MULTIPLICATIVE LATTICES WITH ABSORBING FACTORIZATION

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Abstract. In [22], Yassine et al. introduced the notion of 1-absorbing prime ideals in commutative rings with nonzero identity. In this article, we examine the concept of 1-absorbing prime elements in C-lattices. We investigate the C-lattices in which every element is a finite product of 1-absorbing prime elements (we denote them as OAFLs for short). Moreover, we study C-lattices having 2-absorbing factorization (we denote them as TAFLs for short).

1. Introduction

Let L be a set together with an inner binary operation $\cdot$ on L and a partial order $\leq$ on L such that $(L,\cdot)$ is a monoid (i.e., $(L,\cdot)$ a commutative semigroup with identity) and $(L,\leq)$ is a complete lattice (i.e., each subset of L has both a supremum and an infimum with respect to $\leq$). For each subset $E\subseteq L$, we let $\bigvee E$ denote the supremum of E, called the join of E and we let $\bigwedge E$ denote the infimum of E, called the meet of E. For elements $a,b\in L$, let $a\vee b=\bigvee\{a,b\}$ and $a\wedge b=\bigwedge\{a,b\}$. Moreover, set $1=\bigvee L$ and set $0=\bigwedge L$. We say that $(L,\cdot,\leq)$ is a *multiplicative lattice* if for all $x\in L$ and $E\subseteq L$, it follows that $1x=x$ and $x\bigvee E=\bigvee\{xe\mid e\in E\}$.

We recall a few important situations in which multiplicative lattices occur. In what follows, we use the definitions of star operations, ideal systems and the specific star operations/ideal systems v, t and w without further mention. For more information on star operations see [12] and for more information on ideal systems see [14]. A profound introduction and study of the w-operation can be found in [21].

- It is well-known that if R is a commutative ring with identity, L is the set of ideals of R and $\cdot: L\times L\to L$ is the ideal multiplication on L, then $(L,\cdot,\subseteq)$ is a multiplicative lattice.
- Let D be an integral domain and let $*$ be a star operation on D. Let L be the set of $*$-ideals of D together with the $*$-multiplication $\cdot_*: L\times L\to L$. Then $(L,\cdot_*,\subseteq)$ is a multiplicative lattice.
- Let H be a commutative cancellative monoid and let r be an ideal system on H. Let L be the set of r-ideals of H and let $\cdot_r: L\times L\to L$ be the r-multiplication. Then $(L,\cdot_r,\subseteq)$ is a multiplicative lattice.

Let L be a multiplicative lattice and let $e\in L$. For $a,b\in L$, we set $(a:b)=\bigvee\{x\in L\mid xb\leq a\}$. Then e is called *weak meet principal* if $a\wedge e=(a:e)e$ for each $a\in L$ and e is called *weak join principal* if $(be:e)=(0:e)\vee b$ for each $b\in L$. Furthermore, e is said to be *meet principal* if $a\wedge be=((a:e)\wedge b)e$ for all $a,b\in L$ and e is said to be *join principal* if $((a\vee be):e)=(a:e)\vee b$ for all $a,b\in L$. We say that e is *weak principal* if e is both weak meet principal and weak join principal. Finally, e is said to be *principal* ([9]) if e is both meet principal and join principal. An element $a\in L$ is said to be *compact* if for each subset $F\subseteq L$ with $a\leq\bigvee F$, it follows that $a\leq\bigvee E$ for some finite subset E of F. A subset $C\subseteq L$ is called *multiplicatively closed* if $1\in C$ and $xy\in C$ for each $x,y\in C$. A multiplicative lattice L is called a *C-lattice* if L is generated under joins by a multiplicatively closed subset C of compact elements. Note that a finite product of compact elements in a C-lattice is again compact. By L_* we denote the set of all compact elements of L. We say that L is *principally generated* if every element of L is the join of a set of principal elements of L. It is well-known (see [3, Theorem 1.3]) that each principal element of a

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C-lattice is compact. Moreover, L is said to be *join-principally generated* if each element of L is the join of a set of join principal elements of L. Additionally, a lattice L is called a principal element lattice if every element in L is principal [4].

Let R be a commutative ring with identity, let D be an integral domain, let H be a commutative cancellative monoid, let $*$ be a star operation on D and let r be an ideal system on H. Note that the lattice of ideals of R is a principally generated C-lattice. The lattice of $*$-ideals of D is a C-lattice if and only if $*$ is a star operation of finite type. In analogy, it follows that the lattice of r-ideals of H is a C-lattice if and only if r is a finitary ideal system. Observe that the lattice of v-ideals of D (or of H) can fail to be a C-lattice. Also note that even if $*$ is of finite type (resp. r is finitary), then the lattice of $*$-ideals of D (resp. the lattice of r-ideals of H) need not be (join-)principally generated. For instance, the t-operation is of finite type (resp. the t-system is finitary), but the lattice of t-ideals is (in general) not (join-)principally generated. We also want to emphasize that the lattice of w-ideals of D (resp. of H) is a principally generated C-lattice.

An element $a \in L$ is said to be *proper* if $a < 1$, it is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$ and it is called *comparable* if $a \leq b$ or $b \leq a$ for each $b \in L$. For each $a \in L$, $L/a = \{b \in L \mid a \leq b\}$ is a multiplicative lattice with the multiplication $c \circ d = (cd) \vee a$ for elements $c, d \in L/a$. A proper element $p \in L$ is called *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. A proper element $m \in L$ is said to be *maximal* in L if for each $x \in L$, $m < x \leq 1$ implies $x = 1$. One can easily see that maximal elements are prime. For each $a \in L$, let $\min(a)$ be the set of prime elements of L that are minimal above a. The lattice L is called a *lattice domain* if 0 is a prime element. $\mathrm{J}(L)$ is defined as the meet of all maximal elements of L. For $a \in L$, we define $\sqrt{a} = \bigwedge\{p \in L \mid p \text{ is prime and } a \leq p\}$. Note that in a C-lattice L, $\sqrt{a} = \bigwedge\{p \in L \mid a \leq p \text{ is a minimal prime over } a\} = \bigvee\{x \in L_* \mid x^n \leq a \text{ for some } n \in \mathbb{N}\}$. A proper element $q \in L$ is called *primary* if $ab \leq q$ implies $a \leq q$ or $b \leq \sqrt{q}$ for every $a, b \in L$. It is well-known that C-lattices can be localized at arbitrary multiplicatively closed subsets S of compact elements as follows. The localization of $a \in L$ at S is defined as $a_S = \bigvee\{x \in L \mid xs \leq a \text{ for some } s \in S\}$. The multiplication on $L_S = \{a_S \mid a \in L\}$ is defined by $a \circ_S b = (ab)_S$ for all $a, b \in L_S$. Let $p \in L$ be a prime element and $S = \{x \in L_* \mid x \nleq p\}$. Then the set S is a multiplicatively closed subset of L. In this case, the localization L_S is denoted by L_p. It is well-known that $(L_p)_* = \{a_p \in L_p \mid a \in L_*\}$. Using this, it can be shown that if L is a (principally generated) C-lattice, then L_p is also a (principally generated) C-lattice for any prime element $p \in L$ (see [2, Theorem 2.9]). It can also be proved that in a C-lattice L, for all $a, b \in L$, $(ab)_m = (a_m b_m)_m$ for each maximal element $m \in L$ and also, $a = b$ if and only if $a_n = b_n$ for all maximal elements $n \in L$. For more information on localization, see [2, 3, 9, 17].

In [22], the authors introduced the concept of 1-absorbing prime ideals in commutative rings with identity. These ideals are generalizations of prime ideals and many authors studied them from different points of view (see [8]). The first aim of this paper is to study 1-absorbing prime elements in C-lattices. Another (well-known) generalization of 1-absorbing prime ideals are 2-absorbing ideals. They have first been mentioned in [7] and in [18], the authors introduced 2-absorbing elements in multiplicative lattices.
The aforementioned concepts are part of the more general definition, namely that of n-absorbing ideals. These types of ideals were introduced and studied by Anderson and Badawi (see [6]). It turns out that n-absorbing ideals are not just interesting objects in multiplicative ideal theory, but also in factorization theory. For instance, there is an important connection between n-absorbing ideals and the ω-invariant in factorization theory (see [6]). For a profound discussion of the ω-invariant, we refer to [13].
We want to emphasize that the commutative rings in which each ideal is a finite product of 1-absorbing prime ideals (resp. 2-absorbing ideals, resp. n-absorbing ideals) have already been studied (see [1, 11, 18]). The main goal of this paper is to consider principally generated C-lattices in which various types of elements can be written as finite products of 1-absorbing prime elements or 2-absorbing elements.

We continue with a few more basic definitions that will be needed in the sequel. L is said to be a *field* if $L = \{0, 1\}$ and L is called a *quasi-local* lattice if 1 is compact and L has a unique maximal element.

The dimension of L, denoted by $\dim(L)$, is defined to be $\sup\{n \in \mathbb{N} \mid$ there exists a strict chain of prime elements of L of length $n\}$. If $\dim(L) = 0$, then L is said to be a *zero-dimensional* lattice. Note that L is a zero-dimensional lattice if and only if every prime element of L is maximal. We say that a multiplicative lattice is *Noetherian* if every element of L is compact (see [16, page 352]). A multiplicative lattice is said to be *Prüfer* lattice if every compact element of L is principal. (For more information about Prüfer lattices, see [3, Theorem 3.4].) A *ZPI-lattice* is a multiplicative lattice in which every element is a finite product of prime elements [15]. A multiplicative lattice L is said to be a *Q-lattice* if every element is a finite product of primary elements [20]. A principally generated lattice domain L is called *unique factorization lattice domain* if every principal element of L is a finite product of principal prime elements.

