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Impedance Identification by POD Model Reduction Techniques

Impedanz-Identifikation mittels POD Modellreduktion

Stefan Volkwein and Achim Hepberger

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Proper orthogonal decomposition (POD) is a powerful technique for model reduction of nonlinear systems. It is based on a Galerkin type discretization with basis elements created from the nonlinear system itself. In this paper an identification problem arising in car vehicle accoustics is considered. To estimate the impedance from point-wise sound pressure values the identification problem is formulated in terms of an optimal control problem, which is solved by a globalized quasi-Newton method. For the efficient and fast numerical realization, a POD based model reduction is utilized.

Proper orthogonal decomposition (POD) ist eine effiziente Methode zur Modellreduktion bei nichtlinearen Systemen. Das Verfahren basiert auf einer Galerkin-artigen Diskretisierung mit Basiselementen, die durch Lösungen des nichtlinearen Systems erzeugt werden. In diesem Beitrag wird die Impedanz aus punktweisen Messungen des Schalldruckes identifiziert. Um die Impedanz von punktweisen Schaldruck-Meßwerten zu schätzen, wird das Problem als Optimalsteueraufgabe formuliert, welches mit einem gobalisiertem Quasi-Newton Verfahren gelöst wird. Für die effiziente und schnelle numerische Realisierung wird POD Modellreduktion eingesetzt.

Keywords: Model reduction, proper orthogonal decomposition, Helmholtz equation, parameter identification, quasi-Newton method.

Schlagwörter: Modellreduktion, Proper Orthogonal Decomposition, Helmholtz Gleichung, Parameter-Identifikation, Quasi-Newton Verfahren.

1 Introduction

The acoustical impedance of a component or trim part is one of its most important characteristics. The trim and its absorption behavior contributes significantly to the comfort inside the car. Therefore, correct impedance values are needed when acoustical simulations of car interior noise are carried out.

A generally used methodology to determine the acoustical impedance is to use cut-out round samples of the material in question and measure the acoustic characteristic in the impedance tube. As a result values for the normal impedance and absorption coefficients can be obtained for this material. Disadvantages of this method are that the measurement considers normal acoustic waves, only, that some materials are inappropriate for the impedance tube and that the effects of the shape of the whole part have to be neglected. Therefore efforts have been made to develop methods for impedance mearurements of entire trim parts, such as carpets, dashboards or seats.

In this paper we formulate the identification problem as an optimal control problem, where the cost functional contains a regularization term as well as a least-squares term for the difference of the mearurements and the sound pressure p computed by solving the Helmholtz equation. In contrast to [6] we identify the admittance $A \in \mathbb{C}$ instead of the impedance $Z = 1/A$. Due to the the term Ap in the Helmholtz equation (see (11b)) the obtained optimal control problem has a bilinear structure, whereas in [6] the non-linearity is of the form p/Z. If the admittance A has been estimated, then $Z = 1/A$

is an estimate for the impedance. The optimal control problem is solved by a globalized quasi-Newton method with BFGS update of the Hessian [16]. Furthermore, a discretization based on proper orthogonal decomposition (POD) is utilized for the solution of the Helmholtz equation. POD is a powerful technique for model reduction of nonlinear systems. It is based on a Galerkin type discretization with basis elements created from solutions to the Helmholtz equation itself. is successfully used in different fields including signal analysis and pattern recognition (see, e.g., [8]), fluid dynamics and coherent structures (see, e.g., [9, 20]) and more recently in control theory (see, e.g., [14]). The relationship between POD and balancing is considered in [13, 19, 23]. In contrast to POD approximations, reduced-basis element methods for parameter dependent elliptic are investigated in [1, 15, 18], for instance.

Let us mention that in [6] a standard finite element discretization for the Helmholtz equation is applied. Alternatively, the wave based technique (WBT) is used in [3, 5]. A-posteriori analysis in utilized in [21] to determine the number of POD ansatz functions in the POD Galerkin projection for an optimal control problem governed by the Helmholtz equation.

The paper is organized in the following manner: In Section 2 we review the POD method. We discuss its close relationship to singular value decomposition (SVD) and introduce a continuous variant of POD. The identification problem for the impedance and admittance is formulated in terms of an optimal control problem in Section 3. In Section 4 the reduced-order modeling is carried out illustrated by numerical examples.

2 POD method

In this section we review the POD basis. In Section 2.1 the close connection between POD and SVD is discussed; see also [11]. A continuous version of the POD method is introduced in Section 2.2. For a detailed survey on POD in the context of optimal control and related references we refer to [7], for instance.

2.1 POD and SVD

Let $y_1, \ldots, y_n \in \mathbb{R}^m$ be given vectors and set $\mathcal{V} = \operatorname{span}\{y_j\}_{j=1}^n$ with $d = \dim \mathcal{V} \le m$. On $\mathbb{R}^m$ we use the inner product

$$\langle u, v\rangle_M = u^T W v \quad \text{for } u, v \in \mathbb{R}^m$$

with a symmetric, positive definite weighting matrix $W \in \mathbb{R}^{m\times m}$ and its induced norm $\|u\|_W = (u^T W u)^{1/2}$. Then, for an arbitrary $\ell \le d$ we consider the minimization problem

$$\min_{\psi_1,\ldots,\psi_\ell} \sum_{j=1}^n \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i\rangle_W \psi_i \Big\|_W^2 \quad \text{subject to } \langle \psi_i, \psi_j\rangle_W = \delta_{ij} \text{ for } 1 \le i, j \le \ell, \tag{1}$$

where $\{\alpha_j\}_{j=1}^n$ are nonnegative weights, δ_{ij} stands for the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \ne i$. A solution to (1) is called a *POD basis of rank ℓ*.

