,v_d}). \end{cases} \tag{3.13}$$

In the second step, we fit the measured data in a least squares sense and treat the relations between variables v, m and p as explicit constraints, where we replace (3.1) by

$$\tilde{I}^{\beta,v_d}(x)\, v' + m = 0$$

with $\tilde{I}^{\beta,v_d}(x)$ from step 1. Thus v, m and p are considered as independent variables which are related by certain constraints, and which are identified simultaneously. Moreover, we have to take into account that we do not have continuous dependence of m and p on v. It is easy to construct a sequence (v_n, m_n, p_n) and (v, m, p) in $H^1 \times H^2 \times L^2$ satisfying the constraints for which $v_n \to v$ in H^1 but $p_n \not\to p$ in L^2. Moreover it is not realistic to assume that the inclination v can be measured in the H^1-norm with reasonable accuracy. Error estimates for the measurement are only available in the L^2-norm; hence, we have to consider L^2 as the natural space for v if we solve the inverse problem. We deal with the problem of ill-posedness by adding Tikhonov regularization terms for m and p to the least-squares data fit term

in the cost functional. (cf.Engl, Hahnke and Neubauer [7]). Thus we come to the following formulation.

Step 2: Solve the linear least-squares problem

$$\text{minimize } \tilde{\mathcal{J}}(v,m,p) := \|v-v_d\|_{L^2}^2 + \alpha_1\|m\|_{H_0^1}^2 + \alpha_2\|Dp\|_{L^2}^2 \quad \text{over } (v,m,p) \in H^1 \times H_0^1 \times L^2 \tag{3.14a}$$

$$\text{such that } e_1(m,p) := m'' + p = 0 \text{ in } H^{-1}, \tag{3.14b}$$

$$\tilde{e}_2(v,m) := \tilde{I}^{\beta,v_d}(x)v' + m = 0 \text{ in } L^2, \tag{3.14c}$$

$$\int p\,dx = \int x\,p\,dx = 0, \tag{3.14d}$$

$$\int v\,dx = 0, \tag{3.14e}$$

where $\alpha_1, \alpha_2 > 0$ are small regularization parameters, and $D : L^2 \to L^2$ is a bounded linear operator. We assume that D satisfies

(H1) if (p_n) is bounded in H^{-1} and (Dp_n) is bounded in L^2 then (p_n) is bounded in L^2

(H2) $\ker D$ is finite dimensional and $\operatorname{rg} D$ is closed.

Note that we do not use the precalculated w^{β,v_d} as data for the further identification of m and p. We only specify $\tilde{I}^{\beta,v_d}$ from w^{β,v_d} and use the original data v_d in step 2.

It can be shown (cf. [12]) that hypothesis (H2) is equivalent to the fact that the weak convergence $p_n \rightharpoonup p$ in L^2 and the convergence $\|Dp_n\|_{L^2} \to \|Dp\|_{L^2}$ implies the strong convergence $p_n \to p$ in L^2. We introduce the operator D in the regularization term for p to avoid strong overregularizing effects which could occur in case of a full L^2-regularization with a large regularization parameter α_2. It is, for instance, reasonable to choose a high-pass filter for D which penalizes only high frequencies.

In the following we assume that a set of "exact" data

$$(v_0, w_0, m_0, p_0) \in L^2 \times BV \times H_0^1 \cap H^2 \times L^2 \tag{3.15a}$$

is given satisfying:

$$Kw_0 = v_0, \text{ i.e., } v_0' = w_0 \text{ and } \int v_0\,dx = 0, \tag{3.15b}$$

$$\tilde{I}^0(x)w_0 + m_0 = 0, \tag{3.15c}$$

where

$$J_u^0 = J_u(w_0) = \{x \in (0,1) : |w_0(x)| \le w_m\}, \tag{3.15d}$$

$$J_b^0 = J_b(w_0) = \{x \in (0,1) : |w_0(x)| > w_m\}, \tag{3.15e}$$

and

$$\tilde{I}^0(x) = \begin{cases} I_1 & \text{for } x \in J_u^0, \\ I_2 & \text{for } x \in J_b^0, \end{cases} \tag{3.15f}$$

$$m_0'' + p_0 = 0, \quad \int p_0 \, dx = \int x \, p \, dx = 0. \tag{3.15g}$$

Moreover, we assume that hypothesis (A) is satisfied for w_0 and m_0 respectively.

4. Existence, Stability, and Convergence Analysis

We first discuss problem (3.10) as posed in step 1. We shall prove solvability, stability with respect to data and convergence of the solution of the regularized problem towards the exact solution if the noise on the data and the regularization parameter go to 0. These results are special cases of the stability and convergence results given in Acar and Vogel [1]. For completeness we also present proofs here. Let us first recall some properties of functions of bounded variation.

Proposition 2. 1. *If $\{u_j\}_{j=1}^\infty \subset BV$ and $u_j \to u$ in L^1 with $u \in BV$, then*

$$\int |u'| \le \liminf_{j\to\infty} \int |u_j'| \,. \tag{4.1}$$

2. *For every bounded sequence $\{u_j\}_{j=1}^\infty \subset BV$ and for every $p \in [1,\infty)$ there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ and a function $u \in BV$ such that $u_{j_k} \to u$ in L^p.*
3. *There exists a constant C such that*

$$\|u - \tilde{u}\|_\infty \le C \int |u'|, \text{ for all } u \in BV, \tag{4.2}$$

with $\tilde{u} = \int u \, dx$ (Sobolev Inequality).

