GROUP-THEORETIC AND TOPOLOGICAL INVARIANTS OF COMPLETELY INTEGRALLY CLOSED PRÜFER DOMAINS

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Abstract. We consider the lattice-ordered groups $\mathrm{Inv}(R)$ and $\mathrm{Div}(R)$ of invertible and divisorial fractional ideals of a completely integrally closed Prüfer domain. We prove that $\mathrm{Div}(R)$ is the completion of the group $\mathrm{Inv}(R)$, and we show there is a faithfully flat extension S of R such that S is a completely integrally closed Bézout domain with $\mathrm{Div}(R) \cong \mathrm{Inv}(S)$. Among the class of completely integrally closed Prüfer domains, we focus on the one-dimensional Prüfer domains. This class includes Dedekind domains, the latter being the one-dimensional Prüfer domains whose maximal ideals are finitely generated. However, numerous interesting examples show that the class of one-dimensional Prüfer domains includes domains that differ quite significantly from Dedekind domains by a number of measures, both group-theoretic (involving $\mathrm{Inv}(R)$ and $\mathrm{Div}(R)$) and topological (involving the maximal spectrum of R). We examine these invariants in connection with factorization properties of the ideals of one-dimensional Prüfer domains, putting special emphasis on the class of almost Dedekind domains, those domains for which every localization at a maximal ideal is a rank one discrete valuation domain, as well as the class of SP-domains, those domains for which every proper ideal is a product of radical ideals. For this last class of domains, we show that if in addition the ring has nonzero Jacobson radical, then the lattice-ordered groups $\mathrm{Inv}(R)$ and $\mathrm{Div}(R)$ are determined entirely by the topology of the maximal spectrum of R, and that the Cantor-Bendixson derivatives of the maximal spectrum reflect the distribution of sharp and dull maximal ideals.

Key words: Prüfer domain, almost Dedekind domain, SP-domain, ℓ-group, Boolean space, Cantor-Bendixson derivative.

1. Introduction

All rings considered in this article are commutative with unity. An integral domain R with quotient field F is *completely integrally closed* if for each $x \in F \setminus R$ and $0 \neq r \in R$, there exists $n > 0$ such that $rx^n \notin R$. A *Prüfer domain* R is an integral domain for which every nonzero finitely generated ideal is invertible; equivalently, for each maximal ideal M of R, R_M is a valuation domain. If in addition every invertible ideal is principal, then R is a *Bézout domain.* The class of completely integrally

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closed Prüfer domains includes a number of prominent examples in non-Noetherian commutative ring theory, such as the ring of integer-valued polynomials [9, Proposition VI.2.1, p. 129], the ring of entire functions [14, Proposition 8.1.1(6), p. 276], the real holomorphy ring of a function field [39, Corollary 3.6], and the Kronecker function ring of a field extension of at most countable transcendence degree [26, Corollary 3.9]. For a completely integrally closed Prüfer domain R, the set $\mathrm{Div}(R)$ of nonzero divisorial fractional ideals of R is a lattice-ordered group (ℓ-group) with respect to ideal multiplication, as is the group $\mathrm{Inv}(R)$ of invertible fractional ideals of R. We prove in Proposition 3.1 that $\mathrm{Div}(R)$ is the completion of $\mathrm{Inv}(R)$, a consequence of which is that the group $\mathrm{Div}(R)$ can be calculated solely from the ℓ-group $\mathrm{Inv}(R)$. We show also in Theorem 3.7 that for each completely integrally closed Prüfer domain R, there exists a faithfully flat extension S of R such that S is a completely integrally closed Bézout domain with $\mathrm{Div}(S) = \mathrm{Inv}(S) \cong \mathrm{Inv}(R)$. For this ring S, every divisorial ideal is principal.

After treating the general case of completely integrally closed Prüfer domains in Section 3, we focus for the rest of the article on the special case of one-dimensional Prüfer domains. While this class includes the class of Dedekind domains, even among the class of one-dimensional Prüfer domains, the Dedekind domains are quite special. For example, consider a nonzero proper ideal I of a one-dimensional Prüfer domain R. If R is a Dedekind domain, then $\mathrm{Spec}(R/I)$ is a finite set. If R is not necessarily Dedekind, it can only be asserted that $\mathrm{Spec}(R/I)$ is a *Boolean space*, that is, a compact Hausdorff zero-dimensional space [27, p. 198]. It follows from [38, Section 3] that every Boolean space arises as $\mathrm{Spec}(R/I)$ for an appropriate choice of one-dimensional Prüfer domain R with nonzero Jacobson radical I. By Stone Duality, the category of Boolean spaces is dual to the category of Boolean algebras. Thus the class of one-dimensional Prüfer domains is at least as rich and varied as the category of Boolean algebras.

In fact, the construction in [38, Section 3] shows that each Boolean space can be realized in this fashion by an almost Dedekind domain R with nonzero Jacobson radical. (A domain R is *almost Dedekind* if R_M is a rank one discrete valuation ring (DVR) for each maximal ideal M of R.) The ring R in this case is even an *SP-domain*, a domain for which every proper ideal is a product of radical ideals. The isolated points in the maximal spectrum $\mathrm{Max}(R)$ of R are precisely the finitely generated (hence invertible) maximal ideals of R [38, Lemma 3.1]. Thus, if $\mathrm{Max}(R)$ is infinite and the Jacobson radical of R is nonzero, then the compactness of $\mathrm{Max}(R)$ implies that R will have maximal ideals that are not finitely generated. Because a great deal of variation is possible in the distribution of isolated points in a Boolean space, it follows that even among the almost Dedekind domains for which every proper ideal is a product of radical ideals, a wide range of behavior with respect to finite generation of maximal ideals is possible.

The aforementioned existence results rely on the Krull-Kaplansky-Jaffard-Ohm Theorem to realize a certain lattice-ordered group as the group of divisibility of a

Bézout domain, that is, a domain for which every finitely generated ideal is principal. As such they give existence in what might be considered the somewhat exotic setting of overrings of polynomial rings with as many variables as there are elements of the group. However, much of this same wide range of behavior can be seen in number-theoretic inspired contexts involving infinite towers of finite field extensions [8, 17, 22, 24, 31, 32, 43, 46, 47]. Interesting almost Dedekind domains also appear in the context of holomorphy rings [40].

Motivated by the diversity of these examples, we focus on (1) the group-theoretic invariants $\operatorname{Inv}(R)$ and $\operatorname{Div}(R)$, which encode much of the multiplicative ideal theory of the domain, and (2) the topology of the maximal spectrum $\operatorname{Max}(R)$, which encodes information on the density of finitely generated maximal ideals among the collection of all maximal ideals of R. We are especially interested in the case in which R is an SP-domain with nonzero Jacobson radical, and in this case the group $\operatorname{Inv}(R)$ is entirely determined by $\operatorname{Max}(R)$ and vice versa (see Theorem 5.1). In this sense, much of the multiplicative ideal theory of the domain R is accounted for simply from the topology of $\operatorname{Max}(R)$ alone. For example, not only is $\operatorname{Inv}(R)$ determined by $\operatorname{Max}(R)$, the group $\operatorname{Div}(R)$ of nonzero fractional divisorial ideals of R can also be extracted from the topology of $\operatorname{Max}(R)$ by using the Gleason cover of the topological space $\operatorname{Max}(R)$ (Theorem 5.3). Similarly, we show in Section 6 how to reinterpret Loper and Lucas' notions of sharp and dull degrees from [32] for one-dimensional Prüfer domains in purely topological terms. In doing so, we obtain some of their existence results as simple consequences of well-known topological results, and we extend these results from finite numbers to infinite ordinals, giving another measure of complexity of this class of rings.

The outline of the paper is as follows. After a brief discussion of background material in Section 2, we develop in Section 3 a multiplicative "completion" of a completely integrally closed Prüfer domain by showing that for such a domain R, the group $\operatorname{Div}(R)$ is the lattice-ordered group completion of $\operatorname{Inv}(R)$ and can be realized as $\operatorname{Inv}(S)$ for a Bézout domain S extending R (Theorem 3.7).

In Section 4 we characterize SP-domains with emphasis on principal rather than invertible ideals, and we note in Theorem 4.8 that if a one-dimensional domain R has nonzero Jacobson radical $J(R)$ such that $J(R)$ is invertible, then R is an SP-domain. If also $J(R)$ is principal, then R is a Bézout domain.

In Section 5 we show that for SP-domains with nonzero Jacobson radical, the groups $\operatorname{Inv}(R)$ and $\operatorname{Div}(R)$ are topological invariants of $\operatorname{Max}(R)$ (Theorems 5.1 and 5.3), and as a consequence we obtain that they are both free groups (Corollary 5.5). This leads naturally to consideration of the group $\operatorname{Div}(R)/\operatorname{Inv}(R)$, and we show that it is torsion-free, but that it is divisible only when $\operatorname{Div}(R) = \operatorname{Inv}(R)$, with the latter condition characterized topologically also in terms of $\operatorname{Max}(R)$ (Theorem 5.6).

In Section 6 we reframe Loper and Lucas' notion of the sharp and dull degrees of a one-dimensional Prüfer domain in topological terms and use this observation to obtain existence results for such domains with prescribed sharp degrees.

2. Preliminaries and Notation

Let R be an integral domain with quotient field F. We denote by $J(R)$ the Jacobson radical of R.

(2.1) *The group of invertible fractional ideals.* We denote by $\mathrm{Inv}(R)$ the group of invertible fractional ideals of R, and by $\mathrm{Prin}(R)$ the subgroup of $\mathrm{Inv}(R)$ consisting of the nonzero principal fractional ideals of R; i.e., $\mathrm{Prin}(R)$ is the *group of divisibility* of R. For background on the group of divisibility of a domain, see [21, Section 16]. The *Picard group* of R is the quotient group $\mathrm{Pic}(R) := \mathrm{Inv}(R)/\mathrm{Prin}(R)$.

(2.2) *The group of nonzero divisorial fractional ideals.* The set of nonzero divisorial fractional ideals of R is denoted $\mathrm{Div}(R)$. (Recall that a fractional ideal I of R is *divisorial* if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in F : xI \subseteq R\}$.) The domain R is completely integrally closed if and only if $\mathrm{Div}(R)$ is a group with the binary operation given by $I \star J := ((IJ)^{-1})^{-1}$ for all $I, J \in \mathrm{Div}(R)$ [21, Theorem 34.3]. If I, J are invertible fractional ideals of R, then $I \star J = IJ$. Thus both $\mathrm{Prin}(R)$ and $\mathrm{Inv}(R)$ are subgroups of $\mathrm{Div}(R)$. The quotient group $\mathrm{Cl}(R) = \mathrm{Div}(R)/\mathrm{Prin}(R)$ is the *class group* of R, and $\mathrm{Pic}(R)$ is a subgroup of $\mathrm{Cl}(R)$.

(2.3) *Lattice-ordered groups.* An abelian group $(G, +)$ is a *lattice ordered group (ℓ-group)* if it is partially ordered with a relation $\leq$ such that the meet $x \wedge y$ and join $x \vee y$ of $x, y \in G$ exist and such that for all $x, y, a \in G$, $x \leq y$ implies $a + x \leq a + y$. Throughout the paper, by an "ℓ-group" we mean an abelian ℓ-group. An *ℓ-homomorphism* of ℓ-groups $\phi : G \to G'$ is a group homomorphism such that for all $x, y \in G$, $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ (and thus $\phi(x \vee y) = \phi(x) \vee \phi(y)$).