A solution (1) is characterized by the *first-order necessary optimality conditions*

$$Y D Y^T W \psi_i = \lambda_i \psi_i, \quad 1 \le i \le \ell. \tag{2}$$

where $D \in \mathbb{R}^{n\times n}$ denotes the diagonal matrix containing the weights $\alpha_1, \ldots, \alpha_n$ as diagonal elements. It follows that

$$Y D Y^T W \psi = \sum_{j=1}^n \alpha_j \langle y_j, \psi\rangle_W \psi =: \mathcal{R}^n \psi \tag{3}$$

for any $\psi \in \mathbb{R}^m$, where the linear operator $\mathcal{R}^n$ depends on n.

Let $D^{1/2} = \operatorname{diag}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$. Since W is a symmetric and positive definite matrix, $W^{1/2}$ is also defined via the eigenvalue decomposition of W. Moreover, $W^{1/2}$ is positive definite, therefore invertible,and its inverse is denoted by $W^{-1/2}$. Setting $\hat{Y} = W^{1/2} Y D^{1/2}$ and $\hat{\psi}_i = W^{1/2}\psi_i$ for $i = 1, \ldots, \ell$ we derive from (2) the $m \times m$ symmetric eigenvalue problem

$$\hat{Y}\hat{Y}^T \hat{\psi}_i = \lambda_i \hat{\psi}_i, \quad 1 \le i \le \ell, \tag{4}$$

where we assume $\lambda_1 \ge \ldots \ge \lambda_\ell \ge \ldots \ge \lambda_d > 0$. If $\hat{\psi}_i$ is computed, we obtain ψ_i by $\psi_i = W^{-1/2}\hat{\psi}_i$. Note that if $W = I$ is the identity matrix and $\alpha_j = 1$ for $j = 1, \ldots, n$ holds, we simply have $\hat{Y} = Y$. It can be shown that the solution $\{\psi_i\}_{i=1}^{\ell}$ to the optimality conditions (2) is already a solution to (1); see, e.g., [2, 22].

Next we turn to the practical computation of the POD basis of rank ℓ, in particular, if $n < m$ holds. Due to SVD [4], we can determine $\hat{\psi}_i$ also as follows: solve the $n \times n$ symmetric eigenvalue problem

$$\hat{Y}^T\hat{Y}\hat{v}_i = \sigma_i^2 \hat{v}_i, \quad 1 \le i \le \ell, \tag{5}$$

where $\lambda_i = \sigma_i^2$ and $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_\ell > 0$ are the ℓ largest singular values of $\hat{Y}$. Then, $\hat{\psi}_i = \hat{Y}\hat{v}_i/\sigma_i$. Note that $\hat{Y}^T\hat{Y} = D^{1/2} Y^T W Y D^{1/2}$ and

$$\psi_i = W^{-1/2}\hat{\psi}_i = \frac{1}{\sigma_i} W^{-1/2}\hat{Y}\hat{v}_i = \frac{1}{\sigma_i} Y D^{1/2}\hat{v}_i. \tag{6}$$

Hence, the computation of ψ_i via (5)-(6) does not require the evaluation of $W^{1/2}$ and its inverse. In particular, if W is not a diagonal matrix, this is an advantage compared to the realization of $\hat{Y}\hat{Y}^T$ and $\psi_i = W^{-1/2}\hat{\psi}_i$.

For the application of POD to concrete problems the choice of ℓ is certainly of central importance for applying POD. It appears that no general a-priori rules are available. Rather the choice of ℓ is based on heuristic considerations combined with observing the ratio of the modeled to the total energy contained in the system Y, which is expressed by

$$\mathcal{E}(\ell) = \frac{\sum_{i=1}^{\ell} \lambda_i}{\sum_{i=1}^{d} \lambda_i} = \frac{\sum_{i=1}^{\ell} \lambda_i}{\operatorname{trace}(\hat{Y}^T\hat{Y})}.$$

Let us mention that POD is also called *Principal Component Analysis* (PCA) *Empirical Orthogonal Eigenfunctions*, and *Karhunen-Loève Decomposition*.

2.2 Continuous POD method

Suppose that $\mathcal{D} \subset \mathbb{R}^N$ is a given parameter set and the snapshot set is given by

$$\mathcal{V} = \{y(\mu) \in \mathbb{R}^m \mid \mu \in \mathcal{D}\}$$

with $d = \dim \mathcal{V} \le m$. Mathematically, y is a function defined on $\mathcal{D}$ with values in $\mathbb{R}^m$. Instead of (1) we consider the problem

$$\begin{aligned} &\min_{\psi_1,\dots,\psi_\ell} \int_{\mathcal{D}} \Big\| y(\mu) - \sum_{i=1}^{\ell} \langle y(\mu), \psi_i \rangle_W \psi_i \Big\|_W^2 \, \mathrm{d}\mu \\ &\text{subject to } \langle \psi_i, \psi_j \rangle_W = \delta_{ij} \text{ for } 1 \le i, j \le \ell. \end{aligned} \tag{7}$$

Let $\{\mu_j\}_{j=1}^n$ a set of disjunct (interpolation) points in $\mathcal{D}$ and set $y_j = y(\mu_j)$. Choosing appropriate weights α_j we can interprete the cost functional in (1) as a (e.g., trapezoidal) approximation for the integral in (7).

First-order necessary optimality conditions are given by

$$\mathcal{R}\psi_i = \lambda_i \psi_i, \quad 1 \le i \le \ell, \tag{8}$$

with the linear, symmetric and nonnegative operator

$$\mathcal{R}\psi = \int_{\mathcal{D}} \langle y(t), \psi \rangle_W \, y(t) \, \mathrm{d}\mu, \quad \psi \in \mathbb{R}^m. \tag{9}$$

In contrast to the operator $\mathcal{R}^n$ introduced in (3) the eigenvectors $\{\psi_i\}_{i=1}^\ell$ and corresponding eigenvectors $\{\lambda_i\}_{i=1}^\ell$ in (8) do not depend on n. Thus, we denote by $\{(\lambda_i^n, \psi_i^n\}_{i=1}^\ell$ the first ℓ eigenvalue-eigenvectors pairs of the operator $\mathcal{R}^n$. Utilizing perturbation theory [10] one can prove that

$$\lambda_i^n \to \lambda_i \text{ and } \psi_i^n \to \psi_i \text{ for } n \to \infty, \; i = 1, \dots, \ell \tag{10}$$

provided the weights $\{\alpha_j\}_{j=1}^n$ are chosen in such a way that they ensure the operator convergence

$$\lim_{n\to\infty} \sup_{\|\psi\|_W = 1} \|\mathcal{R}^n\psi - \mathcal{R}\psi\|_W = 0.$$

From (10) an asymptotic error analysis for POD Galerkin discretizations is obtained in [12].