For a proof of Proposition 2 we refer to Giusti [8] or Chavent and Kunisch [4].

The following Lemma will be useful in the subsequent considerations.

Lemma 1. *Let $K : L^2 \to L^2$ be as defined in (3.8), suppose that $\{w_n\}_{n=1}^\infty$ is given such that $\{\int |w_n'|\}_{n=1}^\infty$ is bounded in $\mathbb{R}$ and $\{Kw_n\}_{n=1}^\infty$ is bounded in L^2, and suppose that $p \in [1,\infty)$. Then there exists a subsequence, which we denote again by the same expression $\{w_n\}_{n=1}^\infty$, and $w^* \in BV$ satisfying $w_n \to w^*$ in L^p and $\int |w^{*\prime}| \le \liminf_{n\to\infty} \int |w_n'|$.*

Proof. From the Sobolev inequality (4.2) we conclude that $\{w_n - \tilde{w}_n\}_{n=1}^\infty$ is bounded in L^2 with $\tilde{w}_n = \int w_n \, dx$. We next prove that the sequence of constant functions $\{\tilde{w}_n\}_{n=1}^\infty$ is also bounded. Suppose otherwise that $|\tilde{w}_n| \to \infty$ as $n \to \infty$. Then

$$\|K\tilde{w}_n\|_{L^2} = |\tilde{w}_n| \, \|\cdot - \frac{1}{2}\|_{L^2} = \frac{|\tilde{w}_n|}{12} \to \infty.$$

Thus $\|Kw_n\|_{L^2} \geq \|K\tilde{w}_n\|_{L^2} - \|K(w_n - \tilde{w}_n)\|_{L^2}$ would be unbounded, which cannot hold. We therefore conclude that $\{w_n\}_{n=1}^{\infty}$ is bounded in L^2 and hence also in L^1. This together with the boundedness of $\{\int |w_n'|\}_{n=1}^{\infty}$ is equivalent to the fact that $\{w_n\}_{n=1}^{\infty}$ is bounded in BV. Application of Proposition 2.1 and 2.2 completes the proof. □

Subsequently we shall often have the situation where we have convergence for certain subsequences of a sequence and we want to conclude convergence of the whole sequence. The following elementary lemma will be useful in this context.

Lemma 2. *Suppose X is a normed vector space, and that $\{x_n\}_{n=1}^{\infty} \subset X$ and $x_\infty \in X$ have the property that every subsequence of $\{x_n\}_{n=1}^{\infty}$ contains a convergent subsequence with limit x_∞. Then $x_n \to x_\infty$ as $n \to \infty$.*

Proof. Suppose otherwise that $x_n \not\to x_\infty$. Then a subsequence exists which lies entirely outside a certain neighborhood of x_∞. From this subsequence, however, it is not possible to choose a convergent subsequence with limit x_∞, contradicting our assumption. □

For the inverse problem (3.10) we have the following solvability result.

Proposition 3 (Existence)**.** *For any $v_d \in L^2$ and $\beta > 0$, Problem* (3.10) *has a unique solution $w^{\beta,v_d} \in BV$.*

Proof. Suppose $\{w_j\}_{j=1}^{\infty}$ is a minimizing sequence for Problem (3.10). Then obviously $\{Kw_j\}_{j=1}^{\infty}$ is bounded in L^2, $\{\int |w_j'|\}_{j=1}^{\infty}$ is bounded in $\mathbb{R}$, and we can apply Lemma 1 to find a subsequence — again denoted by $\{w_j\}_{j=1}^{\infty}$ — and $w^{\beta,v_d} \in BV$ such that $w_j \to w^{\beta,v_d}$ in L^2 and $\int |(w^{\beta,v_d})'| \leq \liminf_{j\to\infty} \int |w_j'|$. Using the continuity of K we find

$$\begin{aligned}\|Kw^{\beta,v_d} - v_d\|_{L^2}^2 + \beta \int |(w^{\beta,v_d})'| &\leq \liminf_{j\to\infty} \left(\|Kw_j - v_d\|_{L^2}^2 + \beta \int |w_j'| \right) \\ &= \inf_{\tilde{w}\in BV} \left(\|K\tilde{w} - v_d\|_{L^2}^2 + \beta \int |\tilde{w}'| \right),\end{aligned}$$

which proves that w^{β,v_d} is in fact a solution to (3.10).

To prove uniqueness of the minimizer, it is sufficient to show that the cost functional in (3.10) is strictly convex. This, however, is an immediate consequence of the injectivity of K and the strict convexity of $\|\cdot\|_{L^2}^2$. □

We prove that the regularized problem (3.10) is well-posed in the sense that we have continuous dependence of the solution on the data.

