IDENTIFICATION OF A CORE FROM BOUNDARY DATA*

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Abstract. The problem of determining the interface separating regions of constant density from boundary data of a solution of the corresponding potential equation is considered. An equivalent formulation as a nonlinear integral equation is obtained. Fourier methods are used to analyze and implement the problem. Numerical experiments based on a regularized least-squares method are presented.

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1. Introduction. Suppose (r, φ) denote polar coordinates in the plane and $r = a(\varphi)$ is a continuous function, with graph of a contained in some domain $\Omega \subset \mathbb{R}^2$. The subject of this work is the following problem:

Determine the interface $r = a(\varphi)$ *in*

$$\Delta u = f_a(r, \varphi) = \begin{cases} \mu_1 & \text{for} \quad 0 \le r < a(\varphi) \\ \mu_2 & \text{for} \quad a(\varphi) \le r \end{cases} \tag{1}$$

on $\Omega \subset \mathbb{R}^2$ *from known boundary data* $z_1 = u|_{\partial\Omega}$ *and* $z_2 = \frac{\partial u}{\partial \mathfrak{n}}|_{\partial\Omega}$.

We consider only the situation where $\Omega = U$, the open unit disk in $\mathbb{R}^2$. The following introductory remarks, however, do not only apply to U, but to any simply connected open subset of the plane with smooth boundary.

Problem (1) occurs in the context of recovering an unknown mass distribution inside a planar body from measurements of the potential at the boundary. Suppose a mass distribution consisting of two layers is given on Ω: a region of constant density μ_1 inside Ω (a core) and an annular region (a mantle) of constant density μ_2 enclosing the core, and suppose that the interface between the two regions is given by the curve $r = a(\varphi)$. The gravitational potential u generated by this mass distribution is then a solution of the Poisson equation $\Delta u = f_a$. Therefore (1) can be interpreted as the problem of finding the curve which separates the regions of constant density from measurements of the potential at the boundary of Ω.

Closely related to (1) is the inverse domain recognition problem in potential theory:

Find the domain $\tilde\Omega \subset \Omega$ *in* $\displaystyle\int_{\tilde\Omega} E(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi} = g(\boldsymbol{x})$; $\boldsymbol{x} \in \partial\Omega$, *from known* $g : \partial\Omega \to \mathbb{R}$. (2)

Here $\tilde\Omega$ denotes some open subset of Ω and

$$E(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{4\pi} \log |\boldsymbol{x} - \boldsymbol{\xi}|^2.$$

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In fact, (1) and (2) are equivalent in the following sense. Suppose (1) holds for a, z_1 and z_2. Using Green's formula and well known identities from the theory of boundary integral equations (c.f. Chen and Zhou [4, p 223, (6.43) and p 224, (6.45)]), we obtain

$$\int_{\Omega} f_a(\boldsymbol{\xi})\, E(\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{z_1(\boldsymbol{x})}{2} - \int_{\partial\Omega} z_1(\boldsymbol{\xi}) \frac{\partial}{\partial \mathfrak{n}_\xi} E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi + \int_{\partial\Omega} z_2(\boldsymbol{\xi}) E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi \tag{3}$$

and hence, with Ω_a denoting the core enclosed by a, we find

$$\begin{aligned}\int_{\Omega_a} E(\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{1}{\mu_1-\mu_2}\Big\{ &\frac{z_1(\boldsymbol{x})}{2} - \int_{\partial\Omega} z_1(\boldsymbol{\xi}) \frac{\partial}{\partial \mathfrak{n}_\xi} E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi \\ &+ \int_{\partial\Omega} z_2(\boldsymbol{\xi}) E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi - \mu_2 \int_{\Omega} E(\boldsymbol{x}\, \boldsymbol{\xi})\, d\boldsymbol{\xi} \Big\}\end{aligned} \tag{4}$$

for $\boldsymbol{x} \in \partial\Omega$. This shows that Ω_a (and hence a) can be obtained from z_1 and z_2 as a solution of (2) with g given by the right hand side of (4).

On the other hand, the Newtonian potential $w(\boldsymbol{x}) = \int_{\Omega_a} E(\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{\xi}$ satisfies $\Delta w = f_a$ for $\mu_1 = 1$, $\mu_2 = 0$. We set $z_1 := g = w|_{\partial\Omega}$ and $z_2 := \frac{\partial w}{\partial \mathfrak{n}}|_{\partial\Omega}$. Starting with (2), only z_1 but not z_2 is known. However, if we apply Green's formula and well known integral identities [4, p 223, (6.43) and p 224, (6.45)] to $w(\cdot)$ and $E(\boldsymbol{x},\cdot)$, we obtain the integral equation

$$\int_{\partial\Omega} z_2(\boldsymbol{\xi})\, E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi = \frac{z_1(\boldsymbol{x})}{2} + \int_{\partial\Omega} z_1(\boldsymbol{\xi})\, \frac{\partial}{\partial \mathfrak{n}_\xi} E(\boldsymbol{x},\boldsymbol{\xi})\, d\sigma_\xi \quad \text{for } \boldsymbol{x} \in \partial\Omega. \tag{5}$$

The right-hand side of (5) defines a linear operator $\mathcal{L}_1 : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ acting on z_2, which is Fredholm of index 0. In case $\Omega = U$, the kernel on $\mathcal{L}_1$ consists of the constant functions on $\partial\Omega$, hence z_2 is determined by (5) up to an additive constant c. (c.f. [4] for the facts mentioned above.) Using Gauss' divergence theorem, we obtain the additional relation

$$\int_{\partial\Omega} z_2(\boldsymbol{\xi})\, d\sigma_\xi = \int_{\partial\Omega} \frac{\partial w}{\partial \mathfrak{n}}(\boldsymbol{\xi})\, d\sigma_\xi = \int_{\Omega} f_a(\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

from which the constant c can be uniquely determined. Hence, we arrive at (1) with known boundary data z_1 and z_2 and unknown a.

The domain recognition problem (2) was studied extensively in the past. We mention Novikov [14], Prilepko [15] [16], Cabayan and Belford [3], Brodsky [2], Strakhov and Brodsky [19], the book by Isakov [9] and the literature cited there.

The starting point of yet another equivalent formulation is the boundary value problem

$$\begin{cases} \Delta u = f_a & \text{on } \Omega \\ u|_{\partial\Omega} = z_1 \end{cases} . \tag{6}$$

With $u = u(a, z_1)$ denoting the solution of (6), we can define the operator

$$G(a, z_1) = \frac{\partial}{\partial \mathfrak{n}} u(a, z_1)\big|_{\partial\Omega}$$

and we could solve

$$G(a, z_1) = z_2 \tag{7}$$

for a. A disadvantage of (7) arises if we have to deal with noisy data. Suppose z_1 and z_2 are replaced by inaccurate measurements z_1^i and z_2^i. Then (7) becomes

$$G(a, z_1^i) = z_2^i.$$

This means that we have to deal not only with perturbed data on the right-hand side but also with a perturbed operator $G^i = G(\,\cdot\,, z_1^i)$, which makes the analysis of the problem rather complicated.

A completely different, optimization theoretic approach to a problem similar to (1) is presented in Kunisch and Pan [11]. There the augmented Lagrangian method was used to obtain a formulation of the identification problem as a constrained minimization problem.

In this paper we present an approach to (1) which is very similar to the considerations which led to (4), with the difference that we use the Green's function for the Dirichlet problem on U instead of the fundamental solution $E(\boldsymbol{x}, \boldsymbol{\xi})$. As in (4) we obtain a nonlinear operator equation

$$F(a) = z \tag{8}$$

where F is an integral operator over the variable domain Ω_a and $z = z(z_1, z_2)$. This method is restricted to those domains for which a Green's function is known. The loss of generality is however rewarded by a very compact form of the resulting integral equation for a. Moreover the choices $\Omega = U$ and $a = a(\varphi)$ make it possible to use Fourier analysis, which will turn out to be a very powerful tool for analyzing and implementing the problem. We show that (8) can be reduced to an infinite system of nonlinear functionals of a, attaining prescribed values which depend on the data z_1 and z_2. We derive results on smoothness, differentiability, injectivity and ill-posedness for (8). To deal with the ill-posedness of the problem, we approximate (8) by a regularized least-squares problem, and we check conditions which ensure stability and convergence for the approximating problem. Finally we present numerical results for different interfaces and different choices of the regularization term.

2. Preliminaries. We now fix the terminology for the subsequent sections and we provide some basic technicalities which are needed henceforth. The notation (r, φ) and (ρ, ψ) is used for polar coordinates of points $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{\xi} \in \mathbb{R}^2$ respectively. $U = B(0,1)$ denotes the open unit disk in the plane $\mathbb{R}^2$. $H^1(U)$ is the Sobolev space of all functions $u \in L^2(U)$ for which $\frac{\partial u}{\partial x_i} \in L^2(U)$ for $i = 1, 2$, endowed with the inner product

$$\langle u, v\rangle_{H^1(U)} = \langle u, v\rangle_{L^2(U)} + \Big\langle \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1}\Big\rangle_{L^2(U)} + \Big\langle \frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\Big\rangle_{L^2(U)}.$$

Moreover we define the Hilbert space

$$H(\Delta, U) = \{v \in H^1(U) \,:\, \Delta v \in L^2(U)\}, \tag{9}$$

with inner product

$$\langle u, v\rangle_{H(\Delta,U)} = \langle u, v\rangle_{H^1(U)} + \langle \Delta u, \Delta v\rangle_{L^2(U)}. \tag{10}$$

Points on the boundary of U are identified with the corresponding angle in polar coordinates, i.e. we set $\partial U = \{\varphi + 2\pi\mathbb{Z} \,:\, \varphi \in \mathbb{R}\} = \mathbb{R}/2\pi\mathbb{Z} =: \mathbb{T}$. For $l \geq 0$, the Sobolev space $H^l(\mathbb{T})$ is defined by

$$H^l(\mathbb{T}) = \{f \in L^2(\mathbb{T}) \,:\, \sum_{n\in\mathbb{Z}} (1 + n^2)^l \, |\hat{f}(n)|^2 < \infty\}, \tag{11}$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi)\, e^{-in\psi}\, d\psi \tag{12}$$

denotes the Fourier transform of f. $H^l(\mathbb{T})$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{H^l(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1+n^2)^l\, \hat{f}(n)\overline{\hat{g}(n)}. \tag{13}$$

By $H^{-l}(\mathbb{T})$ ($l \geq 0$) we denote the dual of $H^l(\mathbb{T})$. With the Fourier transform defined on $H^{-l}(\mathbb{T})$ by $\hat{f}(n) := \langle f, e^{-in\varphi}\rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})}$ we find

$$\langle f, g\rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, \hat{g}(-n) \tag{14}$$

for the duality pairing $\langle\cdot,\cdot\rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})}$ and for $f \in H^{-l}(\mathbb{T})$ and $g \in H^l(\mathbb{T})$. Moreover, $H^{-l}(\mathbb{T})$ is characterized by (11) and the inner product on $H^{-l}(\mathbb{T})$ is given by (14), in both cases with l replaced by $-l$. We have

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{in\varphi} \tag{15}$$

for $f \in H^l(\mathbb{T})$, $l \in \mathbb{R}$, where the series (15) converges in $H^l(\mathbb{T})$. The embeddings $H^l(\mathbb{T}) \hookrightarrow H^m(\mathbb{T})$ and $\mathcal{C}(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ are compact and the images are dense for $l > m$ and $s > \frac{1}{2}$. We define the differential operator $D : H^l(\mathbb{T}) \to H^{l-1}(\mathbb{T})$ by

$$Df = \sum_{n\in\mathbb{Z}} in\, \hat{f}(n)\, e^{in\varphi}. \tag{16}$$

It is a bounded linear operator with $\ker D$ given by the constant functions on $\mathbb{T}$. The space $\mathcal{C}^\infty(\mathbb{T})$ of all infinitely differentiable functions on $\mathbb{T}$ is characterized by

$$\mathcal{C}^\infty(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} \,:\, |n|^k|\hat{f}(n)| \to 0 \text{ as } n \to \infty \text{ for all } k \in \mathbb{N}\}. \tag{17}$$

The preceding facts follow as a simple special case from theorems on Sobolev spaces on smooth compact manifolds as presented e.g. in Wloka [20].

The trace map $\gamma_0 : H^1(U) \to H^{\frac{1}{2}}(\mathbb{T})$ is the unique continuous extension of the operator $\gamma_0 : \mathcal{C}^\infty(\overline{U}) \to \mathcal{C}^\infty(\mathbb{T})$ defined by $\gamma_0 u = u|_\Gamma$ from $\mathcal{C}^\infty(\overline{U})$ to $H^1(U)$. γ_0 is a bounded linear surjection from $H^1(U)$ on $H^{\frac{1}{2}}(\mathbb{T})$. Analogously we define $\gamma_1 : H(\Delta, U) \to H^{-\frac{1}{2}}(\mathbb{T})$ to be the unique extension of $\gamma_1 : \mathcal{C}^\infty(\overline{U}) \to \mathcal{C}^\infty(\mathbb{T})$, where $\gamma_1 u = \frac{\partial u}{\partial \mathfrak{n}}\big|_{\mathbb{T}}$, and $\mathfrak{n}$ denotes the unit outward normal vector to the boundary $\mathbb{T}$. If there is no danger of misinterpretation, we write $\gamma_0 u = u$ and $\gamma_1 u = \frac{\partial u}{\partial \mathfrak{n}}$. (c.f. [20, p 130, Satz 8.7] and Dautray and Lions [5, p 381, Lemma 1] for these facts).

