ON INTEGRAL DOMAINS THAT ARE C-MONOIDS

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Abstract. C-monoids are a special class of Mori monoids which play a central role in arithmetical investigations of higher dimensional noetherian domains which are not integrally closed. Mori domains R with complete integral closure $\widehat{R}$ and conductor $\mathfrak{f}$ are C-monoids, provided that both the v-class group $\mathcal{C}_v(\widehat{R})$ and the residue class ring $\widehat{R}/\mathfrak{f}$, are finite. We provide characterizations for several classes of integral domains to be C-monoids.

1. Introduction

C-monoids are suitably defined submonoids of factorial monoids, and they have been introduced as multiplicative models to study the arithmetic of higher dimensional noetherian domains which are not integrally closed. C-monoids satisfy a bunch of nice algebraic and arithmetic properties. Among others, if H is a C-monoid, then (see [19] for an overview, or [20] for details)

- **Algebraic Properties.** H is a Mori monoid (in other words, v-noetherian), its complete integral closure $\widehat{H}$ is a Krull monoid with finite v-class group, and the conductor $\mathcal{F}_{\widehat{H}/H}$ is non-trivial.
- **Arithmetical Properties.** H is locally tame, has finite catenary degree, and H satisfies the Structure Theorem for sets of lengths.

The class of C-monoids includes all Krull monoids (hence Krull domains) with finite v-class group and congruence monoids in Krull domains satisfying some natural finiteness conditions. In particular, let R be a Mori domain with complete integral closure $\widehat{R}$ and conductor $\mathfrak{f} = \mathcal{F}_{\widehat{R}/R} \neq \{0\}$. If the v-class group $\mathcal{C}_v(\widehat{R})$ and the residue class ring $\widehat{R}/\mathfrak{f}$ are both finite, then R is a C-monoid. The concept of C-monoids has turned out to be quite successful for the arithmetical study of Mori domains (see [14, 15, 16, 18, 27, 29]), and it has recently been generalized in various directions ([22, 31]).

On the other hand, the question which integral domains are actually C-monoids remained wide open, and the present paper is devoted to this question. Let R be an integral domain that is a C-monoid. Then the above mentioned algebraic properties of C-monoids show that R is a Mori domain whose complete integral closure is a Krull domain with finite v-class group and conductor $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. But the converse is far from being true, even in the case of one-dimensional noetherian domains. Our main results (presented in Section 3) provide characterizations of semilocal integral domains R that are C-monoids under various additional conditions. It will turn out that such domains are closely connected with Cohen-Kaplansky domains ([2]) and with domains whose group of divisibility is finitely generated ([1, 24, 33]). In order to establish our characterization results we will have to study Mori domains and monoids in greater detail. The study of abstract Mori domains was initiated not before the 1970s ([35]), but since then it has found a lot of attention (see for example [7, 9, 10, 11, 13, 32, 34, 35, 36, 37], and in particular the survey article of V. Barucci [8]).

This paper is organized as follows: In the next section we study the relations between the Mori property and seminormality. We develop the theory for Mori monoids as far as it is needed in the proceeding sections. Especially we show that the monoid of multipliers of a prime s-ideal of a Mori monoid is a Mori

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monoid.
In section 3 we provide further important preliminary results for our main theorems. We summarize crucial properties of integral domains whose group of divisibility is finitely generated and investigate the structure of C-monoids in general.
In section 4 we present the main results of this paper. We prove that the seminormal closure and the root closure of a weakly Krull monoid whose complete integral closure is a Krull monoid is a Mori monoid. This is a partial converse to a theorem of V. Barucci [7], which states that the complete integral closure of a seminormal Mori domain is a Krull domain. We characterize G-domains that are C-monoids. Moreover we classify C-monoids in the case of non-local semilocal noetherian domains.
In the last section we provide some examples that show that even integral domains satisfying "nice" properties need not be C-monoids. We present a monoid theoretical analogue to a result regarding primary integral domains. Moreover we discuss the strongness of this result. By the way we strengthen a result of G. Angermüller [6] concerning root closed integral domains.

2. Mori monoids and seminormality

In the following a monoid is always a multiplicative commutative monoid with an identity 1 and a zero element 0, where every non-zero element is cancellative. A submonoid of a monoid always contains the identity and the zero element of the extension monoid. Every monoid H possesses a (essentially unique) quotient monoid, i.e. a monoid K such that H is a submonoid of K, where every non-zero of H is invertible in K and every element of K can be written as a quotient of an element of H and a non-zero element of H. A subset X of a monoid is called multiplicatively closed if $0 \notin X$, $1 \in X$ and $xy \in X$ for all $x, y \in X$. The prime subsets of a monoid are the complements of multiplicatively closed subsets.
Now let H be a monoid and K a quotient monoid of H. We set $H^\bullet = H \backslash \{0\}$. Note that every integral domain is a monoid and every quotient field of an integral domain is a quotient monoid of the domain. Our goal is to characterize special domains and thus it is convenient to assume that all monoids possess a zero element. Our definition of a monoid differs slightly from the definition in [20].

A domain satisfying some property A as a monoid will be called an A-domain.

Let $H \subseteq S \subseteq K$ be an intermediate monoid and $X, Y, Z \subseteq K$. We set

- $(X :_Z Y) = \{z \in Z \mid zY \subseteq X\}$.
- $X^{-1} = (H :_K X)$.
- $\mathcal{R}(X) = (X :_K X)$, called the monoid of multipliers of X.
- $\mathcal{F}_{S/H} = (H :_K S)$, called the conductor of S over H.

To simplify the notation we omit the indices in the majority of cases.
Now we introduce the most significant types of closures for monoids.

- Let $H' = \{x \in K \mid$ there exists some $k \in \mathbb{N}$ such that $x^n \in H$ for all $n \in \mathbb{N}_{\geq k}\}$ (seminormal closure of H). H is called seminormal if $H = H'$.
- Let $\widetilde{H} = \{x \in K \mid x^k \in H$ for some $k \in \mathbb{N}\}$ (root closure of H). H is called root closed if $H = \widetilde{H}$.
- Let $\widehat{H} = \{x \in K \mid$ there exists some $c \in H^\bullet$ such that $cx^n \in H$ for all $n \in \mathbb{N}\}$ (complete integral closure of H). H is called completely integrally closed if $H = \widehat{H}$.

We briefly recall the concept of ideal systems. For a profound introduction into ideal systems see [26].
Let $r : \mathbb{P}(H) \to \mathbb{P}(H)$ be a map. r is called ideal system on H if the following conditions are satisfied for all $X, Y \subseteq H$ and $c \in H$.

1. $XH \cup \{0\} \subseteq r(X)$.
2. $r(cX) = cr(X)$.
3. If $X \subseteq r(Y)$, then $r(X) \subseteq r(Y)$.