3 Admittance and impedance identification

Suppose that $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is an acoustic domain with boundary $\Gamma = \partial\Omega$. For given complex impedance $Z = Z_{\Re} + \jmath Z_{\Im} \neq 0$ the admittance is defined as $A = A_{\Re} + \jmath A_{\Im} = 1/Z$, where $\jmath$ is the imaginary unit. The associated sound pressure $p : \Omega \to \mathbb{C}$, $p = p_{\Re} + \jmath p_{\Im}$, is governed by the Helmholtz equation

$$\Delta p(\mathbf{x}) + k^2 p(\mathbf{x}) = -\jmath\omega\varrho_\circ q \delta_{\mathbf{x}_q} \quad \text{for all } \mathbf{x} \in \Omega, \tag{11a}$$

where $\mathbf{x} = (x, y)$ for $d = 2$ or $\mathbf{x} = (x, y, z)$ for $d = 3$ hold, $c = 343.799\big[\frac{m}{s}\big]$, denotes the sound of speed, $\varrho_\circ = 1.19985\big[\frac{kg}{m^3}\big]$ is an ambient density, $f \ge 50[Hz]$ stands for the frequency, $\omega = 2\pi f$ is the circle frequency and $k = \frac{\omega}{c}$ is the wave number. The point $\mathbf{x}_q \in \Omega$ is the position of the acoustic source q (e.g., a loud speaker) and $\delta_{\mathbf{x}_q}$ is the Dirac delta distribution satisfying

$$\langle \delta_{\mathbf{x}_q}, \varphi \rangle = \varphi(\mathbf{x}_q) \quad \text{for any continuous } \varphi : \Omega \to \mathbb{C}.$$

Furthermore, Δ is the Laplace operator. The boundary Γ is split into two measurable disjunct parts Γ_R and Γ_N. On Γ_R we impose a normal impedance boundary

$$\frac{\jmath}{\varrho_\circ\omega} \frac{\partial p(\mathbf{x})}{\partial n} = \frac{p(\mathbf{x})}{Z} = Ap(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Gamma_R, \tag{11b}$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the outward normal direction. All other parts on the boundary are assumed to be perfectly rigid, i.e.,

$$\frac{\jmath}{\varrho_\circ\omega} \frac{\partial p(\mathbf{x})}{\partial n} = 0 \quad \text{for all } \mathbf{x} \in \Gamma_N. \tag{11c}$$

We suppose that for any $A \in \mathbb{C}$ and for all f in the frequency range under consideration, (11) admits a unique solution. Due to Fredholm theory [17] we can ensure existence of a soluion provided k^2 is not an eigenvalue of $-\Delta$ considered on Ω with Neumann and Robin boundary conditions on Γ_N respectively Γ_R.

Let p_i^{m}, $i = 1, \dots, N$, be given measurements for the sound pressure at N different observation points $\mathbf{x}_i \in \Omega \cup \Gamma_N$, $1 \le i \le N$. The goal of the parameter identification is to find the complex-valued admittance $A = 1/Z$ such that the difference between the solution p to (11) evaluated at the points $\mathbf{x}_i$, $1 \le i \le N$, and the corresponding measurements p_i^{m} is minimized. Therefore, we introduce the quadratic cost functional

$$J(p, A) = \frac{\gamma}{2} \sum_{i=1}^{N} \big|p(\mathbf{x}_i) - p_i^{\mathrm{m}}\big|^2 + \frac{\sigma}{2} \big|A - A_\circ\big|^2, \tag{12}$$

where $\gamma \ge 0$ is a weighting parameter, $\sigma > 0$ a regularization parameter and $A_\circ \in \mathbb{C}$ is a chosen nominal or estimated value for the admittance. Furthermore, $|A| = (A\overline{A})^{1/2}$ stands for the complex absolute value and the complex conjugate of A is denoted by $\overline{A}$. The parameter identification can be formulated in terms of an optimal control problem:

$$\min_{(p,A)} J(p, A) \quad \text{subject to} \quad (p, A) \text{ solves (11).} \tag{\textbf{P}}$$

Let us mention that (**P**) is a constrained, non-convex optimization problem, where the solution space for the sound pressure is a function space, i.e., an infinite-dimensional space, which has to be discretized for numerical purposes. Throughout the paper we suppose that (**P**) admits a local solution (p^*, A^*). This solution is characterized by first-order necessary optimality conditions in such a way that there exists a Lagrange multiplier

$\lambda^* : \Omega \to \mathbb{C}$ satisfying the following *adjoint* or *dual problem*:

$$\Delta\lambda^*(\mathbf{x}) + k^2\lambda^*(\mathbf{x}) = \gamma \sum_{i=1}^{N} (p_i^{\mathrm{m}} - p^*(\mathbf{x}_i))\delta_{\mathbf{x}_i} \tag{13a}$$

for all $\mathbf{x} \in \Omega$ together with the boundary conditions

$$\frac{\jmath}{\varrho_\circ\omega} \frac{\partial\lambda^*(\mathbf{x})}{\partial n} + \overline{A^*}\lambda^*(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in \Gamma_R, \tag{13b}$$

$$\frac{\jmath}{\varrho_\circ\omega} \frac{\partial\lambda^*(\mathbf{x})}{\partial n} = 0 \quad \text{for all } \mathbf{x} \in \Gamma_N. \tag{13c}$$

In (13a) we denote by $\delta_{\mathbf{x}_i}$, $1 \le i \le N$, the Dirac delta distributions satisfying

$$\langle \delta_{\mathbf{x}_i}, \varphi \rangle = \varphi(\mathbf{x}_i) \quad \text{for any continuous } \varphi : \Omega \cup \Gamma_N \to \mathbb{C}.$$