Our paper is organized as follows. In Section 2, we study the concept of 1-absorbing prime elements (OA-elements). The relationships among prime elements, primary elements, and TA-elements are studied in Examples 2.2 and 2.3. Propositions 2.7 and 2.8, along with Corollary 2.9, demonstrate that the concepts of prime elements and OA-elements coincide in C-lattices that are not quasi-local. In Section 3, we study C-lattices whose elements have a TA-factorization. We call a C-lattices a *TA-factorization lattice* (abbreviated as TAFL) if every element possesses a TA-factorization. In Proposition 3.4, we get that $\dim(L) \leq 1$ if L is a principally generated TAFL. In Theorem 3.5, we obtain that a TAFL is ZPI-lattice domain if it is a Prüfer lattice domain. Then, we study the factorization of C-lattices by assuming that all compact elements of L have a factorization into TA-elements, denoted by *CTAFL*. Finally, we explore the factorization of C-lattices by assuming that all principal elements of L have a factorization into TA-elements, denoted by *PTAFL*. In Theorem 3.11, we have that if (L, m) is a quasi-local principally generated C-lattice domain, then L is a TAFL if and only if L is a PTAFL and $\dim(L) \leq 1$. In Section 4, we study the factorization of L with respect to the OA-element concept, similar to Section 3. We study C-lattices as a *OA-factorization lattice* (abbreviated as OAFL) if every element possesses an OA-factorization. Then, we examine the factorization of C-lattices by assuming that all compact elements of L have a factorization into OA-elements, denoted by *COAFL*. Finally, we explore the factorization of C-lattices by assuming that all principal elements of L have a factorization into OA-elements, denoted by *POAFL*. Among the many results, in Theorem 4.13, we characterize OAFL, COAFL and lattices which of the join of any two principal elements has an OA-factorization. In Theorem 4.13, we also see that if L is an OAFL, then it satisfies one of the following conditions.

- **i.** L is a ZPI-lattice.
- **ii.** L is a quasi-local lattice, m^2 is comparable and m is a nilpotent element.
- **iii.** L is a quasi-local lattice domain, m^2 is comparable and $\bigwedge_{n\in\mathbb{N}} m^n = 0$.

In Theorem 4.14, we conclude that the following statements are equivalent: L is a ZPI-lattice if and only if L is a Prüfer OAFL if and only if L is a Prüfer POAFL. Theorem 4.15 establishes relationships among the concepts of OAFL, COAFL, TAFL and CTAFL.

2. On 1-absorbing prime elements of C-lattices

Definition 2.1. Let L be a C-lattice. A proper element $x \in L$ is called a 1*-absorbing prime element* or an *OA-element* if for all $a, b, c \in L \setminus \{1\}$, $abc \leq x$ implies that $ab \leq x$ or $c \leq x$.

It follows immediately from the definition that every OA-element is both a TA-element and a primary element. Moreover, every prime element is an OA-element. We infer that the class of OA-elements of L lies between the classes of prime elements and TA-elements and also between the classes of prime elements and primary elements.
Let L be a C-lattice and let $a \in L$. We obtain the following irreversible right arrows:

(1) a is a prime element $\Rightarrow$ a is an OA-element $\Rightarrow$ a is a primary element.
(2) a is a prime element $\Rightarrow$ a is an OA-element $\Rightarrow$ a is a TA-element.

We give some examples to show that these arrows are not reversible.

Example 2.2. {This example is inspired by [10, Example 7]}. Let L be a C-lattice, which having underlying set $\{0,1,a,b,c,d\}$ ordered by $a \leq b \leq d$ and $a \leq c \leq d$, with multiplication $xy = a$ for all $x, y \in \{a,b,c,d\}$. The prime elements of L are 0 and d. Moreover, L is a quasi-local lattice. Note that b is an OA-element of L that is not a prime element. In particular, b is a primary TA-element of L.

Example 2.3. We demonstrate that, in general, neither TA-elements nor primary elements are OA-elements. Let $\mathrm{I}(\mathbb{Z})$ be the lattice of ideals of $\mathbb{Z}$. Note that (15) is a TA-element of $\mathrm{I}(\mathbb{Z})$ that is not an OA-element of $\mathrm{I}(\mathbb{Z})$. Furthermore, (8) is a primary element of $\mathrm{I}(\mathbb{Z})$ that fails to be an OA-element of $\mathrm{I}(\mathbb{Z})$.

Lemma 2.4. *Let L be a C-lattice. An element $x \in L$ is an OA-element if and only if for all $a, b, c \in L_* \setminus \{1\}$, $abc \leq x$ implies that $ab \leq x$ or $c \leq x$.*

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) Let $abc \leq x$ and $ab \nleq x$ for some $a, b, c \in L \setminus \{1\}$. We show that $c \leq x$ to complete the proof. Since $abc \leq x$, then $a'b'c' \leq x$ for all compact elements $a\prime, b', c' \in L$ with $a' \leq a$, $b' \leq b$ and $c' \leq c$. Since $ab \nleq x$, then there are some compact elements $a_1, b_1 \in L$ such that $a_1 \leq a$, $b_1 \leq b$ and $a_1 b_1 \nleq x$. Let $a_2 = a' \vee a_1$ and $b_2 = b' \vee b_1$. It is clear that a_2 and b_2 are compact. Obviously, there is a compact element $c^* \in L$ with $c^* \leq c$. Note that $(a' \vee a_1)(b' \vee b_1)c^* \leq x$ and $(a' \vee a_1)(b' \vee b_1) \nleq x$. We obtain that $c^* \leq x$, and thus $c \leq x$. Therefore, x is an OA-element. □

Proposition 2.5. *Let L be a C-lattice and let $x \in L$.*

(1) *If x is an OA-element of L, then $\sqrt{x}$ is a prime element of L with $(\sqrt{x})^2 \leq x$.*
(2) *If x is an OA-element of L, then $(x : a)$ is a prime element of L for each $a \in L$ with $a \nleq x$.*
(3) *If $(p^2 : a) \leq x$ for every compact element $a \leq p$, $a \nleq x$ and x is a p-primary element of L, then x is an OA-element of L.*

Proof. (1) Let x be an OA-element of L and let $a, b \in L$ be such that $ab \leq \sqrt{x}$. There is a positive integer n such that $a^n b^n \leq x$. We can write $a^m a^{n-m} b^n \leq x$ for a positive integer m with $m < n$. By the assumption, $a^n \leq x$ or $b^n \leq x$. Then $a \leq \sqrt{x}$ or $b \leq \sqrt{x}$, and thus $\sqrt{x}$ is prime. Now we will show that $(\sqrt{x})^2 \leq x$. Let $a, b \in L$ be such that $a, b \leq \sqrt{x}$. Then there is an $n \in \mathbb{N}$ with $a^n \leq x$. If $n = 1$, then we are done. Let $n > 2$. Then $a^{n-2}aa \leq x$, and so $a^2 \leq x$. Similarly, we have that $b^2 \leq x$. Note that $a(a \vee b)b \leq x$. Then $ab \leq a(a \vee b) \leq x$ or $ab \leq b \leq x$. In any case, we have that $ab \leq x$. Therefore, $(\sqrt{x})^2 \leq x$.

(2) Let x be an OA-element of L and let $b, c \in L$ be such that $bc \leq (x : a)$. Then $abc \leq x$. By the assumption, $ab \leq x$ or $c \leq x$. Therefore, $b \leq (x : a)$ or $c \leq (x : a)$.

(3) Let $a, b, c \in L_*$ be such that $abc \leq x$ and $a \nleq x$. By assumption, $bc \leq \sqrt{x} = p$. Therefore, we obtain that $abc \leq p^2$, and thus $bc \leq (p^2 : a) \leq p$. We infer that $bc \leq x$, and hence x is an OA-element. □

Lemma 2.6. *Let L be a C-lattice. If $w \vee u \neq 1$ for some distinct proper elements $u, w \in L$, then L is quasi-local.*

Proof. Let $w \vee u \neq 1$ be distinct proper elements $u, w \in L$. Assume that L is not a quasi-local lattice. There are at least two distinct maximal elements $m_1, m_2 \in L$ such that $m_1 \vee m_2 = 1$, a contradiction. Therefore, L is quasi-local. □

Proposition 2.7. *Let L be a C-lattice and let $x \in L$. If x is an OA-element of L that is not prime, then L is quasi-local.*

Proof. Let x be an OA-element of L that is not a prime. By the assumption, $cd \leq x$ for some $c, d \in L$ implies neither $c \leq x$ nor $d \leq x$. If $w \vee u \neq 1$ for each distinct proper elements $w, u \in L$, then we are done by Lemma 2.6. Assume that $w \vee u = 1$ for two distinct proper elements $w, u \in L$. Since $wcd \leq x$ and $d \nleq x$, then $wc \leq x$ and similarly, $ucd \leq x$ and $d \nleq x$, then $uc \leq x$. We obtain that $wc \vee uc = (w \vee u)c \leq x$, and hence $c = 1c = (w \vee u)c \leq x$, a contradiction. Therefore, L is quasi-local. □

Proposition 2.8. *Let (L, m) be a quasi-local C-lattice and let $x \in L$ be proper. Then x is an OA-element if and only if x is a prime element or $m^2 \leq x < m$.*

Proof. ($\Rightarrow$) Without restriction, we can assume that x is not a prime element of L. Clearly, there are two proper elements $a, b \in L$ such that $ab \leq x$, $a \nleq x$ and $b \nleq x$. Set $y = m^2$. Note that $yab \leq ab \leq x$. Since a, b and y are proper elements of L and $b \nleq x$, we have that $ya \leq x$, and hence $mma \leq x$. Moreover, since a and m are proper elements of L and $a \nleq x$, this implies that $m^2 = mm \leq x$. Since x is not a prime element of L, it is obvious that $x < m$.
($\Leftarrow$) If x is a prime element of L, then clearly x is an OA-element of L. Now let $m^2 \leq x < m$. Let $a, b, c \in L$ be proper such that $abc \leq x$ and $c \nleq x$. Note that $a \leq m$ and $b \leq m$. We obtain that $ab \leq m^2 \leq x$. Therefore, x is an OA-element. □

As another consequence of Propositions 2.7 and 2.8, we give the following corollary without proof.