The Poisson kernel $P : U \to \mathbb{C}$ is defined by

$$P(r, \varphi) = \sum_{n\in\mathbb{Z}} r^{|n|}\, e^{in\varphi}. \tag{18}$$

The solution $u \in H^1(U)$ of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ \gamma_0 u = f \in H^{\frac{1}{2}}(\mathbb{T}) \end{cases} \tag{19}$$

is given by the Poisson integral

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\varphi-\psi)\, f(\psi)\, d\psi = \sum_{n\in\mathbb{Z}} r^{|n|}\, \hat{f}(n)\, e^{in\varphi}. \tag{20}$$

For a proof of this, observe that $u_N(r,\varphi) := \sum_{|n|\le N} r^{|n|}\, \hat{f}(n)\, e^{in\varphi} \to u(r,\varphi)$ in $H^1(U)$ and $\gamma_0 u_N = \sum_{|n|\le N} \hat{f}(n)\, e^{in\varphi} \to f$ in $H^{\frac{1}{2}}(\mathbb{T})$ as $N \to \infty$. Hence $\gamma_0 u = f$ and obviously $\Delta u = 0$ on U. This proves (20). Since $\Delta u_N = \Delta u = 0$ on U, it is also clear that $u_N \to u$ in $H(\Delta, U)$ as $N \to \infty$. By the trace theorem, we find $\gamma_1 u_N = \sum_{|n|\le N} |n|\, \hat{f}(n)\, e^{in\varphi} \to \gamma_1 u$ in $H^{-\frac{1}{2}}(\mathbb{T})$. We can therefore state the following

Proposition 2.1. *Suppose given $u \in H^1(U)$, $f \in H^{\frac{1}{2}}(\mathbb{T})$ and $g \in H^{-\frac{1}{2}}(\mathbb{T})$ fulfill*

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad U \\ \gamma_0 u = f \\ \gamma_1 u = g \end{cases}.$$

Then

$$g = \sum_{n\in\mathbb{Z}} |n|\, \hat{f}(n)\, e^{in\varphi},$$

and hence $\hat{g}(n) = |n|\hat{f}(n)$ for all $n \in \mathbb{Z}$.

Remark 2.2. For $f \in H^l(\mathbb{T})$, $l \in \mathbb{R}$ we define

$$\tilde{f} = \sum_{n\in\mathbb{Z}} -i\ \mathrm{sign}(n)\, \hat{f}(n)\, e^{in\varphi}.$$

$\tilde{f}$ is called the function conjugate to f, or the Hilbert transform of f. The Hilbert transform is a bounded linear operator on $H^l(\mathbb{T})$ with kernel given by the constants. (c.f. Edwards [7] for further details on conjugate functions.) A short computation shows that

$$\sum_{n\in\mathbb{Z}} |n|\, \hat{f}(n)\, e^{in\varphi} = (Df)^{\sim}, \tag{21}$$

where D is the (tangential) derivative operator defined in (16). With this notation we can write the conclusion of Proposition 2.1 in the form

$$\gamma_1 u = (D\, \gamma_0 u)^{\sim}. \tag{22}$$

3. An Alternative Approach. Let us return to problem (1) with $\Omega = U$ and given data $z_1 \in H^{\frac{1}{2}}(\mathbb{T})$ and $z_2 \in H^{-\frac{1}{2}}(\mathbb{T})$. We assume that $\mu_i > 0$ for $i = 1, 2$, $\mu_1 \neq \mu_2$ and $a \in H^1(\mathbb{T})$ with $0 \le a(\varphi) \le 1$ for all $\varphi \in \mathbb{T}$. Our goal is to formulate the identification problem for a as an inverse problem

$$F(a) = z,$$

where z is a function on $\mathbb{T}$ depending on the data z_1 and z_2. However, this cannot be done in a naive way since we have to take care of the fact that there is an infinite variety of possible boundary values (z_1, z_2) for a solution $u(a)$ of $\Delta u = f_a$, for one and the same a. Hence a mapping $a \mapsto u(a) \mapsto \big(\gamma_0 u(a), \gamma_1 u(a)\big)$ would be, without additional assumptions, a *multivalued function.* We therefore seek to construct an *invariant* z depending on z_1 and z_2 which is the same function *for all pairs* (z_1, z_2) within the set of all possible boundary values of $u(a)$, i.e. z should not depend on the special choice of z_1 and z_2 but only on the interface a, and we must be able to construct z from any given pair (z_1, z_2). This can be done by means of the following

PROPOSITION 3.1. *Suppose $a \in H^1(\mathbb{T})$ with $0 \le a(\varphi) \le 1$ for all $\varphi \in \mathbb{T}$ and let $u, v \in H^1(U)$ fulfill $\Delta u = \Delta v = f_a$ on U. Then*

$$\gamma_1 u - (D\gamma_0 u)^{\sim} - \frac{\mu_2}{2} = \gamma_1 v - (D\gamma_0 v)^{\sim} - \frac{\mu_2}{2} \tag{23}$$

in $H^{-\frac{1}{2}}(\mathbb{T})$.

Proof. Since $w = u - v \in H^1(U)$ and $\Delta w = 0$ on U, we can apply Proposition 2.1 to obtain

$$\big(D\gamma_0(u-v)\big)^{\sim} = \gamma_1(u-v),$$

and consequently

$$\gamma_1 u - (D\gamma_0 u)^{\sim} - \frac{\mu_2}{2} = \gamma_1 v - (D\gamma_0 v)^{\sim} - \frac{\mu_2}{2}$$

in $H^{-\frac{1}{2}}(\mathbb{T})$. $\square$

Proposition 3.1 implies that we can take *any solution* $u(a)$ of $\Delta u = f_a$ and define the mapping

$$a \mapsto F(a) := \gamma_1 u(a) - \big(D\gamma_0 u(a)\big)^{\sim} - \frac{\mu_2}{2} \tag{24}$$

since this function is independent of the special choice of $u(a)$. We can therefore express (1) in the form:

$$\textit{Solve} \quad F(a) = z_2 - (Dz_1)^{\sim} - \frac{\mu_2}{2} \quad \textit{for} \quad a. \tag{25}$$

The $\frac{\mu_2}{2}$-term on the right-hand side of (25) could of course be brought to the left and be included in the definition of F. We prefer, however, to split it from F in order to obtain a more compact representation of F, as will become apparent later.

In order to derive an integral representation for the operator F, we use the (unique) solution $u_a \in H^1(U)$ of the Dirichlet problem

$$\begin{cases} \Delta u = f_a & \text{on } U \\ \gamma_0 u = 0 \end{cases} . \tag{26}$$

From the definition (24) we obtain

$$F(a) = \gamma_1 u_a - \frac{\mu_2}{2}. \tag{27}$$

Note that the boundary values in (26) were chosen such that the $(D\gamma_0 u)^{\sim}$-term vanishes. Our aim is now to calculate the Neumann trace of u_a. We define $P_N(\rho, \psi) =$

$\sum_{|n|\leq N} \rho^{|n|} e^{in\psi}$ for $N \in \mathbb{N}$ and we use the notation $P_N(\boldsymbol{\xi},\varphi) = P_N(\rho,\varphi-\psi)$. Operators acting on functions with respect to the variable $\boldsymbol{\xi}$ are indicated by the corresponding subscript. Green's formula

$$\int_U \big[\Delta u_a(\boldsymbol{\xi})\, P_N(\boldsymbol{\xi},\varphi) - u_a(\boldsymbol{\xi})\,\Delta_\xi\, P_N(\boldsymbol{\xi},\varphi)\big]\, d\boldsymbol{\xi} = \\ \int_{\partial U} \big[\frac{\partial}{\partial \mathfrak{n}} u_a(\boldsymbol{\xi})\, P_N(\boldsymbol{\xi},\varphi) - u_a(\boldsymbol{\xi})\,\frac{\partial}{\partial \mathfrak{n}_\xi} P_N(\boldsymbol{\xi},\varphi)\big]\, d\sigma_\xi$$

together with $\Delta u_a = f_a$, $\Delta_\xi P_N(\boldsymbol{\xi},\varphi) = 0$ and $\gamma_0 u_a = 0$ implies

$$(\mu_1 - \mu_2)\int_0^{2\pi}\int_0^{a(\psi)} P_N(\rho,\phi-\psi)\,\rho\, d\rho\, d\psi + \mu_2\pi = \int_0^{2\pi} \gamma_1 u_a(\psi) \sum_{|n|\leq N} e^{in(\varphi-\psi)}\, d\psi,$$

and hence

$$\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi}\int_0^{a(\psi)} P_N(\rho,\phi-\psi)\,\rho\, d\rho\, d\psi \\ = \sum_{|n|\leq N} \frac{1}{2\pi}\int_0^{2\pi} \gamma_1 u_a(\psi)\, e^{-in\psi}\, d\psi\ e^{in\varphi} - \frac{\mu_2}{2}. \tag{28}$$

The sum on the right-hand side of (28) is a truncated Fourier series which converges to $\gamma_1 u_a$ in $H^{-\frac{1}{2}}(\mathbb{T})$. (Recall that $u_a \in H(\Delta,U)$ and by the trace theorem $\gamma_1 u_a \in H^{-\frac{1}{2}}(\mathbb{T})$.) $P_N(\boldsymbol{\xi},\varphi)$ converges with respect to ξ pointwise on U towards the Poisson kernel $P(\boldsymbol{\xi},\varphi)$, and it is bounded by the integrable function $Q_\varphi(\rho,\psi) = 4/|1-\rho\, e^{i(\varphi-\psi)}|$. It follows that limit and integral can be transposed in (28) for $N\to\infty$. We can therefore combine (28), (27) and (25) to obtain

$$F(a) = \frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi}\int_0^{a(\psi)} P(\rho,\phi-\psi)\,\rho\, d\rho\, d\psi \tag{29}$$

and

$$\boxed{\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi}\int_0^{a(\psi)} P(\rho,\varphi-\psi)\,\rho\, d\rho\, d\psi = z_2 - (Dz_1)^{\sim} - \frac{\mu_2}{2},} \tag{30}$$

as an equation for a which is equivalent to (1).

Let us return to (28). Carrying out the integration with respect to ρ and taking the limit $N\to\infty$, we find

$$F(a) = \sum_{n\in\mathbb{Z}} \frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi} \frac{a(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi\ e^{in\varphi}. \tag{31}$$

From this representation we immediately obtain

$$\big(F(a)\big)^{\widehat{}}(n) = \frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi} \frac{a(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi, \tag{32}$$

and therefore, following (30),

$$\boxed{\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi} \frac{a(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi = \hat{z}_2(n) - |n|\,\hat{z}_1(n) - \frac{\mu_2}{2}\,\delta_{0,n}} \tag{33}$$

for all $n \in \mathbb{Z}$, where $\delta_{0,n} = 1$ if $n = 0$ and $\delta_{0,n} = 0$ else. This is a system of infinitely many (nonlinear) functionals which we have to solve simultaneously for a.

REMARK 3.2. In view of generalization to domains of arbitrary geometry we shall now give a brief explanation of the above formula (30) from a different perspective. Let $G(\boldsymbol{x}, \boldsymbol{\xi})$ denote the Green's function for the Dirichlet problem (6) on U, i.e. assume that the solution of (6) can be written in the form

$$u(\boldsymbol{x}) = \int_U G(\boldsymbol{x}, \boldsymbol{\xi})\, f_a(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int_{\partial U} u(\boldsymbol{\xi})\, \frac{\partial}{\partial \mathfrak{n}_\xi} G(\boldsymbol{x}, \boldsymbol{\xi})\, d\sigma_\xi \tag{34}$$

for all $\boldsymbol{x} \in U$. We (formally) take now first order traces on both sides of (34), where we differentiate under the integrals on the right-hand side and evaluate at $\boldsymbol{x} \in \partial U$. A rigorous justification of these operations will be given at some other place. We obtain

$$\int_U \frac{\partial}{\partial \mathfrak{n}_x} G(\boldsymbol{x}, \boldsymbol{\xi})\, f_a(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{\partial}{\partial \mathfrak{n}} u(\boldsymbol{x}) - \int_{\partial U} u(\boldsymbol{\xi})\, \frac{\partial^2}{\partial \mathfrak{n}_x \partial \mathfrak{n}_\xi}\, G(\boldsymbol{x}, \boldsymbol{\xi})\, d\sigma_\xi \tag{35}$$

for all $\boldsymbol{x} \in \partial U$. It is well known (c.f. Myint-U [13, p 289]) that G is given by

$$G(r, \varphi; \rho, \psi) = \frac{1}{4\pi} \log[r^2 + \rho^2 - 2r\rho\cos(\varphi - \psi)] - \frac{1}{4\pi} \log[1 + r^2\rho^2 - 2r\rho\cos(\varphi - \psi)].$$

A short calculation then gives

$$\frac{\partial}{\partial \mathfrak{n}_\xi} G(\boldsymbol{x}, \boldsymbol{\xi})\Big|_{\xi \in \partial U} = \frac{\partial}{\partial \rho} G(r, \varphi; 1, \psi) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\varphi - \psi)} = \frac{1}{2\pi} P(r, \varphi - \psi). \tag{36}$$

For the last equality we refer to Rudin [17, p 233, (2)]. From the symmetry of G it follows that

$$\frac{\partial}{\partial \mathfrak{n}_x} G(\boldsymbol{x}, \boldsymbol{\xi})\Big|_{x \in \partial U} = \frac{1}{2\pi} P(\rho, \varphi - \psi).$$

With this (35) has the form

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 P(\rho, \varphi - \psi)\, f_a(\rho, \psi)\, \rho d\rho\, d\psi \\
&\qquad = \gamma_1 u(\varphi) - \frac{1}{2\pi} \int_0^{2\pi} \gamma_0 u(\psi)\, \frac{\partial}{\partial r} P(r, \varphi - \psi)\Big|_{r=1} d\psi \\
&\qquad = \gamma_1 u(\varphi) + \frac{1}{2\pi} \sum_{n \in \mathbb{N}} |n| \int_0^{2\pi} \gamma_0 u(\psi)\, e^{-in\psi}\, d\psi\; e^{in\varphi} \\
&\qquad = \gamma_1 u(\varphi) - (D\gamma_0 u)\;\; sptilde(\varphi).
\end{aligned} \tag{37}$$

Since the left-hand side of (37) can be written as

$$\frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} \int_0^{a(\psi)} P(\rho, \varphi - \psi)\, \rho\, d\rho\, d\psi + \frac{\mu_2}{2},$$

we have derived (30) from the relation (35). Obviously (35) also makes sense if U is replaced by some arbitrary $\Omega \subset \mathbb{R}^2$, hence we can consider (35) as a generalization of (30) to domains Ω different from the unit disk, provided that the Green's function for the Dirichlet problem on Ω is known.