(2.4) *The ℓ-group of invertible fractional ideals of a Prüfer domain.* The group $\mathrm{Inv}(R)$ admits a partial order given for all $I, J \in \mathrm{Inv}(R)$ by $I \leq J$ if and only if $I \supseteq J$. When R is a Prüfer domain, this partial order gives $\mathrm{Inv}(R)$ the structure of an ℓ-group, with meet and join given, respectively, as $I \wedge J = I + J$ and $I \vee J = I \cap J$ for all $I, J \in \mathrm{Inv}(R)$ [7, Theorem 2]. (In a Prüfer domain, finite intersections and finite sums of invertible ideals are invertible. In fact from [21, Theorem 25.2(d)], we have $(I + J)(I \cap J) = IJ$. Now since IJ is invertible, $I \cap J$ is also invertible.) When also R is a Bézout domain, $\mathrm{Inv}(R) = \mathrm{Prin}(R)$, and hence in this case $\mathrm{Prin}(R)$, the group of divisibility of R, is an ℓ-group. By the Krull-Kaplansky-Jaffard-Ohm Theorem [16, Theorem 5.3, p. 113], each ℓ-group is isomorphic to the group of divisibility of a Bézout domain.

3. Completely integrally closed Prüfer domains

As discussed in (2.4), the group $\mathrm{Inv}(R)$ of invertible fractional ideals of a Prüfer domain R is an ℓ-group. We show in Proposition 3.1 that if also R is a completely integrally closed domain, then $\mathrm{Inv}(R)$ is an archimedean ℓ-group, and hence admits a completion that proves to be the group $\mathrm{Div}(R)$ of nonzero divisiorial fractional ideals of R. We develop a ring-theoretic analogue of this by showing that every completely integrally closed Prüfer domain densely embeds in a pseudo-Dedekind Bézout domain. The *pseudo-Dedekind domains*, which play the role of the "complete" objects here, are the domains for which every nonzero divisorial ideal is invertible. (The terminology of "pseudo-Dedekind," which is due to Anderson and Kang [1], is motivated by Bourbaki's use of "pseudo-principal" to describe the domains for which every divisorial ideal is principal [6, Exercise VII.1.21]. Zafrullah [48] uses the term "generalized Dedekind" for what we term "pseudo-Dedekind" here.) In analogy with the fact that only archimedean ℓ-groups have a completion, Proposition 3.1 shows that we must restrict to completely integrally closed Prüfer domains. We use throughout this section that a Prüfer domain is completely integrally closed if and only if $\bigcap_{n>0} I^n = 0$ for all proper finitely generated ideals I [19, Theorem 8].

We recall first the notion of the (conditional) completion of an ℓ-group; for additional details, see [5, pp. 312–313] and [11]. An ℓ-group G is *complete* if every set of elements of G that is bounded below has a greatest lower bound (equivalently, every set of elements that is bounded above has a least upper bound). A complete ℓ-group is necessarily *archimedean* [5, Lemma 5, p. 291], meaning that whenever $a, b \in G$ with $na \leq b$ for all $n > 0$, then $a \leq 0$.

An ℓ-subgroup G is *dense* in an ℓ-group H if for $0 < h \in H$, there is $g \in G$ such that $0 < g \leq h$. The *completion* of an ℓ-group G is the smallest complete ℓ-group H that contains G. Each archimedean ℓ-group G has a completion H, which is characterized by the properties that (a) H is a complete ℓ-group containing G as subgroup, and (b) for each $0 < h \in H$, there exist $g_1, g_2 \in G$ such that $0 < g_1 \leq h \leq g_2$ [11, Theorem 2.4]. In particular, G is dense in its completion.

In the next proposition we apply these ideas to the ℓ-group $\mathrm{Inv}(R)$ for R a Prüfer domain. Note that $\mathrm{Inv}(R)$ is archimedean if whenever $I, J \in \mathrm{Inv}(R)$ with $J \subseteq I^n$ for all $n > 0$, then $R \subseteq I$.

Proposition 3.1. *A Prüfer domain R is completely integrally closed if and only if $\mathrm{Inv}(R)$ is an archimedean ℓ-group. If this is the case, the ℓ-group $\mathrm{Div}(R)$ is the completion of the ℓ-group $\mathrm{Inv}(R)$.*

Proof. Suppose that R is completely integrally closed with $I, J \in \mathrm{Inv}(R)$ such that $J \subseteq I^n$ for all $n > 0$. Since R is a Prüfer domain, $I^n \cap R = (I \cap R)^n$ [20, Theorem 2.2], and hence $J \cap R \subseteq (I \cap R)^n$ for all $n > 0$. Therefore, since R is completely integrally closed and $J \cap R \neq 0$, it must be that $I \cap R = R$, and hence $R \subseteq I$. Thus $\mathrm{Inv}(R)$ is an archimedean ℓ-group. Conversely, if I is a proper finitely generated ideal of R and $\mathrm{Inv}(R)$ is archimedean, then $\bigcap_{n>0} I^n = 0$, and hence R is completely integrally closed.

Now suppose R is completely integrally closed. Using [11, Theorem 2.4] as discussed before the proposition, $\mathrm{Div}(R)$ is the completion of $\mathrm{Inv}(R)$ if and only if

(a) $\mathrm{Div}(R)$ is complete, and (b) whenever I is a proper divisorial ideal of R, there exist invertible ideals J and K such that $K \subseteq I \subseteq J \subsetneq R$. To see that $\mathrm{Div}(R)$ is complete, let $\{I_\alpha\}$ be a collection of ideals in $\mathrm{Div}(R)$ that is bounded below with respect to set inclusion. Then there exists a nonzero divisorial ideal J contained in $I := \bigcap_\alpha I_\alpha$. As an intersection of divisorial ideals, I is divisorial, and since J is nonzero, so is I. Therefore, $I \in \mathrm{Div}(R)$ and it follows that I is the greatest lower bound of $\{I_\alpha\}$. This verifies (a).

Next, to verify (b), let $I \in \mathrm{Div}(R)$ with $I \subsetneq R$, and write $I = \bigcap_\alpha q_\alpha R$, where the q_α are elements of the quotient field of R. Then there is α such that $I \subseteq R \cap q_\alpha R \subsetneq R$. Since R is a Prüfer domain, the ideal $J := R \cap q_\alpha R$ is invertible (cf. [21, Theorem 25.2(d)]). Let K be any nonzero principal ideal of R contained in the nonzero ideal I. Then $K \subseteq I \subseteq J \subsetneq R$, which verifies (b). □

Definition 3.2. We say that for an extension $R \subseteq S$ of Prüfer domains, R is *dense* in S if the extension $R \subseteq S$ is faithfully flat and for each finitely generated proper ideal J of S there is a finitely generated proper ideal I of R such that $J \subseteq IS$.

Lemma 3.3. *An extension $R \subseteq S$ of Prüfer domains is dense if and only if the ℓ-homomorphism $\gamma : \mathrm{Inv}(R) \to \mathrm{Inv}(S) : I \mapsto IS$ is a dense embedding of ℓ-groups.*

Proof. Suppose that R is dense in S. To see that γ is injective, let I and J be invertible fractional ideals of R such that $IS = JS$. Then $IJ^{-1}S = S$. Since I and J^{-1} are fractional ideals of R, there exists $0 \neq r \in R$ such that rIJ^{-1} is an ideal of R. Since S is flat, S commutes with finite intersections of ideals of R [33, Theorem 7.4(ii), p. 48], so that

$$(rR \cap rIJ^{-1})S = rS \cap rIJ^{-1}S = rS.$$

Thus, since r is a nonzerodivisor in S, $(R \cap IJ^{-1})S = S$. Since S is faithfully flat, every proper ideal of R survives in S [33, Theorem 7.2(iii), p. 47], which implies that $R \cap IJ^{-1} = R$. Thus $R \subseteq IJ^{-1}$ and, since J is invertible, $J \subseteq I$. Similarly, from $IS = JS$ we deduce that $S = I^{-1}JS$ and the same argument with the roles of I and J reversed shows that $I \subseteq J$. Thus $I = J$ and $\gamma : \mathrm{Inv}(R) \to \mathrm{Inv}(S)$ is injective.

To see that γ is a dense embedding of ℓ-groups, observe that if J is a proper invertible ideal of S, then by assumption there exists a proper finitely generated ideal I of R with $J \subseteq IS \subsetneq S$. Thus γ is a dense embedding. Moreover, γ is an ℓ-group homomorphism since for all invertible fractional ideals I and J of R, $(I + J)S = IS + JS$ and (by the flatness of S) $(I \cap J)S = IS \cap JS$ [33, Theorem 7.4, p. 48].

Conversely, suppose that γ is a dense embedding. Since γ is injective, every proper invertible ideal I of R survives in S. If I is a not necessarily invertible proper ideal of S such that $IS = S$, then there is a finitely generated, hence invertible, proper ideal J of R with $J \subseteq I$ and $JS = S$, a contradiction. Thus every proper ideal of R survives in S. Since every torsion-free module over a Prüfer domain is flat

[16, Theorem 9.10, p. 233], it follows that S is a faithfully flat extension of R [33, Theorem 7.2, p. 47]. Moreover, if J is a proper finitely generated ideal of S, then, since γ is a dense embedding, there exists a proper finitely generated ideal I of R such that $J \subseteq \gamma(I) = IS \subsetneq S$. $\square$

Remark 3.4. Let $R \subseteq S$ be an extension of domains with R a Prüfer domain. The extension $R \subseteq S$ is faithfully flat if and only if $R = S \cap Q(R)$, where $Q(R)$ is the quotient field of R. Indeed, every torsion-free module over a Prüfer domain is flat [16, Theorem 9.10, p. 233], so it suffices to observe that $R = S \cap Q(R)$ if and only if every maximal ideal of R survives in S. If $R = S \cap Q(R)$ and M is a maximal ideal of R, then since M is a flat R-module, $M = MS \cap MQ(R) = MS \cap Q(R)$ [33, Theorem 7.4(i), p. 48], and hence $S \neq MS$. Conversely, if a maximal ideal of R survives in S, then it survives in $S \cap Q(R)$, so $S \cap Q(R)$ is a faithfully flat ring between R and $Q(R)$, which forces $R = S \cap Q(R)$ [33, Exercise 7.2, p. 53].

Let R be a domain, and let T be an indeterminate for R. The *Nagata function ring* $R(T)$ of R is given by

$$R(T) := \left\{ \frac{f}{g} : f, g \in R[T] \text{ and } c(g) = R \right\},$$

where $c(g)$ denotes the content ideal of g in R. The ring $R(T)$ is a Bézout domain if and only if R is a Prüfer domain [21, Theorem 33.4]. In Theorem 3.7 we apply the Nagata function ring construction to embed a Prüfer domain inside a Bézout domain with the same group of invertible fractional ideals. This is done via the following lemma, which can be deduced from more general results (see for example [36, Proposition 3.1.23, p. 144] and [37, p. 589]). For lack of an explicit reference, we give a proof here.

Lemma 3.5. *If R is a Prüfer domain, then the mapping*

$$\phi : \mathrm{Inv}(R) \to \mathrm{Prin}(R(T)) : I \mapsto IR(T)$$

is an isomorphism of ℓ-groups.

Proof. Since $R(T)$ is a Bézout domain, ϕ is well-defined. The extension $R \subseteq R(T)$ is faithfully flat [15, p. 39], so ϕ is injective and preserves meets and joins [16, Theorem 9.9 and Lemma 9.11]. Thus ϕ is also an ℓ-group homomorphism. To show that ϕ is onto, let $S = R(T)$, and let $J \in \mathrm{Prin}(S)$. Then $J = fg^{-1}S$ for some $f, g \in R[T]$ with $c(g) = R$. Now $fS = c(f)S$ [21, Theorem 32.7] and g is a unit in S, so $c(f)S = fg^{-1}S = J$. Thus ϕ is onto. $\square$

The next lemma is an application of an existence theorem due to Rump and Yang [45, Corollary 3] that shows that if R is a Bézout domain and $f : \mathrm{Inv}(R) \to H$ is an embedding of ℓ-groups, then there exists an extension S of R such that S is a Bézout domain with $\mathrm{Prin}(S) \cong H$ as ℓ-groups. The proof of the lemma uses the notion of a Bézout valuation from [45]. Let K be a field, and let G be an ℓ-group.