Proposition 4 (Stability)**.** *Let $v_n \to v_\infty$ in L^2 as $n \to \infty$ and, for fixed $\beta > 0$, let $w_n \in BV$ and $w_\infty \in BV$ denote the solutions to* (3.10) *with data v_n and v_∞ respectively. Then $w_n \to w_\infty$ in L^p for all $p \in [1, \infty)$ and $\int |w_n'| \to \int |w_\infty'|$ as $n \to \infty$.*

Proof. Throughout the proof subsequences of given sequences are alway denoted by the same symbol as the original sequence. We choose an arbitrary subsequence of $\{w_n, v_n\}_{n=1}^{\infty}$. Since we have

$$\|Kw_n - v_n\|_{L^2}^2 + \beta \int |w_n'| \leq \|v_n\|_{L^2}^2$$

and $v_n \to v_\infty$ in L^2, we conclude that $\{Kw_n\}_{n=1}^{\infty}$ is bounded in L^2 and $\{\int |w_n'|\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$. Let $p \geq 2$ be fixed. Using Lemma 1, we can find a subsequence $\{w_n\}_{n=1}^{\infty}$ and $w^* \in BV$ satisfying $w_n \to w^*$ in L^p and $\int |w^{*\prime}| \leq \liminf_{n\to\infty} \int |w_n'|$. Using the boundedness of K in L^2, the continuous embedding $L^p \hookrightarrow L^2$ on the bounded interval $(0,1)$ and the optimality of w_n, we get

$$\begin{aligned}
\|Kw^* - v_\infty\|_{L^2}^2 + \beta \int |w^{*\prime}| &\leq \liminf_{n\to\infty} \left(\|Kw_n - v_n\|_{L^2}^2 + \beta \int |w_n'| \right) \\
&\leq \lim_{n\to\infty} \left(\|Kw_\infty - v_n\|_{L^2}^2 + \beta \int |w_\infty'| \right) \\
&= \|Kw_\infty - v_\infty\|_{L^2}^2 + \beta \int |w_\infty'|.
\end{aligned}$$

Since w_∞ is the unique solution to (3.10) with data v_∞, we have $w^* = w_\infty$.

Since every subsequence of $\{w_n\}_{n=1}^{\infty}$ has an L^p-convergent subsequence with unique limit w_∞, we can apply Lemma 2 to conclude that the whole original sequence $\{w_n\}_{n=1}^{\infty}$ is convergent. The convergence in L^p for $p < 2$ follows from the continuous embedding $L^q \hookrightarrow L^p$ for $q \geq p$ on the bounded interval $(0,1)$. □

Suppose, we are given data which we consider as noisy measurements of exact data v_0 (as described in (3.15)) with noise-level given by $\delta > 0$. We shall prove a result, which guarantees the convergence of the solution of the regularized problem (3.10) towards the exact w_0 if the noise-level goes to zero, provided that the regularization parameter β is chosen in an appropriate way.

Proposition 5 (Convergence)**.** *Let $v_0 \in L^2$, $w_0 \in BV$ be given with $Kw_0 = v_0$. Assume moreover that $v_n \in L^2$ is given satisfying $\|v_n - v_0\|_{L^2} \leq \delta_n$ with $\delta_n \to 0$ as $n \to \infty$. Let w_n denote the solution to the problem*

$$\min \|Kw - v_n\|_{L^2}^2 + \beta_n \int |w'| \text{ over } w \in BV,$$

where $\beta_n = \beta(\delta_n)$ is chosen such that

$$\frac{\delta_n^2}{\beta_n} \to 0 \text{ as } n \to \infty. \tag{4.3}$$

Then we have $w_n \to w_0$ in L^p for every $p \in [1, \infty)$ and $\int |w_n'| \to \int |w_0'|$ as $n \to \infty$.

Proof. We denote again subsequences of given sequences by the same symbol as the original sequence. Let us choose an arbitrary subsequence $\{w_n\}_{n=1}^{\infty}$. Since we have

$$\beta_n \int |w_n'| \leq \|Kw_0 - v_n\|_{L^2}^2 + \beta_n \int |w_0'|$$

we find

$$\int |w_n'| \leq \frac{\delta_n^2}{\beta_n} + \int |w_0'|. \tag{4.4}$$

From assumption (4.3) it follows that $\{\int |w_n'|\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$. It is easy to argue that $\{Kw_n\}_{n=1}^{\infty}$ is bounded in L^2. Let $p \in [2, \infty)$ be given. Using Lemma 1 we can find a subsequence $\{w_n\}_{n=1}^{\infty}$ and $w^* \in BV$ satisfying $w_n \to w^*$ in L^p and $\int |w_n'| \to \int |w^{*\prime}|$. Since $K : L^p \to L^2$ is bounded, we have $Kw_n \to Kw^*$. Thus we have

$$\|Kw^* - v_0\|_{L^2}^2 = \lim_{n\to\infty} \|Kw_n - v_n\|_{L^2}^2 \leq \lim_{n\to\infty} \|Kw_0 - v_n\|_{L^2}^2 + \beta_n \int |w_0'| = 0.$$

Hence $Kw^* = v_0$ or equivalently $w^* = v_0' = w_0$. Since every arbitrary subsequence of the original sequence contains an L^p-convergent subsequence with limit w_0, we can use Lemma 2 to conclude convergence $w_n \to w_0$ for the whole original sequence in L^p for $p \in [2, \infty)$. The convergence in L^r with $r \in [1, 2)$ follows from the continuous embedding $L^p \hookrightarrow L^r$ for $p > r$.

With a similar argument it is also shown that $\int |w_n'| \to \int |w_0'|$.

□

The convergence result in Proposition 5 extends to the moment of inertia defined in (3.13).