4. Smoothness, Differentiability and Uniqueness. In this section we consider certain properties of the operator F as defined in (29). We investigate smoothness properties of $\big[F(a)\big](\varphi)$ and calculate the Fréchet derivative $F'(a)$. The domain for F is defined as the set

$$\operatorname{dom} F = \{a \in H^1(\mathbb{T}) \,:\, 0 \le a(\varphi) \le 1 \text{ for all } \varphi \in \mathbb{T}\}. \tag{38}$$

Frequently we shall consider the interior $(\operatorname{dom} F)^\circ$ of $\operatorname{dom} F$ in $H^1(\mathbb{T})$, given by

$$(\operatorname{dom} F)^\circ = \{a \in H^1(\mathbb{T}) \,:\, 0 < a(\varphi) < 1 \text{ for all } \varphi \in \mathbb{T}\}. \tag{39}$$

PROPOSITION 4.1. *Let F be as defined in (29) and suppose that* $\operatorname{dom} F$ *and* $(\operatorname{dom} F)^\circ$ *are given as in (38) and (39) respectively. Then*

$$F(a) \in H^l(\mathbb{T})$$

for every $a \in \operatorname{dom} F$ and $l < \frac{1}{2}$. If $a \in (\operatorname{dom} F)^\circ$, then

$$F(a) \in \mathcal{C}^\infty(\mathbb{T}).$$

Moreover, for every $a \in (\operatorname{dom} F)^\circ$, there exists a neighbourhood $U(a)$ of a in $H^1(\mathbb{T})$ such that

$$F : U(a) \subset H^1(\mathbb{T}) \to H^l(\mathbb{T})$$

is Lipschitz continuous for every $l \in \mathbb{R}$, with Lipschitz constant given by $k = |\mu_1 - \mu_2|\, \tilde{c}$, where $\tilde{c}$ is some constant depending on $U(a)$ and l.

Proof. From (14) and (29) follows

$$\|F(a)\|^2_{H^l(\mathbb{T})} = \left|\frac{\mu_1 - \mu_2}{2\pi}\right|^2 \sum_{n\in\mathbb{Z}} (1+n^2)^l \left|\int_0^{2\pi} \frac{a(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi\right|^2 \tag{40}$$

$$\le |\mu_1 - \mu_2|^2 \sum_{n\in\mathbb{Z}} \frac{(1+n^2)^l}{(|n|+2)^2}. \tag{41}$$

Since $\frac{(1+n^2)^l}{(|n|+2)^2} \le 2^l |n|^{2l-2}$ for all $l \ge 0$ and $|n| \ge 1$, the series (40) converges if $2l - 2 < -1$, i.e. if $l < \frac{1}{2}$.

Let us now suppose that $a \in (\operatorname{dom} F)^\circ$. Since $a \in H^1(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$, it follows that $\sup_{\varphi\in\mathbb{T}} |a(\varphi)| = \|a\|_\infty < 1$. We have

$$|n|^k \left|\big(F(a)\big)\widehat{\ }(n)\right| = |n|^k \left|\frac{\mu_1-\mu_2}{2\pi}\right| \left|\int_0^{2\pi} \frac{a(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi\right|$$

$$\le \frac{|n|^k}{|n|+2} |\mu_1 - \mu_2|\, \|a\|_\infty^{|n|+2} \to 0 \tag{42}$$

as $|n| \to \infty$ since $\|a\|_\infty < 1$, and hence $F(a) \in \mathcal{C}^\infty(\mathbb{T})$, due to (17).

The boundedness of the embedding $H^1(\mathbb{T}) \hookrightarrow \mathcal{C}(\mathbb{T})$ implies that we can find a neighbourhood $U(a)$ of $a \in (\operatorname{dom} F)^\circ$ in $H^1(\mathbb{T})$ such that $m := \sup\{\, \|b\|_\infty \,:\, b \in U(a)\,\} < 1$. We consider

$$F : U(a) \subset H^1(\mathbb{T}) \to \mathcal{C}^\infty(\mathbb{T}) \subset H^l(\mathbb{T})$$

with $l \in \mathbb{R}$. Then

$$
\begin{aligned}
&\|F(a_1) - F(a_2)\|_{H^l(\mathbb{T})} \\
&\quad = \Big(\sum_{n\in\mathbb{Z}} (1+n^2)^l \left| \frac{\mu_1-\mu_2}{2\pi} \int_0^{2\pi} \frac{a_1(\psi)^{|n|+2} - a_2(\psi)^{|n|+2}}{|n|+2} e^{-in\psi}\, d\psi \right|^2 \Big)^{\frac{1}{2}} \\
&\le \frac{|\mu_1-\mu_2|}{2\pi} \Big(\sum_{n\in\mathbb{Z}} \frac{(1+n^2)^l}{(|n|+2)^2} \Big(\int_0^{2\pi} |a_1(\psi) - a_2(\psi)| \Big| \sum_{k=0}^{|n|+1} a_1(\psi)^k a_2(\psi)^{|n|+1-k} \Big| \, d\psi \Big)^2 \Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
\le |\mu_1-\mu_2| \Big(\sum_{n\in\mathbb{Z}} (1+n^2)^l\, m^{2(|n|+1)} \Big)^{\frac{1}{2}} \|a_1 - a_2\|_\infty \le k\, \|a_1-a_2\|_{H^1(\mathbb{T})}, \tag{43}
$$

where

$$
k := |\mu_1-\mu_2| \Big(\sum_{n\in\mathbb{Z}} (1+n^2)^l\, m^{2(|n|+1)} \Big)^{\frac{1}{2}} c_1 < \infty
$$

with c_1 such that $\|f\|_\infty \le c_1 \|f\|_{H^1(\mathbb{T})}$ for all $f \in H^1(\mathbb{T})$. The convergence of the series in (43) is easily seen by the quotient criterion. Hence we have proved that

$$
F : U(a) \subset H^1(\mathbb{T}) \to H^l(\mathbb{T})
$$

is locally Lipschitz continuous for every $l \in \mathbb{R}$. ☐

Remark 4.2. If (25) has a solution $a \in (\text{dom } F)^\circ$, then Proposition 4.1 and (25) imply that although each of the boundary data z_2 and $(Dz_2)^\sim$ is only in $H^{-\frac{1}{2}}(\mathbb{T})$ (and, in general, will not be in any space $H^l(\mathbb{T})$ with $l > -\frac{1}{2}$) *their difference* is in $\mathcal{C}^\infty(\mathbb{T})$.

We also point out that in Proposition 4.1 we have also proved the Lipschitz continuity of F with respect to the $\mathcal{C}(\mathbb{T})$-norm on $(\text{dom } F)^\circ$.

Proposition 4.3. *$F : (\text{dom } F)^\circ \subset H^1(\mathbb{T}) \to H^l(\mathbb{T})$, with $l \in \mathbb{R}$, is Fréchet differentiable at every point $a \in (\text{dom } F)^\circ$ with derivative given by*

$$
[F'(a)\,h](\varphi) = \frac{\mu_1-\mu_2}{2\pi} \int_0^{2\pi} a(\psi)\, P(a(\psi), \varphi - \psi)\, h(\psi)\, d\psi \tag{44}
$$

for all $h \in H^1(\mathbb{T})$. The Fourier transform of $F'(a)\,h$ is given by

$$
[F'(a)\,h]^\frown(n) = \frac{\mu_1-\mu_2}{2\pi} \int_0^{2\pi} a(\psi)^{|n|+1}\, h(\psi)\, e^{-in\psi}\, d\psi \tag{45}
$$

for $h \in H^1(\mathbb{T})$ and $n \in \mathbb{N}$.

Proof. First of all we show that the Fourier transform of (44) is in fact given by (45). Since for $a \in (\text{dom } F)^\circ$

$$
\big| P\big(a(\psi), \varphi - \psi\big) \big| \le \frac{2}{1 - \|a\|_\infty} < \infty, \tag{46}
$$

we can exchange summation and integral in (44) and obtain

$$
F'(a)h = \frac{\mu_1-\mu_2}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} a(\psi)^{|n|+1}\, h(\psi)\, e^{-in\psi}\, d\psi\, e^{in\varphi}
$$

which proves that (45) is correct.

To prove Fréchet differentiability it is sufficient (c.f. Berger [1, p 68, Th 2.1.13]) to ensure that

1. $F'(a) : H^1(\mathbb{T}) \to H^l(\mathbb{T})$ is a bounded linear Operator.
2. F' depends locally Lipschitz continuously on a, i.e. there exists an $L > 0$ (depending on a) such that

$$\left\|F'(a) - F'(b)\right\|_{\mathcal{L}(H^1(\mathbb{T}),H^l(\mathbb{T}))} \leq L\|a-b\|_{H^1(\mathbb{T})} \tag{47}$$

for all b in some neighbourhood of a. (This is even more than required in [1].)

3. $F'(a)$ is the Gateaux derivative of F at a, i.e.

$$\lim_{\vartheta\to 0} \|F(a+\vartheta h) - F(a) - \vartheta\, F'(a)\, h\|_{H^l(\mathbb{T})} = 0,$$

where $\vartheta \in [0,\infty)$ for every $h \in H^l(\mathbb{T})$.

We start with 1. and show that F' as defined in (44) is a bounded linear operator from $H^1(\mathbb{T})$ into $H^l(\mathbb{T})$ for some arbitrary but fixed $l \in \mathbb{R}$. Using (45) we find

$$\begin{aligned}
\|F'(a)\, h\|_{H^l(\mathbb{T})} &= \Bigg(\sum_{n\in\mathbb{Z}}(1+n^2)^l\Big|\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi} a(\psi)^{|n|+1}\, h(\psi)\, e^{-in\psi}\, d\psi\Big|^2\Bigg)^{\frac{1}{2}}\\
&\leq (\mu_1-\mu_2)\Big(\sum_{n\in\mathbb{Z}}(1+n^2)^l\, \|a\|_\infty^{2(|n|+1)}\, |\hat{h}(n)|^2\Big)^{\frac{1}{2}}\\
&\leq c\,\Big(\sum_{n\in\mathbb{Z}}|\hat{h}(n)|^2\Big)^{\frac{1}{2}} = c\,\|h\|_{L^2(\mathbb{T})}
\end{aligned}$$

where c is chosen large enough such that $(\mu_1-\mu_2)\,(1+n^2)^{\frac{l}{2}}\,\|a\|_\infty^{|n|+1} \leq c$ for all $n \in \mathbb{Z}$. (Recall that $a \in (\text{dom } F)^\circ$ and therefore $\|a\|_\infty < 1$.) Hence $F'(a) : H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \to H^l(\mathbb{T})$ is a bounded linear operator. We note that this is a stronger result than required in 1. and that $F'(a)$ can be uniquely extended to a bounded linear operator from $L^2(\mathbb{T})$ into $H^l(\mathbb{T})$.

For the proof of 2. we choose a neighbourhood $V(a)$ of a in $H^1(\mathbb{T})$ such that $\|b\|_\infty \leq m < 1$ for all $b \in V(a)$. Then we have

$$\begin{aligned}
&\|F'(a)\, h - F'(b)\, h\|_{H^l(\mathbb{T})}\\
&= \Bigg(\sum_{n\in\mathbb{Z}}(1+n^2)^l\Big|\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi}\big(a(\psi)-b(\psi)\big)\,\Big[\sum_{k=0}^{|n|} a(\psi)^k b(\psi)^{|n|-k}\Big]\, h(\psi)\, e^{-in\psi}\, d\psi\Big|^2\Bigg)^{\frac{1}{2}}\\
&\leq (\mu_1-\mu_2)\Big(\sum_{n\in\mathbb{Z}}(1+n^2)^l(|n|+1)^2 m^{2|n|}\,|\hat{h}(n)|^2\Big)^{\frac{1}{2}}\,\|a-b\|_\infty.
\end{aligned}$$

If we choose constants c_1, c_2 such that $\|a-b\|_\infty \leq c_1\|a-b\|_{H^1(\mathbb{T})}$ for all $b \in V(a)$ and

$$(1+n^2)^{l-1}\,(|n|+1)^2 m^{2|n|} \leq c_2^2$$

for all $n \in \mathbb{Z}$, we obtain

$$\begin{aligned}
\|F'(a)\, h - F'(b)\, h\|_{H^l(\mathbb{T})} &\leq (\mu_1-\mu_2)\, c_1 c_2\, \|a-b\|_{H^1(\mathbb{T})}\Big(\sum_{n\in\mathbb{Z}}(1+n^2)\,|\hat{h}(n)|^2\Big)^{\frac{1}{2}}\\
&= L\,\|a-b\|_{H^1(\mathbb{T})}\,\|h\|_{H^1(\mathbb{T})},
\end{aligned}$$

with

$$L = (\mu_1 - \mu_2)\, c_1 c_2 \tag{48}$$

and consequently

$$\left\| F'(a) - F'(b) \right\|_{\mathcal{L}(H^1(\mathbb{T}), H^l(\mathbb{T}))} \le L \, \|a - b\|_{H^1(\mathbb{T})}$$

for all $b \in V(a)$.