We adjoin an element ∞ to G such that $\infty > g$ and $\infty + g = \infty$ for all $g \in G$. A map $\nu : K \to G \cup \{\infty\}$ is a *Bézout valuation* if for all $x, y \in K$,

(a) $\nu(0) = \infty$, $\nu(1) = 0$,
(b) $\nu(xy) = \nu(x) + \nu(y)$,
(c) $\nu(x) \wedge \nu(y) \leq \nu(x + y)$, and
(d) $\nu(x) \wedge \nu(y) = \nu(xz + yw)$ for some $z, w \in K$ with $\nu(z), \nu(w) \geq 0$.

Lemma 3.6. *Let R be a Prüfer domain, and let $f : \mathrm{Inv}(R) \to H$ be an embedding of ℓ-groups. Then there is a ring extension S of R such that S is a Bézout domain and there is an isomorphism of ℓ-groups $\psi : \mathrm{Inv}(S) \to H$ for which $\psi^{-1}(f(I)) = IS$ for all $I \in \mathrm{Inv}(R)$.*

Proof. Let $A = R(T)$, so that A is a Bézout domain, and let $\phi : \mathrm{Inv}(R) \to \mathrm{Inv}(A) : I \to IA$ be the ℓ-group isomorphism given in Lemma 3.5. Let K be the quotient field of A. Then there is a surjective Bézout valuation $\nu_A : K \to \mathrm{Inv}(A) \cup \{\infty\}$ such that $A = \{x \in K : \nu_A(x) \geq 0\}$ [45, Proposition 1]. Denote by g the map from $\mathrm{Inv}(A) \cup \{\infty\}$ to $H \cup \{\infty\}$ such that $g|_{\mathrm{Inv}(A)} = f \circ \phi^{-1}$ and $g(\infty) = \infty$. By [45, Corollary 3] there is an extension field L of K and a surjective Bézout valuation $\nu : L \to H \cup \{\infty\}$ such that the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{\nu_A} & \mathrm{Inv}(A) \cup \{\infty\} \\ \big\downarrow{\scriptstyle\subseteq} & & \big\downarrow{\scriptstyle g} \\ L & \xrightarrow{\nu} & H \cup \{\infty\}. \end{array}$$

Let $S = \{x \in L : \nu(x) \geq 0\}$. Then S is a Bézout domain with quotient field L [45, Proposition 1]. Define $\psi : \mathrm{Inv}(S) \to H$ by $\psi(xS) = \nu(x)$ for each $0 \neq x \in L$. That ψ is well defined and injective follows from the fact that for $x, y \in L$, $xS = yS$ if and only if $\nu(x) = \nu(y)$. Since ν is surjective, so is ψ. Also, from the fact that $\nu(xy) = \nu(x) + \nu(y)$ for all $0 \neq x, y \in L$, it follows that ψ is a group homomorphism.

To see that ψ is an ℓ-group homomorphism, let $0 \neq x, y \in L$. Since ν is a Bézout valuation, there exists $z \in xS + yS$ such that $\nu(x) \wedge \nu(y) = \nu(z)$, and hence $xS + yS = zS$ (see the proof of [45, Proposition 1]). Therefore,

$$\psi(xS \wedge yS) = \psi(xS + yS) = \psi(zS) = \nu(z) = \nu(x) \wedge \nu(y) = \psi(xS) \wedge \psi(yS),$$

proving that ψ is an ℓ-group isomorphism.

Finally, let $I \in \mathrm{Inv}(R)$. Since A is a Bézout domain, there exists $x \in K$ such that $IA = xA$. Since ϕ is an isomorphism, $I = \phi^{-1}(xA)$, and hence

$$\psi^{-1}(f(I)) = \psi^{-1}(f(\phi^{-1}(xA))) = \psi^{-1}(\nu(x)) = xS = IS.$$

Therefore, $\psi^{-1} \circ f$ is the canonical map from $\mathrm{Inv}(R)$ to $\mathrm{Inv}(S)$. □

We apply the lemma in the case in which R is completely integrally closed and $H = \mathrm{Div}(R)$. We also make use of the fact that a Prüfer domain R is pseudo-Dedekind if and only if $\mathrm{Inv}(R)$ is a complete ℓ-group [1, p. 327].

Theorem 3.7. *A Prüfer domain R is completely integrally closed if and only if R is a dense subring of a pseudo-Dedekind Bézout domain S. Moreover, S can be chosen such that* $\mathrm{Inv}(S) \cong \mathrm{Div}(R)$ *as ℓ-groups.*

Proof. Suppose that R is completely integrally closed. By Proposition 3.1, the inclusion mapping $f : \mathrm{Inv}(R) \to \mathrm{Div}(R)$ is a dense embedding of ℓ-groups. Since R is a Prüfer domain, Lemma 3.6 implies there exists a Bézout domain S having R as a subring such that there is an isomorphism of ℓ-groups $\psi : \mathrm{Inv}(S) \to \mathrm{Div}(R)$ with $\gamma := \psi^{-1} \circ f$ the canonical ℓ-group homomorphism that sends $I \in \mathrm{Inv}(R)$ to $IS \in \mathrm{Inv}(S)$. Since, by Proposition 3.1, $\mathrm{Div}(R)$ is complete, so is $\mathrm{Inv}(S)$. As a complete ℓ-group, $\mathrm{Inv}(S)$ is archimedean. Thus, by Proposition 3.1, S is a completely integrally closed Bézout domain with $\mathrm{Inv}(S) = \mathrm{Div}(S)$. Therefore, S is a pseudo-Dedekind domain.

To show that R is dense in S, it suffices by Proposition 3.1 to show that γ is a dense embedding. Since γ is a composition of injective maps, γ is injective. To see that γ is dense, let $I \in \mathrm{Inv}(S)$ such that $I \subsetneq S$. Then, since ψ is an isomorphism of ℓ-groups, $\psi(I) \subsetneq R$. Since the inclusion mapping f is a dense embedding, there exists $J \in \mathrm{Inv}(R)$ such that $\psi(I) \subseteq f(J) \subsetneq R$. Therefore, $I \subseteq \psi^{-1}(f(J)) \subsetneq S$, and since $\gamma(J) = \psi^{-1}(f(J))$, this proves γ is a dense embedding.

Conversely, if R is dense in a pseudo-Dedekind Bézout domain S and I is a proper finitely generated ideal of R, then since I survives in the completely integrally closed Bézout domain S, we have $\bigcap_{n>0} I^n \subseteq \bigcap_{n>0} I^n S = 0$. Hence R is completely integrally closed. $\square$

In Section 5 we apply Theorem 3.7 to SP-domains with nonzero Jacobson radical. In that case the groups $\mathrm{Inv}(R)$ and $\mathrm{Div}(R)$ are topological invariants of the maximal spectrum of R. Thus the embedding in Theorem 3.7 ultimately depends on $\mathrm{Max}(R)$.

4. SP-domains

Following Vaughan and Yeagy [46], we say that a ring R for which every proper ideal is a product of radical ideals (i.e., "semi-prime" ideals) is an *SP-ring.* These rings have been studied by several authors; see for example [8, 13, 24, 35, 38, 42, 46, 47]. An SP-domain is necessarily an *almost Dedekind domain*; that is, R_M is a Dedekind domain for each maximal ideal M of R [46, Theorem 2.4]. In particular, an SP-domain is a one-dimensional Prüfer domain. Of course there exist SP-domains that are not Bézout domains, since Dedekind domains are SP-domains. However, there are more subtle examples also.

It is possible to construct a variety of SP-domains by using integral extensions of Dedekind domains. More precisely, one can obtain examples of SP-domains that are neither Dedekind domains nor Bézout domains by using the following method (which was already known to Grams [17]).

Example 4.1. Let R be a Dedekind domain with quotient field K such that $\mathrm{Max}(R)$ is countable. Let $\{K_i\}_{i\geq 0}$ be an ascending chain of extension fields of K such that $K_0 = K$, and $[K_i : K] < \infty$ for all $i \geq 0$. For $i \geq 0$ let R_i be the integral closure of R in K_i. For each $i \geq 0$, R_i is a Dedekind domain with countably many maximal ideals. Set $S = \bigcup_{i\geq 0} R_i$ (which is the integral closure of R in $\bigcup_{i\geq 0} K_i$). Suppose that $\{M_i\}_{i\geq 0}$ is the sequence of all maximal ideals of R.

(1) If for every $i > 0$ and every maximal ideal Q of R_i lying over some M_j with $j \leq i$, it follows that QR_{i+1} is a square-free product of maximal ideals of R_{i+1}, then S is an SP-domain (see [43, Proposition 6.5.2]).
(2) If there is an ascending chain $\{Q_j\}_{j\geq 0}$, where Q_{i+1} is a maximal ideal of R_{i+1} lying over the maximal ideal Q_i of R_i and $Q_{i+1} \neq Q_i R_{i+1}$ for each $i \geq 0$, then S is not a Dedekind domain (see [43, Proposition 6.5.5]).
(3) Suppose that $M_0 \subseteq \bigcup_{i>0} M_i$. If for every $i > 0$, QR_{i+1} is a maximal ideal of R_{i+1} whenever Q is a maximal ideal of R_i lying over some M_j with $0 < j \leq i$, then S is a domain that satisfies the ascending chain condition for principal ideals (ACCP) (see [43, Proposition 6.5.4]).

If all the residue fields of R are finite, M_0 is not principal and the Picard group of R is torsion-free, then it is possible to construct an ascending chain $\{K_i\}_{i\geq 0}$ of extension fields of K such that (1), (2), and (3) are satisfied and $[K_{i+1} : K_i] = 2$ for each $i \geq 0$ (see [21, Theorem 42.5]). (Observe that $M_0 \subseteq \bigcup_{i>0} M_i$ since $\mathrm{Pic}(R)$ is torsion-free and M_0 is not principal.)

To give an explicit example, we consider the domain $\overline{D}$ in [17, Example 1], which can be obtained by using the previous construction. By the prior remarks it is evident that $\overline{D}$ is an SP-domain that satisfies ACCP and that is not a Dedekind domain. In particular, $\overline{D}$ is not a Bézout domain, because a Bézout domain that satisfies ACCP is a PID. (If a Bézout domain satisfies ACCP, then it satisfies ACC for finitely generated ideals, which is equivalent to being Noetherian, hence it is a PID.)

SP-domains are characterized in several ways in [38, Theorem 2.1]. We strengthen some of the characterizations in the following lemma. Recall that a domain is *treed* if any two prime ideals contained in a maximal ideal of R are comparable with respect to set inclusion.

Lemma 4.2. *The following are equivalent for a domain R that is not a field.*

(1) *R is an SP-domain.*
(2) *R is an almost Dedekind domain for which each maximal ideal of R contains a finitely generated ideal that is not contained in the square of any maximal ideal of R.*
(3) *Each proper nonzero ideal A of R can be represented uniquely as a product $A = J_1 J_2 \cdots J_n$ of radical ideals J_i such that $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$.*

(4) *R is a treed domain for which the radical of each proper nonzero principal ideal is invertible.*

Proof. (1) $\Rightarrow$ (2) An SP-domain is necessarily an almost Dedekind domain [46, Theorem 2.4]. Thus (2) follows from [38, Theorem 2.1].

(2) $\Rightarrow$ (3) This is proved in [38, Theorem 2.1].

(3) $\Rightarrow$ (4) It is clear that (3) implies R is an SP-domain. Since an SP-domain that is not a field is an almost Dedekind domain, the only nonmaximal prime ideal is (0), so that R is treed. By [38, Theorem 2.1], an SP-domain has the property that the radical of every nonzero finitely generated ideal is invertible.