Let r be an ideal system on H. For $X \subseteq H$ let $X_r = r(X)$. r is called finitary if $X_r = \bigcup_{E \subseteq X, |E| < \infty} E_r$ for all $X \subseteq H$. Note that $s : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $s(\emptyset) = \{0\}$ and $s(X) = XH$ for all $\emptyset \neq X \subseteq H$ is

a finitary ideal system on H. Moreover if $H^\bullet \neq H^\times$, then $v : \mathbb{P}(H) \to \mathbb{P}(H)$ defined by $v(X) = (X^{-1})^{-1}$ for all $X \subseteq H$ is an ideal system on H. If R is an integral domain, then $d : \mathbb{P}(R) \to \mathbb{P}(R)$ defined by $d(X) = (X)_R$ for all $X \subseteq H$ is a finitary ideal system on R.
The elements of $\{X \subseteq H \mid X_r = X\}$ are called r-ideals of H. The v-ideals are called divisorial ideals. Let r-max(H) be the set of all maximal elements of $\{X \subsetneq H \mid X_r = X\}$ and r-spec(H) the set of all prime r-ideals of H. Let r-spec$(H)^\bullet = r$-spec$(H)\backslash\{\{0\}\}$. Note that r-max$(H) \subseteq r$-spec(H). Let $\mathfrak{X}(H)$ be the set of all minimal elements of s-spec$(H)^\bullet$. If $X \subseteq H$, then we set $\mathcal{V}_r(X) = \{P \in r\text{-spec}(H) \mid X \subseteq P\}$.
Next we recall the definition of important types of monoids.

- H is called v-noetherian monoid if it satisfies the ascending chain condition on divisorial ideals. v-noetherian monoids are also called Mori monoids. In the following we use the term "Mori monoid".
- H is called Krull monoid if it is a completely integrally closed Mori monoid.

We repeat the concept of class semigroups which is crucial for the definition of C-monoids. Let F be a monoid such that $H \subseteq F$ is a submonoid. For $x, y \in F^\bullet$ we set $x \sim_H y$ if $x^{-1}H \cap F = y^{-1}H \cap F$. This defines a congruence relation on $F^\bullet$, and we denote by $[x]_H^F$ the equivalence class of x. Then $\mathcal{C}(H, F) = \{[x]_H^F \mid x \in F^\bullet\}$ resp. $\mathcal{C}^*(H, F) = \{[x]_H^F \mid x \in (F^\bullet\backslash F^\times) \cup \{1\}\}$ are called class semigroup of H in F resp. reduced class semigroup of H in F.
The monoid H is said to be a C-monoid if it is a submonoid of a factorial monoid F such that $F^\times \cap H = H^\times$ and $\mathcal{C}^*(H, F)$ is finite. If this holds, then we say more precisely that H is a C-monoid defined in F. If H is a C-monoid, then there exists a canonical choice for F. (For details see [20, Theorem 2.9.11.]).
We introduce the concept of Mori sets to facilitate additional characterizations of Mori monoids. Let K be a quotient monoid of H. A subset $L \subseteq K$ is called Mori set if for all $(a_i)_{i\in\mathbb{N}} \in L^{\mathbb{N}}$ there is some $k \in \mathbb{N}$ such that $(L :_K \{a_i \mid i \in \mathbb{N}_{\leq k}\}) = (L :_K \{a_i \mid i \in \mathbb{N}_{\leq n}\})$ for all $n \in \mathbb{N}_{\geq k}$. The concept of Mori sets stems from M. Roitman (see [36, 37]) who introduced them to study Mori domains.

Lemma 2.1. *Let H be a monoid, K a quotient monoid of H, $A \subseteq H$ a Mori set and $B \subseteq H$.*

1. *If B is a Mori set, then $A \cap B$ is a Mori set.*
2. *$(A : B)$ is a Mori set.*

Proof. **1.** Let B be a Mori set and $(a_i)_{i\in\mathbb{N}} \in (A \cap B)^{\mathbb{N}}$. There are some $k_1, k_2 \in \mathbb{N}$ such that $(A : \{a_i \mid i \in \mathbb{N}_{\leq k_1}\}) = (A : \{a_i \mid i \in \mathbb{N}_{\leq n}\})$ and $(B : \{a_i \mid i \in \mathbb{N}_{\leq k_2}\}) = (B : \{a_i \mid i \in \mathbb{N}_{\leq l}\})$ for all $n \in \mathbb{N}_{\geq k_1}$ and $l \in \mathbb{N}_{\geq k_2}$. Let $k = \max(k_1, k_2)$ and $n \in \mathbb{N}_{\geq k}$. Then $(A \cap B : \{a_i \mid i \in \mathbb{N}_{\leq n}\}) = (A : \{a_i \mid i \in \mathbb{N}_{\leq n}\}) \cap (B : \{a_i \mid i \in \mathbb{N}_{\leq n}\}) = (A : \{a_i \mid i \in \mathbb{N}_{\leq k}\}) \cap (B : \{a_i \mid i \in \mathbb{N}_{\leq k}\}) = (A \cap B : \{a_i \mid i \in \mathbb{N}_{\leq k}\})$.
2. Assume that $(A : B)$ is not a Mori set. Then there are some $(a_i)_{i\in\mathbb{N}} \in (A : B)^{\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have that $a_{n+1}x_n \notin (A : B)$ and $a_i x_n \in (A : B)$ for all $i \in \mathbb{N}_{\leq n}$. There exists some $(b_i)_{i\in\mathbb{N}} \in B^{\mathbb{N}}$ such that $b_n a_{n+1} x_n \notin A$ for all $n \in \mathbb{N}$. Since $(b_i a_{i+1})_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ there exists some $k \in \mathbb{N}$ such that $(A : \{b_i a_{i+1} \mid i \in \mathbb{N}_{\leq k}\}) = (A : \{b_i a_{i+1} \mid i \in \mathbb{N}_{\leq k+1}\})$. Since $b_i a_{i+1} x_{k+1} \in A$ for all $i \in \mathbb{N}_{\leq k}$ it follows that $b_{k+1} a_{k+2} x_{k+1} \in A$, a contradiction. □

Proposition 2.2. *Let H be a monoid, K a quotient monoid of H and $M = H\backslash H^\times$. The following conditions are equivalent*:

1. *H is a Mori monoid.*
2. *H is a Mori set.*
3. *M is a Mori set.*

Proof. **1.** $\Rightarrow$ **2.**: Trivial. **2.** $\Rightarrow$ **3.**: Let H be a Mori set. Assume that M is not a Mori set. Then there exist some $(a_i)_{i\in\mathbb{N}} \in M^{\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have that $a_{n+1}x_n \notin M$ and $a_i x_n \in M$ for all $i \in \mathbb{N}_{\leq n}$. There is some $k \in \mathbb{N}$ such that $(H : \{a_i \mid i \in \mathbb{N}_{\leq k}\}) = (H : \{a_i \mid i \in \mathbb{N}_{\leq k+2}\})$, hence $a_{k+1}x_k, a_{k+2}x_k \in H$, and thus $a_{k+1}x_k \in H^\times$. Consequently, $a_{k+2}x_{k+1} = \frac{a_{k+2}x_k}{a_{k+1}x_k} a_{k+1}x_{k+1} \in M$, a contradiction.
3. $\Rightarrow$ **1.**: Let M be a Mori set. Assume that H is not a Mori monoid. Then there exists some sequence $(I_i)_{i\in\mathbb{N}}$ of divisorial ideals of H such that $I_i \subsetneq I_{i+1}$ for all $i \in \mathbb{N}$. Consequently, there is some $(a_i)_{i\in\mathbb{N}} \in M^{\mathbb{N}}$

such that $a_i \in I_{i+1} \backslash I_i$ for all $i \in \mathbb{N}$. Assume that there exists some $l \in \mathbb{N}$ such that $(H : \{a_i \mid i \in \mathbb{N}_{\leq l}\}) = (H : \{a_i \mid i \in \mathbb{N}_{\leq l+1}\})$. Then $a_{l+1} \in \{a_i \mid i \in \mathbb{N}_{\leq l+1}\}_v = \{a_i \mid i \in \mathbb{N}_{\leq l}\}_v \subseteq I_{l+1}$, a contradiction. Consequently, there is some $(x_i)_{i \in \mathbb{N}} \in K^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have that $a_{n+1}x_n \notin H$ and $a_i x_n \in H$ for all $i \in \mathbb{N}_{\leq n}$. There exists some $k \in \mathbb{N}$ such that $(M : \{a_i \mid i \in \mathbb{N}_{\leq k}\}) = (M : \{a_i \mid i \in \mathbb{N}_{\leq k+2}\})$. If there exists some $i \in \mathbb{N}_{\leq k}$ such that $a_i x_{k+1} \notin M$, then $a_i x_{k+1} \in H^{\times}$, hence $a_{k+1}x_k = \frac{a_{k+1}x_{k+1}}{a_i x_{k+1}} a_i x_k \in H$, a contradiction. Therefore $a_i x_{k+1} \in M$ for all $i \in \mathbb{N}_{\leq k}$, hence $a_{k+2}x_{k+1} \in M \subseteq H$, a contradiction. □