It remains to prove that F' is the Gateaux derivative of F. We find

$$\begin{aligned}
&\|F(a + \vartheta h) - F(a) - \vartheta \, F'(a)\, h\|_{H^l(\mathbb{T})} \\
&= \Bigg(\sum_{n\in\mathbb{Z}} (1+n^2)^l \left| \frac{\mu_1 - \mu_2}{2\pi} \right|^2 \\
&\qquad \left| \int_0^{2\pi} \Big(\frac{\big(a(\psi) + \vartheta h(\psi)\big)^{|n|+2} - a(\psi)^{|n|+2}}{|n|+2} - \vartheta \, a(\psi)^{|n|+1} h(\psi) \Big) e^{-in\psi} \, d\psi \right|^2 \Bigg)^{\frac{1}{2}} \\
&= \frac{\mu_1 - \mu_2}{2\pi} \\
&\qquad \Bigg(\sum_{n\in\mathbb{Z}} (1+n^2)^l \left| \int_0^{2\pi} \frac{1}{|n|+2} \Big[\sum_{k=2}^{|n|+2} \tbinom{|n|+2}{k} \vartheta^k h(\psi)^k \, a(\psi)^{|n|+2-k} \Big] e^{-in\psi} \, d\psi \right|^2 \Bigg)^{\frac{1}{2}} \\
&\le \vartheta \, (\mu_1 - \mu_2) \Bigg(\sum_{n\in\mathbb{Z}} (1+n^2)^l \Big(\frac{1}{|n|+2} \, \|a + \sqrt{\vartheta} |h| \|_\infty^{|n|+2} \Big)^2 \Bigg)^{\frac{1}{2}},
\end{aligned}$$

where we used $\vartheta^{k-1} \le \vartheta^{\frac{k}{2}}$ for all $k \ge 2$ and $\vartheta \in (0,1)$. The last series above converges if $\|a + \sqrt{\vartheta}\, |h|\, \|_\infty < 1$ by the quotient criterion, hence the whole expression tends to 0 if $\vartheta \to 0$. ◻

We now address the problem of uniqueness of a solution $a \in \text{dom } F$ of $F(a) = z$ for $z \in \text{rg } F$. In Section 1. we proved that every solution $a \in \mathcal{C}(\mathbb{T})$ with $0 \le a(\varphi) \le 1$ is also a solution of the equivalent inverse potential problem (3). Hence, to prove injectivity of the operator F, it is sufficient to prove unique solvability of (3). The question of uniqueness for (3) is a well investigated topic in the literature. It was first shown by Novikov [14] that $\tilde{\Omega}$ in (2) is uniquely determined by the potential $u|_{\partial\Omega}$ in the class of star-shaped open sets with smooth boundary. Note that $\Omega_a = \{(r,\varphi) : r < a(\varphi)\}$ is star shaped with respect to the origin. A proof of a unique-solvability result for (2) in $\mathbb{R}^d$ $(d \ge 2)$ for star-shaped regions can be found in [9, p 55, Th 3.1.1]. The following theorem is an immediate consequence of this result.

THEOREM 4.4 (Uniqueness). *Suppose that $a, b \in \text{dom } F$, satisfying $a(\varphi) > 0$ or $b(\varphi) > 0$ for all $\varphi \in \mathbb{T}$ and suppose that*

$$F(a) = F(b)$$

holds in $H^l(\mathbb{T})$ with $l < \frac{1}{2}$. Then

$$a = b$$

in $H^1(\mathbb{T})$.

Next we shall prove the injectivity of F'.

THEOREM 4.5. *The operator $F'(a) : H^1(\mathbb{T}) \to H^l(\mathbb{T})$ with $l \in \mathbb{R}$ is injective for all $a \in (\mathrm{dom}\ F)^\circ$.*

Proof. Suppose that $f \in \ker F'(a) \subset H^1(\mathbb{T})$, i.e. (c.f. (45))

$$\int_0^{2\pi} a(\varphi)^{|n|+1}\, f(\varphi)\, e^{in\varphi}\, d\varphi = 0 \quad \text{for all } n \in \mathbb{Z}. \tag{49}$$

We shall first prove that (49) implies

$$\int_0^{2\pi} a(\varphi)\, f(\varphi)\, h\big(a(\varphi), \varphi\big)\, d\varphi = 0 \tag{50}$$

for all functions h which are harmonic on some neighbourhood V of $\overline{\Omega_a} = \big\{(r,\varphi) : r \le a(\varphi)\big\}$. For this purpose fix h harmonic on V and let $h_c : V \to \mathbb{R}$ be a harmonic conjugate of h, i.e. suppose h_c is chosen such that the function

$$H(x+iy) = h(x,y) + ih_c(x,y)$$

is analytic on V. Here we have changed for the moment from the usual polar coordinates (r,φ) to Cartesian coordinates (x,y). For a proof of the existence of a harmonic conjugate to a given harmonic function see Kellogg [10, p 345, Th III]. H can be approximated by polynomials

$$Q_n(z) = \sum_{k=0}^{m(n)} c_{k,n}\, z^k,$$

with $z = x+iy$, $c_{k,n} \in \mathbb{C}$ and $m(n) \in \mathbb{N}$, uniformly on $\overline{\Omega_a}$ by Runge's theorem (c.f. [17, p 271, Th 13.7]). Then

$$\mathrm{Re}\,\big(Q_n(z) - H(z)\big) = \frac{1}{2}\Big(\sum_{k=0}^{m(n)} c_{k,n}\, z^k + \overline{c_{k,n}}\,\overline{z^k}\Big) - h(z). \tag{51}$$

We set

$$d_{0,n} = \frac{1}{2}(c_{0,n} + \overline{c_{0,n}})$$

and

$$d_{k,n} \begin{cases} \frac{1}{2}c_{k,n} & \text{for } k \ge 1 \\ \frac{1}{2}\overline{c_{k,n}} & \text{for } k \le -1 \end{cases} .$$

With $z = r\, e^{in\varphi}$ we can write (51) as

$$\mathrm{Re}\,(Q_n - H) = \sum_{k=-m(n)}^{m(n)} d_{k,n}\, r^{|k|}\, e^{ik\varphi} - h(r,\varphi). \tag{52}$$

Since $|\mathrm{Re}(z)| \le |z|$ for every complex number z, (52) implies that h can be approximated by a sequence of 'harmonic polynomials'

$$q_n(r,\varphi) = \sum_{k=-m(n)}^{m(n)} d_{k,n}\, r^{|k|}\, e^{ik\varphi}$$

uniformly on $\overline{\Omega_a}$. Hence we obtain

$$\Big|\int_0^{2\pi} a(\varphi)\, f(\varphi)\, h\big(a(\varphi),\varphi\big)\, d\varphi\Big| \le \int_0^{2\pi} \big|a(\varphi)\, f(\varphi)\big|\, \big|h\big(a(\varphi),\varphi\big) - q_n\big(a(\varphi),\varphi\big)\big|\, d\varphi$$
$$+ \sum_{k=-m(n)}^{m(n)} d_{k,n} \Big| \int_0^{2\pi} a(\varphi)^{|n|+1} f(\varphi)\, e^{in\varphi}\, d\varphi\Big|. \tag{53}$$

The second term on the right-hand side of (53) is 0 due to (49) and the first one tends to 0 since $p_n \to h$ uniformly on $\partial\Omega_a$. Hence the left-hand side of (53) must be 0 and (50) is proved.

We consider now the Dirichlet problem

$$\begin{cases} \Delta w = 0 \quad \text{on } \Omega_a \\ w\big|_{\partial\Omega_a} = w\big(a(\varphi),\varphi\big) = \overline{f(\varphi)} \end{cases} . \tag{54}$$

since $a \in (\operatorname{dom} F)^\circ$, there exists $\delta > 0$ satisfying $a(\varphi) \ge \delta$ for all $\varphi \in \mathbb{T}$. This together with $f \in H^1(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$ implies that $\overline{f} : \partial\Omega_a \to \mathbb{C}$ in (54) is a continuous function on $\partial\Omega_a$. From this and from geometric properties of the boundary $\partial\Omega_a$ (c.f. [6, p 336, Th 1 and p 340, Prop 4]) the existence of a classical solution w for (54) follows, that is to say

$$\Delta w = 0 \quad \text{on } \Omega_a \qquad \text{and} \qquad \lim_{\boldsymbol{x}\to\boldsymbol{z}} w(\boldsymbol{x}) = \overline{f(\boldsymbol{z})}$$

for $\boldsymbol{x} \in \Omega_a$ and $\boldsymbol{z} \in \partial\Omega_a$. We define $w_k(r,\varphi) = w\big((1-\frac{1}{k})r,\varphi\big)$ for $k \in \mathbb{N}$. From the properties of a classical solution it follows that $w_k\big(a(\varphi),\varphi\big) \to \overline{f(\varphi)}$ pointwise on $\mathbb{T}$ and by the maximum modulus principle $w_k\big(a(\varphi),\varphi\big)$ is bounded by $|f(\varphi)|$. Hence, by the dominated convergence theorem, $w_k\big(a(\varphi),\varphi\big) \to \overline{f(\varphi)}$ in $L^1(\mathbb{T})$. Moreover, w_k is harmonic on $V_k = (1+\frac{1}{k})\Omega_a$. Employing (50) and the last arguments we obtain

$$0 = \int_0^{2\pi} a(\varphi)\, f(\varphi)\, w_k\big(a(\varphi),\varphi\big)\, d\varphi \to \int_0^{2\pi} a(\varphi)\, |f(\varphi)|^2\, d\varphi \ge 2\pi\,\delta\, \|f\|^2_{L^2(\mathbb{T})},$$

where $0 < \delta \le a(\varphi)$ for all $\varphi \in \mathbb{T}$. Hence we find $f = 0$ in $H^1(\mathbb{T})$. □

REMARK 4.6. The fact that $\int_0^{2\pi} a(\varphi)\, f(\varphi)\, a(\varphi)^{|n|}\, e^{in\varphi}\, d\varphi = 0$ for all $n \in \mathbb{Z}$ implies that $f = 0$ in $H^1(\mathbb{T})$ is equivalent to the assertion that the functions $\{a(\varphi)^{|n|} e^{in\varphi}\}_{n\in\mathbb{Z}}$ are dense in $H^1(\mathbb{T})$ (and hence in $L^2(\mathbb{T})$). It would be interesting to prove this consequence of Theorem 4.5 directly and not by solving boundary value problems which seems somewhat unnatural in this context.

5. Ill-Posedness, Least-Squares Formulation, and Regularization. In this section we consider the problem

$$F(a) = z, \tag{55}$$

within the concept of least-squares methods. Solutions of (55) do not depend continuously on the data z, not even if we replace the parameter space $H^1(\mathbb{T})$ by $L^2(\mathbb{T})$ as the following example shows. We consider the sequence of functions $\{a_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} F$ defined by

$$a_n(\varphi) = \frac{1}{2} + 2\,\epsilon\, \cos n\varphi = \frac{1}{2} + \epsilon\, (e^{in\varphi} + e^{-in\varphi})$$

with $0 < \epsilon < \frac{1}{4}$ and the constant function

$$a(\varphi) = \sqrt{\frac{1}{4} + 2\,\epsilon^2}.$$

It is easy to see that

$$\|a_n - a\|^2_{L^2(\mathbb{T})} = \Big(\frac{1}{2} - \sqrt{\frac{1}{4} + 2\,\epsilon^2}\Big)^2 + 2\,\epsilon^2 = \text{const} > 0,$$

and hence $a_n \not\to a$ for $n \to \infty$ in $L^2(\mathbb{T})$. On the other hand we find

$$\big(F(a_n) - F(a)\big)\widehat{\ }(0) = \frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} \frac{1}{2}\Big[\Big(\frac{1}{2} + \epsilon(e^{in\psi} + e^{-in\psi})\Big)^2 - \frac{1}{4} - 2\,\epsilon^2\Big]\,d\psi = 0$$

for all $n \geq 1$, and (recall that $F(a)$ is constant)

$$\big(F(a_n) - F(a)\big)\widehat{\ }(k) = \frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} \frac{1}{|k|+2}\Big(\frac{1}{2} + \epsilon\,(e^{in\psi} + e^{-in\psi})\Big)^{|k|+2} e^{-ik\psi}\,d\psi$$

for all $k \neq 0$ and $n \geq 1$. The term $\big(\frac{1}{2} + \epsilon\,(e^{in\psi} + e^{-in\psi})\big)^{|k|+2}$ can be written in the form $\sum_{m\in\mathbb{Z}} c_{m,n}\, e^{imn\psi}$ with coefficients $c_{m,n} \in \mathbb{C}$, $c_{m,n} \neq 0$ only for finitely many $m \in \mathbb{Z}$. Hence the orthogonality of the functions $e^{ik\psi}$ in $L^2(\mathbb{T})$ implies that $\big(F(a_n)\big)\widehat{\ }(k) = 0$ for all $k \in \mathbb{Z}$ with $1 \leq |k| < n$. For $|k| \geq n$ we have

$$\big|\big(F(a_n)\big)\widehat{\ }(k)\big| \leq \frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} \frac{1}{|k|+2}\Big(\frac{1}{2} + 2\,\epsilon\Big)^{|k|+2} d\psi = \frac{\mu_1 - \mu_2}{|k|+2}\, q^{|k|+2}$$

where $q = \frac{1}{2} + 2\,\epsilon < 1$. We therefore obtain

$$\begin{aligned} \|F(a_n) - F(a)\|^2_{H^l(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} (1 + k^2)^l\, \big|\big(F(a_n)\big)\widehat{\ }(k) - \big(F(a)\big)\widehat{\ }(k)\big| \\ &\leq (\mu_1 - \mu_2) \sum_{|k|\geq n} (1 + k^2)^l\, \frac{q^{|k|+2}}{|k|+2}. \end{aligned}$$

Using the quotient criterion it is easy to see that the series $\sum_{k\in\mathbb{Z}} (1 + k^2)^l\, \frac{q^{|k|+2}}{|k|+2}$ converges. Thus we have

$$\sum_{|k|\geq n} (1 + k^2)^l\, \frac{q^{|k|+2}}{|k|+2} \to 0$$

for $n \to \infty$ and consequently $F(a_n) \to F(a)$ in $H^l(\mathbb{T})$. Since the k^{th} derivative D^k is a continuous operator from $H^{k+1}(\mathbb{T})$ into $H^1(\mathbb{T})$ it follows that $D^k F(a_n) \to D^k F(a)$ in $H^1(\mathbb{T})$ and due to the compact embedding $H^1(\mathbb{T}) \hookrightarrow \mathcal{C}(\mathbb{T})$ we have

$$D^k F(a_n) \to D^k F(a)$$

uniformly on $\mathbb{T}$ for all $k \in \mathbb{N}$. From this we can conclude that $F(a_n) \to F(a)$ even in the topology on $\mathcal{C}^\infty(\mathbb{T})$.