Proposition 6. *Suppose v_0, w_0, J_u^0, J_b^0 and I^0 are given as described in* (3.15). *We assume that hypothesis* (A) *holds for w_0. Let v_n, w_n, δ_n and β_n be given as in Proposition 5 and let $J_u^n = J_u(w_n)$ and $J_b^n = J_b(w_n)$ be defined according to* (3.11) *and* (3.12). *We set*

$$\tilde{I}^n(x) = \begin{cases} I_1 & \text{for } x \in J_u^n, \\ I_2 & \text{for } x \in J_b^n. \end{cases} \tag{4.5}$$

Then we have

$$\operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)) \leq \frac{2}{I_1 - I_2} \|w_n - w_0\|_{L^1} \tag{4.6}$$

and, therefore, following Proposition 5,

$$\operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)) \to 0 \tag{4.7}$$

as $n \to \infty$. With this we have

$$\tilde{I}^n \to \tilde{I}^0 \text{ in } L^p \text{ for every } p \in [1, \infty) \text{ as } n \to \infty. \tag{4.8}$$

Proof. Obviously $(J_u^0 \setminus J_u^n) \cap (J_u^n \setminus J_u^0) = \emptyset$ and thus

$$\operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)) = \operatorname{meas}(J_u^0 \setminus J_u^n) + \operatorname{meas}(J_u^n \setminus J_u^0).$$

We have

$$\begin{aligned}\operatorname{meas}(J_u^0 \setminus J_u^n) &= \operatorname{meas}(J_u^0 \cap J_b^n) \\ &= \operatorname{meas}\{x \in (0,1) \,:\, |w_0(x)| \le w_m;\, |w_n(x)| > w_m\} \\ &= \operatorname{meas}\{x \in (0,1) \,:\, |w_0(x)| \le w_l;\, |w_n(x)| > w_m\},\end{aligned}$$

due to hypothesis (A). (Recall the definition of w_m, w_l and w_u in (3.3).) Similarly we obtain

$$\operatorname{meas}(J_u^n \setminus J_u^0) = \operatorname{meas}\{x \in (0,1) \,:\, |w_0(x)| \ge w_u;\, |w_n(x)| \le w_m\}.$$

Thus we have

$$\begin{aligned}\|w_n - w_0\|_{L^1} &\ge \int_{J_u^0 \setminus J_u^n} |w_n(x) - w_0(x)|\, dx + \int_{J_u^n \setminus J_u^0} |w_n(x) - w_0(x)|\, dx \\ &\ge \operatorname{meas}(J_u^0 \setminus J_u^n)\,(w_m - w_l) + \operatorname{meas}(J_u^n \setminus J_u^0)\,(w_u - w_m) \\ &= \frac{1}{2}(I_1 - I_2)\operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)).\end{aligned}$$

Application of Proposition 5 proves (4.7).

Moreover we have

$$\begin{aligned}\|\tilde{I}^n - \tilde{I}^0\|_{L^p}^p &= \int_{J_u^0 \setminus J_u^n} |I_1 - I_2|^p\, dx + \int_{J_u^n \setminus J_u^0} |I_1 - I_2|^p\, dx \\ &= |I_1 - I_2|^p \operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)) \to 0\end{aligned}$$

as $n \to \infty$ and hence (4.8) is proved. □

We now investigate convergence of the linear inverse problem (3.14) in step 2. We omit the discussion of solvability and stability with respect to data which can be done in a rather standard way (cf. Engl, Hanke, and Neubauer [7, sec.5.4]), and address directly the problem of convergence of the solution to (3.14) if the noise in the data goes to 0.

Theorem 1. *We suppose that a set of exact data $(v_0, w_0, m_0, p_0) \in L^2 \times BV \times H_0^1 \cap H^2 \times L^2$ and $I^0(x)$ are given according to* (3.15) *and hypothesis* (A). *We assume also that v_n, w_n, β_n and δ_n are chosen as in Proposition 5.*

Let $(v^n, m^n, p^n) \in H^1 \times H_0^1 \times L^2$ denote the solution of the constrained optimization problem (3.14) *with data v_d replaced by v_n and $\tilde{I}^{\beta, v_d}$ replaced by $\tilde{I}^n = \tilde{I}^{\beta_n, v_n}$. We assume that the regularization parameters $\alpha_{1,n}$ and $\alpha_{2,n}$ in* (3.14a) *are chosen such that*

$$\frac{\|w_n - w_0\|_{L^1}}{\alpha_{1,n}} \le M; \quad \frac{\delta_n^2}{\alpha_{1,n}} \le M \text{ and } \alpha_{2,n} = k\alpha_{1,n} \tag{4.9}$$

with positive constants $M, k > 0$.

Then we have

$$(v^n, m^n, p^n) \rightharpoonup (v_0, m_0, p_0) \tag{4.10}$$

weakly in $H^1 \times H^2 \times L^2$ *and* $Dp^n \rightharpoonup z^*$ *weakly in* L^2. *If*

$$\frac{\|w_n - w_0\|_{L^1}}{\alpha_{1,n}} \to 0 \text{ and } \frac{\delta_n^2}{\alpha_{1,n}} \to 0$$

as $n \to \infty$, *then we also have* $p^n \to p_0$ *in* L^2.

Proof. We take an arbitrary subsequence of $\{(v^n, m^n, p^n)\}_{n=1}^\infty$ which we denote again by the same symbol.