A monoid H is called G-monoid if $\bigcap_{P \in s\text{-spec}(H)^{\bullet}} P \neq \{0\}$. By [20, Theorem 2.10.2.5.] an integral domain R is a G-domain if and only if $\bigcap_{P \in \text{spec}(R)^{\bullet}} P \neq \{0\}$ (which is the usual definition as it can be found in Gilmer's book [23]).

Corollary 2.3. *Let H be a Mori monoid and K a quotient monoid of H.*

1. *P is a Mori set and $\mathcal{R}(P)$ is a Mori monoid for all $P \in s$-spec(H).*
2. *If H is a G-monoid, then all radical s-ideals of H are Mori sets.*

Proof. **1.** Let $P \in s$-spec(H). Then H_P is a Mori monoid. Consequently, H and P_P are Mori sets by Proposition 2.2. and hence $P = P_P \cap H$ is a Mori set by Lemma 2.1.1.. Therefore Lemma 2.1.2. implies that $\mathcal{R}(P)$ is a Mori set. Since $\mathcal{R}(P)$ is a monoid and K is a quotient monoid of $\mathcal{R}(P)$ it follows by Proposition 2.2. that $\mathcal{R}(P)$ is a Mori monoid.
2. By [20, Theorem 2.7.9.] it follows that s-spec(H) is finite. Let I be a radical s-ideal of H. Then $I = \bigcap_{P \in \mathcal{V}_s(I)} P$. Since $\mathcal{V}_s(I) \subseteq s$-spec$(H)$ is finite, we have that I is a Mori set by 1. and Lemma 2.1.1.. □

If H is a monoid, K is a quotient monoid of H and $x \in K$, then let $H[x]$ denote the smallest submonoid of K that contains $H \cup \{x\}$.

Lemma 2.4. *Let H be a monoid, K a quotient monoid of H and $S \subseteq H^{\bullet}$ multiplicatively closed.*

1. *The following conditions are equivalent*:
   - **a.** *H is seminormal.*
   - **b.** *If $x \in K$ such that $x^2, x^3 \in H$, then $x \in H$.*
   - **c.** *If $x \in \widetilde{H}$, then $\mathcal{F}_{H[x]/H}$ is a radical s-ideal of $H[x]$.*
2. *H' is seminormal.*
3. *If H is seminormal, then $S^{-1}H$ is seminormal.*
4. *$(S^{-1}H)' = S^{-1}H'$.*

Proof. **1.** This is well known.
2. By 1. it is sufficient to show that for all $x \in K$ such that $x^2, x^3 \in H'$ we have that $x \in H'$. Let $x \in K$ be such that $x^2, x^3 \in H'$. Then there exists some $k \in \mathbb{N}$ such that $x^{2n}, x^{3n} \in H$ for all $n \in \mathbb{N}_{\geq k}$. It is sufficient to show that $x^n \in H$ for all $n \in \mathbb{N}_{\geq 5k+3}$. Let $n \in \mathbb{N}_{\geq 5k+3}$.
Case 1: n is even: There is some $l \in \mathbb{N}_{\geq k}$ such that $n = 2l$, hence $x^n = x^{2l} \in H$.
Case 2: n is odd: Case A: k is even: It follows that $3(k+1)$ is odd, hence there exists some $l \in \mathbb{N}_{\geq k}$ such that $n - 3(k+1) = 2l$. Therefore $x^n = x^{3(k+1)}x^{2l} \in H$. Case B: k is odd: Since $3k$ is odd, there exists some $l \in \mathbb{N}_{\geq k}$ such that $n - 3k = 2l$. Consequently, $x^n = x^{3k}x^{2l} \in H$.
3. Let H be seminormal. By 1. it is sufficient to show that for all $x \in K$ such that $x^2, x^3 \in S^{-1}H$ it follows that $x \in S^{-1}H$. Let $x \in K$ be such that $x^2, x^3 \in S^{-1}H$. Then there exist some $s, t \in S$ such that $tx^2, sx^3 \in H$. It follows that $(stx)^2, (stx)^3 \in H$, hence $stx \in H$. Therefore $x \in S^{-1}H$.
4. "$\subseteq$": By 2. and 3. it follows that $S^{-1}H'$ is seminormal. Since $S^{-1}H \subseteq S^{-1}H'$ we have that $(S^{-1}H)' \subseteq (S^{-1}H')' = S^{-1}H'$. "$\supseteq$": Let $x \in S^{-1}H'$. Then there exist some $t \in S$ and $k \in \mathbb{N}$ such that $(tx)^n \in H$ for all $n \in \mathbb{N}_{\geq k}$. Therefore $x^n \in S^{-1}H$ for all $n \in \mathbb{N}_{\geq k}$. This implies that $x \in (S^{-1}H)'$. □

In the following Lemma 2.4. will be used without citation. A monoid H is called primary if $H^{\bullet} \neq H^{\times}$ and s-spec$(H) = \{\{0\}, H \backslash H^{\times}\}$. Observe that an integral domain is primary if and only if it is local and 1-dimensional (see [20, Proposition 2.10.7.1.]).

Lemma 2.5. *Let H be a seminormal monoid.*

1. $\mathcal{F}_{\widehat{H}/H}$ *is a radical s-ideal of $\widehat{H}$.*
2. *If H is a G-monoid, then $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$.*
3. *If H is primary and $H \neq \widehat{H}$, then $\mathcal{F}_{\widehat{H}/H} = H \setminus H^{\times}$.*

Proof. **1.** Of course $\mathcal{F}_{\widehat{H}/H}$ is an s-ideal of $\widehat{H}$. Let $x \in \sqrt[\widehat{H}]{\mathcal{F}_{\widehat{H}/H}}$. Then there exists some $k \in \mathbb{N}$ such that $x^k \widehat{H} \subseteq H$. Let $y \in \widehat{H}$. Then $(xy)^l = x^k x^{l-k} y^l \in H$ for all $l \in \mathbb{N}_{\geq k}$. Since H is seminormal we have that $xy \in H$. Hence $x \in \mathcal{F}_{\widehat{H}/H}$.
2. See [21, Proposition 4.8.2.].
3. Let H be primary and $H \neq \widehat{H}$. By 1. and 2. it follows that $\mathcal{F}_{\widehat{H}/H} = \bigcap_{P \in \mathcal{V}_s(\mathcal{F}_{\widehat{H}/H})} P = H \setminus H^{\times}$. □

A monoid H is called weakly Krull monoid if $\bigcap_{P \in \mathfrak{X}(H)} H_P = H$ and $\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite for all $x \in H^{\bullet}$. Note that a weakly Krull monoid H is a Mori monoid if and only if H_P is a Mori monoid for all $P \in \mathfrak{X}(H)$. Weakly Krull domains have been introduced by Anderson, Mott and Zafrullah in [3]. Their ideal and divisor theoretic properties are presented in [26, Chapters 22 and 24.5]. Clearly, every 1-dimensional Mori domain is weakly Krull, and so are noetherian Cohen-Macaulay domains. Every primary monoid is a weakly Krull monoid.

