OPTIMIZATION OF THE SHAPE OF AN ELECTROMAGNET: REGULARITY RESULTS

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Abstract. In this paper we study the regularity of the solution of an elliptic partial differential equation with discontinuous coefficients. This result is used to establish the continuous Fréchet differentiability of a shape functional arising in the optimal shape design of an electromagnet.

1. Introduction

The optimization of the shape of an electromagnet is one of the classical problems in shape optimization which has been considered by several authors (see [3], [1] and the references therein). To our knowledge most of the papers focus on various aspects of the discretized problem whereas the infinite dimensional problem is less well analyzed. The objective of this paper is to provide concise conditions which ensure the differentiable dependence of the objective functional on the free part of the shape of the electromagnet. In a follow-up paper we utilize the present results and techniques to study the convergence of the shape gradient of the discretized problem.

In Section 2 we formulate the shape optimization problem and establish the continuous dependence of the state on the shape of the electromagnet in the H^1-topology. Section 3 is devoted to the study of the regularity of the state. In particular it is shown that the restrictions of the vector potential to subdomains corresponding to the magnet and to the surrounding air respectively are smooth, uniformly bounded in H^2 with respect to a specified class of admissible shapes and moreover depend Hölder continuously on the shape in the H^2-topology. The differentiability of the objective functional with respect to the shape is discussed in Section 5.

2. Problem formulation

We consider the problem of optimal design of an electromagnet the cross-section of which is shown in Figure 1. The device consists of an iron core occupying the spatial region Ω_1 and a coil in W_1 and W_2. A current J flows in the coil, pointing outwards the cross-section plane at W_1 and inwards at W_2. The current density j is supposed to be constant over the cross-section of the coil. In order to determine the electromagnetic field generated in this situation we must specify the magnetic reluctivities ν_1 and ν_2 in iron and copper,

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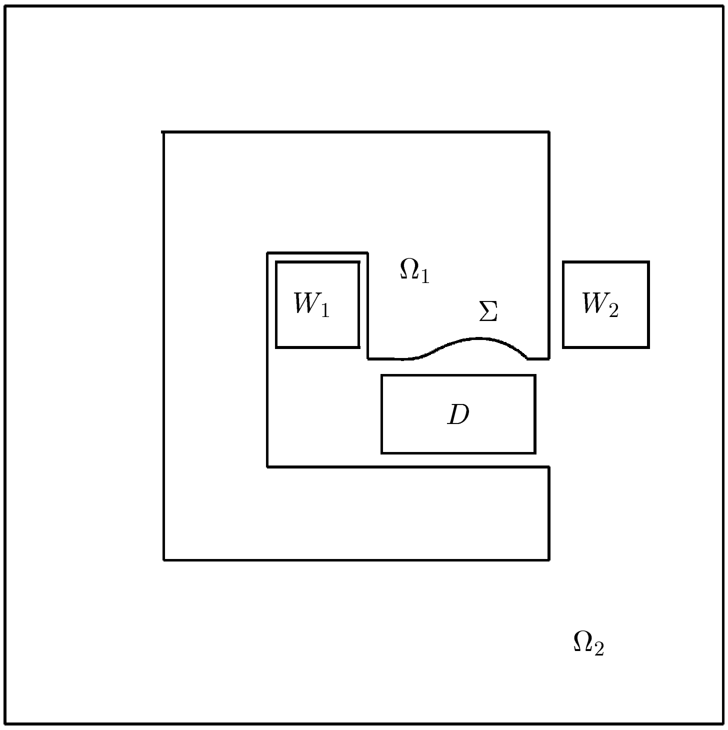

FIGURE 1. Cross-Section of the Electromagnet

respectively air. We assume that both values are constant, although in a more realistic scenario one would have to assume that ν_1 depends on the magnetic field. The (planar) magnetic field $\mathbf{B} = (B_1, B_2, 0)$ is determined by $\mathbf{B} = \operatorname{curl} \mathbf{A}$, $\mathbf{A} = (0, 0, A)$ and A satisfies the elliptic equation

$$-\operatorname{div}(\nu(\mathbf{x})\nabla A) = J(\mathbf{x}). \tag{1}$$

(c.f. [3]). Here

$$\nu(\mathbf{x}) = \begin{cases} \nu_1 & \text{on} \quad \Omega_1 \\ \nu_2 & \text{else} \end{cases} \tag{2}$$

and

$$J(\mathbf{x}) = \begin{cases} j & \text{on} \quad W_1 \\ -j & \text{on} \quad W_2 \\ 0 & \text{else} \end{cases}.$$

We also introduce a domain Ω containing Ω_1 which is large enough such that it is physically reasonable to assume that

$$A|_{\partial\Omega} = 0 \tag{3}$$

holds.

Our goal is, to design the shape of the pole of the magnet in such a way that the magnetic field in some measurement region D is as homogenous (in x_2-direction) as possible and attains a prescriped value c. The independent variables, which should be adjusted in order to obtain an optimal magnetic field are the current density j and the shape of the part of the boundary of Ω_1 denoted by Σ. (c.f. Figure 1). Note that Σ is not the whole curve-segment that connects the edges of the pole, but that there are (fixed) horizontal parts of the boundary of Ω_1 to the left and to the right of Σ. We suppose that Σ can be parametrized as

$$\Sigma = \big\{(x_1, x_2) \colon x_2 = \sigma(x_1), x_1 \in I = [0, l]\big\} \tag{4}$$

for some

$$\begin{aligned}\sigma \in \mathcal{B}_{ad} \;&=\; \{f \in W_2^\infty(0,l) \colon f(0) = f'(0) = f(l) = f'(l) = 0;\\ &\quad\; -a \leq f(x_1) \leq b \text{ for all } x_1 \in I\}.\end{aligned}$$

The constraints $a, b > 0$ are chosen in such a way that for any $\sigma \in \mathcal{B}_{ad}$ the corresponding Σ does neither intersect the remaining part of $\partial\Omega$ nor the measurement region D at any point.

The weak formulation of the boundary value problem (1) and (2) is the following: *Find a function* $A \in H_0^1(\Omega)$ *such that*

$$\int_\Omega \nu \, \nabla A \, (\nabla \varphi)^* \, d\mathbf{x} = j \left(\int_{W_1} \varphi \, d\mathbf{x} - \int_{W_2} \varphi \, d\mathbf{x} \right) \tag{5}$$

holds for all $\varphi \in H_0^1(\Omega)$ (throughout the paper * denotes the algebraic transpose). From standard variational arguments it follows that the boundary value problem (2) and (3) has a unique weak solution $A \in H_0^1(\Omega)$. Here and in the following, the space $H_0^1(\Omega)$ is equipped with the inner product

$$\langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \, (\nabla v)^* \, d\mathbf{x},$$

and we define the inner product on $H^1(\Omega)$ by

$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega u \, v \, d\mathbf{x} + \langle u, v \rangle_{H_0^1(\Omega)}.$$

The coefficient ν defined in (2) depends on the decomposition of Ω into the two subdomains Ω_1 and $\Omega_2 = \Omega \setminus \Omega_1$, and hence on σ. We therefore usually write $\nu = \nu^\sigma$ to indicate the underlying geometry. The solution A of (5) will often be written as $A = A(\sigma, j) = A^{\sigma,j}$ to emphasize its dependence on the parameters σ and j. Frequently only A^σ is used if the dependence on j is not of importance. In general we add a superscript 'σ' to any expression that depends on σ and has to be distinguished from another expression of the same kind but referring to a different geometry. Moreover, using the bilinear form

$$B^\sigma(\varphi_1, \varphi_2) = \int_\Omega \nu^\sigma \, \nabla \varphi_1 \, (\nabla \varphi_2)^* \, d\mathbf{x}, \quad \varphi_1, \varphi_2 \in H_0^1(\Omega), \tag{6}$$

equation (5) can be equivalently written as

$$B^\sigma(A^\sigma, \varphi) = (J, \varphi), \quad \varphi \in H_0^1(\Omega), \sigma \in \mathcal{B}_{ad}.$$

A cost functional that expresses the desired requirements for the optimal magnetic field is given by

$$K(\sigma) = K(A(\sigma, j)) = \frac{1}{2} \int_D \left[A_{x_2}^2 + (A_{x_1} + c)^2 \right] d\mathbf{x}. \tag{7}$$

We consider the optimization problem

$$\text{minimize } K(A(\sigma,j)) \text{ over } (\sigma,j)\in\mathcal{B}_{ad}\times\mathbb{R}, \quad A(\sigma,j) \text{ solves (5).} \tag{8}$$

Before we focus on the problem of existence of a minimizer for (8) and differentiability of the cost functional (7), we consider the regularity of the solution of (5) on the subdomains Ω_i and its continuous dependence on σ.

Suppose j is fixed. Setting $C_1 = (\min(\nu_1,\nu_2))^{-1}$ we find

$$\|A^\sigma\|^2_{H^1_0(\Omega)} \le C_1\, B^\sigma(A^\sigma, A^\sigma) = C_1 \int_\Omega J(\mathbf{x})\, A^\sigma(\mathbf{x})\, d\mathbf{x} \le |j| C_1 L_1\, \|A^\sigma\|_{H^1_0(\Omega)}$$

with some constant L_1 depending only on the domain Ω. Hence we have

$$\|A^\sigma\|_{H^1_0(\Omega)} \le M_1 \tag{9}$$

with a constant M_1 independent of σ.