The triple $(v, m, p) = (0, 0, 0)$ is feasible with respect to the constraints (3.14b)–(3.14e) for every $\tilde{I}^n$, thus, we have

$$\|v^n - v_n\|_{L^2} \le \|v_n\|_{L^2},$$

which shows that $\{v^n\}_{n=1}^\infty$ is bounded in L^2. We set

$$v_{0,n} = -\frac{1}{\tilde{I}^n(x)} m_0.$$

It is easy to see that $(v_{0,n}, m_0, p_0)$ is feasible with respect to (3.14) where $\tilde{I}^n$ is used in the constraint (3.14c). Therefore we have

$$\begin{aligned}
\|v_{n,0} - v_0\|_{L^2}^2 &= \left\| \left(\frac{1}{\tilde{I}^0(x)} - \frac{1}{\tilde{I}^n(x)} \right) K m_0 \right\|_{L^2}^2 \\
&\le \|K m_0\|_\infty^2 \left\| \frac{1}{\tilde{I}^0(x)} - \frac{1}{\tilde{I}^n(x)} \right\|_{L^2}^2 \\
&= \|K m_0\|_\infty^2 \left(\frac{1}{I_2} - \frac{1}{I_1} \right)^2 \operatorname{meas}((J_u^0 \setminus J_u^n) \cup (J_u^n \setminus J_u^0)) \\
&\le \frac{2(I_1 - I_2)}{(I_1 I_2)^2} \|K m_0\|_\infty^2 \|w_n - w_0\|_{L^1} = C_1 \|w_n - w_0\|_{L^1} \to 0
\end{aligned} \tag{4.11}$$

as $n \to \infty$ due to Proposition 5 and Proposition 4 respectively. Here we also used that $K m_0 \in H^2 \subset \mathcal{C}$ in dimension 1. If we use the optimality of (v^n, m^n, p^n), the feasibility of $(v_{0,n}, m_0, p_0)$, (4.11), and (4.9), we obtain

$$\begin{aligned}
\|m^n\|_{H_0^1}^2 &\le \frac{\|v_{0,n} - v_n\|_{L^2}}{\alpha_{1,n}} + \|m_0\|_{H_0^1}^2 + k\|Dp_0\|_{L^2}^2 \\
&\le \frac{2(\|v_{0,n} - v_0\|_{L^2}^2 + \|v_0 - v_n\|_{L^2}^2)}{\alpha_{1,n}} + \|m_0\|_{H_0^1}^2 + k\|Dp_0\|_{L^2}^2 \\
&\le 2\frac{C_1\|w_n - w_0\|_{L^1} + \delta_n^2}{\alpha_{1,n}} + \|m_0\|_{H_0^1}^2 + k\|Dp_0\|_{L^2}^2 \\
&\le 2M(C_1 + 1) + \|m_0\|_{H_0^1}^2 + k\|Dp_0\|_{L^2}^2,
\end{aligned}$$

and, very similarly,

$$\begin{aligned}\|Dp^n\|_{L^2}^2 &\le \frac{\|v_{0,n}-v_n\|_{L^2}}{k\alpha_{1,n}} + \frac{1}{k}\|m_0\|_{H_0^1}^2 + \|Dp_0\|_{L^2}^2\\ &\le 2M\frac{C_1+1}{k} + \frac{1}{k}\|m_0\|_{H_0^1}^2 + \|Dp_0\|_{L^2}^2.\end{aligned}$$

Thus $\{m^n\}_{n=1}^\infty$ is bounded in H_0^1 and $\{Dp^n\}_{n=0}^\infty$ is bounded in L^2. From (3.14b) it follows that $\{p^n\}_{n=1}^\infty$ is bounded in H^{-1} and, by hypothesis (H1), we conclude that $\{p^n\}_{n=1}^\infty$ is bounded in L^2.

Let us consider the constraint (3.14c). Since $\{m^n\}_{n=1}^\infty$ is bounded in L^2 and $\tilde I^n$ is bounded in L^∞, it follows that $\{v^{n\prime}\}_{n=1}^\infty$ is bounded in L^2; hence, $\{v^n\}_{n=1}^\infty$ is bounded in H^1. We can therefore choose a subsequence (as usual denoted by the same symbol) satisfying

$$(v^n, m^n, p^n) \rightharpoonup (v^*, m^*, p^*)$$

weakly in $H^1 \times H_0^1 \times L^2$ and $Dp^n \rightharpoonup z^*$ weakly in L^2. Since D is weakly closed, we have $z^* = Dp^*$. Note that $m^{n\prime\prime} = -p^n \rightharpoonup p^*$ weakly in L^2 implies the weak convergence of $\{m^n\}_{n=1}^\infty$ in H^2.

Let us prove that (v^*, m^*, p^*) satisfies the constraints. It is easy to see that

$$m^{*\prime\prime} + p^* = 0$$

and

$$\int p^*\,dx = \int x\,p^*\,dx = \int v\,dx = 0.$$

Since $\tilde I^n \to \tilde I^0$ in L^2, due to Proposition 6, and $v^{n\prime} \rightharpoonup v^{*\prime}$ weakly in L^2, we have $\tilde I^n v^{n\prime} \rightharpoonup \tilde I^0 v^{*\prime}$ weakly in L^2. Therefore we have

$$0 = \tilde I^n v^{n\prime} + m^n \rightharpoonup \tilde I^0 v^{*\prime} + m^*$$

and the constraint (3.14c) is satisfied in the limit.