The lack of continuous dependence of the interface a on the observation z in (55) is the reason why we consider the inverse problem of determining a from z as 'ill-posed' (in the sense of Hadamard). An efficient and well investigated tool to deal with ill-posed problems is Tikhonov's regularization method. (c.f. Engl, Kunisch and Neubauer [8], Seidman and Vogel [18], the work [12].) The method consists in approximating solutions of (55) by minimizers of the Tikhonov functional

$$\min_{a \in \mathrm{dom}\ F} \Big\{ \|F(a) - z\|^2_{H^{-\frac{1}{2}}(\mathbb{T})} + \alpha \, \|D(a - a^*)\|^2_{L^2(\mathbb{T})} \Big\}, \tag{56}$$

where $D : H^1(\mathbb{T}) \to L^2(\mathbb{T})$ and α is some (small) regularization parameter. The function a^* in the regularization term plays the rôle of an 'origin' with respect to which the semidistance $\|D(\cdot - a^*)\|_{L^2(\mathbb{T})}$ is measured. It should be chosen from some a-priori information close to the actual solution of (55).

In the following we assume that $a \in \mathrm{dom}\ F \subset H^1(\mathbb{T})$ and $z \in H^{-\frac{1}{2}}(\mathbb{T})$. Although we could use any $H^l(\mathbb{T})$-space with $l \in (-\infty, \frac{1}{2})$ as output-space (c.f. Proposition 4.1), we consider only the case where $l = -\frac{1}{2}$. This is motivated by the fact that the data on the right-hand side of (55) have the form $z = z_2 - (Dz_1)^{\sim} - \frac{\mu_2}{2}$, where the functions z_2 and $(Dz_1)^{\sim}$ are in $H^{-\frac{1}{2}}(\mathbb{T})$, only their difference is smoother, e.g. in $\mathcal{C}^\infty(\mathbb{T})$ if $a \in (\mathrm{dom}\ F)^\circ$. If $(Dz_1)^{\sim}$ and z_2 result from real data, we cannot expect the difference $z_2 - (Dz_1)^{\sim}$ to be reasonable accurate in the $\mathcal{C}^\infty(\mathbb{T})$-topology or in any $H^l(\mathbb{T})$-norm with large l but only in the topology which is 'natural' to both terms of the difference, which is $H^{-\frac{1}{2}}(\mathbb{T})$. Moreover, we assume that z is given by some inaccurate measurement $z_\delta \in H^{-\frac{1}{2}}(\mathbb{T})$ of 'unperturbed' data $z_0 \in \mathrm{rg}\ F \subset H^{-\frac{1}{2}}(\mathbb{T})$, with

$$\|z_0 - z_\delta\|_{H^{-\frac{1}{2}}(\mathbb{T})} \le \delta, \tag{57}$$

where we suppose that $a_0 \in (\mathrm{dom}\ F)^\circ$ exists, satisfying $F(a_0) = z_0$. By Theorem 4.4, a_0 is then uniquely determined by z_0.

In [12] conditions were obtained which ensure solvability, stability with respect to noisy data, and convergence of solutions of (56) — with a certain convergence rate — towards a solution of (55) if the noise level δ and the parameter α go to 0. Applying the results [12] leads to the following

THEOREM 5.1. *Assume that $F : \mathrm{dom}\ F \subset H^1(\mathbb{T}) \to H^{-\frac{1}{2}}(\mathbb{T})$ is given as in (29) and (38) respectively, and $a^* \in H^1(\mathbb{T})$, $z \in H^{-\frac{1}{2}}(\mathbb{T})$ and $\alpha > 0$. Then there exists a solution $a = a(\alpha, z) \in \mathrm{dom}\ F$ of (56). Solutions of (56) are stable in the sense that for $z_n \to z$ in $H^{-\frac{1}{2}}(\mathbb{T})$, the sequence $a_n = a(\alpha, z_n)$ has a $H^1(\mathbb{T})$-convergent subsequence, and the limit of every weakly convergent subsequence of $\{a_n\}$ is solution of*

$$\min_{a \in \mathrm{dom}\ F} \Big\{ \|F(a) - z\|^2_{H^{-\frac{1}{2}}(\mathbb{T})} + \alpha \, \|D(a - a^*)\|^2_{L^2(\mathbb{T})} \Big\}.$$

If $z_0 = F(a_0)$ and $\|z_0 - z_\delta\|_{H^{-\frac{1}{2}}(\mathbb{T})} \le \delta$ and if we choose $\alpha = \alpha(\delta) \sim \delta$ then $a(\alpha(\delta), z_\delta) \to a_0$ in $H^1(\mathbb{T})$. Moreover, if there exists $\omega \in H^{\frac{1}{2}}(\mathbb{T})$ such that

$$-\big[D^2(a_0 - a^*)\big](\psi) = \frac{\mu_1 - \mu_2}{2\pi} \, a_0(\psi) \int_0^{2\pi} P\big(a_0(\psi), \varphi - \psi\big)\, \omega(\varphi)\, d\varphi \tag{58}$$

or equivalently

$$\sum_{n \in \mathbb{Z}} n^2 \, (a_0 - a^*)\hat{\ }(n)\, e^{in\psi} = (\mu_1 - \mu_2) \sum_{n \in \mathbb{Z}} \hat{\omega}(n)\, a_0(\psi)^{|n|+1}\, e^{in\psi}. \tag{59}$$

and if $\|\omega\|_{H^{\frac{1}{2}}(\mathbb{T})}$ is bounded by a certain constant M depending on a_0, then

$$\|a(\alpha(\delta), z_\delta) - a_0\|_{H^1(\mathbb{T})} = O(\sqrt{\delta}) \quad \textit{and} \quad \|F\big[a(\alpha(\delta), z_\delta)\big] - z_0\|_{H^{-\frac{1}{2}}(\mathbb{T})} = O(\delta).$$

REMARK 5.2. A detailed description of the dependence of the constant M on the solution a_0 is given in the Appendix. Condition (58) is in fact the explicit form of $D^*D(a_0 - a^*) = \big(F'(a_0)\big)^*\omega$. We demonstrate that (58) includes certain smoothness requirements on the difference $a^* - a_0$. From (59) we see that the range of $\big(F'(a_0)\big)^*$ consists of functions which can be expressed as series in terms of $a_0(\psi)^{|n|+1}e^{in\psi}$ with coefficients c_n satisfying $\sum_{n\in\mathbb{Z}} |n|\,|c_n|^2 < \infty$. Using the constant function $\frac{1}{\mu_1-\mu_2}$ we find that a_0 is always in the range of $\big(F'(a_0)\big)^*$. It is easily seen that the series on the right-hand side of (59) converges uniformly, therefore $\big(F'(a_0)\big)^* g \in \mathcal{C}(\mathbb{T})$ for all $g \in H^{\frac{1}{2}}(\mathbb{T})$. Thus, if (58) is fulfilled, we have $D^2(a_0 - a^*) \in \mathcal{C}(\mathbb{T})$ and hence $a_0 - a^* \in \mathcal{C}^2(\mathbb{T})$. This, however, is only the minimal smoothness requirement as the following example shows. Suppose that $a_0 = q < 1$ is constant. Then

$$\big(F'(a_0)\big)^*\, g = (\mu_1 - \mu_2) \sum_{n\in\mathbb{Z}} \hat{g}(n)\, q^{|n|+1}\, e^{in\psi} \in \mathcal{C}^\infty(\mathbb{T})$$

for all $g \in H^{\frac{1}{2}}(\mathbb{T})$. Hence in this case we have rg $\big(F'(a_0)\big)^* \subset \mathcal{C}^\infty(\mathbb{T})$.

6. Numerical Results. As a starting point for the numerical implementation we use the system of equations (33). We introduce the set S_N of real-valued trigonometric polynomials of degree less or equal than $d = \frac{N-1}{2}$ given by

$$S_N = \big\{f^N = \sum_{j=-d}^{d} c_j^N\, e^{ij\varphi} \;:\; c_0^N \in \mathbb{R},\;\; c_j^N \in \mathbb{C},\;\; \overline{c_j^N} = c_{-j}^N \text{ for } j \neq 0\big\}$$

with odd $N \in \mathbb{N}$. We use the abbreviations $\varphi_k^N := \frac{2\pi}{N}k$ and $f_k^N := f^N(\varphi_k^N)$ for $f^N \in S_N$ and $k = 0, \cdots, N-1$. Given $f^N \in S_N$ there is a one-to-one correspondence between the Fourier-coefficients $\{c_0^N, \cdots, c_d^N\}$ and the interpolation points $\{f_0^N, \cdots, f_{N-1}^N\}$ via the discrete Fourier transform (c.f. [5, pp 60, 61]), hence we can use $\{f_0^N, \cdots, f_{N-1}^N\}$ as coordinate-vector for $f^N \in S_N$ instead of $\{c_{-d}^N, \cdots, c_d^N\}$. In the following it is assumed that F acts on S_N, i.e. we consider $F(a^N)$ and $\big[F(a^N)\big]\hat{}\,(n)$ for $a^N \in S_N$. Since the exact evaluation of the integral in (33) for $a^N \in S_N$ and large values of $|n|$ is very complicated, we used a numerical quadrature by means of the rectangular rule for the computation of the integral

$$\big[F(a^N)\big]\hat{}\,(n) \simeq \frac{\mu_1 - \mu_2}{N} \sum_{k=0}^{N-1} \frac{(a_k^N)^{|n|+2}}{|n|+2}\, e^{-in\varphi_k^N}. \tag{60}$$

Note that the fact that $a^N(0) = a^N(2\pi)$ implies that the rectangular rule and the trapezoid rule give the same numerical quadrature formula (60). We also mention that the dimension N of S_N also defines the mesh-width and therefore also the accuracy of the numerical integration (60).

It follows from the definition that the operator F depends linearly on the difference $\tau = \mu_1 - \mu_2$, hence (56) can be transformed into

$$\min_{a\in\text{dom } F} \tau^2 \Big[\|F_e(a) - z_e\|^2_{H^{-\frac{1}{2}}(\mathbb{T})} + \alpha_e\, \|D(a - a^*)\|^2_{L^2(\mathbb{T})}\Big],$$

where $F_e = \frac{1}{\tau}F$, $z_e = \frac{1}{\tau}z$ and $\alpha_e = \frac{\alpha}{\tau^2}$ are independent of τ. Therefore, if $\tau \neq 0$, the optimization problem (56) depends on τ only as a scaling factor. This justifies to fix

$$\mu_1 = 2 \quad \text{and} \quad \mu_2 = 1$$

for all numerical examples.

In order to obtain numerical data for the right-hand side of (33) from some known interface function a, we also used the numerical integration (60). We set

$$\hat{z}(n) \simeq \frac{\mu_1 - \mu_2}{N_d} \sum_{k=0}^{N_d-1} \frac{(a_k^{N_d})^{|n|+2}}{|n|+2} e^{-in\varphi_k^{N_d}} =: \hat{z}^{N_d}(n),$$

for all $|n| \leq \frac{N_d-1}{2}$, and $\hat{z}^{N_d}(n) = 0$ for $|n| > \frac{N_d-1}{2}$. For this (forward) calculation large values for N_d (usually $N_d = 3003$) were used to obtain reasonable accurate 'real' data, while for the calculation of $\{a_k^N\}_{k=0}^N$ *from the data* $\hat{z}^{N_d}$, mostly $N = 91$ was chosen. The accuracy of the data was then artificially destroyed by adding a random noise $z_\eta^{N_d}$, where

$$z_\eta^{N_d}(\varphi_k^{N_d}) = \eta\Big[\textsc{Rand}(0,1) - \frac{1}{2}\Big]$$

with $\textsc{Rand}(0,1)$ a random number in $[0,1]$ and the noise level η is chosen as one of the values in $\{0.05, 0.01, 0.005, 0.001, 0.0005\}$. All interface functions under consideration were centered around $r = 0.8$. This gave data z with $\|z\|_\infty \sim 0.32$. Hence the noise level varied between 0.075% and 7.5% with respect to the $\|\cdot\|_\infty$-norm. Moreover, the random function $z_\eta^{N_d}$ was the same for all interface functions.

A different possibility to construct data z from a known interface a would be to solve the forward problem

$$\Delta u = f_a$$

by means of the fundamental solution $F(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{2\pi}\log|\boldsymbol{x} - \boldsymbol{\xi}|$ and to evaluate both Dirichlet and Neumann boundary data of the solution $u(\boldsymbol{x}) = \int_\Omega F(\boldsymbol{x}, \boldsymbol{\xi})\, f_a(\boldsymbol{\xi})\, d\boldsymbol{\xi}$ at the points $\boldsymbol{x}_k = (\cos\varphi_k^{N_d}, \sin\varphi_k^{N_d})$. We could then use the discrete Fourier transform to obtain the Fourier transform of the data $z = z_2 - (Dz_1)^\sim - \frac{\mu_2}{2}$.