Furthermore, the following relation holds

$$\sigma\big(A^* - A_\circ\big) - \jmath\varrho_\circ\omega \int_{\Gamma_R} \lambda^*(\mathbf{x})\overline{p^*(\mathbf{x})}\,\mathrm{d}\mathbf{x} = 0. \tag{14}$$

Let $p(A)$ denote the (unique) solution to (11) for given admittance A; in particular, $p^* = p(A^*)$ holds. Introducing the so-called cost functional

$$\hat{J}(A) = J(p(A), A) \quad \text{for } A \in \mathbb{C},$$

we can replace ($\mathbf{P}$) by the unconstrained optimization problem

$$\min_{A\in\mathbb{C}} \hat{J}(A). \tag{$\hat{\mathbf{P}}$}$$

Note that

$$\hat{J}'(A^*) = J_p(p^*, A^*)p'(A^*)A + J_A(p^*, A^*)A$$

for any directon $A \in \mathbb{C}$, where J_p and $_A$ denote the partial dersivative of J with resect to p and A, respectively. A first-order necessary optimality condition for ($\hat{\mathbf{P}}$) is $\hat{J}'(A^*) = 0$, where

$$\hat{J}'(A^*) = \sigma\big(A^* - A_\circ\big) - \jmath\varrho_\circ\omega \int_{\Gamma_R} \lambda^*(\mathbf{x})\overline{p^*(\mathbf{x})}\,\mathrm{d}\mathbf{x} \tag{15}$$

and λ^* solves (13); compare [6] and (14). Since $A \in \mathbb{C}$ holds, ($\hat{\mathbf{P}}$) is considered as an optimization problem in $\mathbb{R}^2$, i.e., in the real and imaginary part of A.

The quasi-Newton method with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update for the 2×2 Hessian and a Wolfe-Powell line search globalization is utilized to solve ($\hat{\mathbf{P}}$) numerically; see, e.g., [16].

Remark 1. *In [6] the impedance $Z = 1/A$ is identified from point-wise measurements of the sound pressure. In this case the reduced cost functional is of the form $\hat{J}(Z) = J(p(Z), Z)$ and its gradient at an optimal solution Z^* has the form*

$$\hat{J}'(Z^*) = \sigma\big(Z^* - Z_\circ\big) + \frac{\jmath\varrho_\circ\omega}{(\overline{Z^*})^2} \int_{\Gamma_R} \lambda^*(\mathbf{x})\overline{p^*(\mathbf{x})}\,\mathrm{d}\mathbf{x}.$$

with $p^* = p(Z^*)$ and λ^* solves

$$\begin{aligned} \Delta\lambda^* + k^2\lambda^* &= \gamma \sum_{i=1}^{N} (p_i^{\mathrm{m}} - p^*(\mathbf{x}_i))\delta_{\mathbf{x}_i} && \text{in } \Omega, \\ \frac{\jmath}{\varrho_\circ\omega} \frac{\partial\lambda^*(\mathbf{x})}{\partial n} &= -\frac{\lambda^*(\mathbf{x})}{\overline{Z^*}} && \text{in } \Gamma_R, \\ \frac{\jmath}{\varrho_\circ\omega} \frac{\partial\lambda^*(\mathbf{x})}{\partial n} &= 0 && \text{in } \Gamma_N. \end{aligned}$$

It turns out that due to the non-linear term $1/(\overline{Z^*})^2$ the quasi-Newton method requires more iterations (i.e., more CPU-time) to reach convergence than identifying the admittance.

The optimization method is described in Algorithm 1.

Algorithm 1 Quasi-Newton method.

1: Choose starting values $A^0 = (A^0_{\Re}, A^0_{\Im}) \in \mathbb{R}^2$ as well as stopping tolerance $\varepsilon > 0$, set $H^0 = \sigma I \in \mathbb{R}^{2\times 2}$ and $i = 0$.
2: **repeat**
3: Compute $\hat{J}(A^i)$ and $\hat{J}'(A^i) \in \mathbb{R}^2$.
4: Solve for $\delta A^i \in \mathbb{R}^2$ the quasi-Newton system

$$H^i \delta A^i = -\hat{J}'(A^i).$$

5: Determine a stepsize parameter $s > 0$ by the Wolfe-Powell linesearch.
6: Update the admittance $A^{i+1} = A^i + s\delta A^i$.
7: Compute new Hessian by te BFGS formula.
8: Set $i = i + 1$.
9: **until** $|\hat{J}'(A^i)| < \varepsilon$

4 Reduced-order modeling (ROM)

In this section we describe the POD reduced-order approach.

4.1 POD basis for the Helmholtz equation

The acoustic domain is plotted in Figure 1. The impedance boundary is $\Gamma_R = \{(x, 0)\,|\,0.5 \le x \le 2.5\}$ and the loud speaker is located in $\mathbf{x}_q = (0.21, 1.28) \in \Omega$. We apply a standard piecewise linear finite element (FE) discretization with $m = 2108$ degrees of freedom. Let $\{\varphi_i\}_{i=1}^m$ denote the piecewise linear finite element ansatz functions. Then, a finite element function is described by a coefficient vector in $\mathbb{R}^m$ containing the values of the finite element function at each grid points.