Lemma 2.6. *Let H be a seminormal weakly Krull monoid such that $\widehat{H}$ is a Krull monoid. Then H is a Mori monoid.*

Proof. Case 1: H is primary: Without restriction we can assume that $H \neq \widehat{H}$. We have that H is a G-monoid. Therefore $\widehat{H}$ is a G-monoid. By Lemma 2.5.1. and Lemma 2.5.3. it follows that $\mathcal{F}_{\widehat{H}/H} = H \setminus H^{\times}$ is a radical s-ideal of $\widehat{H}$ and thus $H \setminus H^{\times}$ is a Mori set by Corollary 2.3.2. Consequently, H is a Mori monoid by Proposition 2.2..
Case 2: H is not necessarily primary: It is sufficient to show that H_P is a Mori monoid for all $P \in \mathfrak{X}(H)$. Let $P \in \mathfrak{X}(H)$. Then H_P is a seminormal, primary monoid and $\widehat{H}_P$ is a Krull monoid. Since $H_P \subseteq \widehat{H}_P \subseteq \widehat{H_P}$, we have that $\widehat{H_P} = \widehat{H}_P$. Therefore H_P is a Mori monoid by Case 1. □

Proposition 2.7. *Let F be a monoid and $H \subseteq F$ a submonoid such that F is a root extension of H. Let $f : s\text{-spec}(F) \to s\text{-spec}(H)$ be defined by $f(Q) = Q \cap H$ for all $Q \in s\text{-spec}(F)$.*

1. *f is bijective.*
2. *$f(\mathfrak{X}(F)) = \mathfrak{X}(H)$.*
3. *$F_Q = F_{Q \cap H}$ for all $Q \in \mathfrak{X}(F)$.*
4. *If $\{P \in \mathfrak{X}(H) \mid x \in P\}$ is finite for all $x \in H^{\bullet}$, then $\{Q \in \mathfrak{X}(H) \mid y \in Q\}$ is finite for all $y \in F^{\bullet}$.*

Proof. Let $g : s\text{-spec}(H) \to s\text{-spec}(F)$ be defined by $g(P) = \{x \in F \mid x^n \in P \text{ for some } n \in \mathbb{N}\}$ for all $P \in s\text{-spec}(H)$.
1. Since f and g are mutually inverse maps, it follows that f is bijective.
2. This is clear since f and g are inclusion preserving maps.
3. Let $Q \in \mathfrak{X}(F)$ and $P = Q \cap H$. By 2. it follows that $s\text{-spec}(F_P) = \{B_P \mid B \in s\text{-spec}(F) \text{ and } B \cap H \setminus P = \emptyset\} = \{\{0\}, Q_P\}$. Therefore $Q_P = F_P \setminus F_P^{\times}$ and thus $F_Q = (F_P)_{Q_P} = F_P$.
4. Let $\{P \in \mathfrak{X}(H) \mid x \in P\}$ be finite for all $x \in H^{\bullet}$ and let $y \in F^{\bullet}$. There is some $k \in \mathbb{N}$ such that $y^k \in H^{\bullet}$. By 1. and 2. we have that $\{Q \cap H \mid Q \in \mathfrak{X}(F) \text{ and } y \in Q\} \subseteq \{P \in \mathfrak{X}(H) \mid y^k \in P\}$ and $|\{Q \cap H \mid Q \in \mathfrak{X}(F) \text{ and } y \in Q\}| = |\{Q \in \mathfrak{X}(F) \mid y \in Q\}|$. Therefore $\{Q \in \mathfrak{X}(F) \mid y \in Q\}$ is finite. □

3. Preliminaries for the main results

Recall that a C-monoid H is a Mori monoid ([20, Theorem 2.9.13.]) and $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$ ([20, Theorem 2.9.11.2.]). A monoid H is called finitely primary if H is primary, $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$ and $\widehat{H}$ is factorial.

Finitely primary monoids have been introduced as multiplicative models of local 1-dimensional noetherian domains. A commutative ring with identity is called local if it has precisely one maximal ideal and it is called semilocal if it has only finitely many maximal ideals.

Lemma 3.1. *Let R be a Mori domain that is not a field and $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$.*

1. *R is 1-dimensional if and only if $\widehat{R}$ is 1-dimensional.*
2. *R is a G-domain if and only if R is semilocal and 1-dimensional.*
3. *If R is semilocal and 1-dimensional, then $\widehat{R}$ is a semilocal principal ideal domain.*

Proof. **1.** This is an immediate consequence of [20, Proposition 2.10.5.].
2. See [20, Proposition 2.10.7.].
3. Let R be semilocal and 1-dimensional. Since R is a G-domain we have that $\widehat{R}$ is a G-domain. On the other hand $\widehat{R}$ is a Krull domain and thus 2. implies that $\widehat{R}$ is semilocal and 1-dimensional. Therefore $\widehat{R}$ is a 1-dimensional Krull domain hence $\widehat{R}$ is a Dedekind domain. Since $\widehat{R}$ is semilocal we have that $\widehat{R}$ is a principal ideal domain. □

If R is a primary Mori domain such that $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, then R is finitely primary (This follows from Lemma 3.1.). This implies that for a Mori domain R that is a weakly Krull domain with $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, the localizations R_P for $P \in \mathfrak{X}(R)$ are finitely primary. If R is a finitely primary noetherian domain, then R is not necessarily a C-monoid (see [28, Example 4.5.] and [20, Corollary 2.9.8.]). If R is an integral domain, then let $\overline{R}$ be the integral closure of R in its quotient field.

Lemma 3.2. *Let H be a C-monoid and $M = H \backslash H^\times$.*

1. $(\widehat{H}^\times : \mathcal{R}(M)^\times) < \infty$.
2. $(\widehat{H}^\times : H^\times) < \infty$ *or* $M_v = M$.