With these approximations for the operator F and the data z, we can enter (56) and calculate the interface a as the minimizer of the approximated Tikhonov functional

$$\begin{aligned}\mathcal{J}^N(a^N) = &\sum_{|n| \leq \frac{M-1}{2}} (1+n^2)^{-\frac{1}{2}} \Big| \frac{\mu_1 - \mu_2}{N} \sum_{k=0}^{N-1} \frac{(a_k^N)^{|n|+2}}{|n|+2} e^{-in\varphi_k^N} - \hat{z}(n) \Big|^2 \\ &+ \alpha \sum_{|n| \leq \frac{M-1}{2}} |n|^2 \left|\hat{a}^N(n) - \hat{a}^*(n)\right|^2. \end{aligned} \tag{61}$$

The first thing to come to mind is perhaps to choose $M = N$ in (61). With this choice it was observed in the numerical computations that large values of M increased the computation time considerably. We were therefore bound to fix M somewhere below 100. However, it turned out that (for fixed M) the value of N (the number of interpolation points) could be increased beyond M without increasing the computational error and without loosing too much computation time. This may be due to the fact that $[F(a)]\hat{}(n)$ decreases exponentially as $|n| \to \infty$ (c.f. (42) with $k = 0$), and hence the influence of the terms for large values of $|n|$ to the minimum of (61) (the stabilized problem!) is small. For this reason we used 'more variables than equations', in fact we fixed $N = 91$ and $M = 45$ for all computations. Moreover for $0 < n \le \frac{M-1}{2}$ we have $\left[F^N(a^N)\right]\hat{}(-n) = \overline{\left[F^N(a^N)\right]\hat{}(n)}$, $\hat{z}^N(-n) = \overline{\hat{z}^N(n)}$ and $(a^N - a^*)\hat{}(-n) = \overline{(a^N - a^*)\hat{}(n)}$, hence each of the terms

$$\left|\frac{\mu_1 - \mu_2}{N} \sum_{k=0}^{N-1} \frac{(a_k^N)^{|n|+2}}{|n|+2}\, e^{-in\varphi_k^N} - \hat{z}(n)\right|^2$$

and

$$\left|\hat{a}^N(n) - \hat{a}^*(n)\right|^2$$

occurs twice in (61). If we identify $a^N \in S^N$ with its coordinate vector $(a_0^N, \cdots a_{N-1}^N) \in \mathbb{R}^N$ and if we ignore the constraints $0 \le a^N \le 1$ we obtain for (61)

$$\begin{aligned} \min_{a^N \in \mathbb{R}^N} \Bigg[& \left|\frac{\mu_1 - \mu_2}{N} \sum_{k=0}^{N-1} \frac{|a_k^N|^2}{2} - \hat{z}(0)\right|^2 \\ & + 2 \sum_{n=1}^{\frac{M-1}{2}} (1 + n^2)^{-\frac{1}{2}} \left|\frac{\mu_1 - \mu_2}{N} \sum_{k=0}^{N-1} \frac{|a_k^N|^{n+2}}{n+2}\, e^{-in\varphi_k^N} - \hat{z}(n)\right|^2 \\ & + 2\,\alpha \sum_{n=1}^{\frac{M-1}{2}} |n|^2 \left|\hat{a}^N(n) - \hat{a}^*(n)\right|^2 \Bigg]. \end{aligned} \tag{62}$$

This optimization problem was implemented using the numerical package MATLAB and the optimization routine `leastsq` provided there, which is based on the Levenberg-Marquard algorithm for nonlinear least-squares problems. The Fourier transform $(a^N - a^*)\hat{}(n)$ occurring in the regularization term was computed by means of MATLAB's fast Fourier transform routine `fft`. Formulation (62) also allows to calculate an analytic gradient which was used in the numerical computations.

As a first example we consider the interface given by the function

$$a_c(\varphi) = 0.8 + 0.1\,\cos\varphi, \tag{63}$$

which we named 'Cosine'. The a-priori guess a^* was chosen 0. In Figure 1 the true interface a_c as a solid line, and the solution-vector $\left\{a_k^N(z,\alpha)\right\}_{k=0}^{N-1}$ of (62) as small crosses, are plotted vs the angle φ. The noise level η was chosen to be 0 and the regularization parameter $\alpha = 10^{-11}$ was (experimentally) determined such that the error

$$\|a^N(z,\alpha) - a_c\|_{L^2(\mathbb{T})} \simeq \Big(\sum_{|n| \le N} \left|(a^N(z,\alpha))\hat{}(n) - \hat{a}_c(n)\right|^2 \Big)^{\frac{1}{2}}$$

is minimal. This fact is indicated in the plot by the word 'optimal'. Also the absolute errors with respect to the $L^2(\mathbb{T})$- and the $H^1(\mathbb{T})$-norm are given. Figure 2 shows the result for the same interface function a_c with perturbed data $z^N + z_\eta^N$, where $\eta = 0.05$ (i.e. 7.5% noise). As in the previous example the regularization parameter α

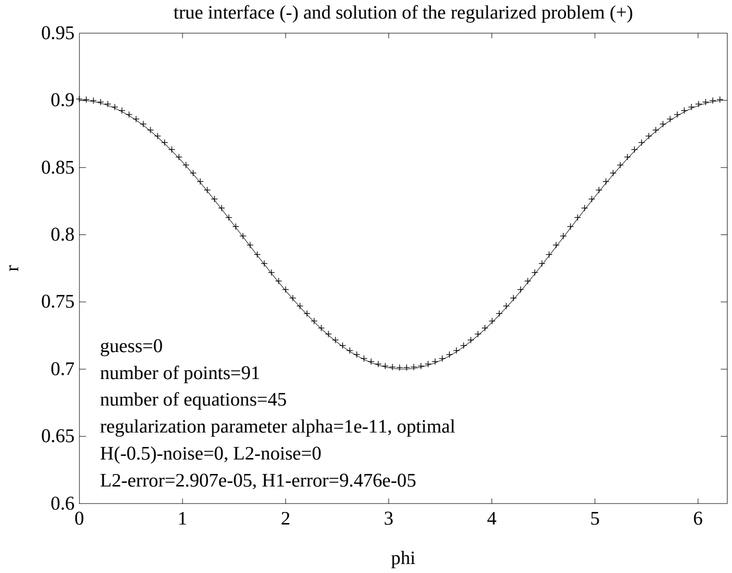

FIG. 1. *Cosine without Noise*

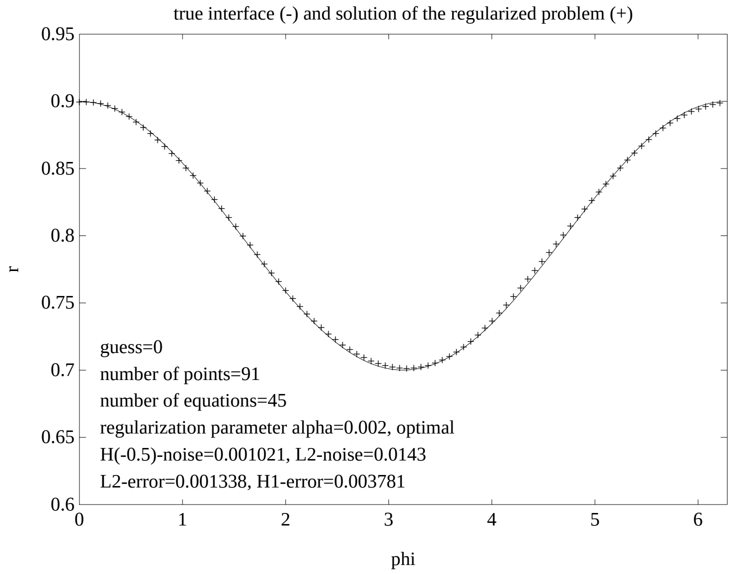

FIG. 2. *Cosine with Noise* ($\eta = 0.05$)

is optimal. Note that the optimal regularization parameter increases for growing noise-level. It is also worth mentioning that the difference between the $L^2(\mathbb{T})$-error and the $H^1(\mathbb{T})$-error is very small which can be attributed to the fact that the representation of a_c as a Fourier series has only two modes ($a_c \in S_3$). An error in the higher modes is therefore not likely to occur since in (62) they are penalized by the regularization term with comparably large α.

The four plots in Figure 3 show numerical experiments with the interface function

$$a_h(\varphi) = \begin{cases} 0.8 & \text{for } \varphi \in [0, \pi - \frac{1}{2}] \cup [\pi + \frac{1}{2}, 2\pi] \\ 0.8 - 0.27 \exp\Big(- \frac{1}{1 - 4(\varphi - \pi)^2} \Big) & \text{for } \varphi \in (\pi - \frac{1}{2}, \pi + \frac{1}{2}) \end{cases} . \tag{64}$$

We called this function 'Hole' and we investigated the behaviour of solutions of (62) for different noise-levels $\eta = 0$, $\eta = 0.0005$, $\eta = 0.005$ and $\eta = 0.05$. According to Theorem 5.1 the regularization parameter α was chosen of order $\alpha \sim \eta$; i.e. we choose $\alpha = 10^{-8}$, $\alpha = 10^{-7}$ and $\alpha = 10^{-6}$ for the values $\eta \neq 0$, where the value $\alpha = 10^{-6}$ for $\eta = 0.05$ is optimal. For this interface the optimal parameter is considerably smaller than for Cosine (with the same noise level), which is due to the fact that higher frequencies are needed to represent a_h, hence it is not good to suppress them by a strong regularization term.

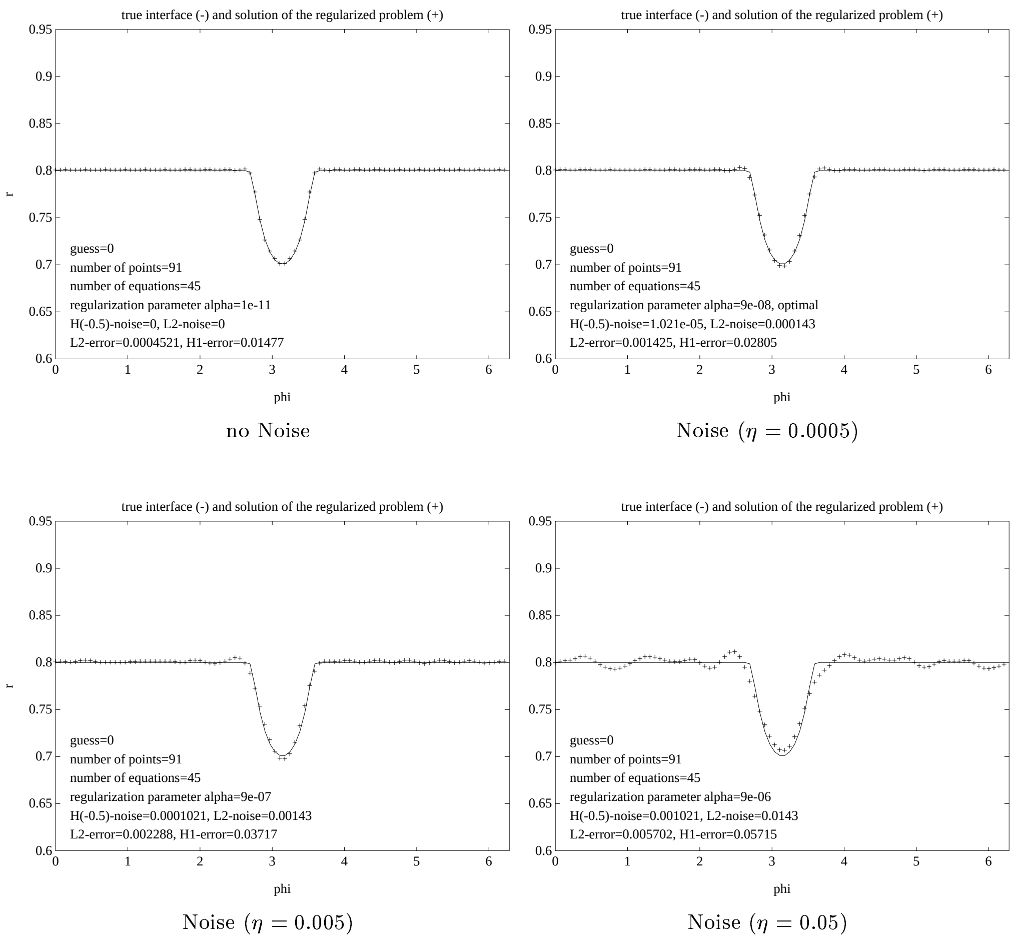

FIG. 3. *Hole with Increasing Noise and* $\alpha \sim \delta$

In Table 1 and Table 2 we listed the $H^{-\frac{1}{2}}(\mathbb{T})$-noise δ, the regularization parameter α, the $H^1(\mathbb{T})$- and the $L^2(\mathbb{T})$-error ϵ and $\tilde{\epsilon}$ of the computed interface, and the rates $\frac{\epsilon^2}{\delta}$ and $\frac{\tilde{\epsilon}^2}{\delta}$ for the preceding examples Cosine and Hole. The noise-levels δ in the $H^{-\frac{1}{2}}(\mathbb{T})$-norm correspond to values for $\eta \in \{0.05, 0.01, 0.005, 0.001, 0.0005\}$. Note that there seems to be a convergence rate $\epsilon \sim \sqrt{\delta}$ for Cosine, while for Hole we can not observe this convergence rate neither in $H^1(\mathbb{T})$ nor in $L^2(\mathbb{T})$. Most likely this can be attributed to a lack of containment of D^*Da_h in rg $\left(F'(a_h)\right)^*$ (c.f. Theorem 5.1

and Appendix, condition (R4)).