Corollary 2.9. *Let L be a C-lattice. Then there is an OA-element of L that is not prime if and only if L is quasi-local with maximal element m such that $m^2 \neq m$.*

Proposition 2.10. *Let L be a principally generated C-lattice and set $m = \mathrm{J}(L)$. The following statements are equivalent.*

(1) *Every proper element of L is an OA-element.*
(2) *Every proper principal element of L is an OA-element.*
(3) *L is quasi-local and $m^2 = 0$.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Assume that L is not a quasi-local lattice. Then each proper principal element is a prime element. Note that L is a lattice domain. Let $x \in L$ be a principal element. It follows that x^2 is a principal prime element. We conclude that $x = x^2$, and thus $1 = x \vee (0 : x)$. Since x is proper, then we have that $x = 0$. Consequently, L is field. But this contradicts the fact that L is not a quasi-local lattice. This implies that L is quasi-local with maximal element m. We infer that 0 is prime or $m^2 = 0$ by Proposition 2.8. Suppose that $m^2 \neq 0$. Then there is a nonzero principal element $c \in L$ with $c \leq m^2$. By Proposition 2.8, we get that c^2 is prime element or $m^2 \leq c^2$. If c^2 is prime, then we have that $c^2 = c$. Let $m^2 \leq c^2$. We have that $m^2 \leq c^2 \leq c \leq m^2$, and hence $c^2 = c$. In any cases, we obtain that $c^2 = c$, and hence $1 = c \vee (0 : c)$, since c is principal. Since L is quasi-local, it follows that $c = 1$, a contradiction. Therefore, $m^2 = 0$.

(3) $\Rightarrow$ (1) This follows from Proposition 2.8. □

Lemma 2.11. *Let L be a join-principally generated C-lattice. If every nonzero element of L is an OA-element, then* $\dim(L) = 0$.

Proof. Let $p \in L$ be a prime element and let $m \in L$ a maximal element with $p < m$. Then there is a join principal element $a \in L$ with $a < m$ and $a \nleq p$. Observe that a is a nonzero element that is not nilpotent. By assumption, a is an OA-element of L. We have that $a^3 = a$ or $a^3 = a^2$, since $0 \neq a^3$ is an OA-element of L. If $a^3 = a$, then it follows that $1 = (0 : a) \vee a^2$.

Since $(0 : a) \neq 1$, we have that $a^2 = 1$, which implies that $m = 1$, a contradiction. We get a similar result when assuming that $a^3 = a^2$. We conclude that $\dim(L) = 0$. □

Proposition 2.12. *Let L be a principally generated C-lattice. If L is quasi-local with maximal element m such that m^2 is comparable, then the following statements are equivalent.*

(1) *Each two principal elements $x, y \in L$ with $m^2 \leq x$ and $m^2 \leq y$ imply that $x \leq y$ or $y \leq x$,*
(2) *If a is an OA-element of L, then a is prime or $a = m^2$.*

Proof. (1) ⇒ (2) Let each two principal elements $x, y \in L$ with $m^2 \leq x$ and $m^2 \leq y$ satisfy $x \leq y$ or $y \leq x$. Let a be an OA-element of L. Suppose that a is not a prime. By Proposition 2.8, we get $m^2 \leq a < m$. Let $m^2 < a$. Clearly, there are two principal elements $c, d \in L$ such that $c \leq x$, $c \nleq m^2$ and $d < m$, $d \nleq a$. Note that $c, d \nleq m^2$. By the assumption, we have that $m^2 \leq c, d$. Consequently, c and d are OA-elements. Since $d \nleq c$, then $c \leq d$. Therefore, there is an element $v \in L$ with $c = vd$ and hence, we deduce $c \leq m^2$, a contradiction. It must be the case that $a = m^2$.

(2) ⇒ (1) This is clear. □

Although we do not derive the result that the meet of two prime elements or two OA-elements yields an OA-element, we deduce the following result.

Lemma 2.13. *Let L be a C-lattice and let $x, y \in L$ be OA-elements that are not prime. Then $x \wedge y$ and $x \vee y$ are OA-elements.*

Proof. Let $x, y \in L$ be OA-elements that are not prime. Then L is quasi-local. By the assumption, we have that $m^2 \leq x, y$, and hence $m^2 \leq x \wedge y$ and $m^2 \leq x \vee y \neq 1$. Since $m^2 \leq x \wedge y \leq x \vee y \leq m$ and L is quasi-local, then $x \wedge y$ and $x \vee y$ are OA-elements. □

Now, we give a relation between OA-elements and lattice domains.

Proposition 2.14. *Let L be a C-lattice. Then 0 is an OA-element of L if and only if L is a lattice domain or L is quasi-local with maximal element m such that $m^2 = 0$.*

Proof. (⇒) Let 0 be an OA-element of L. Let L be not a lattice domain. Then 0 is not prime, and thus L is quasi-local with maximal element m. We infer that $m^2 = 0$.

(⇐) This is obvious. □

Proposition 2.15. *Let L be a C-lattice. Every TA-element of L is an OA-element of L if and only if the following conditions hold.*

(a) *For each two prime elements $p, q \in L$, we have that $p \leq q$ or $q \leq p$. In particular, L is quasi-local.*
(b) *If x is a TA-element of L and $p \in \min(x)$, then $x = p$ or $p = m$.*

Proof. (⇒) Let p and q be prime elements of L. Then $p \wedge q$ is a TA-element of L. By assumption, $p \wedge q$ is an OA-element. Then by Proposition 2.5(1), we have that $\sqrt{p \wedge q} = p \wedge q$ is prime, and hence $p \wedge q = p$ or $p \wedge q = q$ by [18, Lemma 7]. We obtain that $p \leq q$ or $q \leq p$. Therefore, L is quasi-local. Now, assume that p is minimal prime over x. If x is prime, then it is clear that $x = p$. Let x be not a prime element. Then L is quasi-local with maximal element m. Then $m^2 \leq x < p < m$, and thus $p = m$.

(⇐) Suppose that L satisfies (a) and (b). Let x be a TA-element of L and let p be a minimal prime element over x. By [18, Theorem 3(1)], $p^2 \leq x \leq p$. If $x = p$, then x is prime, and thus it is clearly an OA-element. Now let $p = m$. Then $m^2 \leq x \leq m$, and hence x is an OA-element. □

Remark 2.16. Let L be a C-lattice, let $x \in L$ be an OA-element and let $p \in L$ be a prime element of L such that $p \leq x$. Then x is an OA-element of L/p.

Proof. Let $a \circ b \circ c \leq x$ for some $a, b, c \in L/p$. Then $abc \leq x$. By assumption, $ab \leq x$ or $c \leq x$, and hence $a \circ b \leq x$ or $c \leq x$. Consequently, x is an OA-element of L/p. □

Remark 2.17. Let L be a C-lattice and let $p \in L$ be a prime element. If $x \in L$ is an OA-element such that $x \leq p$, then x_p is an OA-element of L_p.

Proof. Let $x \in L$ be an OA-element such that $x \leq p$. Clearly, x_p is a proper element of L_p. Let $a, b, c \in L_*$ be such that $a_p b_p c_p \leq x_p$. Then $abc \leq x_p$, and hence $dabc \leq x$ for some $d \nleq p$. We have that $dab \leq x$ or $c \leq x$. Note that $d_p = 1$ by [17]. Then $a_p b_p \leq x_p$ or $c_p \leq x_p$. Therefore, x_p is an OA-element of L_p. □

Theorem 2.18. *Let L be a principally generated C-lattice. Every nonzero proper element of L is an OA-element if and only if $L \cong L_1 \times L_2$ where L_1, L_2 are fields or L is quasi-local with maximal element m such that $m = \sqrt{0}$ and $m^2 \leq x$ for every nonzero proper principal element $x \in L$.*

Proof. ($\Rightarrow$) Let every nonzero proper element of L be an OA-element. First let L be quasi-local with maximal element m. By assumption, every nonzero proper element of L is a TA-element. Now [18, Theorem 8] completes the proof. Now let L be not quasi-local. Then the concepts of prime elements and OA-elements coincide. Let m_1 and m_2 be two distinct maximal elements of L. Assume that $m_1 \wedge m_2 \neq 0$. By the assumption, $m_1 \wedge m_2$ is prime. It can be shown that $m_1 = m_2$, a contradiction. It follows that $m_1 \wedge m_2 = 0$, and thus $L \cong L/m_1 \times L/m_2$. Note that $L/m_1, L/m_2$ are fields.