We shall prove that $(v^*, m^*, p^*) = (v_0, m_0, p_0)$. We have

$$\begin{aligned}\|v^* - v_0\|_{L^2}^2 &\le \liminf_{n\to\infty}\left(\|v^n - v_n\|_{L^2}^2 + \alpha_{1,n}\|m^n\|_{H_0^1}^2 + \alpha_{2,n}\|Dp^*\|_{L^2}^2\right)\\ &\le \liminf_{n\to\infty}\left(\|v_{0,n} - v_n\|_{L^2}^2 + \alpha_{1,n}\|m_0\|_{H_0^1}^2 + \alpha_{2,n}\|Dp_0\|_{L^2}^2\right) = 0,\end{aligned}$$

and hence $v^* = v_0$. Constraint (3.14c) implies

$$m_0 = -\tilde I^0 v_0' = -\tilde I^0 v^{*\prime} = m^*$$

and (3.14b) implies

$$p_0 = -m_0'' = -m^{*\prime\prime} = p^*.$$

To prove strong convergence of p^n in L^2 we use hypothesis (H2), which is equivalent to the fact that weak convergence $p^n \rightharpoonup p_0$ in L^2 together with

$\|Dp^n\|_{L^2} \to \|Dp_0\|_{L^2}$ implies strong convergence $p^n \to p_0$ in L^2. We have

$$\begin{aligned}
\|m_0\|^2_{H^1_0} + k\|Dp_0\|^2_{L^2} & \\
&\leq \liminf_{n\to\infty} \left(\|m^n\|^2_{H^1_0} + k\|Dp^n\|^2_{L^2} \right) \\
&\leq \limsup_{n\to\infty} \left(\|m^n\|^2_{H^1_0} + k\|Dp^n\|^2_{L^2} \right) \\
&\leq \limsup_{n\to\infty} \left(\frac{\|v_{0,n} - v_n\|^2_{L^2}}{\alpha_1} + \|m_0\|^2_{H^1_0} + k\|Dp_0\|^2_{L^2} \right) \\
&\leq \limsup_{n\to\infty} \left(2\frac{C_1\|w_n - w_0\|_{L^1} + \delta_n^2}{\alpha_{1,n}} + \|m_0\|^2_{H^1_0} + k\|Dp_0\|^2_{L^2} \right) \\
&= \|m_0\|^2_{H^1_0} + k\|Dp_0\|^2_{L^2}
\end{aligned}$$

and thus "lim inf = lim sup = lim" in the above inequalities and we get

$$\|m^n\|^2_{H^1_0} + k\|Dp^n\|^2_{L^2} \to \|m_0\|^2_{H^1_0} + k\|Dp_0\|^2_{L^2}$$

as $n \to \infty$. We have $\|m^n\|_{H^1_0} \to \|m_0\|$ due to the compact embedding $H^2 \hookrightarrow H^1_0$ and the weak convergence of m^n in H^2. Thus it follows that $\|Dp^n\|_{L^2} \to \|Dp_0\|_{L^2}$ and, if we use hypothesis (H2), $p^n \to p_0$ in L^2 as $n \to \infty$.

The usual application of Lemma 2 proves convergence of the whole original sequence $\{(v^n, m^n, p^n)\}_{n=1}^{\infty}$.

□

Remark. The fact that the choice of the regularization parameter α_1 in the convergence result in Theorem 1 depends on the convergence rate of $\|w_n - w_0\|_{L^1}$ is rather unsatisfactory. However, to our knowledge, there is up to now no result giving a-priori estimates for the convergence rate of the solution of (purely) BV-regularized least-squares problems.

5. Numerical examples

In this section we present the results of numerical experiments for two different sets of input data and several different noise-levels.

As a general rule, we discretize differentiable functions (H^1- or H^1_0-functions) using piecewise linear elements and L^2- or BV-functions by piecewise constants. An exact data set (v_0, m_0, p_0) is constructed and the exact input data v_0 are corrupted by artificial (Gaussian) noise to get v_d. The regularization operator D we use is a high pass filter with kernel consisting of constant functions.

For the solution of the non-smooth, convex, unconstrained optimization problem (3.10) we use a trust-region bundle algorithm. (cf. Schramm and Zowe [15]). Our choice in favor of the Schramm-Zowe algorithm was influenced by the fact that the algorithm is capable of dealing with a problem of the given size (∼100 variables) in a reasonable time, that it is robust with respect to data noise and startup value, and that it is easy to implement.

Moreover, it seems that at the current time the development of fast and stable algorithms for the solution of BV-regularized inverse problems is still in progress (cf. Rudin, Osher and Fatemi [13], Ito and Kunisch [11, 9], Dobson and Scherzer [6].)

Once (3.10) is solved and the stiffness function $\tilde{I}$ is defined according to (3.13), the solution of the linearly constrained inverse problem (3.14) is found by solving the respective (linear) optimality system.