We introduce the mass martix $M \in \mathbb{R}^{m\times m}$ with the elements

$$M_{ij} = \int_\Omega \varphi_j(\mathbf{x})\varphi_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad 1 \le i, j \le m,$$

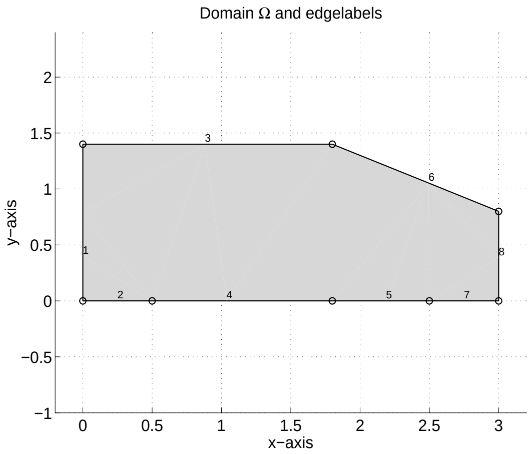

Figure 1: Acoustic domain $\Omega \subset \mathbb{R}^2$, where the impedance boundary Γ_R consists in parts 4 and 5 of Γ.

and the stiffness matrix $S \in \mathbb{R}^{m\times m}$ with the elements

$$S_{ij} = \int_\Omega \sum_{k=1}^{m} \frac{\partial \varphi_j(\mathbf{x})}{\partial x_k} \frac{\partial \varphi_i(\mathbf{x})}{\partial x_k} \,\mathrm{d}\mathbf{x} + M_{ij}, \quad 1 \le i,j \le m.$$

Note that both M and S are symmetric and positive definite. Furthermore, the L^2- and H^1-inner product of two FE functions

$$\varphi(\mathbf{x}) = \sum_{i=1}^{m} c_i \varphi_i(\mathbf{x}), \quad \tilde{\varphi}(\mathbf{x}) = \sum_{i=1}^{m} \tilde{c}_i \varphi_i(\mathbf{x})$$

can be described by

$$\begin{aligned}\langle \varphi, \tilde{\varphi}\rangle_{L^2} &= \int_\Omega \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{i,j=1}^{m} c_i M_{ij} \tilde{c}_j = \underline{c}^T M \underline{\tilde{c}},\\ \langle \varphi, \tilde{\varphi}\rangle_{H^1} &= \langle \varphi, \tilde{\varphi}\rangle_{L^2} + \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \tilde{\varphi}}{\partial x}\right\rangle_{L^2} + \left\langle \frac{\partial \varphi}{\partial y}, \frac{\partial \tilde{\varphi}}{\partial y}\right\rangle_{L^2}\\ &= \underline{c}^T S \underline{\tilde{c}},\end{aligned}$$

where $\underline{c} = (c_1, \ldots, c_L)^T$, $\underline{\tilde{c}} = (\tilde{c}_1, \ldots, \tilde{c}_L)^T$.

We consider a fire resistant form, Melamin $50mm$, as a damping material. The complex impedance in normal direction of this material has been measured with an impedance tube. The measurement data has to be interpolated and smoothed for a quantitative validation. In Figure 2 the impedance and corresponding admittance values are plotted.

Next we compute the FE solution $p^h(f_j) = p^h_{\Re}(f_j) + \jmath p^h_{\Im}(f_j)$ to (11) for the frequencies $f_j = 199 + j$, $j = 1, \ldots, n$ with $n = 251$, and corresponding admittance $A(f_j)$, where

$$p^h_{\Re}(f_j) = \sum_{i=1}^{m} \alpha_i^j \varphi_i, \quad p^h_{\Im}(f_j) = \sum_{i=1}^{m} \beta_i^j \varphi_i, \quad \alpha_i^j, \beta_i^j \in \mathbb{R}$$

and $\underline{p}^j_{\Re} = (\alpha_1^j, \ldots, \alpha_m^j)^T$, $\underline{p}^j_{\Im} = (\beta_1^j, \ldots, \beta_m^j)^T \in \mathbb{R}^m$ are the real respectively imaginary parts of the FE coefficients for p^j_h. Note that also ω and k depend on f. In the context of Section 2.1 we choose the snapshots

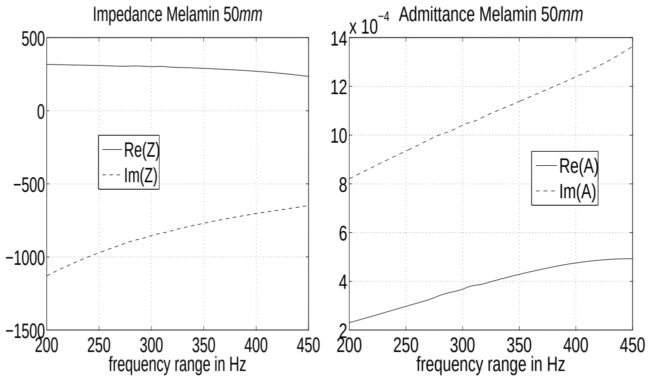

Figure 2: Impedance and admittance values for Melamin $50mm$ in the frequency range from 200 to $450Hz$.

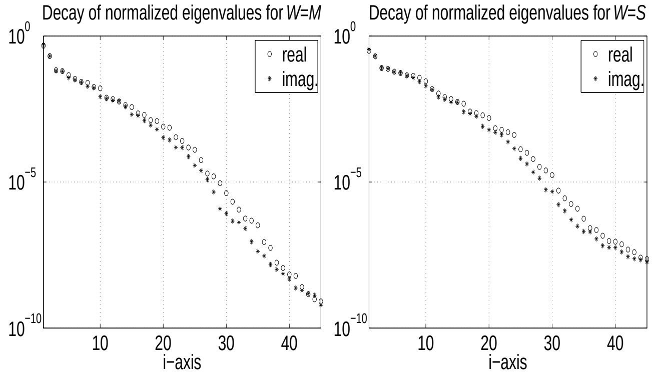

Figure 3: Decay of 45 largest normalized eigenvalues $\lambda_i / \sum_{i=1}^{n} \lambda_i$ or $W = M$ (left plot) and $W = S$ (right plot).

$y_j = \underline{p}^j_{\Re}$, $1 \le j \le n$. For the weighting matrix we study the reasonable choices $W = M$ (L^2-inner product) or $W = S$ (H^1-inner product). Moreover, we choose the weights $\alpha_j = 1$ for all $j = 1, \ldots, n$. Then, we compute the POD basis $\{\psi_i\}_{i=1}^{\ell}$ of rank ℓ for the approximation of the real part of the sound pressure. For the imaginary part we proceed analogously and determine a POD basis $\{\phi_i\}_{i=1}^{\ell}$. In Figure 3 the decay of the largest 45 normalized eigenvalues are presented.