$H^{\frac{1}{2}}$-noise δ	Reg. parameter α	H^1-error ϵ	ϵ^2/δ	L^2-error $\tilde{\epsilon}$	$\tilde{\epsilon}^2/\delta$
0.001021	0.002	0.003781	0.0140	0.001338	0.0018
0.0002043	0.0004	0.001556	0.0119	0.0003865	0.00073
0.0001021	0.0002	0.0009800	0.0094	0.0002247	0.00049
0.00002043	0.00004	0.0003477	0.0059	0.00006623	0.00021
0.00001021	0.00002	0.0002492	0.0061	0.00004551	0.00020

Table 1
Error Rates for Cosine

$H^{-\frac{1}{2}}$-noise δ	Reg. parameter α	H^1-error ϵ	ϵ^2/δ	L^2-error $\tilde{\epsilon}$	$\tilde{\epsilon}^2/\delta$
0.001021	0.000009	0.05715	3.199	0.005702	0.0318
0.0002043	0.0000018	0.04042	7.997	0.002681	0.0352
0.0001021	0.0000009	0.03717	13.532	0.002288	0.0513
0.00002043	0.0000018	0.03089	46.705	0.001685	0.1390
0.00001021	0.00000009	0.02804	77.007	0.001425	0.1989

Table 2
Error Rates for Hole

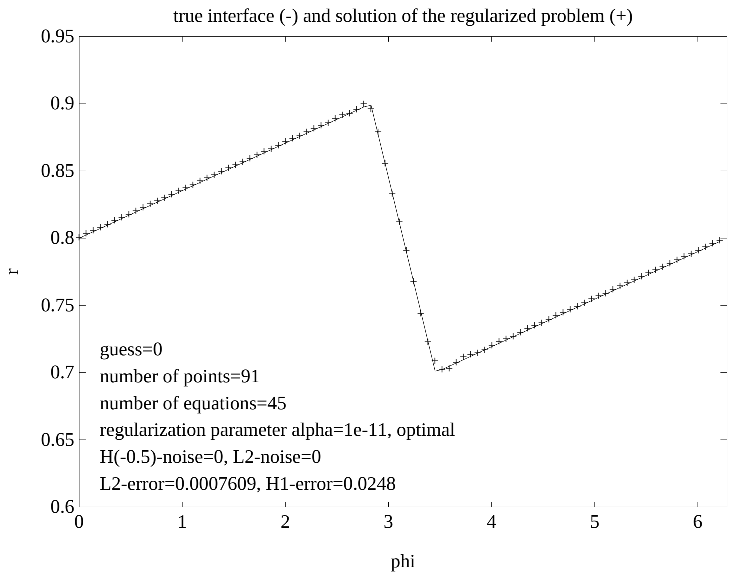

Fig. 4. *Zigzag without Noise*

For the plots in Figures 4–7 we considered the piecewise linear interface function

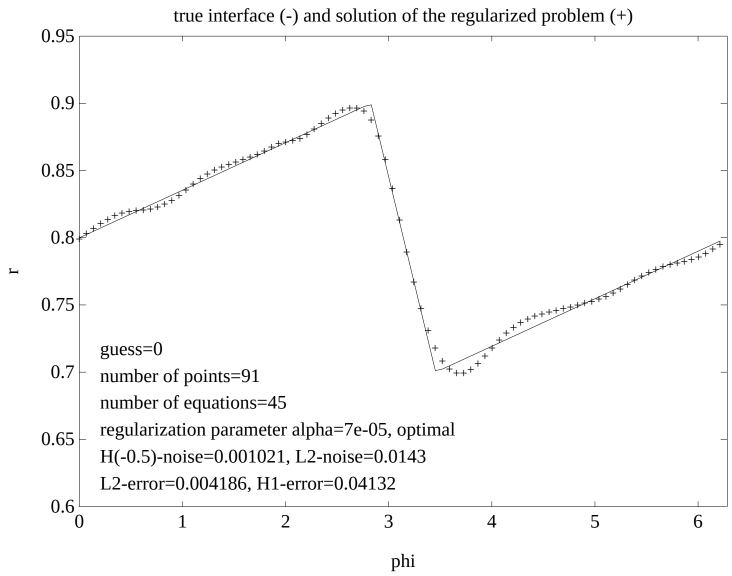

FIG. 5. *Zigzag with Noise* ($\eta = 0.05$)

'Zigzag', defined by

$$a_z(\varphi) = \begin{cases} \frac{4}{5} + \frac{1}{9\pi}\varphi & \text{for } \varphi \in [0, 0.9\pi] \\ \frac{9}{5} - \frac{\varphi}{\pi} & \text{for } \varphi \in (0.9\pi, 1.1\pi) \\ \frac{26}{45} + \frac{1}{9\pi}\varphi & \text{for } \varphi \in [1.1\pi, 2\pi) \end{cases} \tag{65}$$

for noise-levels $\eta = 0$ and $\eta = 0.05$ with optimal regularization parameter α.

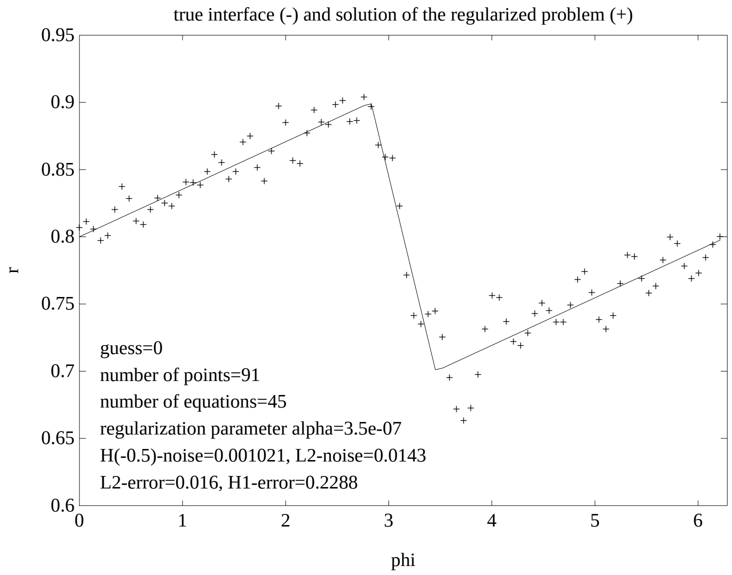

FIG. 6. *Zigzag with Regularization Parameter too Small*

Figures 6 and 7 show what can go wrong if the regularization parameter α is incorrectly chosen. The interface is again Zigzag with $\eta = 0.05$. In Figure 6, the

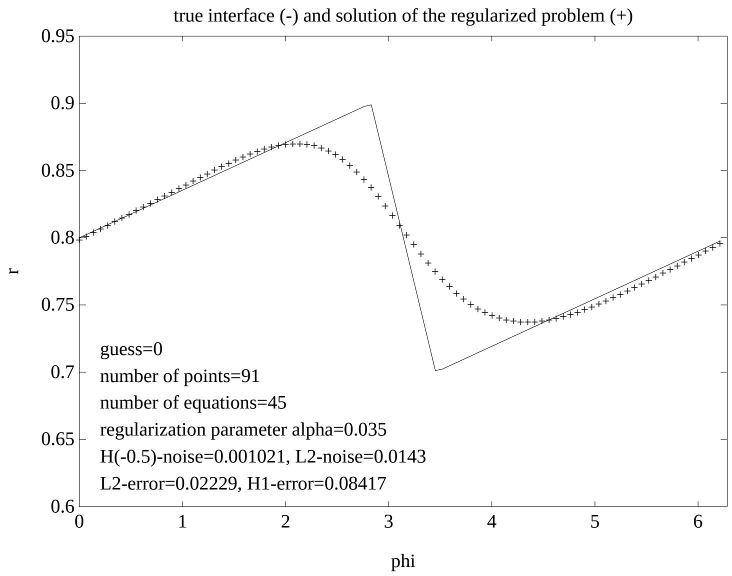

FIG. 7. *Zigzag with Regularization Parameter too Large*

parameter $\alpha = 4.5 \cdot 10^{-6}$ is too small. What should be detected as 'noise' and cut off by the regularization term is considered as 'signal' (or information) and is reflected in the highly oscillating behaviour of the computed solution.

On the other hand if α is chosen too large, part of the information is considered as noise and cut off. Figure 7 shows Zigzag with $\eta = 0.05$ and $\alpha = 0.045$ (too large!). The higher modes which are necessary to build the edges in Zigzag are killed by the dominating regularization term

Figure 8 compares different regularization terms for (62). The dependence of the $L^2(\mathbb{T})$-error on the parameter α is plotted using a logarithmic scale for α. The interface function for all curves is Zigzag with $\eta = 0.05$ and $a^* = 0$. The solid line belongs to our standard regularization term

$$2\alpha \sum_{n=1}^{\frac{M-1}{2}} |n|^2 \, |\hat{a}^N(n)|^2. \tag{66}$$

We also considered regularization with a full $H^1(\mathbb{T})$-norm with regularization term given by

$$\alpha \, |\hat{a}^N(0)|^2 + 2\alpha \sum_{n=1}^{\frac{M-1}{2}} (1+n^2) \, |\hat{a}^N(n)|^2. \tag{67}$$

The corresponding results are represented by the dashed line in Figure 8.

Moreover we considered regularization with a seminorm which penalizes only frequencies $n \geq 4$

$$2\alpha \sum_{n=4}^{\frac{M-1}{2}} |n|^2 \, |\hat{a}^N(n)|^2. \tag{68}$$

This regularization is indicated by the dotted line.

The fourth curve is generated by the '$L^2(\mathbb{T})$-seminorm'

$$2\alpha \sum_{n=1}^{\frac{M-1}{2}} |\hat{a}^N(n)|^2, \tag{69}$$

and is represented by the dashdot line.

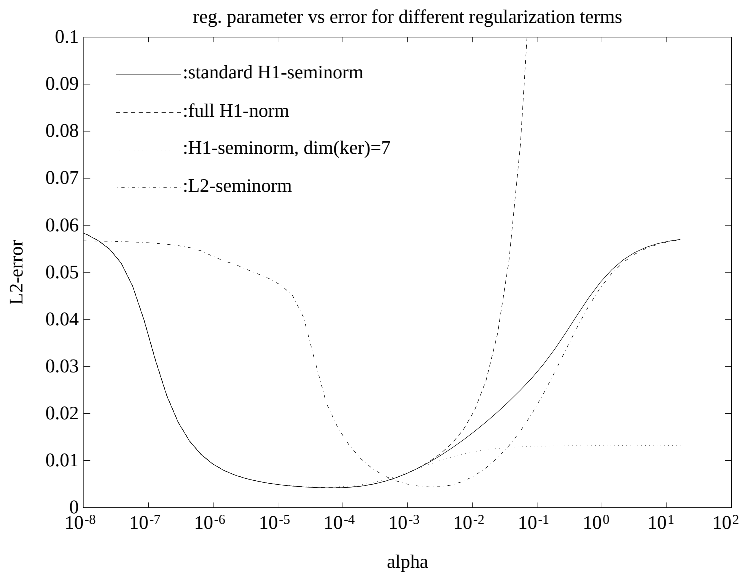

FIG. 8. *Comparison of Different Regularization Terms*

The advantage of the seminorms (66), (68) and (69) is apparent since the error does not blow up for large values of α as the full norm regularization (67) does. Figure 9 shows a typical situation for full-norm regularization with α chosen too large. In this case not only the edges vanish in Zigzag but also (due to the unfortunate choice $a^* = 0$) the whole curve is shifted downward.

A comparison of the seminorms (66) and (69) shows that the optimal error is almost the same in both cases, the major difference lies in the location of the optimum ($\alpha \sim 7 \cdot 10^{-5}$ for (66) and $\alpha \sim 3 \cdot 10^{-3}$ for (69)). Hence (69) generates highly oscillating solutions, as in Figure 6, for (small) values of α for which (66) still gives good results, while the 'overregularizing effect', as in Figure 7, occurs with (66) for parameter values for which (69) is still close to the optimum. We note that the flat left end of the curves for regularization terms (66)–(68) and (much more) (69) in Figure 8 is due to the regularizing effect of the Levenberg-Marquard algorithm which dominates the regularization term for small α. ($\alpha = 10^{-5}$ is already too small to have a significant effect for (69)!)

Finally the regularization term (68) ('$H^1(\mathbb{T})$-seminorm with large kernel') exhibits the same behaviour as (66) and (67) for small values of α. For large values of α however it produces considerably better results than for instance (66) since, due to the lack of the first three frequencies in the regularization term, the overregularization phenomenon is not as strong as for (66).

Appendix A. Appendix.

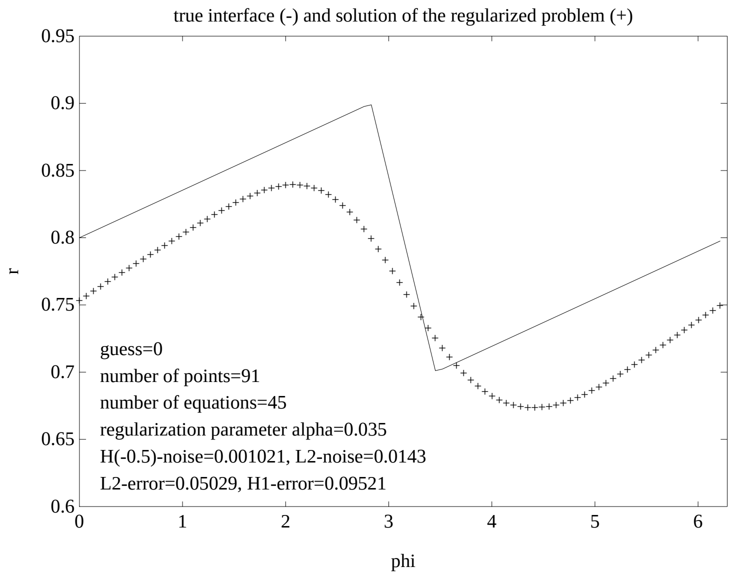

FIG. 9. *Zigzag with Full-Norm Regularization and α too Large*

Following [12, Theorems 2–4, Proposition 5 and Theorem 6], Theorem (5.1) will be proved, once the following list of conditions is satisfied:

(C1): dom $F \subset$ dom D.