Suppose $\sigma,\sigma_0\in\mathcal{B}_{ad}$ with corresponding Σ and Σ_0 (according to (4)). Let

$$S^{\sigma,\sigma_0} = \{(x_1,x_2)\in\Omega\colon \sigma(x_1)\le x_2\le\sigma_0(x_1) \text{ or } \sigma_0(x_1)\le x_2\le\sigma(x_1)\} \tag{10}$$

denote the strip between Σ and $\tilde{\Sigma}$. Then

$$\begin{aligned}
\|A^\sigma - A^{\sigma_0}\|^2_{H^1_0(\Omega)} &\le C_1\, B^{\sigma_0}(A^\sigma - A^{\sigma_0}, A^\sigma - A^{\sigma_0}) \\
&= C_1\Big(B^{\sigma_0}(A^\sigma, A^\sigma - A^{\sigma_0}) - B^\sigma(A^\sigma, A^\sigma - A^{\sigma_0})\Big) \\
&= C_1 \int_\Omega (\nu^{\sigma_0} - \nu^\sigma)\, \nabla A^\sigma\, (\nabla(A^\sigma - A^{\sigma_0}))^*\, d\mathbf{x} \\
&\le C_1|\nu_1-\nu_2| \left(\int_{S^{\sigma,\sigma_0}} |\nabla A^\sigma|^2 d\mathbf{x}\right)^{\frac12} \|A^\sigma - A^{\sigma_0}\|_{H^1_0(\Omega)}
\end{aligned}$$

and therefore

$$\|A^\sigma - A^{\sigma_0}\|_{H^1_0(\Omega)} \le C_1|\nu_1-\nu_2|\, \|\nabla A^\sigma\|_{L^2(S^{\sigma,\sigma_0})}. \tag{11}$$

Here we used $B^{\sigma_0}(A^{\sigma_0}, A^\sigma - A^{\sigma_0}) = B^\sigma(A^\sigma, A^\sigma - A^{\sigma_0})$ and $\nu^{\sigma_0} - \nu^\sigma = 0$ outside of S^{σ,σ_0}.

A standard tool to investigate regularity properties of elliptic PDEs is to consider difference operators

$$D_i^h u(\mathbf{y}) = \frac{u(\mathbf{y}+h\mathbf{e}_i) - u(\mathbf{y})}{h}$$

where $\mathbf{e}_i$ is the i-th unit coordinate vector ($i=1,2$). More precisely we have

Lemma 1. *Let $\mathcal{O}\subset\mathbb{R}^2$ be open, $u\in H^1(\mathcal{O})$, and let $\mathcal{O}'\subset\mathcal{O}$ be compact with* $\operatorname{dist}(\mathcal{O}',\partial\mathcal{O}) = d > 0$. *Then for $i=1,2$, we have $D_i^h u\in L^2(\mathcal{O}')$ and*

$$\|D_i^h u\|_{L^2(\mathcal{O}')} \le \|u_{x_i}\|_{L^2(\mathcal{O})}$$

for all $h < d$.

Proof. cf. [4, pg.321, Lemma 8.48]. □

On the other hand, boundedness of the difference quotients characterizes H^1-functions.

Lemma 2. *Let $\mathcal{O} \subset \mathbb{R}^2$ be open and suppose $u \in L^2(\mathcal{O})$ satisfies for all compact subsets $\mathcal{O}' \subset \mathcal{O}$*

$$\|D_i^h u\|_{L^2(\mathcal{O}')} \leq M, \quad i = 1, 2$$

for all $h < \operatorname{dist}(\mathcal{O}', \partial\mathcal{O})$ with a constant $M > 0$ independent of $\mathcal{O}'$. Then $u_{x_i} \in L^2(\mathcal{O})$ and

$$\|u_{x_i}\|_{L^2(\mathcal{O})} \leq M.$$

Proof. cf. [4, p.321, Lemma 8.49]. □

We also recall the following formulas.

Lemma 3. *Let $\mathcal{O}, \mathcal{O}'$ and d be given as in Lemma 1 and let $u, v \in L^2(\mathcal{O})$ with $v \equiv 0$ on $\mathcal{O} \setminus \mathcal{O}'$. Then, for $h < \operatorname{dist}(\mathcal{O}', \partial\mathcal{O})$ and $i = 1, 2$ we have*

$$D_i^h(uv)(\mathbf{x}) = u(\mathbf{x}) D_i^h v(\mathbf{x}) + D_i^h u(\mathbf{x}) v(\mathbf{x} + h\mathbf{e}_i), \quad x \in \mathcal{O}' \tag{12}$$

and

$$\int_{\mathcal{O}'} D_i^h u(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} = -\int_{\mathcal{O}} u(\mathbf{x})\, D_i^{-h} v(\mathbf{x})\, d\mathbf{x}. \tag{13}$$

3. Regularity

In the following we extend σ outside the interval $[0, l]$ by 0 and we consider σ as defined on some interval $(-\epsilon, l + \epsilon)$. This is possible if ϵ is small enough since $\partial\Omega_1 \setminus \Sigma$ connects to Σ by horizontal lines. Due to the boundary conditions in the definition of $\mathcal{B}_{ad}$, the resulting function is in $W_2^\infty(-\epsilon, l+\epsilon)$. We shall denote the boundary *function* (in $W_2^\infty(-\epsilon, l + \epsilon)$) by a greek lower case letter and the part of the boundary defined in (4) (a subset of $\mathbb{R}^2$) by the corresponding capital letter.

Theorem 1. *Suppose $\sigma_0 \in \mathcal{B}_{ad}$. Then there exists a neighbourhood $\mathcal{U}$ of σ_0 in $W_2^\infty(-\epsilon, l + \epsilon)$ and an open neighbourhood T of Σ_0 in $\mathbb{R}^2$ such that for all $\sigma \in \mathcal{U} \cap \mathcal{B}_{ad}$ the following holds:*

i) $\Sigma \subset T$
ii) $A^\sigma \in H^2(T \cap \Omega_i^\sigma)$ *for $i = 1, 2$*
iii) $\|A^\sigma\|_{H^2(T\cap\Omega_i^\sigma)} \leq K\|A^\sigma\|_{H_0^1(\Omega)}$ *for $i = 1, 2$ with some constant K independent of $\sigma \in \mathcal{B}_{ad}$.*

Proof. Let $\mathcal{Q} \subset \mathbb{R}^2$ be given by

$$\mathcal{Q} = \{(x_1, x_2)\colon \sigma_0(x_1) - 5\delta < x_2 < \sigma_0(x_1) + 5\delta, x_1 \in (-\epsilon, l + \epsilon)\},$$

where ϵ and δ are chosen in such a way that

$$\mathcal{Q}_1 = \{(x_1, x_2) \in \mathcal{Q}\colon x_2 > \sigma_0(x_1)\} \subset \Omega_1^{\sigma_0}$$

and

$$\mathcal{Q}_2 = \{(x_1, x_2) \in \mathcal{Q}\colon x_2 < \sigma_0(x_1)\} \subset \Omega_2^{\sigma_0} \setminus (D \cup W_1 \cup W_2).$$

We define

$$\mathcal{U} = \{\sigma \in W_2^\infty(-\epsilon, l + \epsilon)\colon \|\sigma - \sigma_0\|_\infty < \delta,\ \|\sigma' - \sigma_0'\|_\infty < N_1,\ \|\sigma'' - \sigma_0''\|_\infty < N_2,$$
$$\sigma(x) = 0 \text{ for } x \in (-\epsilon, l + \epsilon) \setminus (0, l)\}$$

with arbitrary (but fixed) constants $N_1, N_2 > 0$ and we set

$$Q^\sigma = \{(x_1, x_2) \colon \sigma(x_1) - 4\delta < x_2 < \sigma(x_1) + 4\delta, x_1 \in (-\epsilon, l + \epsilon)\}$$

for all $\sigma \in \mathcal{U}$. For $\sigma \in \mathcal{U}$ we have $Q^\sigma \subset \mathcal{Q}$ and hence

$$Q_1^\sigma = \{(x_1, x_2) \in Q^\sigma \colon x_2 > \sigma(x_1)\} \subset \Omega_1^\sigma$$

and

$$Q_2^\sigma = \{(x_1, x_2) \in Q^\sigma \colon x_2 < \sigma(x_1)\} \subset \Omega_2^\sigma \setminus (D \cup W_1 \cup W_2)$$

for all $\sigma \in \mathcal{U}$.