Proof. **1.** There is some factorial monoid F such that $H \subseteq F$ is a submonoid, $F^\times \cap H = H^\times$ and $\mathcal{C}^*(H, F)$ is finite. We have that H is a Mori monoid and $\widehat{H}$ is a Krull monoid. By [20, Proposition 2.8.11.] there exists some subgroup V of $F^\times$ such that $(F^\times : V) < \infty$ and $VM \subseteq H$. Assume that $VM \nsubseteq M$. Then there exist some $r \in V$ and $x \in M$ such that $rx \in H^\times$. It follows that $x \in F^\times \cap H = H^\times$, a contradiction. Therefore $VM \subseteq M$. Since F is factorial, it follows that $\widehat{H}$ is a submonoid of F, hence $\widehat{H}^\times$ is a subgroup of $F^\times$. Since $\widehat{H}$ is a Krull monoid and $H \subseteq \mathcal{R}(M) \subseteq \widehat{H}$ is an intermediate monoid we have that $\widehat{\mathcal{R}(M)} = \widehat{H}$. By Corollary 2.3.1. it follows that $\mathcal{R}(M)$ is a Mori monoid. Let $U = V \cap \widehat{H}^\times$. Since $UM \subseteq VM \subseteq M$ and $\mathcal{R}(M)$ is a Mori monoid we have that $U \subseteq \mathcal{R}(M) \cap \widehat{\mathcal{R}(M)}^\times = \mathcal{R}(M)^\times$. Finally this implies that $(\widehat{H}^\times : \mathcal{R}(M)^\times) \leq (\widehat{H}^\times : U) = (V\widehat{H}^\times : V) \leq (F^\times : V) < \infty$.
2. Let $M_v \neq M$. Then $M_v = H$ and thus $H \subseteq \mathcal{R}(M) \subseteq \mathcal{R}(M_v) = \mathcal{R}(H) = H$. Therefore $(\widehat{H}^\times : H^\times) = (\widehat{H}^\times : \mathcal{R}(M)^\times) < \infty$ by 1.. □

Corollary 3.3. *Let R be an integral domain and $M = R \backslash R^\times$. If R is a C-monoid, then $(\widehat{R}^\times : R^\times) < \infty$ or R is local.*

Proof. Let $(\widehat{R}^\times : R^\times) = \infty$, then $M \subseteq (M)_R \subseteq M_v = M$ by Lemma 3.2.2.. Consequently, $(M)_R = M$ and thus R is local. □

Lemma 3.4. *Let R be an integral domain and K a field of quotients of R.*

1. *If $K^\times/R^\times$ is finitely generated, then $\overline{R}$ is a Bézout domain and a finitely generated R-module, $\mathrm{spec}(\overline{R})$ is finite and $(\overline{R}^\times : R^\times) < \infty$.*
2. *$K^\times/R^\times$ is finitely generated if and only if $K^\times/\overline{R}^\times$ is finitely generated and $\overline{R}/\mathcal{F}_{\overline{R}/R}$ is finite.*

Proof. **1.** See [24, Theorem 2.1.] and [24, Theorem 3.9.].
2. See [1, Theorem 3.]. □

Let S be a commutative ring with identity and N an S-module. By $ann_S(N)$ we denote the S-annihilator of N. Note that if N is finite (as a set), then $S/ann_S(N)$ is finite (as a set). Let $\mathcal{J}(S)$ denote the Jacobson radical of S. The following result is an easy consequence of [30, Lemma 2.4.] in the noetherian case. Note that the proof of the equivalence of 1. and 2. in [30, Lemma 2.4.] also works without the noetherian hypothesis. We provide an alternative proof of the following statement.

Proposition 3.5. *Let S be a commutative ring with identity and $R \subseteq S$ a semilocal unitary subring. The following conditions are equivalent*:

1. *$S^\times/R^\times$ is finite and S is a finitely generated R-module.*
2. *S/R is finite.*

Proof. **1.** $\Rightarrow$ **2.**: Let $S^\times/R^\times$ be finite and S a finitely generated R-module.
Claim 1: $\mathcal{J}(S)/\mathcal{J}(R)$ is finite. Clearly, $1+\mathcal{J}(S)$ is a subgroup of $S^\times$ and $(1+\mathcal{J}(S))\cap R^\times = 1+\mathcal{J}(R)$. Therefore $(1+\mathcal{J}(S))/(1+\mathcal{J}(R))$ is isomorphic to a subgroup of $S^\times/R^\times$, hence there are some $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \in \mathcal{J}(S)^{[1,n]}$ such that $(1+\mathcal{J}(S))/(1+\mathcal{J}(R)) = \{(1+a_i)(1+\mathcal{J}(R)) \mid i \in [1,n]\}$.
Claim 1a: For every $x \in \mathcal{J}(S)$ there exists some $i \in [1,n]$ such that $(x+\mathcal{J}(R))_R = (a_i+\mathcal{J}(R))_R$. Let $x \in \mathcal{J}(S)$. There exist some $i \in [1,n]$ and $r \in \mathcal{J}(R)$ such that $1+x = (1+a_i)(1+r)$. It follows that $x = (1+r)a_i + r$, hence $x+\mathcal{J}(R) = (1+r)(a_i+\mathcal{J}(R))$. Since $1+r \in R^\times$ this implies that $(x+\mathcal{J}(R))_R = (a_i+\mathcal{J}(R))_R$.
By claim 1a we have that $\mathcal{J}(S)/\mathcal{J}(R)$ has only finitely many cyclic R-submodules and thus $\mathcal{J}(S)/\mathcal{J}(R)$ has finite R-length. Consequently, there exists some $k \in \mathbb{N}_0$ and some sequence $(N_i)_{i=0}^k$ of R-submodules of $\mathcal{J}(S)$ such that $N_0 = \mathcal{J}(R)$, $N_k = \mathcal{J}(S)$ and such that $N_i \subseteq N_{i+1}$ and N_{i+1}/N_i is a simple R-module for all $i \in [0,k-1]$.
Claim 1b: For every $i \in [0,k-1]$ we have that $N_{i+1}/N_i \subseteq \{a_j + N_i \mid j \in [1,n]\}$. Let $i \in [0,k-1]$ and $y \in N_{i+1}$. There exist some $j \in [1,n]$ and $s \in \mathcal{J}(R)$ such that $1+y = (1+a_j)(1+s)$. By Nakayama's Lemma it follows that $\mathcal{J}(R)(N_{i+1}/N_i) = N_i/N_i$ and thus $\mathcal{J}(R)N_{i+1} \subseteq N_i$. We have that $a_j = (1+s)^{-1}(y-s) \in N_{i+1}$ and since $y = a_j + s + sa_j$ it follows that $y + N_i = a_j + N_i$.
By claim 1b we have that N_{i+1}/N_i is finite for all $i \in [0,k-1]$. Therefore $\mathcal{J}(S)/\mathcal{J}(R)$ is finite.
Claim 2: For every $M \in \max(R)$ it follows that $S/(R+MS)$ is finite. Let $M \in \max(R)$. Case 1: R/M is finite: Since S is a finitely generated R-module we have that S/MS is a finitely generated R/M-module. Therefore S/MS is finite, hence $S/(R+MS)$ is finite. Case 2: R/M is infinite: Since S is semilocal it follows by [1, Lemma 2.] that $S^\times/(R+MS)^\times \cong (S/MS)^\times/((R+MS)/MS)^\times$. Consequently, $(S/MS)^\times/((R+MS)/MS)^\times$ is finite. Since $(R+MS)/MS \cong R/M$ is an infinite field and S/MS is an artinian ring it follows by [25, Lemma 1.6.] that $S/MS = (R+MS)/MS$, hence $S = R+MS$.
Claim 3: $S/(R+\mathcal{J}(S))$ is finite. Let $f : S \to \prod_{M\in\max(R)} S/(R+MS)$ be defined by $f(x) = (x+(R+MS))_{M\in\max(R)}$ for all $x \in S$. Then f is an R-module homomorphism and $ker(f) = \bigcap_{M\in\max(R)}(R+MS)$. If $x \in \bigcap_{M\in\max(R)}(R+MS)$, then there exists some $(r_M)_{M\in\max(R)} \in R^{\max(R)}$ such that $x - r_M \in MS$ for all $M \in \max(R)$. By the Chinese Remainder Theorem there exists some $r \in R$ such that $r - r_M \in M$ for all $M \in \max(R)$. Therefore $x - r \in \bigcap_{M\in\max(R)} MS = \mathcal{J}(R)S$, hence $x \in R+\mathcal{J}(R)S$. This implies that $ker(f) = R+\mathcal{J}(R)S$. Consequently, $S/(R+\mathcal{J}(R)S)$ is isomorphic to an R-submodule of $\prod_{M\in\max(R)} S/(R+MS)$, hence $S/(R+\mathcal{J}(R)S)$ is finite by claim 2. Consequently, $S/(R+\mathcal{J}(S))$ is finite.
By claim 3 we have that $(S/R)/((R+\mathcal{J}(S))/R)$ is finite. Since $(R+\mathcal{J}(S))/R \cong \mathcal{J}(S)/\mathcal{J}(R)$ is finite by claim 1 it follows that S/R is finite.
2. $\Rightarrow$ **1.**: Let S/R be finite. Clearly, S is a finitely generated R-module, hence S is semilocal. Observe that $I := ann_R(S/R)$ is an ideal of R and of S and that R/I is a subring of S/I. By [1, Lemma 2.] it follows that $S^\times/R^\times \cong (S/I)^\times/(R/I)^\times$. Since S/R is finite we have that R/I is finite and thus S/I is finite. Consequently, $S^\times/R^\times$ is finite. □