Since the decay of the eigenvalues for the real is similar to the one for the imaginary part, we choose the same number of POD ansatz functions for the real and the imaginary parts which is not necessary. We summarize the procedure in Algorithm 2.

4.2 ROM for the Helmholtz equation

Next we utilize the computed POD basis functions to derive a POD Galerkin scheme for (11).

Let $\chi_{ik} = \psi_i + \jmath\phi_k : \Omega \to \mathbb{C}$ for $1 \le j,k \le \ell$. Then, we make the ansatz

$$p^\ell(\mathbf{x}) = \sum_{l=1}^{\ell} \alpha_{\ell,l}\psi_l(\mathbf{x}) + \jmath\beta_{\ell,l}\phi_l(\mathbf{x}), \quad \alpha_{\ell,l},\, \beta_{\ell,l} \in \mathbb{R},$$

Algorithm 2 POD basis for the Helmholtz equation.

1: Choose a weighting matrix W (e.g., $W = M$ or S).
2: Fix the number ℓ of POD basis functions for the real as well as for the imaginary part.
3: Choose a reference admittance $A^r(f)$ or impedance $Z^r(f)$ over the frequency range 200 to $450Hz$.
4: **for** $f = 200$ to 450 **do**
5: Compute the FE solution $p^h(f)$ to (11) with $A = A^r(f)$ or $Z = Z^r(f)$.
6: **end for**
6: Compute a POD basis $\{\psi_i\}_{i=1}^{\ell}$ using the snapshots $\{p^h_{\Re}(f)\}_{f=200}^{450}$.
6: Determine a POD basis $\{\phi_i\}_{i=1}^{\ell}$ using the snapshots $\{p^h_{\Im}(f)\}_{f=200}^{450}$.

ℓ	$E_{\Re}(L^2)$	$E_{\Im}(L^2)$	$E_{\Re}(H^1)$	$E_{\Im}(H^1)$	CPU
25	56.64	38.95	56.67	37.63	3.7e-4
28	14.59	12.58	14.28	12.65	2.0e-4
29	9.46	8.25	9.40	8.35	2.9e-4
30	2.85	3.21	2.82	3.28	4.3e-4
35	0.83	0.58	0.86	0.64	3.7e-4
40	0.05	0.28	0.13	0.32	5.1e-4
45	0.03	0.24	0.10	0.28	6.9e-4

Table 1: Relative errors in the reduced-order model compared to the finite element model and CPU times summarized over all $n = 251$ frequencies (times for POD solvers divided by times for FE solver).

$p^\ell = p^\ell_{\Re} + \jmath p^\ell_{\Im}$, multiply (11a) by the test functions $\psi_i + \jmath\phi_k$, $i, k = 1, \ldots, \ell$ and integrate over Ω. Integration by part and the boundary conditions (11b)-(11c) we end up with a linear system in the 2ℓ real coefficients $\alpha_{\ell,l}$, $\beta_{\ell,l}$, $1 \le l \le \ell$, whereas in the FE case we have a linear system of the size $2m = 4216 \gg 2\ell$.

To measure the error between the POD and the FE model we introduce the quantities

$$E_{\Re}(X) = \frac{1}{n}\sum_{j=1}^{n}\sqrt{\frac{\|p^\ell_{\Re}(f_j) - p^h_{\Re}(f_j)\|_X^2}{\|p^h_{\Re}(f_j)\|_X^2}} \cdot 100\%,$$
$$E_{\Im}(X) = \frac{1}{n}\sum_{j=1}^{n}\sqrt{\frac{\|p^\ell_{\Im}(f_j) - p^h_{\Im}(f_j)\|_X^2}{\|p^h_{\Im}(f_j)\|_X^2}} \cdot 100\%,$$

where X stands for $L^2(\Omega)$ or $H^1(\Omega)$ (shortly, L^2 respectively H^1). The obtained quantities are presented in Table 1. As expected, the relative errors decrease with increasing number ℓ of POD basis functions in the Galerkin ansatz.

4.3 ROM for the identification problem

Now we turn to the identification problem. In Figure 4 the $N = 6$ measurement points $\mathbf{x}_i$ for the sound pressure are plotted. The goal is to identify the admittance A^{id} for Melamin $50mm$ in the frequency range from 200 to $450Hz$ (see Figure 2, right plot) from $N = 6$ pointwise sound pressure values p_i^m (compare (12)) associa-

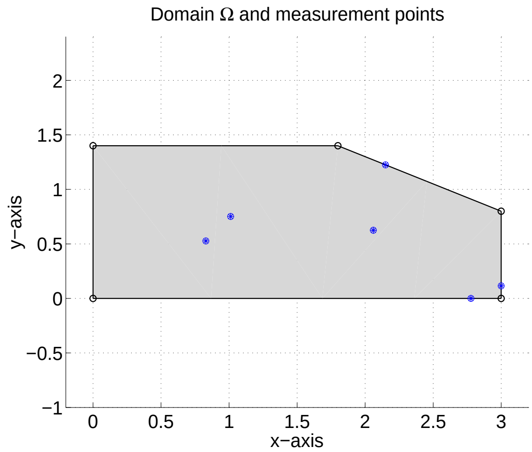

Figure 4: Acoustic domain $\Omega \subset \mathbb{R}^2$ with $N = 6$ measurement points for the sound pressure.