(C2): $\{a_n\} \subset$ dom F; $a_n \rightharpoonup a$ in $H^1(\mathbb{T})$ and $\big(F(a_n), Da_n\big) \rightharpoonup (z, w)$ in $H^{-\frac{1}{2}}(\mathbb{T}) \times L^2(\mathbb{T})$ implies that $a \in$ dom F and $(z, w) = \big(F(a), Da\big)$. (Here '$\rightharpoonup$' denotes weak convergence.)

(C3): If $\{a_n\} \in$ dom F and $\{F(a_n), Da_n\}$ is bounded in $H^{-\frac{1}{2}}(\mathbb{T}) \times L^2(\mathbb{T})$, then $\{a_n\}$ is bounded in $H^1(\mathbb{T})$.

(C4): $\ker D$ is finite dimensional and rg D is closed in $L^2(\mathbb{T})$.

(R1): $F : H^1(\mathbb{T}) \to H^{-\frac{1}{2}}(\mathbb{T})$ is Fréchet differentiable

(R2): $\ker D \cap \ker F'(a_0) = \{0\}$ and there exists a number $\beta > 0$ such that $\|F'(a_0)\, g\|_{H^{-\frac{1}{2}}(\mathbb{T})} \geq \beta \, \|g\|_{H^1(\mathbb{T})}$ for all $g \in \ker D$.

(R3): There exists $L > 0$ and a neighbourhood $V(a_0) \subset H^1(\mathbb{T})$ of a_0 such that

$$\|F'(a_0) - F'(a)\|_{\mathcal{L}(H^1(\mathbb{T}), H^{-\frac{1}{2}}(\mathbb{T}))} \leq L \, \|a_0 - a\|_{H^1(\mathbb{T})}$$

for all $a \in V(a_0)$.

(R4): $a_0 - a^* \in$ dom D^*D and there exists $\omega \in H^{\frac{1}{2}}(\mathbb{T})$ such that $D^*D(a_0 - a^*) = [F'(a_0)]^*\omega$.

(R5): $L \, \|\omega\|_{H^{\frac{1}{2}}(\mathbb{T})} \leq \kappa^2$, where $\kappa > 0$ is a constant satisfying

$$\Big(\|F'(a_0)\, g\|^2_{H^{-\frac{1}{2}}(\mathbb{T})} + \|Dg\|^2_{L^2(\mathbb{T})}\Big)^{\frac{1}{2}} \geq \Big(\|g\|^2_{H^1('T)} + \|Dg\|^2_{L^2(\mathbb{T})}\Big)^{\frac{1}{2}} \tag{70}$$

for all $g \in H^1(\mathbb{T})$.

Conditions (C1)–(C4) are sufficient to guarantee the solvability, stability and convergence results, while for the result on the convergence rate additionally (R1)–(R5) are required to hold.

(C1) is obvious. The boundedness of $\{a_n\}$ in $H^1(\mathbb{T})$ as claimed in (C3) follows from the boundedness of dom F in $L^2(\mathbb{T})$ and the assumption that $\{Da_n\}$ is bounded in $L^2(\mathbb{T})$. (C4) follows from rg $D = \{f \in L^2(\mathbb{T}) : \int_0^{2\pi} f(\varphi)\, d\varphi = 0\}$ and $\ker D = \{f \in L^2(\mathbb{T}) : f = \text{const}\}$. In Proposition (4.3) we proved that (R1) and (R3) hold with L given by (48). Note that L depends on the choice of the neighbourhood $V(a_0) \subset (\text{dom } F)^\circ$.

ad (C2): This condition requires the weak closedness of

$$(F, D) : \text{dom } F \subset H^1(\mathbb{T}) \to H^{-\frac{1}{2}}(\mathbb{T}) \times L^2(\mathbb{T}).$$

As a first step we prove that dom F is closed with respect to the weak topology on $H^1(\mathbb{T})$. Obviously dom F is a closed subset of $\mathcal{C}(\mathbb{T})$. The embedding $H^1(\mathbb{T}) \hookrightarrow \mathcal{C}(\mathbb{T})$ is compact, hence it is continuous with respect to the $\|\cdot\|_\infty$-norm topology on $\mathcal{C}(\mathbb{T})$ and the weak topology on $H^1(\mathbb{T})$. Therefore dom F, being the preimage of a closed set in $\mathcal{C}(\mathbb{T})$, is weakly closed in $H^1(\mathbb{T})$. It follows that $a \in \text{dom } F$. The weak closedness of D implies $w = Da$, thus $y = F(a)$ is the only point which remains to be proved. Since $F(a_k) \rightharpoonup y$ in $H^{-\frac{1}{2}}(\mathbb{T})$ yields that $F(a_k) \to y$ for every $l < -\frac{1}{2}$, by the compactness of $H^{-\frac{1}{2}}(\mathbb{T}) \hookrightarrow H^l(\mathbb{T})$, it is sufficient to prove that $F(a_k) \to F(a)$ in $H^l(\mathbb{T})$ for some $l < -\frac{1}{2}$. The uniqueness of the limit then gives $y = F(a)$. We have

$$\begin{aligned}
\|F(a_k) - F(a)\|^2_{H^l(\mathbb{T})} &= \sum_{n\in\mathbb{Z}} (1+n^2)^l \left| \frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} \frac{a_k(\psi)^{|n|+2} - a(\psi)^{|n|+2}}{|n|+2}\, e^{-in\psi}\, d\psi \right|^2 \\
&\le \sum_{n\in\mathbb{Z}} (1+n^2)^l \left| \frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} |a_k(\psi) - a(\psi)|\, \frac{\sum_{j=0}^{|n|+1} |a_k(\psi)|^j |a(\psi)|^{|n|+1-j}}{|n|+2}\, d\psi \right|^2 \\
&\le \sum_{n\in\mathbb{Z}} (1+n^2)^l\, |\mu_1 - \mu_2|^2\, \|a_k - a\|^2_\infty \to 0,
\end{aligned}$$

for $k \to \infty$. Here we used the fact that $\sum_{n\in\mathbb{Z}} (1+n^2)^l$ converges for all $l < -\frac{1}{2}$ and that $a_k \rightharpoonup a$ in $H^1(\mathbb{T})$ implies that $a_k \to a$ in $\mathcal{C}(\mathbb{T})$.

ad (R2): Since $\ker F'(a_0) = \{0\}$ (Theorem 4.5) $\ker D \cap \ker F'(a_0) = \{0\}$. To obtain the required estimate on $\ker D$ suppose that some constant $c \in \ker D$ is given. Then we have

$$\|F'(a_0)\, c\|_{H^{-\frac{1}{2}}(\mathbb{T})} = \left[\sum_{n\in\mathbb{Z}} (1+n^2)^{-\frac{1}{2}} \Big(\frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} a_0(\psi)^{|n|+1}\, e^{-in\psi}\, d\psi \Big)^2 \right]^{\frac{1}{2}} |c|.$$

Since $|c| = \|c\|_{H^1(\mathbb{T})}$ and since

$$\beta = \left[\sum_{n\in\mathbb{Z}} (1+n^2)^{-\frac{1}{2}} \Big(\frac{\mu_1 - \mu_2}{2\pi} \int_0^{2\pi} a_0(\psi)^{|n|+1}\, e^{-in\psi}\, d\psi \Big)^2 \right]^{\frac{1}{2}} < \infty, \tag{71}$$

we have proved that

$$\|F'(a_0)\, c\|_{H^{-\frac{1}{2}}(\mathbb{T})} \ge \beta \|c\|_{H^1(\mathbb{T})}$$

holds for all $c \in \ker D$.

ad (R4): The adjoint $D^* : L^2(\mathbb{T}) \to H^{-1}(\mathbb{T}) = \big(H^1(\mathbb{T})\big)'$ is given by

$$D^* f = \sum_{n\in\mathbb{Z}} -in\, \hat{f}(n)\, e^{in\varphi} = -Df.$$

Therefore we have $D^*D(a_0 - a^*) = -D^2(a_0 - a^*) = \sum_{n\in\mathbb{Z}} n^2\,(a_0 - a^*)\hat{}(n)\,e^{in\varphi}$.

Since rg $F'(a_0) \subset H^l(\mathbb{T})$ for all $l \in \mathbb{R}$, we can write the action of a functional $g \in H^{\frac{1}{2}}(\mathbb{T})$ on $F'(a_0)\,h$ (with arbitrary $h \in H^1(\mathbb{T})$) in the form

$$
\begin{aligned}
&\langle F'(a_0)\,h, g\rangle_{H^{-\frac{1}{2}}(\mathbb{T}),H^{\frac{1}{2}}(\mathbb{T})} \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{\mu_1-\mu_2}{2\pi}\int_0^{2\pi} a_0(\psi)\,P\big(a_0(\psi),\varphi-\psi\big)\,h(\psi)\,d\psi\Big)g(\varphi)\,d\varphi \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} h(\psi)\,\frac{\mu_1-\mu_2}{2\pi}a_0(\psi)\int_0^{2\pi} P\big(a_0(\psi),\varphi-\psi\big)\,g(\varphi)\,d\varphi\,d\psi. \qquad (72)
\end{aligned}
$$

Here the order of integration can be reversed since we have

$$
\big|a_0(\psi)\,P\big(a_0(\psi),\varphi-\psi\big)\,h(\psi)\,g(\varphi)\big| \le \frac{2\|a_0\|_\infty\|h\|_\infty}{1-\|a_0\|_\infty}\,|g(\varphi)|
$$

for all $(\varphi,\psi) \in \mathbb{T}\times\mathbb{T}$ (c.f. (46)) and $g \in H^{\frac{1}{2}}(\mathbb{T}) \subset L^1(\mathbb{T})$. We set

$$
g^*(\psi) = \frac{\mu_1-\mu_2}{2\pi}\,a_0(\psi)\int_0^{2\pi} P\big(a_0(\psi),\varphi-\psi\big)\,g(\varphi)\,d\varphi. \qquad (73)
$$

From

$$
\big|P\big(a_0(\psi),\varphi-\psi\big)\,g(\varphi)\big| \le \frac{2}{1-\|a_0\|_\infty}\,|g(\varphi)|,
$$

it follows that the order of summation and integration in (73) can be transposed, hence we obtain

$$
\begin{aligned}
g^*(\psi) &= \frac{\mu_1-\mu_2}{2\pi}\sum_{n\in\mathbb{Z}} a_0(\psi)^{|n|+1}\int_0^{2\pi} g(\varphi)\,e^{in\varphi}\,d\varphi\;e^{-in\psi} \\
&= (\mu_1-\mu_2)\sum_{n\in\mathbb{Z}}\hat{g}(n)\,a_0(\psi)^{|n|+1}\,e^{in\psi}. \qquad (74)
\end{aligned}
$$

The following estimates

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}|\hat{g}(n)|\,\|a_0\|_\infty^{|n|+1} &\le \Big(\sum_{n\in\mathbb{Z}}(1+n^2)^{\frac{1}{2}}\,|\hat{g}(n)|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\in\mathbb{Z}}(1+n^2)^{-\frac{1}{2}}\,\|a_0\|_\infty^{2(|n|+1)}\Big)^{\frac{1}{2}} \\
&= \|g\|_{H^{\frac{1}{2}}(\mathbb{T})}\Big(\sum_{n\in\mathbb{Z}}(1+n^2)^{-\frac{1}{2}}\,\|a_0\|_\infty^{2(|n|+1)}\Big)^{\frac{1}{2}} < \infty
\end{aligned}
$$

imply the uniform convergence of the series (74) on $\mathbb{T}$. We therefore conclude that $g^* \in \mathcal{C}(\mathbb{T}) \subset H^{-1}(\mathbb{T})$. Moreover (72) implies that

$$
\langle F'(a_0)\,h, g\rangle_{H^{-\frac{1}{2}}(\mathbb{T}),H^{\frac{1}{2}}(\mathbb{T})} = \langle h, g^*\rangle_{H^1(\mathbb{T}),H^{-1}(\mathbb{T})},
$$

and consequently

$$
\begin{aligned}
\big[\big(F'(a_0)\big)^*\,g\big](\psi) &= \frac{\mu_1-\mu_2}{2\pi}\,a_0(\psi)\int_0^{2\pi} P\big(a_0(\psi),\varphi-\psi\big)\,g(\varphi)\,d\varphi \qquad (75) \\
&= (\mu_1-\mu_2)\sum_{n\in\mathbb{Z}}\hat{g}(n)\,a_0(\psi)^{|n|+1}\,e^{in\psi}. \qquad (76)
\end{aligned}
$$

Hence (R4) reads as: There exists $\omega \in H^{\frac{1}{2}}(\mathbb{T})$ satisfying

$$-\left[D^2(a_0 - a^*)\right](\psi) = \frac{\mu_1 - \mu_2}{2\pi}\, a_0(\psi) \int_0^{2\pi} P\big(a_0(\psi), \varphi - \psi\big)\, \omega(\varphi)\, d\varphi \tag{77}$$

or

$$\sum_{n\in\mathbb{Z}} n^2\, (a_0 - a^*)\hat{}\,(n)\, e^{in\psi} = (\mu_1 - \mu_2) \sum_{n\in\mathbb{Z}} \hat{\omega}(n)\, a_0(\psi)^{|n|+1}\, e^{in\psi}. \tag{78}$$

ad (R5): The existence of a constant κ satisfying (70) follows from (C4), (R2) and Proposition 5 in [12].

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