($\Leftarrow$) If $L \cong L_1 \times L_2$, where L_1 and L_2 are fields, then each nonzero proper element of L is prime, and hence it is an OA-element. Now let L be quasi-local with maximal element m such that $m = \sqrt{0}$ and $m^2 \leq x$ for every nonzero proper principal element $x \in L$. Let y be a nonzero proper element of L. There is some nonzero principal element $c \in L$ with $c \leq y$. We have that $m^2 \leq c \leq y$, and thus y is an OA-element of L. □

Proposition 2.19. *Let L be a principally generated quasi-local Noetherian lattice with maximal element m. Then every OA-element is prime if and only if L is field.*

Proof. ($\Rightarrow$) Since every OA-element is prime, then we obtain that $m^2 = m$. Therefore, $m = 0$ by [3, Theorem 1.4], and thus L is field.

($\Leftarrow$) This is clear. □

3. TAFLs and their generalizations

In this section, we study C-lattices whose elements have a TA-factorization. A TA-factorization of an element $x \in L$ means that x is written as a finite product of TA-elements $(x_k)_{k=1}^n$. (Note that the element 1 is the empty product.) We say that a C-lattice L is a *TA-factorization lattice* (abbreviated as TAFL) if every element of L has a TA-factorization.

In this section, we investigate the factorization of elements into TA-elements. Firstly, we study C-lattices whose elements possess a TA-factorization, called *TAFLs*. Next, we study C-lattices whose compact elements have a factorization into TA-elements. We call them *CTAFLs*. We also explore the C-lattices whose principal elements have a factorization into TA-elements, called *PTAFLs*. Clearly, every TAFL is both a CTAFL and a PTAFL.

Example 3.1. As a simple example, it is clear that each prime element is a TA-element, then every ZPI-lattice is a TAFL. By [19, Example 2.1], we have that the lattice of ideals of $\mathbb{Z}[\sqrt{-7}]$ is not a TAFL.

First, we will present some basic results related to TAFL in the following proposition.

Remark 3.2. Let L be a TAFL, let L_1 and L_2 be C-lattices and let $p \in L$ be a prime element.

(1) $\min(x)$ is finite for each $x \in L$.
(2) $L_1 \times L_2$ is a TAFL if and only if L_1 and L_2 are both TAFLs.
(3) L/p is a TAFL.
(4) L_p is a TAFL.

Proof. (1) Let $x = \prod_{k=1}^n x_k$ be a TA-factorization of x. By [18, Theorem 3], we have that $\min(x_i)$ is finite. Then $\min(x)$ is finite, since $\min(x) \subseteq \bigcup_{i=1}^n \min(x_i)$.

(2) It is well-known by [18] that $p_1 = 1_{L_1}$ and p_2 is a TA-element of L_2 or $p_2 = 1_{L_2}$ and p_1 is a TA-element of L_1 or p_1 and p_2 are prime elements of L_1 and L_2, respectively if and only if (p_1, p_2) is a TA-element of L. The rest now follows easily.

(3) Let $y \in L/p$. By the assumption, $y = \prod_{k=1}^{n} x_k$ where x_i is a TA-element of L for each $i \in [1,k]$. Note that x_i is a TA-element of L/p and $y = (\prod_{k=1}^{n} x_k) \vee p = \bigcirc_{k=1}^{n} x_k$. Consequently, L/p is a TAFL.

(4) Recall that if x is a TA-element of L, then x_p is a TA-element of L_p. Let $a \in L_p$. Then we have that $a = y_p$ for some $y \in L$. By assumption, y has a TA-factorization in L, meaning y is the finite product of some TA-elements $(y_k)_{k=1}^{n}$. We have that $a = y_p = (\prod_{k=1}^{n} y_k)_p = \bigcirc_{k=1}^{n} (y_k)_p$. This completes the proof. □

Lemma 3.3. *Let L be a C-lattice, let $x \in L$ be proper with $\sqrt{x} \in \max(L)$ and let one of the following conditions be satisfied:*

(a) *L is a TAFL.*
(b) *L is a CTAFL and x is compact.*
(c) *L is a PTAFL and x is principal.*

Then $x \leq (\sqrt{x})^2$ or $(\sqrt{x})^2 \leq x$.

Proof. Let L be a TAFL, let $(\sqrt{x})^2 \nleq x$ and $\sqrt{x} = m$. By [18, Theorem 3], x is not a TA-element. By assumption, $x = \prod_{i=1}^{n} x_i$ where x_i is TA-element of L and $n \geq 2$. Since $x \leq x_i$ for each $i \in [1,n]$ and $\sqrt{x} = m \in \max(L)$, we have that $\sqrt{x_i} = m$ for each $i \in [1,n]$. Consequently, $x \leq m^2$.

If L is a CTAFL (resp. a PTAFL) and x is a compact (resp. principal), then this can be shown along the same lines as before. □

Proposition 3.4. *Let L be a principally generated TAFL. Then* $\dim(L) \leq 1$.

Proof. Observe that since $\dim(L) = \sup\{\dim(L_q) \mid q \in L \text{ is a prime element}\}$, we can assume without restriction that L is quasi-local with maximal element $m \neq 0$. It remains to show that each nonmaximal prime element of L is a minimal prime element. Let $p \in L$ be a nonmaximal prime element. Since L is principally generated, there is some principal element $y \in L$ such that $y \leq m$ and $y \nleq p$.

Moreover, p_q is a prime element of L_q, y_q is a principal element of L_q and $q_q \in \min((p \vee y)_q)$. If p_q is a minimal prime element of L_q, then p is a minimal prime element of L. For these reasons, we can assume without restriction that $m \in \min(p \vee y)$. Since L is quasi-local, this implies that $\sqrt{p \vee y} = m$. Next we verify the following claims.

Claim 1: $m \neq m^2$.

Claim 2: $q \leq m^2$ for every prime element $q < m$.

Assume the contrary of claim 1 that $m = m^2$. Then m is the only TA-element whose radical is m. There is a (join) principal element y with $y < m$ and $y \nleq p$. Clearly, $p \vee y = \prod_{i=1}^{k} x_i$ where k is a positive integer and x_i is a TA-element of L for each $i \in [1,k]$. Note that m is minimal over $p \vee y$. Then $m^2 \leq x_i$ for each $i \in [1,k]$ by [18, Lemma 5]. Therefore, we get that $m^{2k} = m \leq p \vee y \leq m^k = m$ by [18, Theorem 3] and Lemma 3.3. Similarly, we have that $m^{2k} = m \leq p \vee y^2 \leq m^k = m$. This implies that $p \vee y = m = p \vee y^2$. Note that $((p \vee y^2) : y) = (p : y) \vee y = p \vee y$. Then $1 = ((p \vee y) : y) = ((p \vee y^2) : y) = p \vee y$, and hence $1 = p \vee y = m$, a contradiction.

To show that the second claim is true, assume that there is a prime element $q < m$ with $q \nleq m^2$. Also, there is a principal element $b \in L$ with $b \leq m$ and $b \nleq q$. Note that $b^n \nleq q$ for a positive integer n. We have that $b \in L$ is a nonzero element that is not nilpotent. Since L is a TAFL and $q \nleq m^2$, we have that $q \vee b^3$ is a TA-element by Lemma 3.3 and [18, Theorem 3]. It follows that $b^2 \leq q \vee b^3$, since $b^3 \leq q \vee b^3$. Note that $1 = ((q \vee b^3) : b^2) = (q : b^2) \vee b$. We conclude that $(q : b^2) = 1$ or $b = 1$. Therefore, $b^2 \leq q$ or $b = 1$, a contradiction. We infer that every prime element $q \in L$ with $q < m$ satisfies $q \leq m^2$.

We will return to the proof of the main part. Since $m \neq m^2$, there is a nonzero principal element $c \in L$ with $c \nleq m^2$. By claim 2, it follows that $\sqrt{c} = m$. By Lemma 3.3, we get $m^2 \leq c$. Let $0 \neq s \leq p$. We have that $s = \prod_{i=1}^{n} y_i$ where y_i is a TA-element of L. Since p is prime, then $y_j \leq p$ for some $j \in [1,n]$.

Since $y_j \leq p < m^2 \leq c$, then $y_j = c\ell_j$ for some $\ell_j \in L$ because c is weak meet principal. Note that $\ell_j \leq p$ because $c \nleq p$. Consequently, $\ell_j \leq p \leq m^2 \leq c$, and thus $\ell_j = ct_j$ for some $t_j \in L$. Therefore, $y_j = c^2 t_j$, and hence $\ell_j = ct_j \leq y_j$, since y_j is a TA-element. We obtain that $y_j = \ell_j$, and so $y_j = cy_j$. Therefore, $s = sc$. Note that $sm \leq s$. Since $s = sc$, we have that $s = sc \leq sm$, and hence $s = sm$. We conclude that $s = 0$ by [3, Theorem 1.4], a contradiction. This implies that $p = 0$. □

Theorem 3.5. *If L is a TAFL and a Prüfer lattice domain, then L is a ZPI-lattice domain.*

Proof. Let L be a Prüfer lattice domain such that L is also a TAFL. First we have that L_m is a TAFL for each maximal element $m \in L$. Let $m \in L$ be maximal. Since L is a Prüfer lattice domain, then L_m is a linearly ordered TAFL. From [18, Theorem 10] recall that if L is a Prüfer lattice domain, then the following statements are equivalent: (1) p is a TA-element, (2) p is a prime element of L or $p = p_1^2$ is a p_1-primary element of L or $p = p_1 \wedge p_2$ where p_1 and p_2 are some nonzero prime elements of L. By using this characterization, we conclude that every TA-element of L_m is a finite product of some prime elements. Therefore, L_m is a ZPI-lattice (since it is a TAFL), and hence every element of L_m is principal and compact. Moreover, $\dim(L_m) \leq 1$. Therefore, $\dim(L) \leq 1$. Since L is a TAFL domain and $\dim(L) \leq 1$, every nonzero element of L is contained in only finitely many maximal elements. Now one can show that every element of L is compact (since every nonzero element is locally compact and contained in only finitely many maximal elements). Therefore, every element of L is compact. In particular, L is a principal element lattice domain (since L is a Prüfer lattice domain), and hence it is a ZPI-lattice domain. □

Next we study CTAFLs.