(4) $\Rightarrow$ (1) We first show that R is an almost Dedekind domain. Let $M \in \mathrm{Max}(R)$, and suppose that P and Q are prime ideals with $P \subsetneq Q \subseteq M$. Let $x \in P$ be nonzero, and let $y \in Q \setminus P$. Then $\sqrt{xR} \subsetneq \sqrt{yR}$ are by assumption invertible ideals. Thus, since R_M is a treed domain, $(\sqrt{xR})R_M \subseteq (\sqrt{yR})R_M$ is a containment of principal prime ideals. Since R_M is a domain, this forces, $(\sqrt{xR})R_M = (\sqrt{yR})R_M$. But then $y \in (\sqrt{xR})R_M \subseteq PR_M$, contrary to the assumption that $y \notin P$. This shows that R_M is a one-dimensional domain. Since also the radical of every nonzero proper principal ideal is invertible, it follows that R_M is a DVR. Therefore, R is an almost Dedekind domain. Finally, from (4) we have that every maximal ideal of R contains an invertible radical ideal, and hence by [35, Proposition 1.3], R is an SP-domain. □

Using the lemma we show next that whether an SP-domain is Bézout depends only on the radicals of principal ideals.

Theorem 4.3. *A domain R is a Bézout SP-domain if and only if R is a treed domain such that the radical of every principal ideal of R is principal.*

Proof. We may assume R is not a field since otherwise the claim is clear. If R is a Bézout SP-domain, then by Lemma 4.2 the radical of each principal ideal of R is principal. Conversely, suppose that R is a treed domain for which the radical of every principal ideal is principal. By Lemma 4.2, R is an SP-domain, so it remains to show that R is a Bézout domain. We do this by proving two claims.

Claim 1: If a, b, c are nonzero in R with aR, bR radical ideals and $a = bc$, then

$$\{M \in \mathrm{Max}(R) : a \in bM\} = \{M \in \mathrm{Max}(R) : a \in M, b \notin M\}.$$

To verify the claim, let $M \in \mathrm{Max}(R)$. Suppose $a \in bM$. Since $a = bc$, we have $c \in MR_M$. Since R_M is a DVR and aR_M is a nonzero radical ideal, it must be that $MR_M = aR_M = bcR_M$. Therefore, since $c \in MR_M$, it cannot be that $b \in M$. The reverse inclusion is clear in light of the fact that $a = bc$.

Claim 2: $\sqrt{xR + yR}$ is principal for all $x, y \in R$.

To prove Claim 2, let $x, y \in R$. We may assume x and y are nonzero. By assumption, there exist $a, b, c \in R$ such that $\sqrt{xyR} = aR$, $\sqrt{xR} = bR$, and $\sqrt{yR} = cR$. Note that $aR = bR \cap cR$. Moreover, there is $d \in R$ such that $\sqrt{(a/b)R} \cap$

$\sqrt{(a/c)R} = dR$. By Claim 1 we have

$$\begin{aligned}\{M \in \mathrm{Max}(R) : d \in M\} &= \{M \in \mathrm{Max}(R) : a \in bM\} \cup \{M \in \mathrm{Max}(R) : a \in cM\} \\ &= \{M \in \mathrm{Max}(R) : a \in M, (b \notin M \text{ or } c \notin M)\}.\end{aligned}$$

Therefore, Claim 1 implies that

$$\begin{aligned}\{M \in \mathrm{Max}(R) : x, y \in M\} &= \{M \in \mathrm{Max}(R) : b, c \in M\} \\ &= \{M \in \mathrm{Max}(R) : a \in M, d \notin M\} \\ &= \{M \in \mathrm{Max}(R) : a \in dM\}.\end{aligned}$$

Consequently, $\sqrt{xR + yR}$ is the radical of the principal ideal $(a/d)R$, and hence by assumption principal.

Using Claim 2 it follows by induction that every finitely generated radical ideal of R is principal. Since R is an SP-domain, every nonzero finitely generated ideal of R is a product of finitely generated, hence principal, radical ideals. Therefore, every finitely generated ideal of R is principal and R is a Bézout domain. $\square$

Corollary 4.4. *A domain R is a pseudo-Dedekind SP-domain if and only if R is a treed domain such that the radical of every nonzero divisorial ideal is invertible. If also the radical of every nonzero divisorial ideal is principal, then R is a Bézout domain.*

Proof. We may assume R is not a field. If R is a pseudo-Dedekind SP-domain, then since every nonzero divisorial ideal is invertible, Lemma 4.2 implies that the radical of every nonzero divisorial ideal is invertible. Conversely, suppose R is a treed domain for which the radical of every nonzero divisorial ideal of R is invertible. By Lemma 4.2, R is an SP-domain. Let I be a nonzero divisorial ideal of R. By Lemma 4.2(3), $I = \sqrt{I}A$ for some ideal A such that $A = R$ or A is a product of radical ideals of R. Since I is divisorial and by assumption $\sqrt{I}$ is invertible, it follows that A is divisorial. An inductive argument (which by Lemma 4.2(3) terminates after finitely many steps) now shows that I is a product of invertible ideals, and hence is itself invertible. Therefore, R is a pseudo-Dedekind domain. The last statement follows from Theorem 4.3. $\square$

Remark 4.5. As noted in the proof of Theorem 4.3, a one-dimensional domain R is an SP-domain if and only if each maximal ideal of R contains an invertible radical ideal (see [35, Proposition 1.3]). However, it is not true that R is a Bézout SP-domain if and only if each maximal ideal of R contains a nonzero principal radical ideal. There exists a Dedekind domain R with no principal maximal ideals and a sequence $\{M_i\}_{i=1}^{\infty}$ of all maximal ideals of R such that M_iM_{i+1} is principal for each i (see [10, Example 3-2]). In particular, R is an SP-domain and every nonzero radical ideal of R contains a nonzero principal radical ideal, yet R is not a Bézout domain.

Remark 4.6. Let R be an integral domain. Recall that R is a factorial domain (or unique factorization domain) if every principal ideal of R is a finite product of principal prime ideals. Furthermore, R is called radical factorial (see [42]) if each of its principal ideals is a finite product of principal radical ideals. These notions are the "principal ideal analogues" of Dedekind domains and SP-domains. We discuss how these classes of domains fit into the theory presented in this section. For instance, it is known that if the radical of every principal ideal of R is principal, then R is radical factorial (see [42, Proposition 2.10]). By using the same methods as in the proof of [42, Proposition 3.11] it can be shown that every treed radical factorial domain is an SP-domain. But even a Dedekind domain can fail to be radical factorial (see [42, Example 4.3]). Clearly, every SP-domain with trivial Picard group is radical factorial (see [42, Proposition 3.10(2)]). Note that R is a Dedekind domain if and only if R is an SP-domain of finite character (i.e., every nonzero element of R is contained in only finitely many maximal ideals). It can be shown that R is factorial if and only if R is of "finite height-one character" (i.e., every nonzero element is contained in only finitely many height-one prime ideals) and the radical of every principal ideal of R is principal (see [42, Theorem 2.14]). On the other hand a radical factorial domain of finite height-one character (and finite character) can fail to be factorial (see [42, Example 4.3]). It is well known that R is a PID if and only if R is a factorial Prüfer domain.

(Since a factorial domain has trivial class group and thus trivial Picard group, it follows that a factorial Prüfer domain is a Bézout domain. Moreover, a factorial domain satisfies ACCP. Therefore, a factorial Prüfer domain is a Bézout domain that satisfies ACCP, and thus it is a PID.)

This shows that a factorial domain that is not a PID (e.g. $\mathbb{Q}[X,Y]$) is an example of an integral domain where the radical of every principal ideal is principal, but that is neither an SP-domain nor a Bézout domain. In particular, we infer that the property "treed" cannot be omitted in Lemma 4.2 and Theorem 4.3.

We characterize next the SP-domains with nonzero Jacobson radical. It is this class of SP-domains with which we will be particularly concerned in the next sections.

Lemma 4.7. *Let R be an SP-domain with $J(R) \neq 0$. If there is $n > 0$ such that $J(R)^n$ is principal, then I^n is principal for each invertible ideal I of R.*

Proof. Let $J = J(R)$, and let $n > 0$ be such that J^n is principal, say $J^n = bR$ for $b \in J$. Since every proper invertible ideal of R is a product of invertible radical ideals, to prove that I^n is principal for each invertible ideal I of R, it suffices to show that L^n is a principal ideal of R for each invertible radical ideal L of R. Let L be an invertible radical ideal of R. Since R/J is a von Neumann regular ring, the finitely generated ideal L/J of R/J is a principal ideal, and hence there exists $x \in L$ nonzero such that $L = xR + J$. Since J is the Jacobson radical of R, a maximal ideal of R contains L if and only if it contains xR. Thus, since R is one-dimensional, $L = \sqrt{xR}$. We claim that $L^n = (x^n + b)R$, and to prove this it suffices to show that for each maximal ideal M of R, $L^n R_M = (x^n + b)R_M$. To this end, let $M \in \mathrm{Max}(R)$. Since R is an almost Dedekind domain, MR_M is a principal ideal of R_M and hence $M^nR_M \neq M^{n+1}R_M$. Thus, since $M^nR_M = J^nR_M = bR_M$, it follows that $b \notin M^{n+1}$ and hence $x^{n+1} + b \in M^n \setminus M^{n+1}$. Thus, if $L \subseteq M$, we have $L^nR_M = M^nR_M = (x^{n+1} + b)R_M$. Otherwise, if $L \nsubseteq M$, we have $x^{n+1} + b \notin M$, so that $(L^n)_M = R_M = (x^{n+1} + b)R_M$. Therefore, $L^n = (x^{n+1} + b)R$. □

Theorem 4.8. *Let R be a domain such that $J(R) \neq 0$. Then R is an SP-domain if and only if R is a one-dimensional domain such that $J(R)$ is an invertible ideal of R. If also $J(R)$ is a principal ideal of R, then R is a Bézout domain.*

Proof. Since $J(R) \neq 0$, R is not a field. If R is an SP-domain, then by Lemma 4.2 the radical of every nonzero principal ideal is invertible, so since $J := J(R)$ is the radical of any nonzero ideal contained in it, J is invertible.

Conversely, suppose R is one-dimensional and J is invertible. For each maximal ideal M of R, JR_M is an invertible, hence principal, ideal of the local ring R_M. Thus, since R_M is one-dimensional, R_M is a DVR, which shows that R is an almost Dedekind domain. Since J is contained in each maximal ideal of R but not in the square of any maximal ideal, Lemma 4.2 implies R is an SP-domain.

Finally, if J is principal, then by Lemmas 4.2 and 4.7, $\sqrt{xR}$ is principal for every $x \in R$. Therefore, by Theorem 4.3, R is a Bézout domain. □

Remark 4.9. (1) Let R be an SP-domain with $J(R) \neq 0$. It follows from Lemma 4.7 and Theorem 4.8 that $\mathrm{Pic}(R)$ is bounded if only if $\mathrm{Pic}(R)$ is a torsion group, if and only if $J(R)^n$ is principal for some $n \in \mathbb{N}$. However, we do not know whether an SP-domain with nonzero Jacobson radical is always a Bézout domain; that is, we do not know whether an SP-domain with $J(R) \neq 0$ always has that $J(R)$ is a principal ideal.

(2) Consider the following three properties for a one-dimensional domain R: (a) R is a Bézout SP-domain, (b) R is radical factorial, and (c) every nonzero maximal ideal of R contains a nonzero principal radical ideal of R. It is known that (a) $\Rightarrow$ (b) $\Rightarrow$ (c); cf. [42, Propositions 2.4(3) and 3.10(2)]. The converse implications do not hold in general (even when R is a Dedekind domain and $\mathrm{Pic}(R)$ is torsion-free [42, Example 4.3]), but the counterexamples to the reverse implications have Jacobson radical 0. We do not know whether (a), (b) and (c) are equivalent when R is a one-dimensional Prüfer domain with nonzero Jacobson radical.