For every $\sigma \in \mathcal{U}$ we transform Q^σ onto the rectangular domain $\tilde{Q} = (-\epsilon, l + \epsilon) \times (-4\delta, 4\delta)$ by the diffeomorphism

$$\mathbf{y} = \Phi^\sigma(\mathbf{x}) = (x_1, x_2 - \sigma(x_1)). \tag{14}$$

The inverse transformation is given by $\Psi^\sigma(\mathbf{y}) = (\Phi^\sigma)^{-1}(\mathbf{y}) = (y_1, y_2 + \sigma(y_1))$. For arbitrary $\varphi \in H^1(Q^\sigma)$, we define the transformed function $\tilde{\varphi} = \varphi \circ \Psi^\sigma$. Then we have

$$\nabla_{\mathbf{x}}\varphi = \nabla_{\mathbf{y}}\tilde{\varphi}\, D\Phi^\sigma \qquad \nabla_{\mathbf{y}}\tilde{\varphi} = \nabla_{\mathbf{x}}\varphi\, D\Psi^\sigma \tag{15}$$

with Jacobi matrices

$$D\Phi^\sigma(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ -\sigma'(x_1) & 1 \end{pmatrix} \quad \text{and} \quad D\Psi^\sigma(\mathbf{y}) = \begin{pmatrix} 1 & 0 \\ \sigma'(y_1) & 1 \end{pmatrix}. \tag{16}$$

In the subsequent integrals $D\Phi^\sigma$ represents $D\Phi^\sigma(\Psi^\sigma(\mathbf{y}))$ and $D\Psi^\sigma$ represents $D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))$. This together with $|\det D\Phi^\sigma| = |\det D\Psi^\sigma| = 1$ implies that

$$\int_{\tilde{Q}} |\nabla_{\mathbf{y}}\tilde{\varphi}|^2\, d\mathbf{y} = \int_{Q^\sigma} \nabla_{\mathbf{x}}\varphi\, D\Psi^\sigma\, (D\Psi^\sigma)^*\, (\nabla_{\mathbf{x}}\varphi)^*\, d\mathbf{x} \le K_1 \int_{Q^\sigma} |\nabla_{\mathbf{x}}\varphi|^2\, d\mathbf{x} \tag{17}$$

and also

$$\int_{Q^\sigma} |\nabla_{\mathbf{x}}\varphi|^2\, d\mathbf{x} = \int_{\tilde{Q}} \nabla_{\mathbf{y}}\tilde{\varphi}\, D\Phi^\sigma\, (D\Phi^\sigma)^*\, (\nabla_{\mathbf{y}}\tilde{\varphi})^*\, d\mathbf{y} \le K_1 \int_{\tilde{Q}} |\nabla_{\mathbf{y}}\tilde{\varphi}|^2\, d\mathbf{y} \tag{18}$$

holds for all $\sigma \in \mathcal{U}$, with $K_1 = 1 + N_1 + N_1^2$. Thus we conclude that $\tilde{\varphi} \in H^1(\tilde{Q})$ if and only if $\varphi \in H^1(Q^\sigma)$. It is obvious that $\tilde{\varphi}|_{\partial\tilde{Q}} = 0$ if and only if $\varphi|_{\partial Q^\sigma} = 0$, hence $\tilde{\varphi} \in H_0^1(\tilde{Q})$ if and only if $\varphi \in H_0^1(Q^\sigma)$. This, together with (15) and (5), implies that $\tilde{A}^\sigma = A^\sigma \circ \Psi^\sigma$ satisfies

$$\int_{\tilde{Q}} \tilde{\nu}\, \nabla_{\mathbf{y}}\tilde{A}^\sigma\, (D\Phi^\sigma)\, (D\Phi^\sigma)^* (\nabla_{\mathbf{y}}\tilde{\varphi})^*\, d\mathbf{y} = 0 \tag{19}$$

for all $\tilde{\varphi} \in H_0^1(\tilde{Q})$, where $\tilde{\nu} = \nu^\sigma \circ \Psi^\sigma$. Note that $Q^\sigma \cap (W_1 \cup W_2) = \emptyset$ for all $\sigma \in \mathcal{U}$, hence the right-hand side in (19) is zero if we restrict ourselfs to test functions in $H_0^1(\tilde{Q})$. Introducing

$$\mathcal{M} = \tilde{\nu}\, D\Phi^\sigma (D\Phi^\sigma)^*$$

the bilinear form in (19) may be written as

$$\left(\nabla\tilde{A}^\sigma \mathcal{M}, \nabla\tilde{\varphi}\right)_{L^2(\tilde{Q})}.$$

Moreover we set $\tilde{T} = (-\frac{\epsilon}{2}, l + \frac{\epsilon}{2}) \times (-2\delta, 2\delta)$. Let ζ be a C^∞-function with compact support in $\tilde{Q}$ satisfying

$$\begin{cases} \zeta \equiv 1 \quad \text{on } \tilde{T}, \\ \operatorname{supp} \zeta \subset \tilde{R} = [-\frac{3}{4}\epsilon, l + \frac{3}{4}\epsilon] \times [-3\delta, 3\delta], \\ 0 \le \zeta(\mathbf{y}) \le 1 \quad \text{for all } \mathbf{y} \in \tilde{Q}. \end{cases} \tag{20}$$

Let

$$v = -D_1^{-h}(\zeta^2 D_1^h \tilde{A}^\sigma)$$

and

$$w = \zeta\, D_1^h \tilde{A}^\sigma \tag{21}$$

By the definition of ζ we have $\zeta^2 D_1^h \tilde{A}^\sigma = 0$ on $\partial\tilde{R}$ and hence $v = 0$ on $\partial\tilde{Q}$ if $|h| < \frac{\epsilon}{4}$. Thus $v \in H_0^1(\tilde{Q})$ and we can use it as a test function in (19) if h is small enough. Applying Lemma 3 one obtains

$$\begin{aligned}
0 &= \big(\nabla\tilde{A}^\sigma\mathcal{M}, \nabla v\big) = -\big(\nabla\tilde{A}^\sigma\mathcal{M}, \nabla D_1^{-h}(\zeta^2 D_1^h\tilde{A}^\sigma)\big) = \big(D_1^h[\nabla\tilde{A}^\sigma\mathcal{M}], \nabla(\zeta w)\big) \\
&= \big([D_1^h\nabla\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), \nabla(\zeta w)\big) + \big([\nabla\tilde{A}^\sigma]D_1^h\mathcal{M}, \nabla(\zeta w)\big) \\
&= \big([D_1^h\nabla\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), \zeta\nabla w\big) + \big([D_1^h\nabla\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), w\nabla\zeta\big) \\
&\quad + \big([\nabla\tilde{A}^\sigma]D_1^h\mathcal{M}, \nabla(\zeta w)\big) \\
&= \big(\nabla[\zeta D_1^h\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w\big) - \big(D_1^h\tilde{A}^\sigma\nabla\zeta\mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w\big) \\
&\quad + \big([D_1^h\nabla\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), w\nabla\zeta\big) + \big([\nabla\tilde{A}^\sigma]D_1^h\mathcal{M}, \nabla(\zeta w)\big)
\end{aligned}$$

and consequently

$$(\nabla w\mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w) = I_1 - I_2 - I_3, \tag{22}$$

where $(\cdot,\cdot)$ stands for the scalar product in $L^2(\tilde{Q}, \mathbb{R}^2)$ and

$$\begin{aligned}
I_1 &= \big(D_1^h\tilde{A}^\sigma\nabla\zeta\mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w\big) \\
I_2 &= \big([D_1^h\nabla\tilde{A}^\sigma]\mathcal{M}(\cdot + h\mathbf{e}_1), w\nabla\zeta\big) \\
I_3 &= \big([\nabla\tilde{A}^\sigma]D_1^h\mathcal{M}, \nabla(\zeta w)\big).
\end{aligned}$$

Since supp $\zeta \subset \tilde{R}$ the integrals in these terms may be carried out over $\tilde{R}$ instead of $\tilde{Q}$.

The discontinuity of $\tilde{\nu}$ is located along the line $y_2 = 0$, hence the tangential difference quotient $D_1^h\tilde{\nu}$ is zero. This leads to the following estimates of the spectral norm of $\mathcal{M}(\mathbf{y})$ and $D_1^h\mathcal{M}(\mathbf{y})$ for all $\mathbf{y} \in \tilde{Q}$ and $h < \frac{\epsilon}{4}$.

$$\|\mathcal{M}(\mathbf{y})\| \le \max(\nu_1, \nu_2)K_1 = \kappa_1$$

and

$$\|D_1^h\mathcal{M}(\mathbf{y})\| \le 2\max(\nu_1, \nu_2)N_2K_1 = \kappa_2$$

We set

$$\kappa_3 = \max(\|\zeta\|_\infty, \|\nabla\zeta\|_\infty, \|\frac{\partial}{\partial y_1}\nabla\zeta\|_\infty).$$

By Lemma 1 and the Cauchy-Schwarz inequality we obtain for $|h| \le \frac{\epsilon}{4}$

$$|I_1| \le \kappa_1\kappa_3\|\tilde{A}^\sigma_{y_1}\|_{L^2(\tilde{Q})}\|\nabla w\|_{L^2(\tilde{Q})}$$