Remark 3.6. Let L be a CTAFL, let L_1 and L_2 be C-lattices and let $p \in L$ be a prime element.

(1) $\min(x)$ is finite for each $x \in L_*$.
(2) $L_1 \times L_2$ is a CTAFL if and only if L_1 and L_2 are both CTAFLs.
(3) If p is compact, then L/p is a CTAFL.
(4) L_p is a CTAFL.

Proof. (1) This can be proved along the same lines as in the proof of Remark 3.2.

(2) It follows from [18] that $p_1 = 1_{L_1}$ and p_2 is a TA-element of L_2 or $p_2 = 1_{L_2}$ and p_1 is a TA-element of L_1 or p_1 and p_2 are prime elements of L_1 and L_2, respectively if and only if (p_1, p_2) is a TA-element of L. The rest is straightforward.

(3) Let p be compact. Observe that every element $a \in L$ with $a \geq p$ is compact in L if and only if a is compact in L/p. Now, let $y \in L/p$ be compact. Then $y \geq p$. By the assumption, $y = \prod_{k=1}^n x_k$ where x_i is a TA-element of L. Note that x_i is a TA-element of L/p. Then we get that $y = (\prod_{k=1}^n x_k) \vee p = \bigcirc_{k=1}^n x_k$. Therefore, L/p is a CTAFL.

(4) This can be shown along similar lines as in Remark 3.2(4). □

Proposition 3.7. *Let L be a principally generated CTAFL that satisfies the ascending chain condition on prime elements. Then* $\dim(L) \leq 1$.

Proof. Observe that for each prime element $q \in L$, we have that L_q is a principally generated CTAFL that satisfies the ascending chain condition on prime elements. Since $\dim(L) = \sup\{\dim(L_q) \mid q \in L$ is a prime element$\}$, we can assume without restriction that L is quasi-local with maximal element $m \neq 0$. It remains to show that each nonmaximal prime element of L is a minimal prime element. Let $p \in L$ be a nonmaximal prime element. Since L is principally generated, there is some principal element $y \in L$ such that $y \leq m$ and $y \nleq p$.

Since L is a C-lattice, there exists some $q \in \min(p \vee y)$ such that $q \leq m$. Clearly, L_q is a principally generated CTAFL with maximal element q_q that satisfies the ascending chain condition on prime elements. Moreover, p_q is a prime element of L_q, y_q is a principal element of L_q and $q_q \in \min((p \vee y)_q)$. If p_q is a minimal prime element of L_q, then p is a minimal prime element of L. For these reasons, we can assume without restriction that $m \in \min(p \vee y)$. Since L is quasi-local, this implies that $\sqrt{p \vee y} = m$. Next we verify the following claims.

Claim 1: $m \neq m^2$.

Claim 2: $q \leq m^2$ for every prime element $q \in L$ with $q < m$.

First we prove claim 1. It follows from Remark 3.6(1) that $\min(x)$ is finite for each compact element $x \in L$. Since L satisfies the ascending chain condition on prime elements, it follows from [16, Theorem 2] that $p = \sqrt{d}$ for some compact element $d \in L$. Observe that $m = \sqrt{d \vee y}$ and $d \vee y$ is compact. Consequently, $d \vee y = \prod_{i=1}^{t} x_i$ for some positive integer t and some TA-elements $x_i \in L$. Clearly, $\sqrt{x_i} = m$ for each $i \in [1, t]$, and thus $m^2 \leq x_i \leq m$ by [18, Theorem 3] for each $i \in [1, t]$. Assume to the contrary that $m = m^2$. Then $d \vee y = m$, and hence $p \vee y = m$. We infer that $m = m^2 \leq p \vee y^2 \leq p \vee y \leq m$, and thus $p \vee y = p \vee y^2$. Since y is (join) principal, we obtain that $1 = ((p \vee y) : y) = ((p \vee y^2) : y) = p \vee y = m$, a contradiction. This implies that $m \neq m^2$. □(Claim 1)

Now we prove claim 2. Assume that there is a prime element $q \in L$ such that $q < m$ and $q \nleq m^2$. Since L is principally generated, there is a principal element $b \in L$ such that $b \leq m$ and $b \nleq q$. Note that $b^2 \nleq q$. Since $q \nleq m^2$ and L is a C-lattice, there is a compact element $a \in L$ such that $a \leq q$ and $a \nleq m^2$. Since a and b are compact, we have that $a \vee b^3$ is compact. Note that $a \vee b^3$ is a TA-element, since L is a CTAFL and $a \nleq m^2$. Since $b^3 \leq a \vee b^2$ and $a \vee b^2$ is a TA-element, we get that $b^2 \leq a \vee b^3$. Note that $1 = ((a \vee b^3) : b^2) = (a : b^2) \vee b$. Since $b \leq m$ and L is quasi-local, we have that $(a : b^2) = 1$. Consequently, $b^2 \leq a \leq q$, a contradiction. □(Claim 2)

It is sufficient to show that $p = 0$. (Then p is a minimal prime element of L and we are done.) Since $m \neq m^2$ by claim 1 and L is principally generated, there is a principal element $c \in L$ such that $c \leq m$ and $c \nleq m^2$. By claim 2, we have that $\sqrt{c} = m$. Furthermore, Lemma 3.3 implies that $m^2 \leq c$. Let $s \in L$ be compact such that $s \leq p$. It follows that $s = \prod_{i=1}^{k} y_i$ for some positive integer k and some TA-elements $y_i \in L$. Obviously, there is some $j \in [1, k]$ such that $y_j \leq p$. Since $y_j \leq p \leq m^2 \leq c$ and c is (weak meet) principal, we infer that $y_j = c\ell$ for some $\ell \in L$. Note that $\ell \leq p$, since $c \nleq p$. Consequently, $\ell \leq p \leq m^2 \leq c$, and thus $\ell = ct$ for some $t \in L$. This implies that $y_j = c^2 t$, and hence $\ell = ct \leq y_j$ (since y_j is a TA-element of L). Therefore, $y_j = \ell$, and thus $y_j = cy_j$. We infer that $s = sc$, and hence $s = sc \leq sm \leq s$. We conclude that $s = sm$. It is an immediate consequence of [3, Theorem 1.4] that $s = 0$. Finally, we have that $p = 0$ (since L is a C-lattice). □

Next we study PTAFLs. We start with a simple observation.

Remark 3.8. Let L be a Prüfer lattice. Then L is a CTAFL if and only if L is a PTAFL.

Proof. This is obvious, since every compact element in a Prüfer lattice is principal. □

Note that Proposition 3.7 does not hold for PTAFLs. To show that, we consider the following example.

Example 3.9. Note that if L is the lattice of ideals of a local two-dimensional unique factorization domain D (e.g. take $D = K[X, Y]_{(X,Y)}$ where K is a field and X and Y are indeterminates over K), then L is a quasi-local principally generated PTAFL that satisfies the ascending chain condition on prime elements and $\dim(L) = 2$.

Theorem 3.10. *Let (L, m) be a quasi-local principally generated C-lattice domain such that* $\dim(L) \leq 1$, *m^2 is comparable and $\bigwedge_{n \in \mathbb{N}} m^n = 0$. Then L is a TAFL.*

Proof. Assume that $m = m^2$. Since L is a lattice domain and $\bigwedge_{n \in \mathbb{N}} m^n = 0$, we infer that $m = 0$. Therefore, L is a field and hence we get the desired properties. Now, suppose that $m \neq m^2$. Then there is a principal element $x \in L$ with $x \nleq m^2$. By the assumption, we get that $m^2 < x$. Since $\dim(L) \leq 1$, then we conclude that x is a TA-element by [18, Theorem 3]. Since x is meet principal, then we conclude that $m^2 = xm$. Let $z \in L$ be proper. If $z = 0$, then it is clear that it has a TA-factorization under the assumption that L is a lattice domain. Now let $z \neq 0$. If $m^2 \leq z$, then the proof is complete by [18, Theorem 3]. Now let $z \leq m^2$. Let n be the largest positive integer satisfying $z \leq m^n$. We conclude that $z \leq m^n = x^{n-1}m$. Consequently, $z \leq x^{n-1}$. Note that $z = x^{n-1}a$ for some $a \in L$ since x^{n-1} is principal. Suppose that $a \leq m^2$. We conclude that $a \leq x$. Moreover, we can write $a = xb$ for some $b \in L$. Consequently, we obtain $z = x^n b \leq x^n m = m^{n+1}$, leading to a contradiction. This implies that $m^2 \leq a$, and thus a is a TA-element. Therefore, z has a TA-factorization. □

Theorem 3.11. *Let (L, m) be a quasi-local principally generated C-lattice domain. Then L is a TAFL if and only if $\dim(L) \leq 1$ and L is a PTAFL. If these equivalent conditions are satisfied, then $\bigwedge_{n\in\mathbb{N}} m^n = 0$ and m^2 is comparable.*

Proof. (⇒): This is follows from Proposition 3.4.