5. Topological aspects of SP-domains

In this section we extend ideas from [35, 38] to give an explicit function-theoretic representation of $\mathrm{Inv}(R)$ and $\mathrm{Div}(R)$ for SP-domains with nonzero Jacobson radical. We show in particular that these groups, and hence also the group $\mathrm{Div}(R)/\mathrm{Inv}(R)$, are topological invariants of the maximal spectrum $\mathrm{Max}(R)$ of R. When R is any one-dimensional domain (e.g. R is an SP-domain) with nonzero Jacobson radical, then $\mathrm{Max}(R)$ is a *Boolean space* (or *Stone space*), that is, a compact Hausdorff space having an open basis consisting of clopen (= closed and open) subsets [27, p. 198]. By Stone Duality, the Boolean spaces are precisely the topological spaces that occur as the space of ultrafilters of a Boolean algebra; see for example [27, 29]. Specifically, every Boolean space is homeomorphic to the space of ultrafilters on the Boolean algebra of its clopen sets.

We associate to each Boolean space X the ℓ-group $C(X,\mathbb{Z})$ consisting of the continuous functions from X to $\mathbb{Z}$. (This group is the *Boolean power* of $\mathbb{Z}$ over X; see [4, 44] for more on this point of view.) Specifically, $(C(X,\mathbb{Z}),+)$ is an abelian group with respect to pointwise addition; that is, for each $f,g \in C(X,\mathbb{Z})$, $(f+g)(x) = f(x)+g(x)$ for all $x \in X$. The group $C(X,\mathbb{Z})$ is lattice-ordered with respect to the pointwise ordering given by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Join and meet are defined, respectively, by $(f \vee g)(x) = \max\{f(x), g(x)\}$ for all $x \in X$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all $x \in X$. Since X is compact, it is straightforward to see that every function $f \in C(X,\mathbb{Z})$ may be represented as $f = a_1\chi_{A_i} + \cdots + a_n\chi_{A_n}$, where each $a_i \in \mathbb{Z}$, $\{A_1, A_2, \ldots, A_n\}$ is a partition of X into clopens A_i, and χ_{A_i} represents the characteristic function of A_i (see for example [4, Lemma 2.1] or [38, Lemma 3.3]).

In [38, Section 3], it is shown that for each Boolean space X, there is a Bézout SP-domain with nonzero Jacobson radical whose group of divisibility is isomorphic as an ℓ-group to $C(X,\mathbb{Z})$. McGovern [35, p. 1781] has shown that this fact characterizes such rings. Namely, a Bézout domain R is an SP-domain with $J(R) \neq 0$ if and only if its group of divisibility $\mathrm{Prin}(R)$ is isomorphic as an ℓ-group to $C(X,\mathbb{Z})$ for some Boolean space X. In the next theorem we extend this result to the group of invertible fractional ideals of a Prüfer domain. Implicit in the proof is an argument that a Prüfer domain R is an SP-domain if and only if the Nagata function ring $R(T)$ of R is an SP-domain.

Theorem 5.1. *A Prüfer domain R is an SP-domain with $J(R) \neq 0$ if and only if there is a Boolean space X (necessarily homeomorphic to* $\mathrm{Max}(R)$*) such that* $\mathrm{Inv}(R) \cong C(X,\mathbb{Z})$ *as ℓ-groups.*

Proof. Since the maximal ideals of the Nagata function ring $R(T)$ are of the form $MR(T)$ with M a maximal ideal of R and $MR(T) \cap R = M$, it follows that $J(R(T)) \cap R = J(R)$. Every ideal of $R(T)$ is extended from R [12, p. 174], and thus $J(R(T)) = 0$ if and only if $J(R) = 0$.

Suppose that R is an SP-domain with $J(R) \neq 0$. Since R is an SP-domain, R is almost Dedekind and so is $R(T)$ [21, Proposition 36.7]. We show that $R(T)$ is an SP-domain via Lemma 4.2(2). Let M be a maximal ideal of R. Then there exists a finitely generated ideal I contained in M such that I is not contained in the square of any maximal ideal of R. If $IR(T)$ is contained in $N^2R(T)$ for some maximal ideal N of R, then $I \subseteq N^2R(T) \cap R = N^2$, where the last equality holds since $R(T)$ is a faithfully flat extension of R. However, I was chosen not in the square of any maximal ideal of R, so this contradiction implies by Lemma 4.2 that

$R(T)$ is an SP-domain. Now $R(T)$ is a Bézout SP-domain with $J(R(T)) \neq 0$, so the group of divisibility $\mathrm{Prin}(R(T))$ of $R(T)$ is isomorphic as an ℓ-group to $C(X,\mathbb{Z})$ for $X = \mathrm{Max}(R)$ [35, p. 1781]. By Lemma 3.5, $\mathrm{Inv}(R)$ is ℓ-isomorphic to $C(X,\mathbb{Z})$.

Conversely, suppose that $\mathrm{Inv}(R)$ is ℓ-isomorphic to $C(X,\mathbb{Z})$ for some Boolean space X. By Lemma 3.5, $\mathrm{Prin}(R(T))$ is ℓ-isomorphic to $C(X,\mathbb{Z})$. Hence $R(T)$ is an SP-domain with $J(R(T)) \neq 0$ [35, p. 1781]. Thus $J(R) \neq 0$ by the above observation that $J(R(T)) = 0$ if and only if $J(R) = 0$. Since $R(T)$ is almost Dedekind, R is almost Dedekind [21, Proposition 36.7]. To see that R is an SP-domain, let M be a maximal ideal of R. By Lemma 4.2 there exists a finitely generated ideal B of $R(T)$ such that $B \subseteq MR(T)$ and B is not contained in the square of any maximal ideal of $R(T)$. Now, since R is a Prüfer domain, $B = AR(T)$ for some finitely generated ideal A of R [12, p. 174], so A is contained in M but not in the square of any maximal ideal of R. Therefore, by Lemma 4.2, R is an SP-domain.

Finally, it is routine to check that every prime ℓ-subgroup of $C(X,\mathbb{Z})$ is both minimal and maximal, and that X is homeomorphic to the space of prime subgroups of $C(X,\mathbb{Z})$; see for example [38, Section 3] (the group G there is $C(X,\mathbb{Z})$). Since the space of minimal prime ℓ-subgroups of the group of divisibility of a Bézout domain is homeomorphic to the maximal spectrum of the domain [30, Lemma 5.7], we conclude that X is homeomorphic to $\mathrm{Max}(R(T))$, and hence by [12, (a), p. 174] homeomorphic to $\mathrm{Max}(R)$. $\square$

Thus the group of invertible ideals of an SP-domain R with nonzero Jacobson radical, and hence much of the ideal theory of R, depends solely on the topological space $\mathrm{Max}(R)$. In [38, Section 4] a number of examples are given to show how the topology of $\mathrm{Max}(R)$ influences the SP-domain R.

Continuing in this line, we show next that $\mathrm{Div}(R)$ is also a topological invariant of the space $\mathrm{Max}(R)$. For this we recall the notion of the Gleason cover of a compact Hausdorff space. A topological space X is *extremally disconnected* if the closure of any open subset of X is clopen; i.e., the regular open subsets coincide with the clopen subsets of X. Gleason [23] has shown that the extremally disconnected Boolean spaces are the projective objects in the category of compact Hausdorff spaces. In particular, for any compact Hausdorff space X, there is a continuous irreducible (i.e., sends proper closed sets to proper closed sets) surjection $j : EX \to X$, where EX is an extremally disconnected Boolean space. This property characterizes EX up to homeomorphism in the category of compact Hausdorff spaces X [23, Theorem 3.2]. The space EX is called the *Gleason cover* of X.

When X is a Boolean space, then $C(EX,\mathbb{Z})$ is a complete ℓ-group [28, Proposition 3.29]. Moreover, the map $\psi : C(X,\mathbb{Z}) \to C(EX,\mathbb{Z})$ defined by $\psi(f) = f \circ j$ for each $f \in C(X,\mathbb{Z})$ is an ℓ-group homomorphism that is injective since j is surjective. We show in the next lemma that under this embedding $C(EX,\mathbb{Z})$ is the completion of $C(X,\mathbb{Z})$.

Lemma 5.2. *If X is a Boolean space, then the mapping $\psi : C(X,\mathbb{Z}) \to C(EX,\mathbb{Z})$ is a dense embedding of ℓ-groups and $C(EX,\mathbb{Z})$ is the completion of the image of $C(X,\mathbb{Z})$.*

Proof. As noted before the lemma, $C(EX,\mathbb{Z})$ is a complete ℓ-group. By [11, Theorem 2.4], it suffices to show that for each $0 < f \in C(EX,\mathbb{Z})$, there exist $g_1, g_2 \in C(X,\mathbb{Z})$ such that $0 < \psi(g_1) \leq f \leq \psi(g_2)$. Let $0 < f \in C(EX,\mathbb{Z})$. As discussed at the beginning of the section, there exist $n_1, \dots, n_l \in \mathbb{N}$ and nonempty disjoint clopen sets $B_1, \dots, B_l$ of EX such that $f = \sum_{i=1}^{l} n_i \chi_{B_i}$. Let $n = \max\{n_1, \dots, n_l\}$, and let $g_2 = n\chi_X$. Then $f \leq \psi(g_2) = n\chi_{EX}$. Thus it remains to show there exists $g_1 \in C(X,\mathbb{Z})$ such that $0 < g_1 \leq f$.

Since $f > 0$ and the B_i are disjoint, there is $1 \leq i \leq l$ such that $n_i\chi_{B_i} > 0$. Thus to complete the proof it suffices to show that if B is a nonempty clopen subset of EX, there exists a nonempty clopen subset A of X such that $0 < \psi(\chi_A) \leq \chi_B$. Let j be the continuous irreducible surjection $EX \to X$. Since $\psi(\chi_A) = \chi_{j^{-1}(A)}$, it suffices to show there is a nonempty clopen subset A of X such that $j^{-1}(A) \subseteq B$. Before proving this, we show that $B = \operatorname{int} j^{-1}(j(B))$.

Since $B \subseteq j^{-1}(j(B))$ and B is open in EX, we have $B \subseteq \operatorname{int} j^{-1}(j(B))$. Hence $j(B) \subseteq j(\operatorname{int} j^{-1}(j(B))) \subseteq j(j^{-1}(j(B))) = j(B)$. Thus $j(B) = j(\operatorname{int} j^{-1}(j(B)))$. Now, since j is irreducible and B is regular closed (in fact, clopen), $j(B)$ is a regular closed subset of X [41, Theorem 6.5(d), pp. 454–455]. Thus $j^{-1}(j(B))$ is closed in EX, so that $\operatorname{int} j^{-1}(j(B))$ is a regular open, hence clopen, subset of EX. Therefore, j maps both of the clopen sets B and $\operatorname{int} j^{-1}(j(B))$ onto $j(B)$. Since j is irreducible, there exists a unique clopen of EX mapping onto the regular closed set $j(B)$ [41, Theorem 6.5(d), pp. 454–455]. Consequently, $B = \operatorname{int} j^{-1}(j(B))$, which proves the claim.

Finally, since B is clopen in EX and j is irreducible, $\operatorname{int} j(B)$ is nonempty [41, Lemma 6.5(b), p. 452]. Since X has a basis of clopens, there exists a nonempty clopen A in $\operatorname{int} j(B)$. Since $A \subseteq j(B)$ and A is open, we have $j^{-1}(A) \subseteq \operatorname{int} j^{-1}(j(B)) = B$, which completes the proof. ☐

Theorem 5.3. *Let R be an SP-domain with $J(R) \neq 0$, and let $X = \operatorname{Max}(R)$. Then there is a commutative diagram,*

$$\begin{array}{ccc} \operatorname{Inv}(R) & \xrightarrow{\subseteq} & \operatorname{Div}(R) \\ \big\downarrow \alpha & & \big\downarrow \beta \\ C(X,\mathbb{Z}) & \xrightarrow{\psi} & C(EX,\mathbb{Z}), \end{array}$$

where the vertical maps are isomorphisms.