For I_2 we find

$$\begin{aligned}|I_2| &= |(w\nabla\zeta\mathcal{M}(\cdot+h\mathbf{e}_1),[D_1^h\nabla\tilde{A}^\sigma])| = |(D_1^{-h}[w\nabla\zeta\mathcal{M}(\cdot+h\mathbf{e}_1)],\nabla\tilde{A}^\sigma)| \\ &= \int_{\tilde{R}}[w\nabla\zeta D_1^{-h}\mathcal{M}(\mathbf{y}+h\mathbf{e}_1)+[D_1^{-h}w]\nabla\zeta(\mathbf{y}-h\mathbf{e}_1)\mathcal{M}(\mathbf{y}) \\ &\quad +w[D_1^{-h}\nabla\zeta](\mathbf{y})\mathcal{M}(\mathbf{y})]\,\nabla(\tilde{\mathbf{A}}^\sigma)^*\,\mathbf{dy} \\ &\le (\kappa_2+2\kappa_1)\kappa_3\|w\|_{H^1(\tilde{Q})}\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}\end{aligned}$$

Similarly one obtains

$$|I_3| \le \kappa_2\kappa_3\|w\|_{H^1(\tilde{Q})}\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}.$$

By Poincaré's inequality, there exists a constant $l_2 > 0$ (independent of h) such that

$$\|w\|_{H^1(\tilde{Q})} \le l_2\|w\|_{H_0^1(\tilde{Q})}.$$

Summarizing the above estimates one gets

$$|(\nabla w\,\mathcal{M}(\cdot+h\mathbf{e}_1),\nabla w^*)| \le \kappa l_2\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}\|w\|_{H_0^1(\tilde{Q})} \tag{23}$$

with a constant $\kappa > 0$ independent of h. The left hand side of (23) is uniformly coercive on $H_0^1(\tilde{Q})$, i.e. there exists a constant $m > 0$ independent of h such that

$$m\,\|u\|_{H_0^1(\tilde{Q})} \le (\nabla u\,\mathcal{M}(\mathbf{y}+h\mathbf{e}_1),\nabla u)_{L^2(\tilde{Q})} \tag{24}$$

for all $u \in H_0^1(\tilde{Q})$. Hence (23) and (24) imply

$$\|w\|_{H_0^1(\tilde{Q})} \le \frac{\kappa l_2}{m}\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}.$$

Since $\|w_{y_i}\|_{L^2(\tilde{T})} \le \|w_{y_i}\|_{L^2(\tilde{Q})}$ and $\zeta \equiv 1$ on $\tilde{T}$ we find (recall (21))

$$\|D_1^h\tilde{A}^\sigma_{y_i}\|_{L^2(\tilde{T})} \le \frac{\kappa l_2}{m}\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}$$

for all $|h| < \frac{\epsilon}{4}$. Lemma 2 implies $\tilde{A}^\sigma_{y_1y_i} \in L^2(\tilde{T})$ and

$$\|\tilde{A}^\sigma_{y_1y_i}\|_{L^2(\tilde{T})} \le \frac{\kappa l_2}{m}\|\nabla\tilde{A}^\sigma\|_{L^2(\tilde{Q})}, \quad i=1,2. \tag{25}$$

Define $\tilde{T}_1 = \tilde{T}\cap\{\mathbf{y} : y_1 > 0\}$ and $\tilde{T}_2 = \tilde{T}\cap\{\mathbf{y} : y_1 < 0\}$. It follows from (19) that $\tilde{A}^\sigma$ is the distributional solution to

$$\tilde{A}^\sigma_{y_1y_1} - 2\sigma'\tilde{A}^\sigma_{y_1y_2} - \sigma''\tilde{A}^\sigma_{y_2} + (1+{\sigma'}^2)\tilde{A}^\sigma_{y_2y_2} = 0$$

on $\tilde{T}_i$ and hence by (25)

$$\tilde{A}^\sigma_{y_2y_2} = \frac{1}{1+{\sigma'}^2}(2\sigma'\tilde{A}^\sigma_{y_1y_2} + \sigma''\tilde{A}^\sigma_{y_2} - \tilde{A}^\sigma_{y_1y_1}) \in L^2(\tilde{T}_i) \tag{26}$$

which entails $\tilde{A}^\sigma \in H^2(\tilde{T}_i)$, $i = 1,2$.

We still have to carry out the backward transformation Ψ^σ to show that A^σ is an H^2-function on an upper (and lower) neighbourhood of the boundary Σ. Since

$$\begin{aligned} A^\sigma_{x_1x_1} &= \tilde{A}^\sigma_{y_1y_1} - 2\sigma' \tilde{A}^\sigma_{y_1y_2} + \sigma'^2 \tilde{A}^\sigma_{y_2y_2} - \sigma'' \tilde{A}^\sigma_{y_2}, \\ A^\sigma_{x_1x_2} &= \tilde{A}^\sigma_{y_1y_2} - \sigma' \tilde{A}^\sigma_{y_2y_2}, \\ A^\sigma_{x_2x_2} &= \tilde{A}^\sigma_{x_2x_2} \end{aligned}$$

we conclude from $\sigma \in W^\infty_2(-\epsilon, l+\epsilon)$ and from $\det(\Psi^\sigma(\mathbf{y})) = 1$ for all $\mathbf{y} \in \tilde{T}$ that $A^\sigma_{x_ix_j} \in L^2(T^\sigma_i)$ where $T^\sigma_i = (\Phi^\sigma)^{-1}(\tilde{T}_i)$, $i = 1, 2$.

We set

$$T = \{(x_1, x_2) \,:\, x_1 \in (-\frac{\epsilon}{2}, l + \frac{\epsilon}{2});\ \sigma_0(x_1) - \delta < x_2 < \sigma_0(x_1) + \delta\}.$$

Then we have $\Sigma \subset T$ and $T \subset \Psi^\sigma(\tilde{T})$ for all $\sigma \in \mathcal{U}$. Thus $T \cap \Omega^\sigma_i \subset T^\sigma_i$ and $A^\sigma \in H^2(T \cap \Omega^\sigma_i)$. The uniform boundedness of $\|A^\sigma\|_{H^2(T\cap\Omega^\sigma_i)}$ follows from (25), (26), and from the uniform boundedness of $\|\sigma'\|_\infty$ and $\|\sigma''\|_\infty$ in $\mathcal{U}$. $\square$

For $\sigma \in \mathcal{B}_{ad}$ we denote the restriction of A^σ to the subdomains Ω^σ_i by A^σ_i, $i = 1, 2$. To be able to compare (in the H^2-norm) $A^{\sigma_0}_i$ and A^σ_i which are defined on different domains we map Q^{σ_0} and Q^σ onto the fixed reference domain $\tilde{Q}$ and introduce

$$\tilde{A}^{\sigma_0}_i = A^{\sigma_0}_i \circ \Psi^{\sigma_0} \qquad \text{respectively } \tilde{A}^\sigma_i = A^\sigma_i \circ \Psi^\sigma.$$

Note that different transformations $(x_1, x_2) = \Psi^{\sigma_0}(y_1, y_2)$ and $(x_1, x_2) = \Psi^\sigma(y_1, y_2)$ are used. However both of them do not affect the first coordinate. One can write

$$D\Phi^\sigma = D\Phi^{\sigma_0} - \mathcal{N}^{\sigma-\sigma_0} \tag{27}$$

where

$$\mathcal{N}^{\sigma-\sigma_0} = (\sigma - \sigma_0)' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{28}$$

Using the technique of tangential displacement which was applied in the proof of Theorem 1 we can sharpen the continuous dependence result (11).

Theorem 2. *Let $\sigma_0 \in \mathcal{B}_{ad}$ and $\hat{\mathcal{U}}$ be defined as $\mathcal{U}$ with δ replaced by some $\hat{\delta} < \frac{2}{5}\delta$ and ε replaced by $\hat{\varepsilon} = \frac{\varepsilon}{2}$ where ε and δ are given in the proof of Theorem 1. Set $\hat{Q}_1 = (-\hat{\varepsilon}, l + \hat{\varepsilon}) \times (0, 4\hat{\delta})$ and $\hat{Q}_2 = (-\hat{\varepsilon}, l + \hat{\varepsilon}) \times (-4\hat{\delta}, 0)$. Then there exists a constant $\rho > 0$ such that $\tilde{A}^\sigma_i \in H^2(\hat{Q}_i)$ and*

$$\|\tilde{A}^{\sigma_0}_i - \tilde{A}^\sigma_i\|_{H^2(\hat{Q}_i)} \le \rho \|\sigma_0 - \sigma\|^{\frac{1}{4}}_{W^{2,\infty}}$$

holds for all $\sigma \in \hat{\mathcal{U}}$.