4. Main results

Theorem 4.1. *Let H be a weakly Krull monoid.*

1. *H' and $\widetilde{H}$ are weakly Krull monoids.*
2. *If $\widehat{H}$ is a Krull monoid, then H' and $\widetilde{H}$ are Mori monoids.*

Proof. **1.** Claim 1: $\bigcap_{P\in\mathfrak{X}(H)} H'_P = H'$. "$\subseteq$": Let $x \in \bigcap_{P\in\mathfrak{X}(H)} H'_P$. There exists some $y \in H^\bullet$ such that $yx \in H$. Let $\mathcal{P} = \{P \in \mathfrak{X}(H) \mid y \in P\}$. Then $\mathcal{P}$ is finite. Since $x \in H'_P = (H_P)'$ for all $P \in \mathcal{P}$, it follows that there exists some $k \in \mathbb{N}$ such that $x^n \in H_P$ for all $n \in \mathbb{N}_{\geq k}$ and $P \in \mathcal{P}$. Since $x = y^{-1}yx \in H_P$ for all $P \in \mathfrak{X}(H)\backslash\mathcal{P}$ we have that $x^n \in \bigcap_{P\in\mathfrak{X}(H)} H_P = H$ for all $n \in \mathbb{N}_{\geq k}$. Consequently, $x \in H'$. "$\supseteq$": Trivial.
Claim 2: $\bigcap_{P\in\mathfrak{X}(H)} \widetilde{H}_P = \widetilde{H}$. "$\subseteq$": Let $x \in \bigcap_{P\in\mathfrak{X}(H)} \widetilde{H}_P$. There is some $y \in H^\bullet$ such that $yx \in H$. Let $\mathcal{P} = \{P \in \mathfrak{X}(H) \mid y \in P\}$. Then $\mathcal{P}$ is finite. Since $x \in \widetilde{H}_P = \widetilde{H_P}$ for all $P \in \mathcal{P}$, we have that there exists some $k \in \mathbb{N}$ such that $x^k \in H_P$ for all $P \in \mathcal{P}$. Since $x = y^{-1}yx \in H_P$ for all $P \in \mathfrak{X}(H)\backslash\mathcal{P}$ it follows that $x^k \in \bigcap_{P\in\mathfrak{X}(H)} H_P = H$, hence $x \in \widetilde{H}$. "$\supseteq$": Trivial.
By claim 1, claim 2 and Proposition 2.7. we have that $\bigcap_{Q\in\mathfrak{X}(H')} H'_Q = \bigcap_{Q\in\mathfrak{X}(H')} H'_{Q\cap H} = \bigcap_{P\in\mathfrak{X}(H)} H'_P = H'$ and $\bigcap_{Q\in\mathfrak{X}(\widetilde{H})} \widetilde{H}_Q = \bigcap_{Q\in\mathfrak{X}(\widetilde{H})} \widetilde{H}_{Q\cap H} = \bigcap_{P\in\mathfrak{X}(H)} \widetilde{H}_P = \widetilde{H}$. Finally Proposition 2.7.4. implies that H' and $\widetilde{H}$ are weakly Krull monoids.
2. Let $\widehat{H}$ be a Krull monoid. Obviously, $\widehat{H} \subseteq \widehat{H'} \subseteq \widehat{\widetilde{H}} \subseteq \widehat{\widehat{H}} = \widehat{H}$ and thus $\widehat{\widetilde{H}} = \widehat{H'} = \widehat{H}$. Since H' and $\widetilde{H}$ are seminormal it follows by 1. and Lemma 2.6. that H' and $\widetilde{H}$ are Mori monoids. □

Next we characterize integral domains that are C-monoids under various additional conditions. Recall that if R is an integral domain and K is a field of quotients of R that $K^\times/R^\times$ is called the group of divisibility of R ([1, 24, 33]).

Theorem 4.2. *Let R be an integral domain that is not a field and K a field of quotients of R. The following conditions are equivalent*:

1. *R is a C-monoid and a G-domain and $(\widehat{R}^\times : R^\times) < \infty$.*
2. *R is a 1-dimensional, semilocal Mori domain, $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$, $(\widehat{R}^\times : R^\times) < \infty$.*
3. *R is a Mori domain and $K^\times/R^\times$ is finitely generated.*
4. *R is noetherian and $K^\times/R^\times$ is finitely generated.*
5. *R is a Mori domain and a G-domain and $\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ is finite.*
6. *R is a Mori domain and a G-domain and $\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ has only finitely many R-submodules.*

Proof. **1.** ⇒ **2.**: Since R is a C-monoid it follows that R is a Mori domain and $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. By Lemma 3.1.2. we have that R is semilocal and 1-dimensional.
2. ⇒ **3.**: By Lemma 3.1.3. it follows that $\widehat{R}$ is a semilocal principal ideal domain. Therefore [2, Corollary 3.5.] implies that $K^\times/\widehat{R}^\times$ is finitely generated. Consequently, $(K^\times/R^\times)/(\widehat{R}^\times/R^\times)$ and $\widehat{R}^\times/R^\times$ are finitely generated and thus $K^\times/R^\times$ is finitely generated.
3. ⇒ **4.**: By Lemma 3.4.1. it follows that $\overline{R}$ is a Bézout domain and a finitely generated R-module and $\mathrm{spec}(\overline{R})$ is finite. Therefore $\mathcal{F}_{\overline{R}/R} \neq \{0\}$ and $\overline{R}$ is a G-domain. Consequently, Lemma 2.5.2. implies that $\mathcal{F}_{\widehat{R}/\overline{R}} \neq \{0\}$, hence $\{0\} \neq \mathcal{F}_{\widehat{R}/\overline{R}}\mathcal{F}_{\overline{R}/R} \subseteq \mathcal{F}_{\widehat{R}/R}$. It follows that $\widehat{R}$ is a Krull domain. Since $\overline{R}$ is a G-domain we have that $\widehat{R}$ is a G-domain. Therefore Lemma 3.1.2. implies that $\dim(\widehat{R}) = 1$ and thus $\widehat{R}$ is a Dedekind domain. Consequently, Lemma 3.1.1. implies that $\dim(R) = 1$. It follows that $\overline{R}$ is a 1-dimensional Bézout domain and thus $\overline{R}$ is completely integrally closed. This implies that $\overline{R} = \widehat{R}$. Finally it follows by the Theorem of Eakin-Nagata that R is noetherian.
4. ⇒ **5.**: By Lemma 3.4.1 we have that $\overline{R}$ is a finitely generated R-module and $\mathrm{spec}(\overline{R})$ is finite. Therefore $\overline{R}$ is a noetherian G-domain and thus Lemma 3.1.2. implies that $\overline{R}$ is semilocal and 1-dimensional. Consequently, R is semilocal and 1-dimensional, hence R is a G-domain. $\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ is finite by Lemma 3.4.2..
5. ⇒ **6.**: Trivial. **6.** ⇒ **1.**: Clearly, $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. By Lemma 3.1. we have that $\widehat{R}$ is a semilocal