Proof. By Theorem 5.1, there is an isomorphism $\alpha : \operatorname{Inv}(R) \to C(X,\mathbb{Z})$, and by Lemma 5.2 the mapping $\psi : C(X,\mathbb{Z}) \to C(EX,\mathbb{Z})$ is a dense embedding, with $C(EX,\mathbb{Z})$ a complete ℓ-group. By Proposition 3.1, $\operatorname{Div}(R)$ is the completion of

$\mathrm{Inv}(R)$, so the mapping α lifts to a (unique) isomorphism $\beta : \mathrm{Div}(R) \to C(EX, \mathbb{Z})$ [11, Theorem 1.1]. □

Combining the theorem with Theorem 3.7, we have the following ring-theoretic analogue of Lemma 5.2.

Corollary 5.4. *An SP-domain R with $J(R) \neq 0$ is a dense subring of a pseudo-Dedekind Bézout SP-domain S with $J(S) \neq 0$.*

Proof. By Theorem 3.7, R is a dense subring of a pseudo-Dedekind Bézout domain S with $\mathrm{Inv}(S) \cong \mathrm{Div}(R)$ as ℓ-groups. By Theorem 5.3, $\mathrm{Inv}(S)$ is isomorphic as an ℓ-group to $C(EX, \mathbb{Z})$, where $X = \mathrm{Max}(R)$. Thus the result of McGovern [35, p. 1781] discussed before Theorem 5.1 shows that S is an SP-domain with $J(S) \neq 0$. □

Bergman [2, Theorem 1.1] has proved that every group of the form $C(X, \mathbb{Z})$, with X a Boolean space, is a free abelian group, so from Theorems 5.1 and 5.3 we obtain the following corollary.

Corollary 5.5. *If R is an SP-domain with $J(R) \neq 0$, then* $\mathrm{Inv}(R)$ *and* $\mathrm{Div}(R)$ *are free abelian groups.* □

In light of the fact that a completely integrally closed domain R is pseudo-Dedekind if and only if $\mathrm{Inv}(R) = \mathrm{Div}(R)$, the group $\mathrm{Div}(R)/\mathrm{Inv}(R)$ can be viewed as a measure of how far the ring R is from being pseudo-Dedekind. We next examine this group for SP-domains. Let n be an integer with $n > 1$. A group G is *n-divisible* if for each $g \in G$, there exists $h \in G$ such that $g = nh$.

Theorem 5.6. *Let R be an SP-domain with $J(R) \neq 0$. Then* $\mathrm{Div}(R)/\mathrm{Inv}(R)$ *is a torsion-free group, and the following statements are equivalent.*

(1) $\mathrm{Div}(R) = \mathrm{Inv}(R)$ *(and hence R is a pseudo-Dedekind domain).*
(2) $\mathrm{Max}(R)$ *is an extremally disconnected space.*
(3) $\mathrm{Div}(R)/\mathrm{Inv}(R)$ *is an n-divisible group for some integer $n > 1$.*

Proof. We first prove $\mathrm{Div}(R)/\mathrm{Inv}(R)$ is torsion-free. By Theorem 5.3, to prove that $\mathrm{Div}(R)/\mathrm{Inv}(R)$ is torsion-free, it is enough to show that with $\psi : C(X, \mathbb{Z}) \to C(EX, \mathbb{Z})$, the group $G = \mathrm{coker}(\psi)$ is torsion-free. Let j be the continuous irreducible surjection $EX \to X$, and let $f \in C(EX, \mathbb{Z})$. Suppose there is an integer $n > 0$ such that $nf \in \mathrm{Im}(\psi)$. Then for some nonempty clopen subsets A_i of X, $i = 1, \ldots, l$, that form a partition of X, we can write $nf = \sum_{i=1}^{l} m_i(\chi_{A_i} \circ j)$. Let $k \in \{1, \ldots, l\}$, and let $x \in A_k$. Then there is $y \in EX$ such that $x = j(y)$. Thus

$$nf(y) = \sum_{i=1}^{l} m_i(\chi_{A_i} \circ j)(y) = \sum_{i=1}^{l} m_i \chi_{A_i}(x) = m_k.$$

This shows that n divides m_k for each $k = 1, \ldots, l$. Therefore,

$$nf = n \sum_{i=1}^{l} \frac{m_i}{n}(\chi_{A_i} \circ j) \in n \cdot \mathrm{Im}(\psi).$$

Since $C(EX, \mathbb{Z})$ is torsion-free, $f \in \operatorname{Im}(\psi)$, which proves that G is torsion-free.

Next, we show that (1), (2) and (3) are equivalent. Let $X = \operatorname{Max}(R)$. Then $\psi : C(X, \mathbb{Z}) \to C(EX, \mathbb{Z})$ is surjective if and only $j : EX \to X$ is a homeomorphism. (Recall that as discussed in the proof of Theorem 5.1, the space of minimal prime ℓ-subgroups of $C(Y, \mathbb{Z})$ is homeomorphic to Y for any Boolean space Y.) The equivalence of (1) and (2) now follows from Theorem 5.3. Thus to prove the theorem, it is enough to prove that $G = \operatorname{coker}(\psi)$ is n-divisible for some integer $n > 1$ if and only if X is extremally disconnected.

If X is extremally disconnected, then G is n-divisible for every $n > 1$ as it is trivial. Now suppose that G is n-divisible for some $n > 1$ and X is not extremally disconnected. Then there exists a regular closed subset V of X that is not clopen. Let $C = \overline{j^{-1}(\operatorname{int} V)}$. Since j is continuous, C is a regular closed set. Since EX is extremally disconnected, C is clopen and $\chi_C \in C(EX, \mathbb{Z})$. Now, since G is n-divisible, there is $f \in C(EX, \mathbb{Z})$ such that $nf - \chi_C = h \circ j$ for some $h \in C(X, \mathbb{Z})$.

Let $V' = \overline{X \setminus V}$. Since V is not clopen, there exists $x \in V \cap V'$. The open set $h^{-1}(h(x)) \subseteq X$ is a neighborhood of x, so since $\operatorname{int} V$ is dense in V, there exists $x_1 \in \operatorname{int} V$ such that $h(x_1) = h(x)$. Let $y_1 \in EX$ such that $j(y_1) = x_1$. Then $y_1 \in j^{-1}(x_1) \subseteq j^{-1}(\operatorname{int} V) \subseteq C$, so that

$$(*) \qquad h(x) = h(x_1) = (h \circ j)(y_1) = (nf - \chi_C)(y_1) = nf(y_1) - 1.$$

Similarly, since $X \setminus V$ is dense in V', there exists $x_2 \in X \setminus V$ such that $h(x_2) = h(x)$. Let $y_2 \in EX$ such that $j(y_2) = x_2$. Then $y_2 \in j^{-1}(x_2) \subseteq j^{-1}(X \setminus V)$. Since j is continuous and V is regular closed, we have

$$C = \overline{j^{-1}(\operatorname{int} V)} \subseteq j^{-1}(\overline{\operatorname{int} V}) = j^{-1}(V).$$

If also $y_2 \in C$, then $y_2 \in j^{-1}(V)$, so that $j(y_2) \in (X \setminus V) \cap V$, a contradiction. We conclude $y_2 \notin C$ and

$$h(x) = h(x_2) = (h \circ j)(y_2) = (nf - \chi_C)(y_2) = nf(y_2).$$

Combining this observation with $(*)$, we have $nf(y_1) - 1 = h(x) = nf(y_2)$, which implies $n(f(y_1) - f(y_2)) = 1$, a contradiction to the fact that $n > 1$ and $f(y_1) - f(y_2) \in \mathbb{Z}$. Therefore, G is not n-divisible. □

Remark 5.7. Let X be a Boolean space. The proof of Theorem 5.6 shows that the cokernel of the embedding $C(X, \mathbb{Z}) \to C(EX, \mathbb{Z})$ is torsion-free, and it is n-divisible for some $n > 1$ if and only if X is extremally disconnected. We do not know (a) whether the cokernel $G(X)$ of the embedding $C(X, \mathbb{Z}) \to C(EX, \mathbb{Z})$ is free, or (b) whether there exists a Boolean space X such that $G(X)$ is finitely generated and nontrivial. With regards to (a), the groups $C(X, \mathbb{Z})$ and $C(EX, \mathbb{Z})$ are free by the theorem of Bergman [2, Theorem 1.1] cited above.

6. Cantor-Bendixson theory for one-dimensional Prüfer domains

A one-dimensional Prüfer domain R is a Dedekind domain if and only if every maximal ideal of R is finitely generated. Thus if R is not Dedekind, then there exist maximal ideals that are not finitely generated. More generally, a maximal ideal of a one-dimensional Prüfer domain need not even be the radical of a finitely generated ideal. For example, if R is an almost Dedekind domain, then it is routine to see that a maximal ideal is finitely generated if and only if it contains a finitely generated ideal that is not contained in any other maximal ideal of R. Following [32], we say a maximal M of a Prüfer domain R is *sharp* if it contains a finitely generated ideal that is not contained in any other maximal ideal; otherwise, M is *dull.* Thus a maximal ideal of a one-dimensional Prüfer domain is sharp if and only if it is the radical of a finitely generated ideal, and a maximal ideal of an almost Dedekind domain is sharp if and only if it is finitely generated.

Recently, Loper and Lucas [32] have introduced the notions of sharp and dull degrees of maximal ideals of one-dimensional Prüfer domains as a measure of how far a maximal ideal is from being sharp. In this section we recast their theory in topological terms and use Cantor-Bendixson theory to shed additional light on the sharp and dull degrees of maximal ideals. We show in particular that the sharp and dull degrees of one-dimensional Prüfer domains are topological invariants of their maximal spectra when viewed with the inverse topology. (When also the Prüfer domain has nonzero Jacobson radical, this topology coincides with the Zariski topology on $\mathrm{Max}(R)$.) This allows us to prove existence results for almost Dedekind domains of various sharp and dull degrees. These results give a different approach, as well as extend, similar existence theorems due to Loper and Lucas. In our context these existence results reduce to quick consequences of well-known facts about Boolean spaces.

Loper and Lucas' definition of sharp and dull degree is motivated by a characterization of sharp maximal ideals: A maximal ideal M of a Prüfer domain R is sharp if and only if $\bigcap_{N\neq M} R_N \not\subseteq R_M$, where N ranges over the maximal ideals of R distinct from M [19, Corollary 2.]; i.e., M is sharp if and only if R_M is irredundant in the representation of R as the intersection of its localizations at maximal ideals. While R_M may not be irredundant in this representation, it could be irredundant in representations of overrings of R, and it is this observation that provides the motivation for Loper and Lucas' definition of sharp and dull degrees of a one-dimensional Prüfer domain. (Alternatively, while M may not be sharp in R, it can extend to a sharp maximal ideal of an overring of R.) We extend their definition using transfinite induction to permit infinite degree also.

Definition 6.1. Let R be a Prüfer domain. Denote by $\mathcal{M}_{\#}(R)$ the set of sharp maximal ideals of R and by $\mathcal{M}_{\dagger}(R)$ the set of dull maximal ideals of R. Define $R_0 = R$. For each ordinal number α, define $R_{\alpha+1} = \bigcap_{M\in\mathcal{M}_{\dagger}(R_\alpha)} R_M$, and for each limit ordinal number λ, define $R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$.