Proof. The condition $\hat{\delta} < \frac{2}{5}\delta$ ensures $\hat{Q}_i \subset \tilde{T}_i$. Hence the first assertion of the Theorem follows from (25) and (26). Moreover one verifies the existence of a constant $\lambda > 0$ such that

$$\|\tilde{A}^\sigma_i\|_{H^2(\hat{Q}_i)} \le \lambda \tag{29}$$

holds for all $\sigma \in \hat{\mathcal{U}}$ and $i = 1, 2$. Throughout the proof $\kappa_i > 0$ will denote a constant which is independent of $\sigma \in \hat{\mathcal{U}}$. We consider the transformed elliptic problems

$$\int_{\hat{Q}} \tilde{\nu}\, \nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}\, D\Phi^{\sigma_0} (D\Phi^{\sigma_0})^* (\nabla_{\mathbf{y}} \tilde{\varphi})^*\, d\mathbf{y} = 0 \tag{30}$$

and

$$\int_{\hat{Q}} \tilde{\nu}\, \nabla_{\mathbf{y}} \tilde{A}^{\sigma} D\Phi^{\sigma} (D\Phi^{\sigma})^* (\nabla_{\mathbf{y}} \tilde{\varphi})^*\, d\mathbf{y} = 0 \tag{31}$$

for all $\tilde{\varphi} \in H_0^1(\hat{Q})$, where $\hat{Q} = (-\hat{\epsilon}, l + \hat{\epsilon}) \times (-4\hat{\delta}, 4\hat{\delta})$. Subtracting (27) and (28) we obtain (note $D\Phi^\sigma \circ \Psi^\sigma = D\Phi^\sigma$, $\mathcal{N}^\eta \circ \Psi^\sigma = \mathcal{N}^\eta$)

$$\begin{aligned}
&\int_{\hat{Q}} \tilde{\nu}\, \nabla(\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})\, D\Phi^{\sigma_0}\, (D\Phi^{\sigma_0})^*\, (\nabla\tilde{\varphi})^*\, d\mathbf{y} \\
&\quad = \int_{\hat{Q}} \tilde{\nu}\, \nabla \tilde{A}^{\sigma} \Big[\mathcal{N}^{\sigma-\sigma_0} (D\Phi^{\sigma_0})^* + D\Phi^{\sigma_0} (\mathcal{N}^{\sigma-\sigma_0})^* \Big] (\nabla\tilde{\varphi})^*\, d\mathbf{y} \\
&\quad - \int_{\hat{Q}} \tilde{\nu}\, \nabla \tilde{A}^{\sigma-\sigma} \mathcal{N}^{\sigma-\sigma_0} (\mathcal{N}^{\sigma-\sigma_0})^* (\nabla\tilde{\varphi})^*\, d\mathbf{y}
\end{aligned} \tag{32}$$

for all $\tilde{\varphi} \in H_0^1(\hat{Q})$. If we choose

$$\tilde{\varphi} = -D_1^{-h}\, (\zeta^2 D_1^h (\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})\,)$$

and

$$w = \zeta D_1^h (\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})$$

with ζ as in (20) and δ, ε replaced by $\hat{\delta}$, $\hat{\varepsilon}$ respectively, then we obtain as in the proof of Theorem 1

$$\begin{aligned}
&\big(\nabla w \mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w\big) = \big([D_1^h(\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})] \nabla\zeta\, \mathcal{M}(\cdot + h\mathbf{e}_1), \nabla w\big) \\
&\quad - \big(w[D_1^h \nabla(\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})] \mathcal{M}(\cdot + h\mathbf{e}_1), \nabla\zeta\big) - \big(\nabla(\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0}) D_1^h \mathcal{M}, \nabla(\zeta w)\big) \\
&\quad + \int_{\hat{R}} D_1^h \{\tilde{\nu}\, \nabla \tilde{A}^{\sigma} [\mathcal{N}^{\sigma-\sigma_0} (D\Phi^{\sigma_0})^* + D\Phi^{\sigma_0} (\mathcal{N}^{\sigma-\sigma_0})^*]\}\, \nabla(\zeta w)^*\, d\mathbf{y} \\
&\quad - \int_{\hat{R}} D_1^h \Big(\tilde{\nu}\, \nabla \tilde{A}^{\sigma} \mathcal{N}^{\sigma-\sigma_0} (\mathcal{N}^{\sigma-\sigma_0})^*\Big) \nabla(\zeta w)^*\, d\mathbf{y} \\
&\quad = \hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_4 + \hat{I}_5
\end{aligned} \tag{33}$$

where $\hat{R} = [-\frac{3}{4}\hat{\varepsilon}, l + \frac{3}{4}\hat{\varepsilon}] \times [-3\hat{\delta}, 3\hat{\delta}]$.

As in the proof of Theorem 1 one derives the estimate

$$|\hat{I}_1 + \hat{I}_2 + \hat{I}_3| \le \kappa l_2\, \|\nabla(\tilde{A}^{\sigma} - \tilde{A}^{\sigma_0})\|_{L^2(\hat{Q})} \|w\|_{H_0^1(\hat{Q})}.$$

Moreover, since $\sigma, \sigma_0 \in \hat{\mathcal{U}}$ and $\tilde{A}^\sigma \in H^2(\hat{Q}_i)$, $i = 1, 2$, we obtain in view of (29) and with Lemma 1

$$\begin{aligned} |\hat{I}_4| &\leq \max(\nu_1, \nu_2)\big(\|\nabla \tilde{A}^\sigma\|_{L^2(\hat{R})} + \|D_1^h \nabla \tilde{A}^\sigma\|_{L^2(\hat{R})}\big) \\ &\quad \cdot (1 + 2(N_1 + N_2))\, \kappa_3 \,\|\sigma - \sigma_0\|_{W^{2,\infty}} \|w\|_{H_0^1(\hat{Q})} \\ &\leq \kappa_4 \,\|\sigma - \sigma_0\|_{W^{2,\infty}} \|w\|_{H_0^1(\hat{Q})} \end{aligned}$$

and

$$\begin{aligned} |\hat{I}_5| &\leq \max(\nu_1, \nu_2) 2\big(\|\nabla \tilde{A}^\sigma\|_{L^2(\hat{R})} + \|D_1^h \nabla \tilde{A}^\sigma\|_{L^2(\hat{R})}\big)\kappa_3 \|\sigma - \sigma_0\|^2_{W^{2,\infty}} \|w\|_{H_0^1(\hat{Q})} \\ &\leq \kappa_5 \,\|\sigma - \sigma_0\|^2_{W^{2,\infty}} \|w\|_{H_0^1(\tilde{Q})}. \end{aligned}$$

Therefore we obtain

$$\|w\|_{H_0^1(\hat{Q})} \leq \frac{\kappa l_2}{m} \|\nabla(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})\|_{L^2(\hat{Q})} + \|\sigma - \sigma_0\|_{W^{2,\infty}} \frac{\kappa_4}{m} + \|\sigma - \sigma_0\|^2_{W^{2,\infty}} \frac{\kappa_5}{m}$$

and hence

$$\begin{aligned} \|(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_1 y_i}\|_{L^2(\hat{T})} &\leq \frac{\kappa l_2}{m} \|\nabla(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})\|_{L^2(\hat{Q})} \\ &\quad + \frac{\kappa_4}{m} \|\sigma - \sigma_0\|_{W^{2,\infty}} + \frac{\kappa_5}{m} \|\sigma - \sigma_0\|^2_{W^{2,\infty}} \end{aligned} \tag{34}$$

for $i = 1, 2$ and $\hat{T} = (-\frac{\hat{\epsilon}}{2}, l + \frac{\hat{\epsilon}}{2}) \times (-2\hat{\delta}, 2\hat{\delta})$.

Next $\|\nabla(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})\|_{L^2(\hat{Q})}$ is estimated. One has

$$\begin{aligned} &\int_{\hat{Q}} |\nabla_{\mathbf{y}}(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})|^2 d\mathbf{y} = \\ &= \int_{\Psi^\sigma(\hat{Q})} \big|\big[\nabla_{\mathbf{x}} A^\sigma - \nabla_{\mathbf{x}}(\tilde{A}^{\sigma_0} \circ \Phi^\sigma)(\mathbf{x})\big] D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))\big|^2 d\mathbf{x} \\ &\leq 2 \int_{\Psi^\sigma(\hat{Q})} \big|\nabla_{\mathbf{x}}(A^\sigma - A^{\sigma_0})(\mathbf{x}) D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))\big|^2 \, d\mathbf{x} \\ &\quad + 2 \int_{\Psi^\sigma(\hat{Q})} \big|[\nabla_{\mathbf{x}} A^{\sigma_0}(\mathbf{x}) - \nabla_{\mathbf{x}}(\tilde{A}^{\sigma_0} \circ \Phi^\sigma)(\mathbf{x})] D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))\big|^2 \, d\mathbf{x} \\ &\leq \kappa_6 \|\nabla_{\mathbf{x}}(A^\sigma - A^{\sigma_0})\|^2_{L^2(\Omega)} + 2 \int_{\Psi^\sigma(\hat{Q})} \big|[\nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}(\Phi^{\sigma_0}(\mathbf{x})) D\Phi^{\sigma_0}(\mathbf{x}) \\ &\quad - \nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}(\Phi^\sigma(\mathbf{x})) D\Phi^\sigma(\mathbf{x})] D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))\big|^2 \, d\mathbf{x} \\ &\leq \kappa_6 \|\nabla_{\mathbf{x}}(A^\sigma - A^{\sigma_0})\|^2_{L^2(\Omega)} + 4 \int_{\Psi^\sigma(\hat{Q})} \big|\nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}(\Phi^\sigma(\mathbf{x})) - \nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}(\Phi^{\sigma_0}(\mathbf{x}))\big|^2 d\mathbf{x} \\ &\quad + 4 \int_{\Psi^\sigma(\hat{Q})} \big|\nabla_{\mathbf{y}} \tilde{A}^{\sigma_0}(\Phi^{\sigma_0}(\mathbf{x}))[D\Phi^\sigma - D\Phi^{\sigma_0}](\mathbf{x}) D\Psi^\sigma(\Phi^\sigma(\mathbf{x}))\big|^2 \, d\mathbf{x} \\ &= \iota_1 + \iota_2 + \iota_3. \end{aligned}$$