(⇐): Let L be a PTAFL such that $\dim(L) \leq 1$. First, assume that $m = m^2$. Since $\dim(L) \leq 1$, then m is the only TA-element whose radical is m. We infer that each proper nonzero element of L is equal to m and we are done.

Now let $m \neq m^2$. Then there is a principal element $x \in L$ such that $x \leq m$ and $x \nleq m^2$. Since $x \nleq m^2$, we infer that x is a TA-element (since x cannot be the product of more than one TA-element). From [18, Theorem 3], we get that $m^2 \leq x$ since $\sqrt{x} = m$. Since x is a (weak meet) principal element, we conclude that $m^2 = xm$. Let $z \in L$ be proper. We have to show that z has a TA-factorization. If $z = 0$, then we are done, since L is lattice domain. Therefore, we can assume without restriction that $z \neq 0$.
Next we show that $\bigwedge_{n\in\mathbb{N}} m^n = 0$. Assume the contrary that $\bigwedge_{n\in\mathbb{N}} m^n \neq 0$. Note that each nonzero TA-element $v \in L$ satisfies $m^2 \leq v$ by [18, Lemma 2]. Clearly, there is some nonzero principal element $x \in L$ such that $x \leq \bigwedge_{n\in\mathbb{N}} m^n$. We have that $x = \prod_{i=1}^{k} a_i$ where a_i is a TA-element for each $i \in [1, k]$. We obtain that $(m^2)^k \leq x \leq (m^4)^k \leq (m^2)^k$, and hence $x = x^2$. Since x is principal, we infer that $1 = x \vee (0 : x)$. Therefore, we obtain that $x = 1$, a contradiction.
First let $z \leq m^2$. Let n be the largest positive integer satisfying $z \leq m^n$. We conclude that $z \leq m^n = x^{n-1}m$, and thus $z \leq x^{n-1}$. Note that $z = x^{n-1}a$ for some $a \in L$ since x^{n-1} is principal. Suppose that $a \leq m^2$. It follows that $a \leq x$. Moreover, we can write $a = xb$ for some $b \in L$. Consequently, we obtain $z = x^n b \leq x^n m = m^{n+1}$, a contradiction. Let $a \nleq m^2$. There is a principal element $a' \in L$ such that $a' \leq a$ but $a' \nleq m^2$. Since $a' \nleq m^2$, we know that a' is a TA-element of L. Therefore, $m^2 \leq a' \leq a$. We conclude that a is a TA-element. This implies that $z = x^{n-1}a$ has a TA-factorization.
Finally, let $z \nleq m^2$. Then there is a principal element $y \in L$ such that $y \leq z$ and $y \nleq m^2$. We have that y is a TA-element, and hence $m^2 \leq y \leq z$. This implies that z is a TA-element. Consequently, L is a TAFL.
It remains to show that m^2 is comparable. Let $x \in L$ be proper such that $x \nleq m^2$. Since L is a TAFL, we conclude that x is a TA-element. From [18, Lemma 2], we get that $m^2 \leq x$ since $\dim(L) \leq 1$. □

Proposition 3.12. *Let (L, m) be a quasi-local principally generated C-lattice such that L is not a lattice domain. Then L is a TAFL if and only if $\dim(L) = 0$ and L is a PTAFL.*

Proof. (⇒): Let L be a TAFL. By proof of Proposition 3.4, we know that if $p \in L$ is a nonmaximal prime element of L, then $p = 0$. Therefore, $\dim(L) \leq 1$. Since L is not a lattice domain, we infer that $\dim(L) = 0$.

(⇐): Let $\dim(L) = 0$ and let L be a PTAFL. Note that m is nilpotent element of L since $0 = \prod_{i=1}^{k} x_i$ where x_i is a TA-element with $\sqrt{x_i} = m$ for each $i \in [1, k]$, and hence $(m^2)^k \leq 0$. This implies that $0 = m^{2k}$, and thus 0 has a TA-factorization. If $m = m^2$, we get that $m = 0$. Now let $m \neq m^2$. Then there is a principal element $z \in L$ such that $z \leq m$ and $z \nleq m^2$. Since $z \nleq m^2$, we infer that z is a TA-element (since z cannot be the product of more than one TA-element). From [18, Lemma 2], we get that $m^2 \leq z$ since $\sqrt{z} = m$. Since z is a (weak meet) principal element, we conclude that $m^2 = zm$. Let

$x \in L$ be nonzero. If $m^2 \leq x$, then x is a TA-element because it is a primary element as shown by [18, Lemma 5].
Next let $x \leq m^2$. Let n be the largest positive integer for which $x \leq m^n$. We conclude that $x \leq m^n = z^{n-1}m$, which implies $x \leq z^{n-1}$. Note that $x = z^{n-1}a$ for some $a \in L$, given that z^{n-1} is a principal element. Assume that $a \leq m^2$. This implies that $a \leq z$, and hence $a = zb$ for some $b \in L$. We conclude $x = z^n b \leq z^n m = m^{n+1}$, a contradiction. Therefore, $a \nleq m^2$. There is a principal element $a' \in L$ such that $a' \leq a$ and $a' \nleq m^2$. Since $a' \nleq m^2$, we have that a' is a TA-element. Therefore, $m^2 \leq a' \leq a$. Consequently, a is a TA-element. It follows that $x = z^{n-1}a$ has a TA-factorization.
Finally, let $x \nleq m^2$. Then there is a principal element $y \in L$ such that $y \nleq m^2$ and $y \leq x$. We have that y is a TA-element, and hence $m^2 \leq y \leq x$. We infer that x is a TA-element. Therefore, L is a TAFL. □

Remark 3.13. Let (L, m) be a quasi-local principal element TAFL domain. Then every proper element of L is a power of m.

Proof. We know that a TA-element of L equals m or m^2 by [18, Theorem 9]. This completes the proof. □

4. OAFLs and their generalizations

In this section, we study the factorization of elements of L with respect to the OA-elements, similar to the previous section. It consists of three parts. The first part involves C-lattices, whose elements possess an OA-factorization, called a *OAFLs*. Next, we examine C-lattices whose compact elements have a factorization into OA-elements, called a *COAFLs*. Finally, we explore the C-lattices whose principal elements have a factorization into OA-elements, called *POAFLs*. It can easily be shown that every OAFL is both a COAFL and a POAFL. We continue by presenting some results related to OAFL.

Example 4.1. Let L be the lattice of ideals of $\mathbb{Z}[2i]$. Note that $(2+2i)$ has no OA-factorization. Therefore, L is not an OAFL.

Remark 4.2. Let L be an OAFL and let $p \in L$ be a prime element.

(1) $\min(x)$ is finite for each $x \in L$.
(2) L is both a Q-lattice and a TAFL.
(3) L_p is an OAFL.
(4) L/p is an OAFL.

Proof. (1) Let $x \in L$ and let $x = \prod_{k=1}^{n} x_k$ be an OA-factorization of x for some OA-elements $x_i \in L$. Recall that each OA-element is a TA-element and $\min(a)$ is finite for a TA-element $a \in L$ by [18, Theorem 3]. Hence $\min(x)$ is finite since $\min(x) \subseteq \bigcup_{i=1}^{n} \min(x_i)$.

(2) Since every OA-element of L is a TA-element and a primary element, we have that L is both a Q-lattice and a TAFL.

(3) Let $y \in L_p$ be proper. Then there is a proper element $x \in L$ such that $y = x_p$. By the assumption, we get such a factorization $x = \prod_{k=1}^{n} x_k$ for some OA-elements $x_i \in L$. By Remark 2.17, we know that $(x_i)_p$ is an OA-element of L_p, and thus $y = x_p = (\prod_{k=1}^{n} x_k)_p = \bigcirc_{k=1}^{n} (x_k)_p$. This completes the proof.

(4) Let $y \in L/p$. By the assumption, $y = \prod_{k=1}^{n} x_k$ where x_i is an OA-element of L for each $i \in [1, k]$. Note that x_i is an OA-element of L/p. Then we get that $y = (\prod_{k=1}^{n} x_k) \vee x = \bigcirc_{k=1}^{n} x_k$. Observe that L/p is an OAFL. □

Corollary 4.3. *Let L be a principally generated OAFL. Then* $\dim(L) \leq 1$.

Proof. This is an immediate consequence of Remark 4.2(2) and Proposition 3.4. □

Lemma 4.4. *Let (L, m) be a quasi-local principally generated C-lattice such that m^2 is comparable and m is nilpotent or L is a lattice domain with $\bigwedge_{n \in \mathbb{N}} m^n = 0$. Then L is an OAFL and every proper principal element of L is a finite product of principal OA-elements.*

Proof. First let $m = m^2$. Since m is nilpotent or L is lattice domain with $\bigwedge_{n\in\mathbb{N}} m^n = 0$, we have that $m = 0$. Therefore, L is a field and hence it satisfies the desired conditions.

Now let that $m \neq m^2$. There is a proper principal element $x \in L$ with $x \nleq m^2$. By the assumption, we get that $m^2 < x$. We conclude that x is an OA-element. Since x is meet principal, then we get that $m^2 = x(m^2 : x)$. We have that $m^2 = xm$. Let $z \in L$ be proper. First let $z = 0$. If L is lattice domain, then it is clear that z is a principal OA-element. Let m is nilpotent. Then there is a positive integer n with $m^n = 0$. Note that $z = x^n$ is a finite product of principal OA-elements.