(1) The ring R has *sharp degree* α if $R_{\alpha+1}$ is the quotient field F of R and $R_\alpha \neq R_{\alpha+1}$; equivalently, R has sharp degree α if $\mathcal{M}_\dagger(R_\alpha) = \emptyset$ and $\mathcal{M}_\dagger(R_\beta) \neq \emptyset$ for all $\beta < \alpha$. We say R *has a sharp degree* if R has sharp degree α for some α.
(2) The ring R has *dull degree* α if $R_\alpha = R_{\alpha+1} \neq F$ and $R_\beta \neq R_\alpha$ for all $\beta < \alpha$. In this case, we say R *has a dull degree*.

Our main result of this section, Theorem 6.6, connects sharp and dull degrees with the Cantor-Bendixson rank of the topological space $\mathrm{Max}(R)$ endowed with the inverse topology. We recall this topology.

Definition 6.2. Let R be a ring. We denote by $\mathrm{Max}^{-1}(R)$ the topological space whose underlying set is $\mathrm{Max}(R)$ and which has a basis of open sets of the form $\{M \in \mathrm{Max}(R) : r_1, \ldots, r_n \in M\}$, where $r_1, \ldots, r_n \in R$; equivalently, the quasicompact open subsets of the space $\mathrm{Max}(R)$ (with respect to the Zariski topology) form a basis of closed of sets for $\mathrm{Max}^{-1}(R)$. This topology on $\mathrm{Max}^{-1}(R)$ is the *inverse topology.*

Lemma 6.3. *Let R be a one-dimensional Prüfer domain.*

(1) *Let X be a subspace of* $\mathrm{Max}^{-1}(R)$. *Then M is a limit point for X if and only if every finitely generated ideal of R contained in M is contained in another maximal ideal in X.*
(2) *A maximal ideal M of R is sharp if and only if M is an isolated point in* $\mathrm{Max}^{-1}(R)$*; M is dull if and only if M is a limit point in* $\mathrm{Max}^{-1}(R)$.
(3) *If I is a nonzero proper ideal of R, then the Zariski and inverse topologies on* $\mathrm{Spec}(R/I)$ *coincide.*
(4) *The Zariski and inverse topologies on* $\mathrm{Max}(R)$ *coincide if and only if $J(R) \neq 0$.*

Proof. (1) Suppose there is a finitely generated ideal I of R contained in M but in no other maximal ideal in X. Then $\{M\} = \{N \in X \cup \{M\} : I \subseteq N\}$ is open in $X \cup \{M\}$, so M is an isolated point in $X \cup \{M\}$. Conversely, if M is an isolated point in $X \cup \{M\}$, then there is a finitely generated ideal J of R such that $\{M\} = \{N \in X \cup \{M\} : J \subseteq N\}$.

(2) This follows from (1).

(3) Without loss of generality we assume that I is a radical ideal of R. For each ideal K of R, let $V_I(K) = \{P \in \mathrm{Spec}(R) : I + K \subseteq P\}$ and $U_I(K) = \{P \in \mathrm{Spec}(R) : I \subseteq P \text{ and } K \not\subseteq P\}$. To prove (3), it suffices to show that for each finitely generated ideal J of R, there is a finitely generated ideal K of R such that $V_I(J) = U_I(K)$. Let J be a finitely generated ideal of R. Since I is a nonzero radical ideal and R/I has Krull dimension 0, R/I is a von Neumann regular ring. Thus $(J + I)/I$ is a summand of R/I, so that there exists an ideal L of R containing I such that $(J + I) \cap L = I$ and $J + L = R$. This implies there is a finitely generated ideal

$K \subseteq L$ such that $J \cap K \subseteq I$ and $J + K = R$. Therefore, $V_I(J) = U_I(K)$, which proves the claim.

(4) Suppose that the Zariski and inverse topologies on $\mathrm{Max}(R)$ coincide. Note that $\mathrm{Max}(R)$ is compact with respect to the Zariski topology. Therefore, $\mathrm{Max}^{-1}(R)$ is compact. Since $\mathrm{Max}(R) = \bigcup_{a \in R \setminus \{0\}} \{M \in \mathrm{Max}(R) : a \in M\}$, we infer that $\mathrm{Max}(R) = \bigcup_{a \in E} \{M \in \mathrm{Max}(R) : a \in M\}$ for some finite nonempty $E \subseteq R \setminus \{0\}$. Observe that $0 \neq \prod_{a \in E} a \in J(R)$.

To prove the converse, suppose that $J(R) \neq 0$. By (3) it follows that the Zariski and inverse topologies on $\mathrm{Spec}(R/J(R))$ coincide, and hence the Zariski and inverse topologies on $\mathrm{Max}(R)$ coincide. $\square$

Definition 6.4. Let X be a topological space. Define $X^0 = X$, and for each ordinal number α let $X^{\alpha+1}$ denote the set of limit points of X^α. For each limit ordinal λ, let $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$. The closed set X^α of X is the α-th *Cantor-Bendixson derivative* of X. The *Cantor-Bendixson rank* of X, rk(X), is the smallest ordinal number α for which $X^\alpha = X^{\alpha+1}$. The topological space X is *scattered* if $X^{\mathrm{rk(X)}} = \emptyset$ (equivalently, every nonempty subspace of X contains an isolated point).

By Lemma 6.3(2), if $X = \mathrm{Max}^{-1}(R)$, the set X^1 of limit points of X is $\mathcal{M}_\dagger(R)$. Consequently, $R_1 = \bigcap_{M \in X^1} R_M$. We extend this to all ordinals in the next lemma. To do so, we recall that if R is a one-dimensional Prüfer domain and S is a ring between R and its quotient field such that S is not a field, then S is also a one-dimensional Prüfer domain with $\mathrm{Max}(S) = \{MS : M \in \mathrm{Max}(R), MS \neq S\}$ [18, Theorem 1].

Lemma 6.5. *Let R be a one-dimensional Prüfer domain with quotient field F, let $X = \mathrm{Max}^{-1}(R)$ and let α be an ordinal number. Then*

(1) $\mathrm{Max}(R_\alpha) \setminus \{0\} = \{MR_\alpha : M \in X^\alpha\}$,
(2) $X^\alpha = \{M \in X : MR_\alpha \neq R_\alpha\}$, *and*
(3) $R_\alpha = \bigcap_{M \in X^\alpha} R_M$.

Thus if β is an ordinal, $X^\alpha = X^\beta$ if and only if $R_\alpha = R_\beta$.

Proof. (1) The proof is by transfinite induction. Suppose (1) holds for an ordinal α. Let $N \in \mathrm{Max}(R_{\alpha+1}) \setminus \{0\}$. Then, as discussed before the lemma, $N = MR_{\alpha+1}$ for some maximal ideal M of R. Thus $MR_\alpha \neq R_\alpha$, and so $MR_\alpha \in \mathrm{Max}(R_\alpha) \setminus \{0\}$, which by the induction hypothesis implies $M \in X^\alpha$. Moreover, since the maximal ideal $MR_{\alpha+1}$ of $R_{\alpha+1}$ is extended from a maximal ideal of R_α, MR_α is necessarily dull in R_α [32, Lemma 2.1]. Therefore, every finitely generated ideal of R that is contained in M is contained in some other maximal ideal of R_α. Since $\mathrm{Max}(R_\alpha) \setminus \{0\} = \{LR_\alpha : L \in X^\alpha\}$, every finitely generated ideal of R that is contained in M is contained in some other maximal ideal in X^α. By Lemma 6.3(1), M is a limit point in X^α, so that $M \in X^{\alpha+1}$. This shows that $\mathrm{Max}(R_{\alpha+1}) \setminus \{0\} \subseteq \{MR_\alpha : M \in X^{\alpha+1}\}$.

Conversely, suppose $M \in X^{\alpha+1}$. Then $M \in X^\alpha$, so since (1) holds for α, MR_α is a maximal ideal of R_α. Since M is a limit point in X^α, Lemma 6.3(1) implies that every finitely generated ideal of R contained in M is contained in some other maximal ideal in X^α. Let $I = (x_1, \ldots, x_n)R_\alpha$ be a finitely generated ideal of R_α contained in MR_α, and let $J = (x_1, \ldots, x_n)R \cap R$. Then J is a finitely generated ideal of R [16, Exercise 1.1, p. 95] and $I = JR_\alpha$ [16, Claim (A), p. 95]. Since J is contained in M, J is contained in some other maximal ideal in X^α. Since (1) holds for α, it follows that $I = JR_\alpha$ is contained in some other maximal ideal in $\mathrm{Max}(R_\alpha)$. Therefore, MR_α is a dull maximal ideal of R_α, so that $MR_{\alpha+1} \neq R_{\alpha+1}$ and hence $MR_{\alpha+1} \in \mathrm{Max}(R_{\alpha+1})\backslash\{0\}$. This proves $\mathrm{Max}(R_{\alpha+1})\backslash\{0\} = \{MR_{\alpha+1} : M \in X^{\alpha+1}\}$, which verifies (1) for $\alpha + 1$.

Next, suppose that λ is a limit ordinal and (1) holds for every $\alpha < \lambda$. Let M be a maximal ideal of R. Then $MR_\lambda \neq R_\lambda$ if and only if $MR_\alpha \neq R_\alpha$ for each $\alpha < \lambda$, if and only if (by (1)) $M \in \bigcap_{\alpha<\lambda} X^\alpha$, if and only if $M \in X^\lambda$. Therefore, $\mathrm{Max}(R_\lambda) \setminus \{0\} = \{MR_\lambda : M \in \mathrm{Max}(R), MR_\lambda \neq R_\lambda\} = \{MR_\lambda : M \in X^\lambda\}$. This completes the induction and the proof of (1).

(2) If $M \in X^\alpha$, then, by (1), $MR_\alpha \neq R_\alpha$. Conversely, if $M \in \mathrm{Max}(R)$ and $MR_\alpha \neq R_\alpha$, then, as discussed before the lemma, MR_α is a maximal ideal of R_α and hence by (1), $M \in X^\alpha$.

(3) This follows from (1).

The final assertion follows from (2) and (3). □

Theorem 6.6. *Let R be a one-dimensional Prüfer domain, let $X = \mathrm{Max}^{-1}(R)$ and let α be an ordinal number.*

1. *The ring R has sharp degree α if and only if $\mathrm{rk(X)} = \alpha + 1$ and $X^{\alpha+1} = \emptyset$. Thus if R has a sharp degree, then X is a scattered space.*
2. *If X is a scattered space and $J(R) \neq 0$, then R has a sharp degree.*
3. *The ring R has dull degree α if and only if $\mathrm{rk(X)} = \alpha$ and $X^\alpha \neq \emptyset$. Thus R has a dull degree if and only if X is not a scattered space.*

Proof. (1) Suppose that R has sharp degree α. Then $R_\alpha \subsetneq R_{\alpha+1} = F$, so that by Lemma 6.5, $X^{\alpha+1} \subsetneq X^\alpha$ and $X^{\alpha+1} = \emptyset$. Hence X is scattered and $\mathrm{rk(X)} = \alpha + 1$. Conversely, if $X^{\alpha+1} = \emptyset$ and $\mathrm{rk(X)} = \alpha + 1$, then Lemma 6.5 implies that $R_\alpha \subsetneq R_{\alpha+1} = F$, and hence R has sharp degree α.

(2) Suppose X is scattered and $J(R) \neq 0$, and let α be the smallest ordinal such that $X^\alpha = \emptyset$. It follows from Lemma 6.3(4) that X is compact. If α is a limit ordinal, then $\bigcap_{\beta<\alpha} X^\beta = \emptyset$, contrary to the fact that X is compact and the X_β are nonempty. Therefore, α is a successor ordinal, say $\alpha = \beta^+$, and $\mathrm{rk(X)} = \alpha = \beta + 1$ with $X^{\beta+1} = \emptyset$. Thus R has sharp degree β.