principal ideal domain, hence $\widehat{R}$ is factorial. Since R is a Mori domain we have that $\widehat{R}^\times \cap R = R^\times$. Obviously, $\{x^{-1}R\cap\widehat{R} \mid x \in \widehat{R}^\bullet\} \subseteq \{N \mid N$ is an R-submodule of $\widehat{R}$ such that $\mathcal{F}_{\widehat{R}/R} \subseteq N\}$. Consequently, $\{x^{-1}R\cap\widehat{R} \mid x \in \widehat{R}^\bullet\}$ is finite and thus $\mathcal{C}(R,\widehat{R})$ is finite. Finally, R is a C-monoid defined in $\widehat{R}$ and by [20, Lemma 2.8.4.] we have that $(\widehat{R}^\times : R^\times) < \infty$. □

To simplify the notation, a noetherian domain whose group of divisibility is finitely generated is called a *QCK*-domain (quasi Cohen-Kaplansky domain). Recall that an atomic integral domain with only finitely many pairwise non-associated atoms is called a Cohen-Kaplansky domain. By [2, Theorem 4.3.] an integral domain R is a Cohen-Kaplansky domain if and only if R is a *QCK*-domain such that $|\max(\overline{R})| = |\max(R)|$. If R is an integral domain, then let $\mathcal{C}_v(R)$ (the v-class group of R) be defined as in [20, Definition 2.1.8.].

Theorem 4.3. *Let R be an integral domain and $M = R\backslash R^\times$. The following conditions are equivalent:*

1. *R is a C-monoid and a G-domain.*
2. *R is a QCK-domain or (R is primary and $\mathcal{R}(M)$ is a QCK-domain).*
3. *R is a QCK-domain or (R is finitely primary and $(\widehat{R}^\times : \mathcal{R}(M)^\times) < \infty$).*

If R is noetherian or a seminormal Mori domain, then the following is additionally equivalent:

4. *$K^\times/R^\times$ is finitely generated or (R is 1-dimensional and $\mathcal{F}_{\widehat{R}/R} = M$).*

Proof. **1.** ⇒ **2.**: By Lemma 3.1.2. we have that R is a semilocal, 1-dimensional Mori domain and $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. Suppose that R is not a *QCK*-domain. Then Theorem 4.2. implies that $(\widehat{R}^\times : R^\times) = \infty$. We have that R is local by Corollary 3.3.. Consequently, R is primary and Lemma 3.2.1. implies that $(\widehat{R}^\times : \mathcal{R}(M)^\times) < \infty$. By Corollary 2.3.1. it follows that $\mathcal{R}(M)$ is a Mori domain. Since R is a G-domain we have that $\mathcal{R}(M)$ is a G-domain. Since $\mathcal{R}(M) \subseteq \widehat{R}$ and $\widehat{R}$ is a Krull domain it follows that $\widehat{\mathcal{R}(M)} = \widehat{R}$ and thus $\{0\} \neq \mathcal{F}_{\widehat{R}/R} \subseteq \mathcal{F}_{\widehat{\mathcal{R}(M)}/\mathcal{R}(M)}$. Therefore $\mathcal{R}(M)$ is a *QCK*-domain by Theorem 4.2..
2. ⇒ **3.**: Let R be primary and $\mathcal{R}(M)$ a *QCK*-domain. By Theorem 4.2. we have that $\mathcal{F}_{\widehat{\mathcal{R}(M)}/\mathcal{R}(M)} \neq \{0\}$. Since $\mathcal{F}_{\mathcal{R}(M)/R} \neq \{0\}$ and $\widehat{R} \subseteq \widehat{\mathcal{R}(M)}$ this implies that $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. Therefore $\widehat{R}$ is completely integrally closed, hence $\widehat{R} = \widehat{\mathcal{R}(M)}$. By Lemma 3.1.3. and Theorem 4.2. it follows that $\widehat{R}$ is a semilocal principal ideal domain and $(\widehat{R}^\times : \mathcal{R}(M)^\times) < \infty$. Obviously, R is finitely primary.
3. ⇒ **1.**: Without restriction let R be finitely primary and $(\widehat{R}^\times : \mathcal{R}(M)^\times) < \infty$. Observe that $\mathcal{R}(M)^\times M \subseteq M$. By [28, Corollary 2.8.2.] and [20, Corollary 2.9.8.] we have that R is a C-monoid.
Now suppose that R is noetherian or a seminormal Mori domain.
2. ⇒ **4.**: Assume that $K^\times/R^\times$ is not finitely generated.
Case 1: R is noetherian: By 2. it follows that R is primary and $\mathcal{R}(M)$ is a *QCK*-domain. Since $\widehat{\mathcal{R}(M)} = \widehat{R}$ we have that $\widehat{R}/\mathcal{F}_{\widehat{R}/\mathcal{R}(M)}$ is finite and $(\widehat{R}^\times : \mathcal{R}(M)^\times) < \infty$ by Theorem 4.2.. Consequently, $\widehat{R}/\mathcal{R}(M)$ is finite and thus $R/ann_R(\widehat{R}/\mathcal{R}(M))$ is finite. Assume that $\mathcal{F}_{\widehat{R}/R} \neq M$. Then $\mathcal{R}(M) \subsetneq \widehat{R}$, hence $ann_R(\widehat{R}/\mathcal{R}(M)) = (\mathcal{R}(M) :_R \widehat{R}) \subseteq M$ and thus R/M is finite. Since $\mathcal{F}_{\mathcal{R}(M)/R} = M \neq \{0\}$, we have that $\mathcal{R}(M)$ is a finitely generated R-module. Consequently, [4, Theorem 7.] implies that $(\mathcal{R}(M)^\times : R^\times) < \infty$. Therefore $(\widehat{R}^\times : R^\times) < \infty$ and hence R is a *QCK*-domain by Theorem 4.2., a contradiction.
Case 2: R is a seminormal Mori domain: R is finitely primary by 3., hence R is 1-dimensional. Since $\widehat{R}$ is a semilocal principal ideal domain we have that $R \neq \widehat{R}$ and thus $\mathcal{F}_{\widehat{R}/R} = M$ by Lemma 2.5.3..
4. ⇒ **2.**: Let R be a 1-dimensional and $\mathcal{F}_{\widehat{R}/R} = M$. Obviously, R is primary, $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ and $\mathcal{R}(M) = \widehat{R}$. By Lemma 3.1. we have that $\widehat{R}$ is a semilocal principal ideal domain, and thus $\mathcal{R}(M)$ is a *QCK*-domain by Theorem 4.2.. □

Theorem 4.4. *Let R be a non-local semilocal integral domain. The following statements are equivalent:*

1. *R is a C-monoid, $\overline{R} = \widehat{R}$ and $\{M \in \max(\overline{R}) \mid |\overline{R}/M| = 2\}$ is finite.*