In the first example the noise-level is chosen to be 1%, which means that we added (0-1) normally distributed Gaussian noise scaled by a factor of magnitude $0.01\|v_d\|_\infty$. The BV-regularization parameter for the first step is set to $\beta = 10^{-8}$. The startup value for w is chosen to be 0. The (optimal) regularization parameters $\alpha_1 = 10^{-14}$ and $\alpha_2 = 10^{-11}$ were found experimentally. Figure 3 shows the reconstruction of w, v, m and p (dashed lines) on the same plot with the exact data w_0, m_0, and p_0 (solid lines). The plot for v shows the noisy input data v_d as solid line and not the exact data v_0. It is seen that the broken and unbroken parts are identified with sufficient accuracy to give a satisfactory reconstruction of m and p.

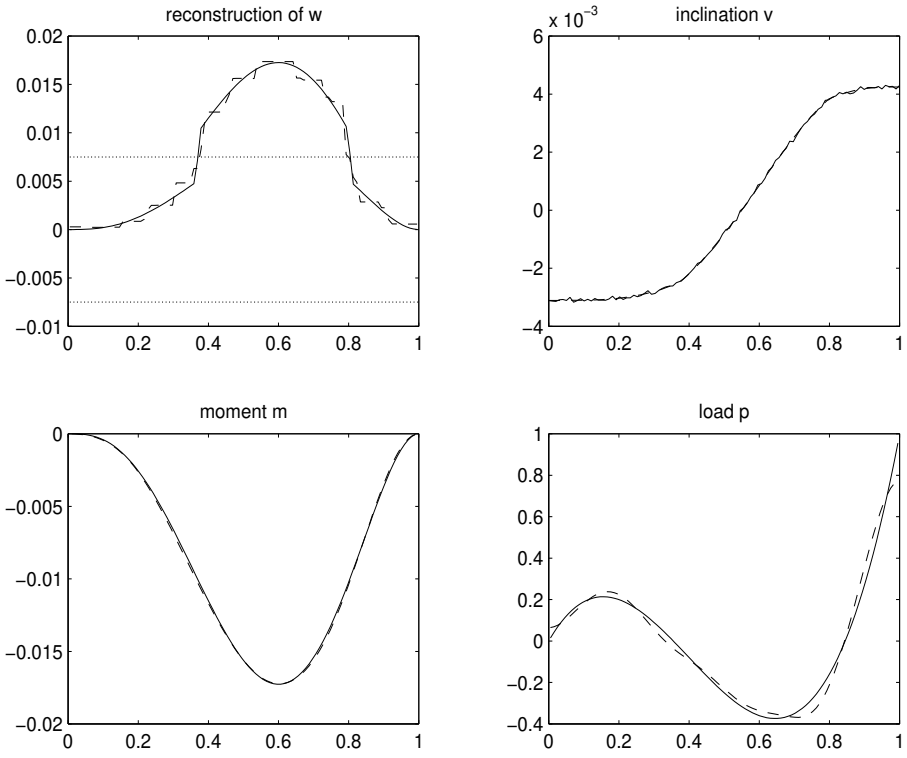

FIGURE 3. Reconstruction with noisy data; 1% noise.

If we increase the noise-level to 3% ($\alpha_1 = 10^{-13}$; $\alpha_2 = 5 \cdot 10^{-10}$; $\beta = 5 \cdot 10^{-8}$) and 10% ($\alpha_1 = 10^{-12}$; $\alpha_2 = 10^{-9}$; $\beta = 2 \cdot 10^{-7}$) we get the results which are shown in Figures 4 and 5.

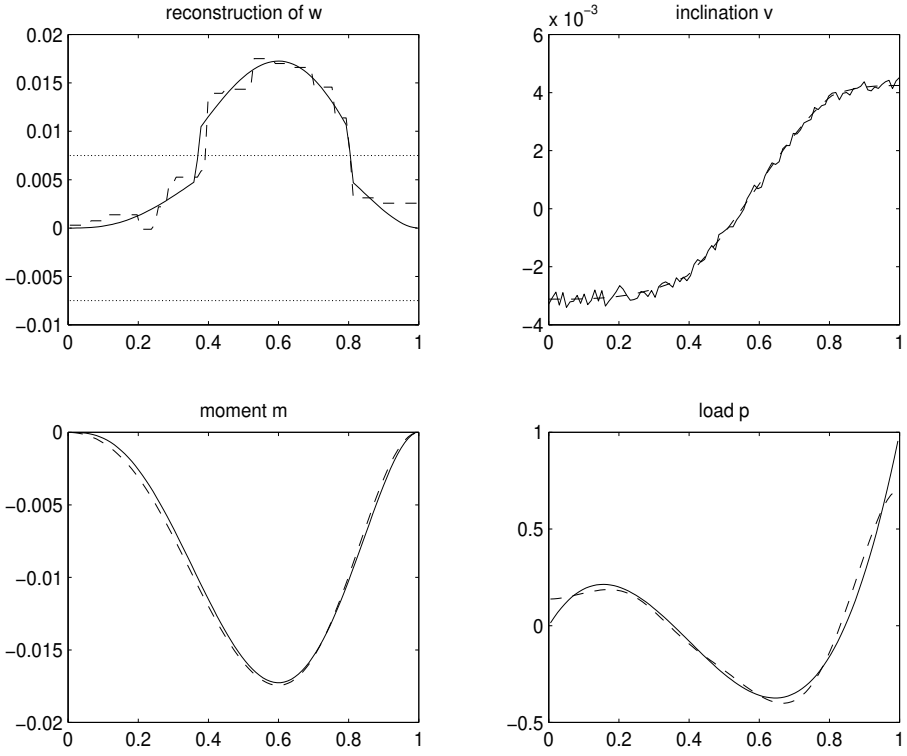

FIGURE 4. Reconstruction with noisy data; 3% noise; 2-step procedure.