(3) Suppose that R has dull degree α. Then $R_\alpha = R_{\alpha+1} \neq F$ and $R_\beta \subsetneq R_\alpha$ for all $\beta < \alpha$. Thus, by Lemma 6.5, $\emptyset \neq X^{\alpha+1} = X^\alpha \subsetneq X_\beta$ for all $\beta < \alpha$. Therefore, $\mathrm{rk(X)} = \alpha$. Conversely, suppose that $\mathrm{rk(X)} = \alpha$ and $X^\alpha \neq \emptyset$. By Lemma 6.5, for each $\beta < \alpha$, $R_\beta \subsetneq R_\alpha = R_{\alpha+1} \neq F$. Thus R has dull degree α. The last statement follows from the first. □

Combining Theorem 6.6 with results from [25], we see in the next corollary that the ideals of almost Dedekind domains possessing a sharp degree can be decomposed as a irredundant intersections of powers of maximal ideals.

Corollary 6.7. *Let R be an almost Dedekind domain that is not a field. If R has a sharp degree, then every nonzero proper ideal of R has a unique representation as an irredundant intersection of powers of distinct maximal ideals.*

Proof. Suppose that R is an almost Dedekind domain with sharp degree α for some ordinal α. By Theorem 6.6, $\mathrm{Max}^{-1}(R)$ is scattered, and hence so is $\mathrm{Max}^{-1}(R/I)$ for every nonzero ideal I of R. Since Lemma 6.3(3) implies that the Zariski and inverse topologies agree on $\mathrm{Max}(R/I)$, we have that $\mathrm{Max}(R/I)$ is scattered. It follows now from [25, Corollary 3.9] that every nonzero proper ideal of R can be represented uniquely as an irredundant intersection of completely irreducible ideals. □

Corollary 6.8. *If R is a one-dimensional Prüfer domain such that $J(R) \neq 0$ and $X = \mathrm{Max}^{-1}(R)$ is countable, then there is a countable ordinal α such that R has sharp degree α and X^{α} is finite.*

Proof. Let $X = \mathrm{Max}^{-1}(R)$. Since the space X is countable, compact and Hausdorff, rk(X) is a countable successor ordinal $\alpha + 1$ such that X^{α} is finite and $X^{\alpha+1} = \emptyset$ [34, p. 18]. By Theorem 6.6, R has sharp degree α. □

Corollary 6.9. *If a one-dimensional Prüfer domain R with $J(R) \neq 0$ has a dull degree, then* $\mathrm{Max}(R)$ *is an uncountable set.*

Proof. If $\mathrm{Max}(R)$ is countable, then, by Corollary 6.8, R has a sharp degree, in which case R does not have dull degree. □

In the next corollary we recover Loper and Lucas' existence results for almost Dedekind domains with specified sharp degrees (see [32, Section 3]). In their case they restrict to finite sharp degrees; in ours, we allow sharp degrees of any ordinal.

Corollary 6.10. *Every ordinal number occurs as the sharp degree of an SP-domain.*

Proof. Note that every Dedekind domain that is not a field is an SP-domain that has sharp degree 0. Let α be a nonzero ordinal number, and let ω denote the first infinite ordinal. The ordinal space (hence scattered space) $X = \omega^{\alpha} + 1$ satisfies rk(X) $= \alpha + 1$ by [3, Theorem 2.6(1)]. Since $\omega^{\alpha} + 1$ is a successor ordinal, the ordinal space X is compact. Thus X is a Boolean space, and by [38, Section 3] there exists an SP-domain R with $J(R) \neq 0$ such that $\mathrm{Max}(R)$ is homeomorphic to X. Therefore, the corollary follows from Theorem 6.6(1). □

By restricting to countable ordinals, we are also able to prescribe the size of the maximal spectrum of the penultimate ring in the sequence of the R_{α}'s.

Corollary 6.11. *For every countable ordinal α and finite positive integer n, there exists an SP-domain R of sharp degree α such that $J(R) \neq 0$,* $\mathrm{Max}(R)$ *is countable and* $|\mathrm{Max}(R_{\alpha})| = n$.

Proof. Let ω denote the first infinite ordinal. The ordinal space $X = \omega^\alpha \cdot n + 1$ is then a countable compact Hausdorff space for which X^α has n elements and $X^{\alpha+1} = \emptyset$ [34]. Every ordinal space is totally disconnected and Hausdorff, and since the ordinal $\omega^\alpha \cdot n + 1$ is a successor ordinal, X is compact. Thus X is a Boolean space and the proof now finishes just as the proof of Corollary 6.10. □

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References

[1] D. D. Anderson and B. G. Kang, Pseudo-Dedekind domains and divisorial ideals in $R[X]_T$, J. Algebra **122** (1989), no. 2, 323–336.

[2] G. M. Bergman, Boolean rings of projection maps, J. London Math. Soc. (2) **4** (1972), 593–598.

[3] G. Bezhanishvili and P. J. Morandi, Scattered and hereditarily irresolvable spaces in modal logic, Arch. Math. Logic **49** (2010), no. 3, 343–365.

[4] G. Bezhanishvili, V. Marra, P. J. Morandi and B. Olberding, Idempotent generated algebras and Boolean powers of commutative rings, Algebra Universalis **73** (2015), no. 2, 183–204.

[5] G. Birkhoff, *Lattice theory.* Corrected reprint of the 1967 third edition. American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, R.I., 1979.

[6] N. Bourbaki, *Commutative Algebra, Chapters 1–7.* Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998.

[7] J. Brewer and L. Klingler, The ordered group of invertible ideals of a Prüfer domain of finite character, Comm. Alg. **33** (2005), 4197–4203.

[8] H. S. Butts and R. W. Yeagy, Finite bases for integral closures, J. Reine Angew. Math **282** (1976), 114–125.

[9] P. J. Cahen and J. L. Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs, 48. American Mathematical Society, Providence, RI, 1997.

[10] L. Claborn, Specified relations in the ideal group, Michigan Math. J. **15** (1968), 249–255.

[11] P. F. Conrad and D. McAlister, The completion of a lattice ordered group, J. Australian Math. Soc. **9** (1969), 182–208.

[12] M. Fontana and K. A. Loper, An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations, in *Multiplicative Ideal Theory in Commutative Algebra* (J.W. Brewer, S. Glaz, W. Heinzer and B. Olberding, Editors), Springer (2006), 169–187.

[13] M. Fontana, E. Houston and T. Lucas, Factoring ideals in integral domains. Lecture Notes of the Unione Matematica Italiana, 14. Springer, Heidelberg; UMI, Bologna, 2013.

[14] M. Fontana, J. Huckaba and I. Papick, *Prüfer domains.* Monographs and Textbooks in Pure and Applied Mathematics, 203. Marcel Dekker, Inc., New York, 1997.

[15] R. M. Fossum, *The divisor class group of a Krull domain*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74, Springer-Verlag, New York-Heidelberg, 1973.

[16] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, Mathematical Surveys and Monographs 84, American Mathematical Society, Providence, RI, 2001.

[17] A. Grams, Atomic rings and the ascending chain condition for principal ideals, Proc. Cambridge Philos. Soc. **75** (1974), 321–329.

[18] R. Gilmer, Overrings of Prüfer domains, J. Algebra **4** (1966), 331–340.

[19] R. Gilmer and W. J. Heinzer, Overrings of Prüfer domains II, J. Algebra **7** (1967), 281–302.

[20] R. Gilmer and A. Grams, The equality $(A \cap B)^n = A^n \cap B^n$ for ideals, Canad. J. Math. **24** (1972), 792–798.

[21] R. Gilmer, *Multiplicative ideal theory.* Corrected reprint of the 1972 edition. *Queen's Papers in Pure and Applied Mathematics*, 90. Queen's University, Kingston, ON, 1992.

[22] R. Gilmer, Prüfer domains and rings of integer-valued polynomials, J. Algebra **129** (1990), no. 2, 502–517.

[23] A. M. Gleason, Projective topological spaces, Illinois. J. Math. **2** (1958), 482–489.

[24] R. Hasenauer, Normsets of almost Dedekind domains and atomicity, J. Commut. Alg., to appear.

[25] W. Heinzer and B. Olberding, Unique irredundant intersections of completely irreducible ideals, J. Algebra **287** (2005), 432–448.

[26] O. A. Heubo-Kwegna, Kronecker function rings of transcendental field extensions, Comm. Algebra **38** (2010), no. 8, 2701–2719.

[27] P. Johnstone, *Stone spaces.* Reprint of the 1982 edition. Cambridge Studies in Advanced Mathematics, 3. Cambridge University Press, Cambridge, 1986.

[28] M. Knox and W. W. McGovern, Rigid extension of ℓ-groups of continuous functions, Czech. Math. J. **58** (2008), no. 4, 993–1014.

[29] S. Koppelberg, *Handbook of Boolean Algebras,* Vol. 1, North Holland, 1989.

[30] G. Leloup, Preorders, rings, lattice-ordered groups and formal power series, in *Valuation theory and its applications II* (F.V. Kuhlmann, S. Kuhlmann, M. Marshall, editors), American Mathematical Society (2003) 157–174.

[31] K. A. Loper, Almost Dedekind domains which are not Dedekind, in *Multiplicative Ideal Theory in Commutative Algebra* (J.W. Brewer, S. Glaz, W. Heinzer and B. Olberding, Editors), Springer (2006), 279–292.

[32] K. A. Loper and T. G. Lucas, Factoring ideals in almost Dedekind domains. J. Reine Angew. Math. **565** (2003), 61–78.

[33] H. Matsumura, *Commutative ring theory,* Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, 1986.

[34] S. Mazurkiewicz and W. Sierpiński, Contribution à la topologie des ensembles dénombrables, Fund. Math. **1** (1920), 17–27.

[35] W. W. McGovern, Bézout SP-Domains, Comm. Alg. **35** (2007), 1777–1781.

[36] J. Močkoř, *Groups of divisibility.* Mathematics and its Applications (East European Series). D. Reidel Publishing Co., Dordrecht, 1983.

[37] J. Ohm, Semi-valuations and groups of divisibility. Canad. J. Math. **21** (1969), 576–591.

[38] B. Olberding, Factorization into Radical Ideals, in *Arithmetical properties of commutative Rings and monoids* (S. Chapman, editor), Lect. Notes in Pure Appl. Math. Vol 241. Chapman & Hall, 363–377, 2005.

[39] B. Olberding, Holomorphy rings in function fields, in *Multiplicative ideal theory in commutative algebra*, 331-348, Springer-Verlag, 2006.

[40] B. Olberding, On Matlis domains and Prüfer sections of Noetherian domains, in *Commutative algebra and its applications*, 321–332, Walter de Gruyter, Berlin, 2009.

[41] J. R. Porter and R. G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer, New York, 1988.

[42] A. Reinhart, Radical factorial monoids and domains, Ann. Sci. Math. Québec **36** (2012), no. 1, 193–229.

[43] A. Reinhart, On monoids and domains whose monadic submonoids are Krull, in *Commutative Algebra - Recent Advances in Commutative Rings, Integer-valued Polynomials, and Polynomial Functions* (M. Fontana, S. Frisch, and S. Glaz, editors), Springer, 2014.

[44] P. Ribenboim, Boolean powers, Fund. Math. **65** (1969), 243–268.

[45] W. Rump and Y. C. Yang, Jaffard-Ohm correspondence and Hochster duality, Bull. London Math. Soc. **40** (2008), no. 2, 263–273.

[46] N. H. Vaughan and R. W. Yeagy, Factoring ideals into semiprime ideals, Canad. J. Math. **30** (1978), no. 6, 1313–1318.

[47] R. W. Yeagy, Semiprime factorizations in unions of Dedekind domains, J. Reine Angew. Math. **310** (1979), 182–186.

[48] M. Zafrullah, On generalized Dedekind domains, Mathematika **33** (1986), no. 2, 285–295.

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