Since $H^1(T\cap\Omega_i^{\sigma_0})$ embeds continuously into $L^4(T\cap\Omega_i^{\sigma_0})$ the estimate (11) entails

$$\begin{aligned}\iota_1 &\le C_1^2|\nu_1-\nu_2|^2\big(\|\nabla A^{\sigma_0}\|_{L^2(S^{\sigma_0,\sigma}\cap\Omega_1^{\sigma_0})}+\|\nabla A^{\sigma_0}\|_{L^2(S^{\sigma_0,\sigma}\cap\Omega_2^{\sigma_0})}\big)\\ &\le C_1^2|\nu_1-\nu_2|^2\mathrm{vol}(S^{\sigma_0,\sigma})^{\frac12}\big(\|\nabla A_1^{\sigma_0}\|^2_{L^4(S^{\sigma_0,\sigma}\cap\Omega_1^{\sigma_0})}+\|\nabla A_2^{\sigma_0}\|^2_{L^4(S^{\sigma_0,\sigma}\cap\Omega_2^{\sigma_0})}\big)\\ &\le C_1^2|\nu_1-\nu_2|^2\|\sigma_0-\sigma\|_\infty^{\frac12}d^2\big(\|A_1^{\sigma_0}\|^2_{H^2(T\cap\Omega_1^{\sigma_0})}+\|A_2^{\sigma_0}\|^2_{H^2(T\cap\Omega_2^{\sigma_0})}\big),\end{aligned}$$

where d denotes an embedding constant. Concerning ι_3 one obtains the bound

$$\begin{aligned}|\iota_3| &\le 4(1+N_1+N_1^2)\int_{\tilde Q}|\nabla_{\mathbf y}\tilde A^{\sigma_0}(\mathbf y)|^2\,d\mathbf y\|\sigma'-\sigma_0'\|_\infty^2\\ &\le \kappa_8\|\sigma'-\sigma_0'\|_\infty^2\end{aligned}$$

where we used $\Phi^{\sigma_0}\big(\Psi^\sigma(\hat Q)\big)\subset(-\hat\varepsilon,l+\hat\varepsilon)\times(-5\hat\delta,5\hat\delta)\subset\tilde Q$. In order to estimate ι_2 one splits I into $I^+=\{x_1\in I\colon \sigma(x_1)\ge\sigma_0(x_1)\}$ and $I^-=I\backslash I^+$. Accordingly ι_2 is written as

$$\iota_2=\iota_2^+ + \iota_2^-.$$

In view of the definition of $\tilde A_i^{\sigma_0}$ this yields

$$\begin{aligned}\iota_2^+ &= \int\limits_{I^+}\int_{\sigma(x_1)-4\hat\delta}^{\sigma(x_1)+4\hat\delta}|\nabla_{\mathbf y}\tilde A^{\sigma_0}(x_1,x_2-\sigma(x_1))-\nabla_{\mathbf y}\tilde A^{\sigma_0}(x_1,x_2-\sigma_0(x_1))|^2\,dx_2dx_1\\ &= \int\limits_{I^+}\int_{\sigma(x_1)-4\hat\delta}^{\sigma_0(x_1)}|\nabla_{\mathbf y}\tilde A_2^{\sigma_0}(x_1,x_2-\sigma(x_1))-\nabla_{\mathbf y}\tilde A_2^{\sigma_0}(x_1,x_2-\sigma_0(x_1))|^2\,dx_2dx_1\\ &+\int\limits_{I^+}\int_{\sigma_0(x_1)}^{\sigma(x_1)}|\nabla_{\mathbf y}\tilde A_2^{\sigma_0}(x_1,x_2-\sigma(x_1))-\nabla_{\mathbf y}\tilde A_1^{\sigma_0}(x_1,x_2-\sigma_0(x_1))|^2\,dx_2dx_1\\ &+\int\limits_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+4\hat\delta}|\nabla_{\mathbf y}\tilde A_1^{\sigma_0}(x_1,x_2-\sigma(x_1))-\nabla_{\mathbf y}\tilde A_1^{\sigma_0}(x_1,x_2-\sigma_0(x_1))|^2\,dx_2dx_1\\ &= \iota_{21}^+ + \iota_{22}^+ + \iota_{23}^+.\end{aligned}$$

An argument analogous to the one applied in the estimate of ι_1 leads to

$$|\iota_{22}^+|\le\kappa_9\|\sigma-\sigma\|_\infty^{\frac12}.$$

Using $\tilde A_2^{\sigma_0}\in H^2(\tilde T_2)$ and Fubini's Theorem one verifies

$$\begin{aligned}|\iota_{21}^+| &\le \int\limits_{I^+}\int_{\sigma(x_1)-4\hat\delta}^{\sigma_0(x_1)}\Big|\int_{x_2-\sigma_0(x_1)}^{x_2-\sigma(x_1)}\frac{\partial}{\partial\eta}\nabla\tilde A_2^{\sigma_0}(x_1,\eta)\,d\eta\Big|^2dx_2dx_1\\ &\le \|\sigma-\sigma_0\|_\infty\int\limits_{I^+}\int_{\sigma(x_1)-4\hat\delta}^{\sigma_0(x_1)}\int\limits_{-5\hat\delta}^{0}\Big|\frac{\partial}{\partial\eta}\nabla\tilde A_2^{\sigma_0}(x_1,\eta)\Big|^2\,d\eta dx_2dx_1\\ &\le \|\sigma-\sigma_0\|_\infty\,4\hat\delta\,\|\tilde A_2^{\sigma_0}\|_{H^2(\tilde T_2)}.\end{aligned}$$

In the last inequality we used $I^+ \times (-5\hat{\delta}, 0) \subset \tilde{T}_2$ which is a consequence of the choice of $\hat{\delta}$. Combining the estimates for $\|\tilde{A}^\sigma - A^{\sigma_0}\|_{L^2(\hat{Q})}$ with (34) one arrives at

$$\|(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_1 y_i}\|_{L^2(\hat{T})} \le \kappa_9 \|\sigma_0 - \sigma\|^{\frac{1}{4}}_{W^{2,\infty}}, \quad i = 1, 2. \tag{35}$$

On $\tilde{T}_i$, $i = 1, 2$ the following relations are satisfied in the distributional sense:

$$(1 + {\sigma_0'}^2)\tilde{A}^{\sigma_0}_{y_2 y_2} = 2\sigma_0' \tilde{A}^{\sigma_0}_{y_1 y_2} + \sigma_0'' \tilde{A}^{\sigma_0}_{y_2} - \tilde{A}^{\sigma_0}_{y_1 y_1}$$
$$(1 + {\sigma'}^2)\tilde{A}^{\sigma}_{y_2 y_2} = 2\sigma' \tilde{A}^{\sigma}_{y_1 y_2} + \sigma'' \tilde{A}^{\sigma}_{y_2} - \tilde{A}^{\sigma}_{y_1 y_1}.$$

This gives rise to

$$(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_2 y_2} = \frac{1}{1 + {\sigma_0'}^2}\big[2\sigma_0'(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_1 y_2} + \sigma_0''(\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_2} - (\tilde{A}^\sigma - \tilde{A}^{\sigma_0})_{y_1 y_1}$$
$$+ 2(\sigma - \sigma_0)' \tilde{A}^\sigma_{y_1 y_2} + (\sigma - \sigma_0)'' \tilde{A}^\sigma_{y_2} - (2\sigma'(\sigma - \sigma_0)' + (\sigma' - \sigma_0')^2)\tilde{A}^\sigma_{y_2 y_2}\big].$$

which together with (35) and (29) completes the proof of the Theorem. □

Theorem 2 implies the existence of the traces $\nabla A_i^\sigma(\cdot, \sigma(\cdot))$ in $L^2(0, l)$ which depend continuously on σ:

Corollary 1. *Assume the hypotheses of Theorem 2. Then there is a constant $k > 0$ such that*

$$\|\nabla A_i^\sigma(\cdot, \sigma(\cdot)) - \nabla A_i^{\sigma_0}(\cdot, \sigma_0(\cdot))\|_{L^2(0,l)} \le k \|\sigma - \sigma_0\|^{\frac{1}{4}}_{W^{2,\infty}}, \quad i = 1, 2$$

holds for all $\sigma \in \hat{\mathcal{U}}$.