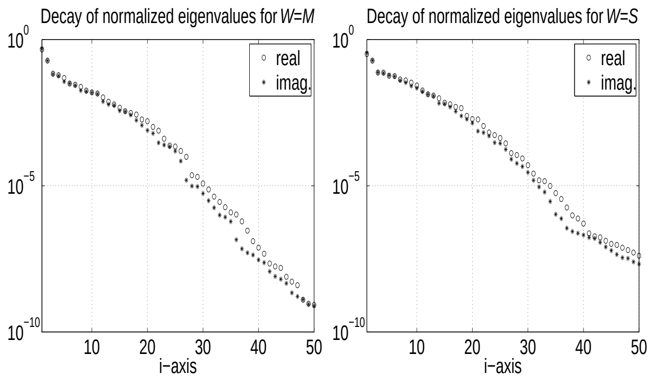

Figure 5: Decay of 50 largest normalized eigenvalues $\lambda_i / \sum_{i=1}^{n}\lambda_i$ or $W = M$ (left plot) and $W = M+S$ (right plot).

ted with the FE solution to (11) using $A = A^{id}$. The POD basis is computed from FE solutions to (11) for the frequencies $f = 200, 201, \ldots, 450$, where for every f we vary the admittance $A = A_{\Re} + \jmath A_{\Im}$ as follows

$$A_{\Re} = 2\cdot 10^{-4}, 4\cdot 10^{-4}, 6\cdot 10^{-4},$$
$$A_{\Im} = 6\cdot 10^{-4}, 10^{-3}, 1.6\cdot 10^{-3},$$

i.e., we have $251 \times 9 = 2259$ snapshots. From Figure 2 we observe that $2\cdot 10^{-4} \le A^{id}_{\Re} \le 6\cdot 10^{-4}$ and $6\cdot 10^{-4} \le A^{id}_{\Im} \le 1.6\cdot 10^{-3}$ hold. For the computation of the snapshots 3331 seconds CPU time is needed, see Table 2. The decay of the eigenvalues are depicted in Figure 5.

In Figures 6-8 we compare the FE solution and its POD approximation at the 6 measurement points, where we use the admittance $A = A^{id}(f)$. We observe that the relative errors are small except for certain frequencies. The error can be decreased if we include more ten 45 POD basis functions in our Galerkin ansatz. However, it turns out that for the identification a more precise approximation is not necessary.

The goal of the identification problem is to recover the admittance by solving ($\hat{\mathbf{P}}$) for the frequencies $f =$

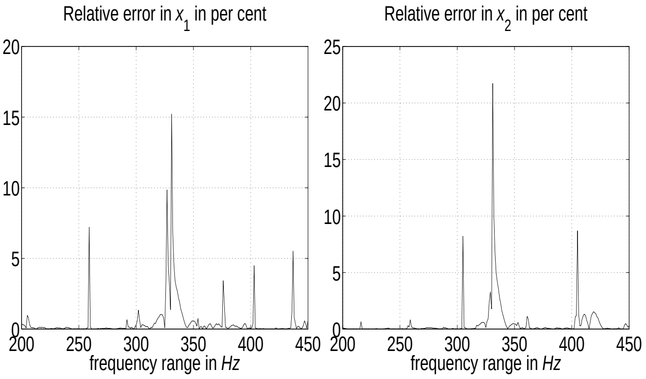

Figure 6: Relative error in the measurement points $\mathbf{x}_1 = (2.15, 1.23)$ and $\mathbf{x}_2 = (2.8, 0)$ in per cent.

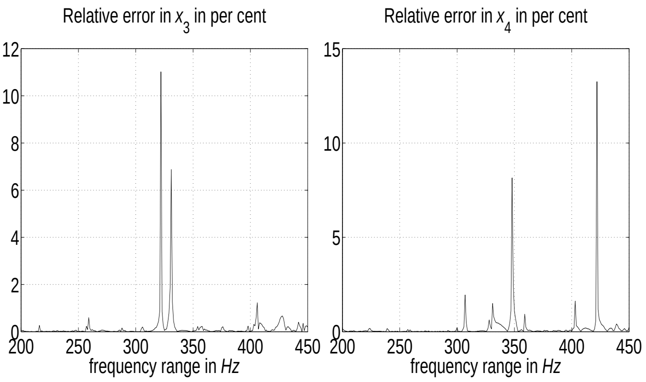

Figure 7: Relative error at the measurement points $\mathbf{x}_3 = (3, 0.11)$ and $\mathbf{x}_4 = (0.83, 0.53)$ in per cent.

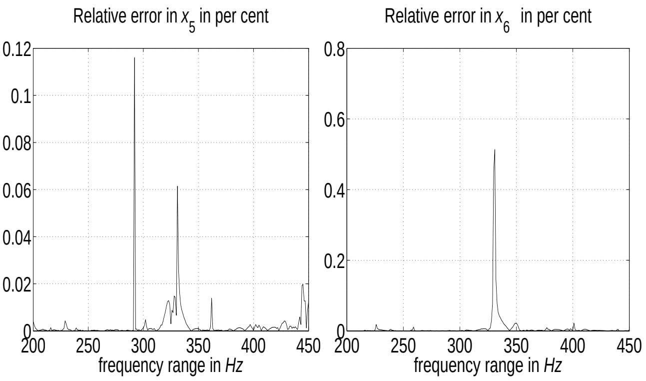

Figure 8: Relative error at the measurement points $\mathbf{x}_5 = (1.0, 0.8)$ and $\mathbf{x}_6 = (2.1, 0.6)$ in per cent.

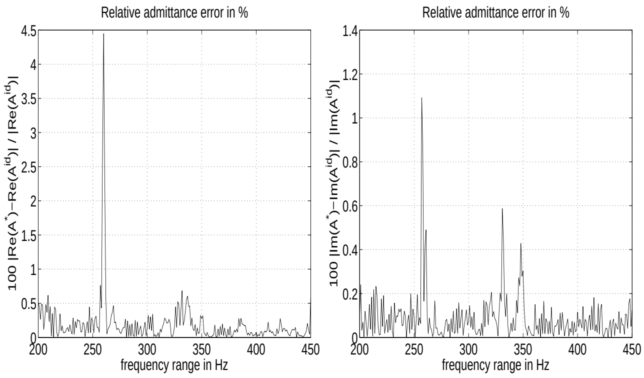

Figure 9: Relative error in the real part of the admittance (left plot) and in the imaginary part of the admittance (right plot) in per cent.