Next let $z \neq 0$. If $m^2 < z$, then z is an OA-element and we are done. Now let $z \leq m^2$. Let n be the largest positive integer satisfying $z \leq m^n$. Therefore, $z \leq m^n = x^{n-1}m$, and thus $z \leq x^{n-1}$. Note that $z = x^{n-1}a$ for some $a \in L$ since x^{n-1} is principal. If z is principal, then we can by [5, Theorem 9] assume that a is principal. Suppose that $a \leq m^2$. We have that $a \leq x$. Moreover, $a = xb$ for some $b \in L$. Consequently, we obtain $z = x^n b \leq x^n m = m^{n+1}$, a contradiction. Therefore, $m^2 \leq a$ and so a is an OA-element by Proposition 2.8. We infer that z has an OA-factorization and if z is principal, then z is a finite product of principal OA-elements. □

Next we study COAFLs.

Remark 4.5. Let L be a COAFL and let $p \in L$ be a prime element.

(1) $\min(x)$ is finite for each $x \in L_*$.
(2) L is a CTAFL.
(3) If p is compact, then L/p is a COAFL.
(4) Then L_p is a COAFL.

Proof. (1) This can be shown along the lines of the proof of Remark 4.2.

(2) This is clear, since every OA-element of L is a TA-element.

(3) Let p be compact. Note that every element $a \in L$ with $a \geq p$ is compact in L if and only if a is compact in L/p. Now, let $y \in L/p$ be compact. Then $y \geq p$. By the assumption, $y = \prod_{k=1}^n x_k$ where x_i is an OA-element of L. Note that x_i is an OA-element of L/p. Then we get that $y = (\prod_{k=1}^n x_k) \vee x = \bigcirc_{k=1}^n x_k$. Consequently, L/p is a COAFL.

(4) This can proved along similar lines as in Remark 2.17. □

Proposition 4.6. *Let L be a quasi-local COAFL. Then each minimal nonmaximal prime element of L is a weak meet principal element.*

Proof. Let $p \in L$ be a minimal nonmaximal prime element of L. First we show that $x = p(x : p)$ for each compact element $x \in L$ with $x \leq p$. Let $x \in L$ be a compact element of L with $x \leq p$. By the assumption, we can write $x = \prod_{i=1}^n x_i$ where n is a positive integer and x_i is an OA-element of L for each $i \in [1, n]$. Since $x \leq p$, we have that $x_i \leq p$ for some $i \in [1, n]$. If x_i is not a prime element, then $m^2 \leq x_i \leq p$ by Proposition 2.8. We infer that $m = p$, a contradiction. Consequently, x_i is a prime element, and hence $x_i = p$. We have that $x = pz$, where $z = \prod_{k=1,k\neq i}^n x_k$. Observe that $x = p(x : p)$.

Let $y \in L$ be such that $y \leq p$. Since L is a C-lattice, we have that

$$y = \bigvee\{v \in L_* \mid v \leq y\} = \bigvee\{p(v : p) \mid v \in L_*, v \leq y\} = p\bigvee\{(v : p) \mid v \in L_*, v \leq y\}.$$

Set $w = \bigvee\{(v : p) \mid v \in L_*, v \leq y\}$. Then $w \in L$ and $y = pw$, and thus p is weak meet principal. □

Next we study POAFLs. We start with a simple observation.

Remark 4.7. Let L be a Prüfer lattice. Then L is a COAFL if and only if L is a POAFL.

Proof. This is obvious, since every compact element in a Prüfer lattice is principal. □

Lemma 4.8. *Let L be a quasi-local principally generated POAFL. Then every minimal nonmaximal prime element of L is a principal element.*

Proof. Let $p \in L$ be a minimal nonmaximal prime element of L. Let $a \in L$ be a principal element of L with $a \leq p$. By the assumption, we can write $a = \prod_{k=1}^{n} q_k$ where q_i is an OA-element of L for $i \in [1, n]$. Since $a \leq p$, then $q_j \leq p$ for some $j \in [1, n]$. If q_j is an OA-element that is not prime, then $m^2 \leq q_j \leq p$. Therefore, $m = p$, a contradiction. Assume that q_i is prime. Then we get that $q_i = p$. We infer that $a = p \prod_{k=1,k\neq i}^{n} q_k$, and thus $a = pb$ where $b = \prod_{k=1,k\neq i}^{n} q_k$. Obviously, $a = p(a : p)$. Now, take an element $c \in L$ with $c \leq p$. By assumption, $c = \bigvee A$ for some set A of principal elements of L. We conclude that $c = \bigvee A = \bigvee\{p(a : p) \mid a \in A\} = p(\bigvee\{(a : p) \mid a \in A\})$. Therefore, p is a weak meet principal element, and hence p is a principal element by [3, Theorem 1.2]. □

Lemma 4.9. *Let (L, m) be a quasi-local principally generated C-lattice. If the join of any two principal elements of L has an OA-factorization, then every nonmaximal prime element of L is a principal element and* $\dim(L) \leq 2$.

Proof. Let the join of any two principal elements of L have an OA-factorization. First, we show that $\dim(L) \leq 2$. Let $p \in L$ be a nonmaximal prime element of L. Consider that L_p is quasi-local and L_p is generated by the set of elements $\{a_p \mid a \in L \text{ is principal}\}$. We can see that the join of any two principal element of L_p has a prime factorization. Because $a_p \vee_p b_p = (a \vee b)_p = (\prod_{i=1}^{n} x_i)_p = \bigcirc_{i=1}^{n} (x_i)_p$ where $(x_i)_p$ is a prime element of L_p. Consequently, L_p is a ZPI-lattice by [15, Theorem 8], and thus $\dim(L_p) \leq 1$. Therefore, $\dim(L) \leq 2$.

Let q be a nonmaximal prime element of L. Assume that $qm = q$. If $q \in \min(0)$, then it is clear that $q = 0$ by Nakayama's Lemma since q is principal. Now let q be not minimal. There is a $p \in \min(0)$ such that $p < q$, and thus there is a principal element $c \in L$ such that $c \nleq p$ and $c \leq q$. By assumption, $c \vee p$ has an OA-factorization since p is principal by Lemma 4.8. Since $\dim(L) \leq 2$, we have that q is minimal over $c \vee p$. Let $c \vee p = \prod_{i=1}^{n} x_i$ where x_i is an OA-element for each $i \in [1, n]$. We get that $x_j \leq q$ for some $j \in [1, n]$. Note that $m^2 \nleq x_j$, and hence $x_j = q$. We infer that $c \vee p = q\ell$ for some $\ell \in L$. We conclude that $(c \vee p)m = c \vee p$ by the assumption, and thus $c \vee p = 0$. Consequently, $c = p = 0$, which contradicts the fact that $c \leq p$. Now assume that $q \neq qm$. Assume that q is a nonminimal and nonmaximal prime element of L. Since L is principally generated, there is a principal element $a \in L$ such that $a \leq q$ and $a \nleq qm$. Since q is nonmaximal and L is principally generated, there is a principal element $b \in L$ such that $b \leq m$ and $b \nleq q$. It remains to show that $x \leq a$ of each principal element $x \in L$ such that $x \leq q$. (Then $q = a$ is principal, since L is principally generated and $a \leq q$.) Let $x \in L$ be principal such that $x \leq q$. Note that xb^2 is principal, and thus $a \vee xb^2$ is the join of two principal elements of L. Consequently, $a \vee xb^2$ has an OA-factorization. Since $a \vee xb^2 \leq p$, there are $v, w \in L$ such that v is an OA-element of L, $v \leq q$ and $a \vee xb^2 = vw$. If $w \neq 1$, then $w \leq m$, and hence $a \leq a \vee xb^2 \leq qm$, a contradiction. Therefore, $a \vee xb^2 = v$ is an OA-element. Assume that $b^2 \leq a \vee xb^2$. Since b^2 is principal, we have that $1 = (a \vee xb^2 : b^2) = (a : b^2) \vee x$, and thus $(a : b^2) = 1$, since L is quasi-local. Consequently, $b^2 \leq a \leq q$, and hence $b \leq q$, a contradiction. This implies that $b^2 \nleq a \vee xb^2$. Since $xb^2 \leq a \vee xb^2$ and $a \vee xb^2$ is an OA-element, we conclude that $x \leq a \vee xb^2$. By [3, Theorem 1.4], it follows that $x \leq a$. □

Proposition 4.10. *Let L be a principally generated C-lattice. If the join of any two principal elements has an OA-factorization, then* $\dim(L) \leq 1$.

Proof. First let every OA-element of L be a prime element. We infer that L is a ZPI-lattice by [15, Theorem 8]. Therefore, $\dim(L) \leq 1$ by [4, Theorem 2.6]. Now let there be an OA-element that is not a prime element. Then L is quasi-local by Proposition 2.7. Let m be the maximal element of L. We conclude by Proposition 2.8 that $m \neq m^2$. There exists some principal element $c \in L$ such that $c \nleq m^2$ and $c \leq m$.

Claim: $p \leq m^2$ for each nonmaximal prime element $p \in L$.