Proof. In view of (15) one obtains (μ stands for a constant independent of $\sigma \in \hat{\mathcal{U}}$)

$$\|\nabla A_i^\sigma(\cdot, \sigma(\cdot)) - \nabla A_i^{\sigma_0}(\cdot, \sigma_0(\cdot))\|_{L^2(0,l)} =$$
$$\int_0^l \left|\nabla_{\mathbf{y}} \tilde{A}_i^\sigma(x, 0) D\Phi^\sigma(x, \sigma(x)) - \nabla_{\mathbf{y}} \tilde{A}_i^{\sigma_0}(x, 0) D\Phi^{\sigma_0}(x, \sigma_0(x))\right|^2 dx$$
$$\le 2 \int_0^l \left|\left[\nabla_{\mathbf{y}} \tilde{A}_i^\sigma(x, 0) - \nabla_{\mathbf{y}} \tilde{A}_i^{\sigma_0}(x, 0)\right] D\Phi^\sigma(x, \sigma(x))\right|^2 dx +$$
$$2 \int_0^l \left|\nabla_{\mathbf{y}} \tilde{A}_i^\sigma(x, 0)\left[D\Phi^\sigma(x, \sigma(x)) - D\Phi^{\sigma_0}(x, \sigma_0(x))\right]\right|^2 dx$$
$$\le \mu\left[\|\tilde{A}_i^\sigma - \tilde{A}_i^{\sigma_0}\|^2_{H^2(\hat{Q}_i)} + \|\tilde{A}_i^{\sigma_0}\|^2_{H^2(\hat{Q}_i)} \|\sigma' - \sigma_0'\|^2_\infty\right],$$

which by Theorem 2 implies the result. □

4. Differentiability

In this section we investigate the differentiability of the cost functional (7) with respect to σ. For this purpose we introduce the adjoint problem: *Find a* *function* $P^\sigma \in H_0^1(\Omega)$ *such that*

$$\int_\Omega \nu^\sigma \nabla P^\sigma (\nabla\psi)^* \, d\mathbf{x} = \int_D \Big(A^\sigma_{x_2}\, \psi_{x_2} + (A^\sigma_{x_1} - c)\, \psi_{x_1} \Big)\, d\mathbf{x} \tag{36}$$

for all $\psi \in H_0^1(\Omega)$, where $A^\sigma \in H_0^1(\Omega)$ is the (known) solution to (5) corresponding to the geometry determined by σ.

The right-hand side of (36) which can be considered as the derivative of the cost functional K with respect to A^σ in direction ψ defines a bounded linear functional in ψ on $H_0^1(\Omega)$, the norm of which is bounded by $|c|\sqrt{\mathrm{vol}(D)} + \|A^\sigma\|_{H_0^1(\Omega)}$. It is known that (36) has a unique solution $P^\sigma \in H_0^1(\Omega)$. The coercivity of the bilinear form B^σ on $H_0^1(\Omega)$ ensures as in (9) the bound

$$\|P^\sigma\|_{H_0^1(\Omega)} \le |c|\sqrt{\mathrm{vol}(D)} + M_1 \tag{37}$$

for all $\sigma \in \mathcal{B}_{ad}$ $\big(M_1$ being the bound in (9)$\big)$. Moreover the continuous dependence result (11) is valid also for P^σ:

$$\|P^\sigma - P^{\tilde\sigma}\|_{H_0^1(\Omega)} \le C_1\, |\nu_1 - \nu_2|\, \|\nabla P^\sigma\|_{L^2(S^{\sigma,\tilde\sigma})}. \tag{38}$$

Concerning the smoothness of P^σ we can derive, with exactly the same proof as for A^σ, the analogous result $P^\sigma \in H^2(T \cup \Omega_i^\sigma)$ for $\sigma \in \mathcal{U}$, where $\mathcal{U}$ and T are choosen as in Theorem 1. Also the uniform boundedness result in Theorem 1 and the continuous dependence result of Theorem 2 hold for P^σ.

Let us now formulate the differentiability result.

Theorem 3. *The cost functional K, as given in (7), is Fréchet differentiable at every $\sigma \in \mathcal{B}_{ad}$ as a functional on the Banach space*

$$\mathcal{B} = \{\sigma \in W_2^\infty(-\epsilon, l+\epsilon) \,:\, \sigma \equiv 0 \quad \text{on } (-\epsilon, 0] \cup [l, l+\epsilon)\,\}.$$

The Fréchet derivative in direction $\eta \in \mathcal{B}$ is given by

$$\begin{aligned}\frac{\partial K}{\partial \sigma}\eta = \frac{\nu_1 - \nu_2}{2} \int_0^l \eta(x_1) \Big(&\nabla P_1^\sigma(x_1, \sigma(x_1))\, (\nabla A_2^\sigma(x_1, \sigma(x_1)))^* \\ &+ \nabla P_2^\sigma(x_1, \sigma(x_1))\, (\nabla A_1^\sigma(x_1, \sigma(x_1)))^* \Big)\, dx_1,\end{aligned} \tag{39}$$

where A^σ and P^σ denote the solutions of (5) and (36) respectively and subscripts $i \in \{1,2\}$ to A^σ and P^σ denote restrictions to Ω_i^σ.

Note that, by the regularity result in Theorem 1, the first order traces in (39) are well defined.

Proof. First we prove that the expression (39) is the Gateaux derivative of K. To this aim, we consider the finite-difference quotient $\frac{1}{h}(K(\sigma + h\,\eta) - K(\sigma))$ for $h \to 0$. We have

$$\begin{aligned}\frac{1}{h}\Big(K(\sigma + h\,\eta) \;&-\; K(\sigma)\Big) = \\ &= \frac{1}{2h} \Big(\int_D A^{\sigma+h\eta}_{x_2}\, (A^{\sigma+h\eta} - A^\sigma)_{x_2} + (A^{\sigma+h\eta}_{x_1} - c)\, (A^{\sigma+h\eta} - A^\sigma)_{x_1}\, d\mathbf{x} \\ &+ \int_D A^\sigma_{x_2}\, (A^{\sigma+h\eta} - A^\sigma)_{x_2} + (A^\sigma_{x_1} - c)\, (A^{\sigma+h\eta} - A^\sigma)_{x_1}\, d\mathbf{x} \Big).\end{aligned}$$

Both terms in the last expression can be considered as right-hand sides of the adjoint equation (36) with $A^{\sigma+h\eta} - A^{\sigma}$ as test function. Replacing them by the corresponding left-hand side results in

$$\begin{aligned}\frac{1}{h}\Big(K(\sigma + h\,\eta) - K(\sigma)\Big) &= \frac{1}{2h}\left(\int_\Omega \nu^{\sigma+h\eta}\,\nabla P^{\sigma+h\eta}\,(\nabla(A^{\sigma+h\eta} - A^{\sigma}))^*\,d\mathbf{x}\right.\\ &+ \left.\int_\Omega \nu^{\sigma}\,\nabla P^{\sigma}\,(\nabla(A^{\sigma+h\eta} - A^{\sigma}))^*\,d\mathbf{x}\right). \qquad (40)\end{aligned}$$

From (5) we obtain

$$\begin{aligned}\int_\Omega \nu^{\sigma+h\eta}\,\nabla P^{\sigma+h\eta}\,(\nabla A^{\sigma+h\eta})^*\,d\mathbf{x} &= j\Big(\int_{W_1} P^{\sigma+h\eta}\,d\mathbf{x} - \int_{W_2} P^{\sigma+h\eta}\,d\mathbf{x}\Big)\\ &= \int_\Omega \nu^{\sigma}\,\nabla P^{\sigma+h\eta}\,(\nabla A^{\sigma})^*\,d\mathbf{x}\end{aligned}$$

and analogously

$$\int_\Omega \nu^{\sigma}\,\nabla P^{\sigma}(\nabla A^{\sigma})^*\,d\mathbf{x} = \int_\Omega \nu^{\sigma+h\eta}\,\nabla P^{\sigma}\,(\nabla A^{\sigma+h\eta})^*\,d\mathbf{x}.$$

Inserting these terms into (40) gives

$$\frac{1}{h}\Big(K(\sigma + h\,\eta) - K(\sigma)\Big) \qquad (41)$$

$$= \frac{1}{2h}\int_\Omega (\nu^{\sigma} - \nu^{\sigma+h\eta})\Big(\nabla P^{\sigma+h\eta}(\nabla A^{\sigma})^* + \nabla P^{\sigma}(\nabla A^{\sigma+h\eta})^*\Big)d\mathbf{x}$$

$$= \frac{\nu_1 - \nu_2}{2h}\int_0^l \int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \Big(\nabla P^{\sigma+h\eta}(\nabla A^{\sigma})^* + \nabla P^{\sigma}(\nabla A^{\sigma+h\eta})^*\Big)\,dx_2\,dx_1 \qquad (42)$$

since $\nu^{\sigma} - \nu^{\sigma+h\eta} = 0$ outside $S^{\sigma,\sigma+h\eta}$. On $S^{\sigma,\sigma+h\eta}$ we have $\nu^{\sigma} - \nu^{\sigma+h\eta} = \pm(\nu_1 - \nu_2)$. The sign depends on whether $\sigma(x_1) + h\eta(x_1) > \sigma(x_1)$ or $\sigma(x_1) + h\eta(x_1) < \sigma(x_1)$.