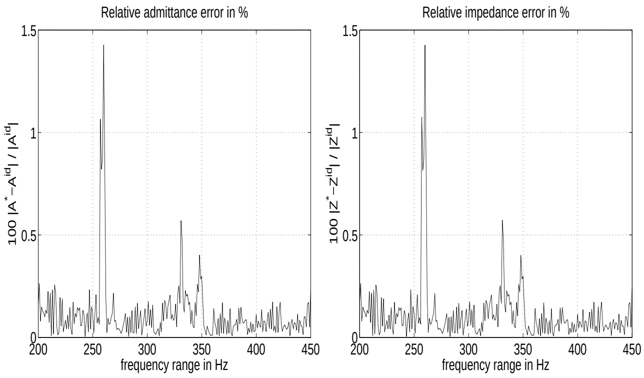

Figure 10: Relative error in the admittance (left plot) and in the impedance (right plot) in per cent.

$200, 201, \ldots, 450$. For every frequency we apply Algorithm 1. In the cost functional (12) we take $\gamma = 1$ and $\sigma = 200$. Denoting by $A^*(f)$ the obtained optimal admittance for the frequency f, we choose the nominal values

$$A_\circ = \begin{cases} 0 & \text{if } f = 200, \\ A^*(f-1) & \text{otherwise.} \end{cases}$$

As starting values for the quasi-Newton method we take

$$A^0 = \begin{cases} 1/Z^0 & \text{if } f = 200, \\ A^*(f-1) & \text{otherwise} \end{cases}$$

with $Z^0 = 100(1 - \jmath)$. For the Hessian we use the start matrix σI. For more details we refer also to [6].

In Figures 9-10 we plot the relative errors between the estimated admittance $A^*(f)$ and the 'true' value $A^{id}(f)$. The error is less than 5%. The CPU times are depicted in Table 2.

Note that the quasi-Newton method based on a POD approximation for the Helmholtz and adjoint equations requires only 43 seconds, whereas the quasi-Newton method based on a FE approximation is 338 times larger. Furthermore, the CPU time for the FE optimizer is significantly larger than for the computation of the snapshots plus for the POD optimizer.

	CPU time
Snapshot computation	3331 s
Computation of the POD basis	12 s
Quasi-Newton method with POD	43 s
Quasi-Newton method with FE	14452 s

Table 2: CPU times for the POD modeling and the optimization.

Literature

[1] M. Barrault, Y. Maday, N.C. Nguyen, and A.T. Patera. An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations. *Comptes Rendus de'l Académie des Sciences Paris*, Ser. I 339:667-672, 2004.

[2] P. Holmes, J.L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge Monographs on Mechanics, Cambridge University Press, 1996.

[3] W. Desmet. *A wave based prediction technique for coupled vibro-acoustic analysis*, PhD thesis, K. U. Leuven, division PMA, Belgium, 2002.

[4] G.H. Golub and C.F. Van Loan. *Matrix Computation*, Oxford University Press, 1996.

[5] A. Hepberger. *Mathematical methods for the prediction of the interior car noise in the middle frequency range.* PhD tesis, TU Graz, Institute for Mathematics, Austria, 2002.

[6] A. Hepberger, S. Volkwein, F. Diwoky, and H.-H. Priebsch. Impedance identification out of pressure datas with a hybrid measurement-simulation methodology up to 1kHz. In *Proceedings of International Conference on Noise and Vibration Engineering*, Leuven, Belgium, 2006.

[7] M. Hinze and S. Volkwein. Proper orthogonal decomposition surrogate models for nonlinear dynamical systems: error estimates and suboptimal control. In Reduction of Large-Scale Systems, P. Benner, V. Mehrmann, D. C. Sorensen (eds.), *Lecture Notes in Computational Science and Engineering*, Vol. 45, 261-306, 2005.

[8] K. Fukuda. *Introduction to Statistical Recognition.* Academic Press, New York, (1990).

[9] P. Holmes, J.L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry.* Cambridge Monographs on Mechanics, Cambridge University Press, 1996.

[10] T. Kato. *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980.

[11] K. Kunisch and S. Volkwein. Control of Burgers' equation by a reduced order approach using proper orthogonal decomposition. *Journal on Optimization Theory and Applications*, 102, 345–371, 1999.

[12] K. Kunisch and S. Volkwein. Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. *SIAM Journal on Numerical Analysis*, 40:492-515, 2002.

[13] S. Lall, J.E. Marsden and S. Glavaski. Empirical model reduction of controlled nonlinear systems. In: *Proceedings of the IFAC Congress*, vol. F, 473-478, 1999.

[14] H.V. Ly and H.T. Tran. Modelling and control of physical processes using proper orthogonal decomposition. *Mathematical and Computer Modeling*, 33:223-236, 2001.

[15] L. Machiels, Y. Maday, and A.T. Patera. Output bounds for reduced-order approximations of elliptic partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 190:3413-3426, 2001.

[16] J. Nocedal and S.J. Wright. *Numerical Optimization*, Springer Series in Operation Research, Second Edition, Springer Verlag, New York, 2006.

[17] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. Volume 1: Functional Analysis*, Academic Press, Inc., Boston, 1980.

[18] Y. Maday and E.M. Rønquist. A reduced-basis element method. *Journal of Scientific Computing*, 17, 1-4, 2002.

[19] C.W. Rowley. Model reduction for fluids, using balanced proper orthogonal decomposition. *International Journal of Bifurcation and Chaos*, 15:997-1013, 2005.

[20] L. Sirovich. Turbulence and the dynamics of coherent structures, parts I-III. *Quarterly of Applied Mathematics*, XLV:561-590, 1987.

[21] F. Tröltzsch and S. Volkwein. POD a-posteriori error estimates for linear-quadratic optimal control problems. Submitted, 2007.

[22] S. Volkwein. *Model Reduction using Proper Orthogonal Decomposition*, Lecture notes, Institute of Mathematics and Scientific Computing, University of Graz. `http://www.uni-graz.at/imawww/volkwein/POD.pdf`

[23] K. Willcox and J. Peraire. Balanced model reduction via the proper orthogonal decomposition. *American Institute of Aeronautics and Astronautics (AIAA)*, 40, 2323-2330, 2002.

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