Let $p \in L$ be a nonmaximal prime element. By Lemma 4.9, we have that $\dim(L) \leq 2$. Therefore, we can assume without restriction that there are no prime elements of L that are properly between p and m (i.e., for each prime element $r \in L$ with $p \leq r \leq m$, it follows that $r \in \{p, m\}$).
Next we show that $p = \bigwedge\{p \vee a \mid a \in L$ is principal and $a \nleq p\}$. Assume to the contrary that $p \neq \bigwedge\{p \vee a \mid a \in L$ is principal and $a \nleq p\}$. Then there is a principal element $y \in L$ such that $y \nleq p$ and $y \leq \bigwedge\{p \vee a \mid a \in L$ is principal and $a \nleq p\}$. Since $p < m$, there is a principal element $b \in L$ such that $b \nleq p$ and $b \leq m$. Since $by \in L$ is principal and $by \nleq p$, we have that $y \leq \bigwedge\{p \vee a \mid a \in L$ is principal and $a \nleq p\} \leq p \vee by$. It follows from [3, Theorem 1.4] that $y \leq p$, a contradiction. Therefore, $p = \bigwedge\{p \vee a \mid a \in L$ is principal and $a \nleq p\}$.
Assume that $p \nleq m^2$. It is sufficient to show that $m^2 \leq p \vee a$ for each proper principal $a \in L$ with $a \nleq p$. (Then $m^2 \leq \bigwedge\{p \vee d \mid d \in L$ is principal and $d \nleq p\} = p$, and hence $p = m$, a contradiction.) Let $a \in L$ be a proper principal element such that $a \nleq p$. Since p is principal by Lemma 4.9, it follows that $p \vee a$ has an OA-factorization in L. Since $p \vee a \nleq m^2$ and $p \vee a$ is proper, we obtain that $p \vee a$ is an OA-element. Clearly, $p \vee a$ is not a nonmaximal prime element, and hence $m^2 \leq p \vee a$ by Proposition 2.8. Consequently, $q \leq m^2$. □(Claim)

It is sufficient to show that $r = 0$ for each nonmaximal prime element $r \in L$. Let $r \in L$ be a nonmaximal prime element. Since c has an OA-factorization and $c \nleq m^2$, we infer that c is an OA-element of L. By the claim, it follows that c is not a nonmaximal prime element of L. Observe that $r \leq m^2 \leq c$ by the claim and by Proposition 2.8. Since c is a (weak meet) principal element of L, r is a prime element of L and $c \nleq r$, we conclude that $r = cr$. Therefore, $r = 0$ by [3, Theorem 1.4]. □

Proposition 4.11. *Let (L, m) be a quasi-local principally generated C-lattice. If the join of any two principal elements of L has an OA-factorization and $\dim(L) = 1$, then L is a domain.*

Proof. Let $p \in L$ be a minimal nonmaximal prime element with $p < m$. If $m^2 = 0$, then $p = m$, a contradiction. Therefore, $m^2 \neq 0$. Since (L, m) is a quasi-local, then $m^2 \neq m$. There is a principal element $x \in L$ such that $x \nleq m^2$. by the assumption, we infer that $m^2 \leq x$ and x is not prime. We show that $p \leq m^2$. Assume the contrary that $p \nleq m^2$. There is a principal element $a \in L$ with $a \nleq p$ and $a \leq m^2$. Note that $a^n \nleq p$ for a positive integer n. Note that $p \vee a^3 \nleq m^2$ because $p \nleq m^2$. By the assumption $p \vee a^3$ is an OA-element. Since $a^3 \leq p \vee a^3$, then we infer that $a^2 \leq p \vee a^3$ or $a \leq p \vee a^3$. Note that we have that $((p \vee a^3) : a^2) = (p : a^2) \vee a$ and $((p \vee a^3) : a) = (p : a) \vee a^2$. If $a^2 \leq p \vee a^3$, then we obtain that $1 = (p : a^2) \vee a$. This implies that $1 = (p : a^2)$ or $a = 1$, and hence $a^2 \leq p$ or $a = 1$, a contradiction. Assume that $a \leq p \vee a^3$. We obtain that $1 = (p : a) \vee a^2$. This implies that $1 = (p : a)$ or $a^2 = 1 \leq a$, and thus $a \leq p$ or $a = 1$, a contradiction. Therefore, $p \leq m^2$. Since $p \leq x$, we get that $px = p$. By Nakayama's Lemma and Lemma 4.8, we have that $p = 0$. □

Proposition 4.12. *Let L be a principally generated C-lattice and set $m = \mathrm{J}(L)$. If the join of any two principal elements of L has an OA-factorization, then L satisfies one of the following conditions.*

(a) *L is a ZPI-lattice.*
(b) *L is a quasi-local lattice, m^2 is comparable and m is a nilpotent element.*
(c) *L is a quasi-local lattice domain, m^2 is comparable and $\bigwedge_{n \in \mathbb{N}} m^n = 0$.*

Proof. Let the join of any two principal elements of L has an OA-factorization. Assume that every OA-element of L is prime. By [15, Theorem 8], it follows that L is a ZPI-lattice. Now, assume that there is an OA-element which is not a prime. We conclude that (L, m) is a quasi-local C-lattice with $m^2 \neq m$ by Propositions 2.7 and 2.8. Then there is a nonzero principal element $x \nleq m^2$. Clearly, x is not the product of more than one OA-element, and thus x is an OA-element.

First let $\dim(L) = 0$. We show that m^2 is comparable. Let $z \in L$ be proper such that $z \nleq m^2$. There is a principal element $a \in L$ with $a \leq z$ and $a \nleq m^2$. Note that a is an OA-element which is not a prime element. We have that $m^2 \leq a$ by Proposition 2.8, and thus $m^2 \leq z$. Consequently, m^2 is comparable.

Clearly, 0 has an OA-factorization. This implies that $m^k \leq 0$ for some positive integer k, and hence $m^k = 0$.

Now let $\dim(L) = 1$. We obtain that (L, m) is a quasi-local lattice domain by Proposition 4.11. To verify that $\bigwedge_{n\in\mathbb{N}} m^n = 0$, assume the contrary that $\bigwedge_{n\in\mathbb{N}} m^n \neq 0$. There is a nonzero principal element $x \in L$ with $x \leq \bigwedge_{n\in\mathbb{N}} m^n$. Since x is a product of k OA-elements of L, we conclude that $m^{2k} \leq x \leq m^{4k} \leq m^{2k}$. In particular, we get that $x = x^2$. Since x is principal, we have that $1 = x \vee (0 : x)$. We infer that $x = 1$, a contradiction. Therefore, $\bigwedge_{n\in\mathbb{N}} m^n = 0$. Finally, we show that m^2 is comparable. Let $z \in L$ be proper such that $z \nleq m^2$. There is a principal element $a \in L$ with $a \leq z$ and $a \nleq m^2$. We get that a is a nonzero OA-element. Therefore, $m^2 \leq a$ (since $\dim(L) = 1$), and thus $m^2 \leq z$. □

Theorem 4.13. *Let L be a principally generated C-lattice and set $m = \mathrm{J}(L)$. The following statements are equivalent.*

1. *L is an OAFL.*
2. *L is a COAFL.*
3. *The join of any two principal elements of L has an OA-factorization.*
4. *L satisfies one of the following conditions.*
   - (a) *L is a ZPI-lattice.*
   - (b) *L is a quasi-local lattice, m^2 is comparable and m is a nilpotent element.*
   - (c) *L is a quasi-local lattice domain, m^2 is comparable and $\bigwedge_{n\in\mathbb{N}} m^n = 0$.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (3) Note that every principal element is compact, and hence the join of each two principal elements is compact. The statement is now immediately clear.

(3) ⇒ (4) This follows from Proposition 4.12.

(4) ⇒ (1) If L is a ZPI-lattice, then clearly L is an OAFL. Now let L be not a ZPI-lattice. It is an immediate consequence of Lemma 4.4 that L is an OAFL. □

Theorem 4.14. *Let L be a principally generated C-lattice. The following statements are equivalent.*

1. *L is a ZPI-lattice.*
2. *L is a Prüfer OAFL.*
3. *L is a Prüfer POAFL.*

Proof. (1)⇒ (2) ⇒ (3) This follows from [15, Theorem 8].

(3) ⇒ (1) Let L be a Prüfer POAFL. If L is not quasi-local, then the prime elements coincide with the OA-elements. By Theorem 4.13, we infer that L is a ZPI-lattice. Assume that (L, m) is a quasi-local lattice with maximal element m. Then m^2 is comparable by Theorem 4.13. We know from Proposition 2.12(2) that each OA-element is either prime or equal to m^2. Therefore, L is a ZPI-lattice. □

Finally, we provide a theorem that connects the various types of factorization lattices for a quasi-local principally generated C-lattice domain.

Theorem 4.15. *Let (L, m) be a quasi-local principally generated C-lattice domain. The following statements are equivalent.*

1. *L is an OAFL.*
2. *L is a TAFL.*
3. *L is a COAFL.*
4. *L is a CTAFL that satisfies the ascending chain condition on prime elements.*
5. $\dim(L) \leq 1$ *and L is a POAFL.*
6. $\dim(L) \leq 1$ *and L is a PTAFL.*
7. $\dim(L) \leq 1$*, m^2 is comparable and $\bigwedge_{n\in\mathbb{N}} m^n = 0$.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (7) This follows from Theorem 4.13.

(1) $\Rightarrow$ (5) $\Rightarrow$ (6) This follows from Corollary 4.3.

(2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) This is an immediate consequence of Theorem 3.11 and Proposition 3.12.

(2) $\Rightarrow$ (4) Clearly, L is a CTAFL. Moreover, $\dim(L) \leq 1$ by Proposition 3.4. It is clear now that L satisfies the ascending chain condition on prime elements.

(4) $\Rightarrow$ (6) Obviously, L is a PTAFL. We infer by Proposition 3.7 that $\dim(L) \leq 1$. □

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