We introduce

$$I^+ = \{x_1 \in [0, l]\,:\,\sigma(x_1) + h\eta(x_1) > \sigma(x_1)\}. \qquad (43)$$

For $x_1 \in I^+$ and $x_2 \in [\sigma(x_1), \sigma(x_1)+h\eta(x_1)]$ one has $P^{\sigma+h\eta}(x_1, x_2) = P_2^{\sigma+h\eta}(x_1, x_2)$, $P^{\sigma}(x_1, x_2) = P_1^{\sigma}(x_1, x_2)$, $A^{\sigma+h\eta}(x_1, x_2) = A_2^{\sigma+h\eta}(x_1, x_2)$ and $A^{\sigma}(x_1, x_2) = A_1^{\sigma}(x_1, x_2)$. Consider the difference

$$\begin{aligned}&\frac{\nu_1 - \nu_2}{2h}\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \Big(\nabla P_2^{\sigma+h\eta}(x_1, x_2)\,(\nabla A_1^{\sigma}(x_1, x_2))^*\\ &\qquad + \nabla P_1^{\sigma}(x_1, x_2)\,(\nabla A_2^{\sigma+h\eta}(x_1, x_2))^*\Big)\,dx_2\,dx_1 \qquad (44)\\ &-\frac{\nu_1 - \nu_2}{2}\int_{I^+}\eta(x_1)\Big(\nabla P_2^{\sigma}(x_1, \sigma(x_1))\,(\nabla A_1^{\sigma}(x_1, \sigma(x_1)))^*\\ &\qquad + \nabla P_1^{\sigma}(x_1, \sigma(x_1))\,(\nabla A_2^{\sigma}(x_1, \sigma(x_1)))^*\Big)\,dx_1\\ &= T_1 + \tilde{T}_1 + T_2 + \tilde{T}_2 + T_3 + \tilde{T}_3,\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \frac{\nu_1-\nu_2}{2h}\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \left[\nabla P_2^{\sigma+h\eta}(x_1,x_2)\,(\nabla A_1^\sigma(x_1,x_2))^*\right.\\
&\qquad \left.-\nabla P_2^{\sigma+h\eta}(x_1,\sigma(x_1))\,(\nabla A_1^\sigma(x_1,\sigma(x_1)))^*\right] dx_2\,dx_1 \qquad (45)\\
\tilde{T}_1 &= \frac{\nu_1-\nu_2}{2h}\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \left[\nabla P_1^{\sigma}(x_1,x_2)\,(\nabla A_2^{\sigma+h\eta}(x_1,x_2))^*\right.\\
&\qquad \left.-\nabla P_1^{\sigma}(x_1,\sigma(x_1))\,(\nabla A_2^{\sigma+h\eta}(x_1,\sigma(x_1)))^*\right] dx_2\,dx_1 \qquad (46)\\
T_2 &= \frac{\nu_1-\nu_2}{2}\int_{I^+}\eta(x_1)\left[\nabla P_2^{\sigma+h\eta}(x_1,\sigma(x_1)) - \nabla P_2^{\sigma+h\eta}(x_1,\sigma(x_1)+h\eta(x_1))\right]\\
&\qquad (\nabla A_1^\sigma(x_1,\sigma(x_1)))^*\,dx_1 \qquad (47)\\
\tilde{T}_2 &= \frac{\nu_1-\nu_2}{2}\int_{I^+}\eta(x_1)\left[\nabla A_2^{\sigma+h\eta}(x_1,\sigma(x_1)) - \nabla A_2^{\sigma+h\eta}(x_1,\sigma(x_1)+h\eta(x_1))\right]\\
&\qquad (\nabla P_1^\sigma(x_1,\sigma(x_1)))^*\,dx_1 \qquad (48)\\
T_3 &= \frac{\nu_1-\nu_2}{2}\int_{I^+}\eta(x_1)\left[\nabla P_2^{\sigma+h\eta}(x_1,\sigma(x_1)+h\eta(x_1)) - \nabla P_2^{\sigma}(x_1,\sigma(x_1))\right]\\
&\qquad (\nabla A_1^\sigma(x_1,\sigma(x_1)))^*\,dx_1 \qquad (49)\\
\tilde{T}_3 &= \frac{\nu_1-\nu_2}{2}\int_{I^+}\eta(x_1)\left[\nabla A_2^{\sigma+h\eta}(x_1,\sigma(x_1)+h\eta(x_1)) - \nabla A_2^{\sigma}(x_1,\sigma(x_1))\right]\\
&\qquad (\nabla P_1^\sigma(x_1,\sigma(x_1)))^*\,dx_1. \qquad (50)
\end{aligned}$$

We turn to the estimate of T_1:

$$\begin{aligned}
|T_1| &= \left|\frac{\nu_1-\nu_2}{2h}\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)}\int_{\sigma(x_1)}^{x_2}\frac{\partial}{\partial\xi}\Big(\nabla P_2^{\sigma+h\eta}(x_1,\xi)\,(\nabla A_1^\sigma(x_1,\xi))^*\Big)d\xi\,dx_2\,dx_1\right|\\
&= \left|\frac{\nu_1-\nu_2}{2h}\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)}((\sigma+h\eta)(x_1)-\xi)\frac{\partial}{\partial\xi}\Big(\nabla P_2^{\sigma+h\eta}(x_1,\xi)(\nabla A_1^\sigma(x_1,\xi))^*\Big)d\xi\,dx_1\right|\\
&\le \frac{|\nu_1-\nu_2|}{2|h|}\,|h|\,\|\eta\|_\infty\int_{I^+}\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)}\left|\frac{\partial}{\partial x_2}\Big(\nabla P_2^{\sigma+h\eta}(x_1,x_2)\,(\nabla A_1^\sigma(x_1,x_2))^*\Big)\right|dx_2\,dx_1\\
&\le \frac{|\nu_1-\nu_2|}{2}\,\|\eta\|_\infty\big\{\|\frac{\partial}{\partial x_2}\nabla P_2^{\sigma+h\eta}\|_{L^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\|\nabla A_1^\sigma\|_{L^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\\
&\qquad +\|\nabla P_2^{\sigma+h\eta}\|_{L^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\|\frac{\partial}{\partial x_2}\nabla A_1^\sigma\|_{L^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\big\}\\
&\le |\nu_1-\nu_2|\,\|\eta\|_\infty\|P_2^{\sigma+h\eta}\|_{H^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\|A_1^\sigma\|_{L^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}.
\end{aligned}$$

Since Theorem 1 holds for P^σ, $\sigma\in\mathcal{B}_{ad}$, we conclude by (37) that $\|P_2^{\sigma+h\eta}\|_{H^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}$ is bounded uniformly in h for all $|h|$ small enough such that $\sigma+h\eta\in\mathcal{U}(\sigma)$. This implies

$$\lim_{h\to 0}|T_1| = 0.$$

In a completely analogous way, it can be shown that $|\tilde{T}_1|\to 0$ as $h\to 0$.

T_2 is estimated as follows:

$$\begin{aligned}
|T_2| &\le \frac{|\nu_1-\nu_2|}{2}\|\eta\|_\infty \int_{I^+}\Big|\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \frac{\partial}{\partial x_2}\nabla P_2^{\sigma+h\eta}(x_1,x_2)dx_2\nabla A_1^\sigma(x_1,\sigma(x_1))^*\Big|dx_1\\
&\le \frac{|\nu_1-\nu_2|}{2}\|\eta\|_\infty\Big(\int_{I^+}\Big|\int_{\sigma(x_1)}^{\sigma(x_1)+h\eta(x_1)} \frac{\partial}{\partial x_2}\nabla P_2^{\sigma+h\eta}(x_1,x_2)\,dx_2\Big|^2 dx_1\Big)^{\frac{1}{2}}\\
&\qquad \Big(\int_{I_+}\Big|\nabla A_1^\sigma(x_1,\sigma(x_1))\Big|^2 dx_1\Big)^{\frac{1}{2}}\\
&\le \frac{|\nu_1-\nu_2|}{2}\|\eta\|_\infty^{\frac{3}{2}}\|P_2^{\sigma+h\eta}\|_{H^2(S^{\sigma,\sigma+h\eta}\cap\Omega_2^{\sigma+h\eta})}\|A_1^\sigma\|_{H^2(T\cap\Omega_1^\sigma)}\,|h|^{\frac{1}{2}}.
\end{aligned}$$

By Theorem 1 and (37) $|T_2|\to 0$ as $h\to 0$ follows. An analogous argument establishes $|\tilde{T}_2|\to 0$ as $h\to 0$.

Finally, T_3 and $\tilde{T}_3$ vanish as $h\to 0$ which is a consequence of Corollary 1.

So far we have proved that (39) is in fact the Gateaux derivative of the cost functional K. For each $\sigma\in\mathcal{B}_{ad}$ the Gateaux derivative (39) defines a bounded linear functional on the Banach space

$$\mathcal{B}=\{\sigma\in W^{2,\infty}(0,l)\colon \sigma(0)=\sigma'(0)=0,\sigma(1)=\sigma'(1)=0\}.$$

Furthermore, Corollary 1 and Theorem 1 imply the existence of constants $\tilde\kappa>0$, $\tilde\delta>0$ such that

$$\|K'(\tilde\sigma)-K'(\sigma)\|_{\mathcal{B}}^*\le\tilde\kappa\,\|\tilde\sigma-\sigma\|_{W^{2,\infty}}^{\frac{1}{4}}$$

holds for all $\tilde\sigma\in\{\eta\in\mathcal{B}\colon \|\eta-\sigma\|_{W^{2,\infty}}\le\tilde\delta\}$. As a consequence, Theorem II-4.3 in ([2]) ensures that $DK(\sigma)$ is in fact the Fréchet derivative of K. $\square$

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