The last example in Figure 6 shows a situation with a piecewise constant load p which is difficult to solve. The choice of the regularization parameter α_2 is critical. If it is chosen too small, more and more artificial oscillations emerge in the reconstruction of p. If it is chosen to large, the amplitude of the function p is damped too much and the (narrow) steps in p can not be resolved.

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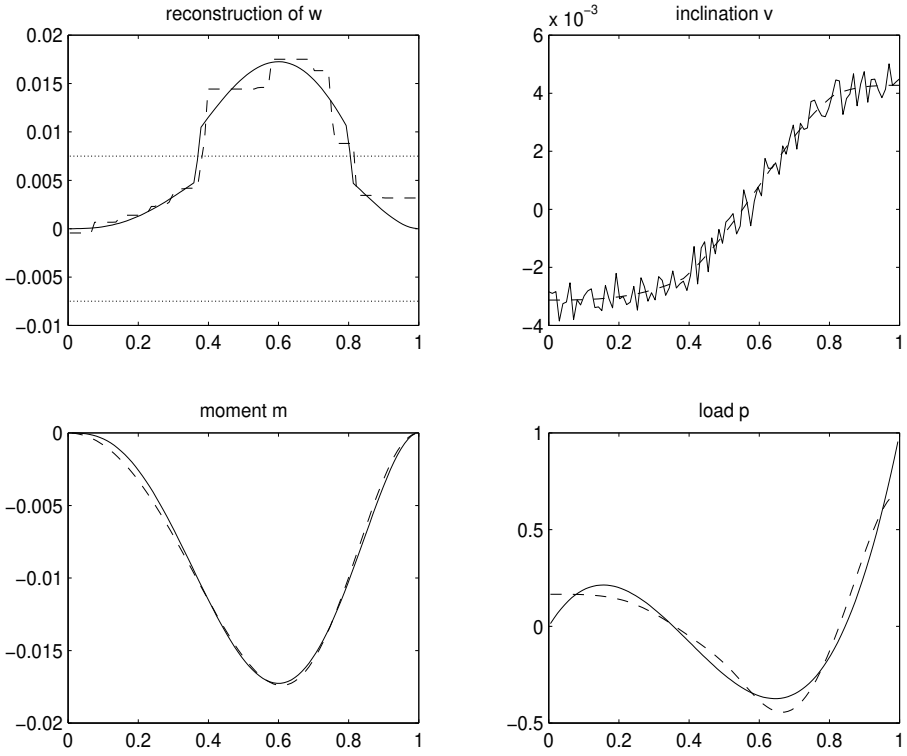

FIGURE 5. Reconstruction with noisy data; 10% noise; 2-step procedure.

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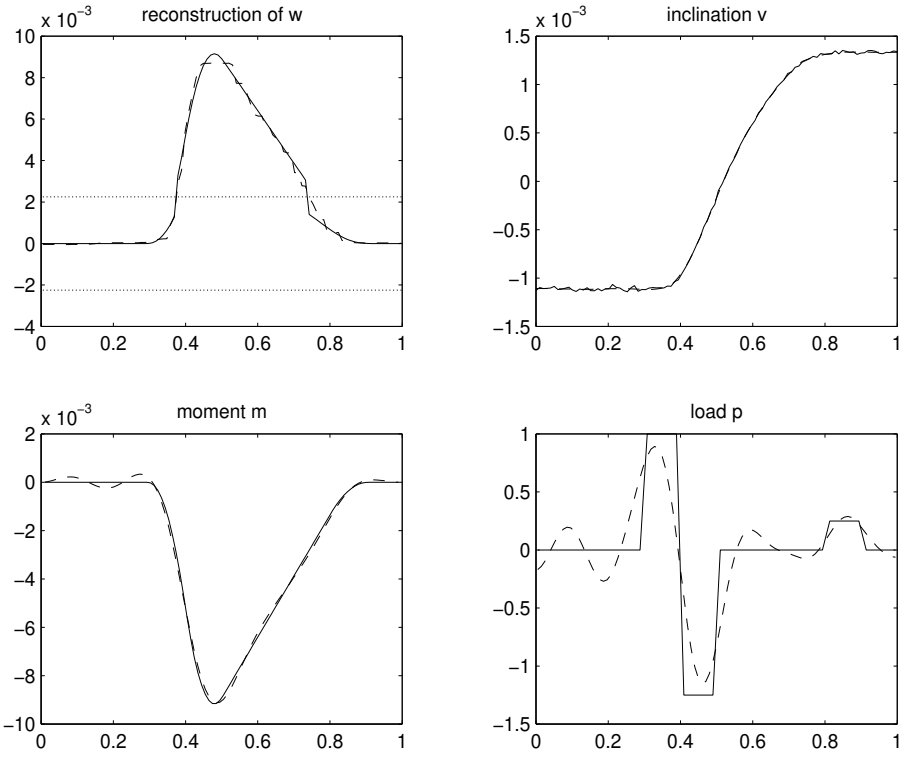

FIGURE 6. Reconstruction of a piecewise constant load.

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