2. *R is a C-monoid and $\widehat{R}$ is a finitely generated R-module.*
3. *R is a Mori domain and $\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ and $\mathcal{C}_v(\widehat{R})$ are both finite.*

Proof. **1.** $\Rightarrow$ **2.**: By Corollary 3.3. it follows that $(\widehat{R}^\times : R^\times) < \infty$. Consequently, by [5, Theorem 10.] we have that $\widehat{R}$ is a finitely generated R-module.
2. $\Rightarrow$ **3.**: By Corollary 3.3. we have that $(\widehat{R}^\times : R^\times) < \infty$. Therefore Proposition 3.5. implies that $\widehat{R}/R$ is finite, hence $R/\mathcal{F}_{\widehat{R}/R}$ is finite and thus $\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ is finite. We have that R is a Mori domain. It is an immediate consequence of [20, Theorem 2.9.11.2.] that $\mathcal{C}_v(\widehat{R})$ is finite.
3. $\Rightarrow$ **1.**: Clearly, $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$. Therefore [20, Theorem 2.11.9.2.] implies that R is a C-monoid. Obviously, $\widehat{R}$ is a finitely generated R-module. Therefore $\overline{R} = \widehat{R}$ and $\overline{R}$ is semilocal. □

Corollary 4.5. *Let R be a non-local semilocal noetherian domain. The following are equivalent*:
1. *R is a C-monoid.*
2. *$\widehat{R}/\mathcal{F}_{\widehat{R}/R}$ and $\mathcal{C}_v(\widehat{R})$ are both finite.*

Proof. We have that $\mathcal{F}_{\widehat{R}/R} \neq \{0\}$ in any case. Since R is noetherian it follows that $\widehat{R}$ is a finitely generated R-module in every case. Consequently, the assertion follows from Theorem 4.4.. □

5. Examples and remarks

If R is an integral domain, X is an indeterminate over R and $A \subseteq R$, then let $A[X]$ be the set of all polynomials whose coefficients are in A. The second example is a generalization of [17, Example 2.6.] but the proof is almost identically. For the convenience of the reader we include detailed proofs.

Example 5.1. *Each of the following properties is satisfied by some 1-dimensional integral domain R such that $|\max(R)| = 2$ and R is not a C-monoid*:
1. *R is root closed and noetherian.*
2. *R is an integrally closed Mori domain.*

Proof. **1.** Let S be a non-integrally closed, root closed order in an algebraic number field K and X an indeterminate over K. Let $\mathfrak{f} = \mathcal{F}_{\overline{S}/S}$. Then there exist some $M, N \in \max(S)$ such that $M \neq N$ and $\mathfrak{f} \subseteq M$. Let $T = S[X] \setminus (M[X] \cup N[X])$ and $R = T^{-1}S[X]$. Then R is a noetherian 1-dimensional domain such that $\max(R) = \{T^{-1}M[X], T^{-1}N[X]\}$, hence $|\max(R)| = 2$. By [12, Theorem 2.] it follows that $S[X]$ is root closed, hence R is root closed. Since $(S/M)[X] \cong S[X]/M[X]$ can be embedded into $R/T^{-1}M[X]$, we have that $R/T^{-1}M[X]$ is infinite. Since $\mathcal{F}_{\overline{R}/R} = T^{-1}(S[X] : \overline{S[X]}) = T^{-1}(S[X] : \overline{S}[X]) = T^{-1}\mathfrak{f}[X] \subseteq T^{-1}M[X]$ it follows that $\overline{R}/\mathcal{F}_{\overline{R}/R}$ is infinite. Therefore Theorem 4.2. and Theorem 4.3. imply that R is not a C-monoid.
2. Let K be a field and L/K a purely transcendental field extension such that $L \neq K$. Let X be an indeterminate over L.
Claim 1: For all $f \in K[X]$ such that $\deg(f) = 1$ it follows that $S = K + fL[X]_{(f)}$ is a non-noetherian, primary, integrally closed Mori domain and $L(X)$ is a field of quotients of S. Let $f \in K[X]$ be such that $\deg(f) = 1$ and let $S = K + fL[X]_{(f)}$.
Claim 1a: $L(X)$ is a field of quotients of S. Let $Q \subseteq L(X)$ be the field of quotients of S. Let $h \in L[X]$. Then $fh, f \in S$, hence $h = \frac{fh}{f} \in Q$. Therefore $L[X] \subseteq Q$ and thus $Q = L(X)$.
Claim 1b: $L[X]_{(f)} = L + fL[X]_{(f)}$ and $L \cap fL[X]_{(f)} = \{0\}$. "$\subseteq$": Let $k \in L[X]_{(f)}$. Then there exist some $g \in L[X]$ and $h \in L[X] \setminus fL[X]$ such that $k = \frac{g}{h}$. There exist some $a_1, a_2 \in L$ and $f_1, f_2 \in L[X]$ such that $g = f_1 f + a_1$ and $h = f_2 f + a_2$. Of course $a_2 \neq 0$ and $k = \frac{a_2^{-1}a_1h + g - a_2^{-1}a_1h}{h} = a_2^{-1}a_1 + \frac{f(f_1 - a_2^{-1}a_1f_2)}{h} \in L + fL[X]_{(f)}$. "$\supseteq$": Trivial. Obviously, $L \cap fL[X]_{(f)} = L \cap L[X] \cap fL[X]_{(f)} = L \cap fL[X] = \{0\}$.
Claim 1c: $\widehat{S} = L[X]_{(f)}$. "$\subseteq$": Observe that $S \subseteq L[X]_{(f)}$. Consequently, $\widehat{S} \subseteq \widehat{L[X]_{(f)}} = \widehat{L[X]}_{(f)} =$

$L[X]_{(f)}$, since $L[X]$ is factorial. "$\supseteq$": It is sufficient to show that $L \subseteq \widehat{S}$. Let $a \in L$. Then $fa^n \in S$ for all $n \in \mathbb{N}$, hence $a \in \widehat{S}$.

Claim 1d: S is integrally closed. Let $h \in \overline{S}$. Then $h \in \widehat{S}$, hence there exist some $a \in L$ and $g \in fL[X]_{(f)}$ such that $h = a + g$. Consequently, $a \in \overline{S}$. It follows that there exist some $n \in \mathbb{N}$, $(a_i)_{i=0}^{n-1} \in K^{[0,n-1]}$ and $(g_i)_{i=0}^{n} \in (fL[X]_{(f)})^{[0,n-1]}$ such that $a^n + \sum_{i=0}^{n-1}(a_i + g_i)a^i = 0$. This implies that $a^n + \sum_{i=0}^{n-1} a_i a^i = -\sum_{i=0}^{n-1} g_i a^i \in L \cap fL[X]_{(f)} = \{0\}$ and thus a is algebraic over K. Hence $a \in K$ and $h \in S$.

Claim 1e: S is not noetherian. Assume to the contrary that S is noetherian. By claim 1b and claim 1d we have that $L \subseteq L[x]_{(f)} = \widehat{S} = \overline{S} = S$. There exists some $a \in L\backslash K$. It follows that there exist some $b \in K$ and $g \in fL[X]_{(f)}$ such that $a = b + g$. Therefore claim 1b implies that $a - b = g \in fL[X]_{(f)} \cap L = \{0\}$, hence $a \in K$, a contradiction.

Claim 1f: S is local. It is sufficient to show that $S\backslash S^\times = fL[X]_{(f)}$. "$\subseteq$": Let $h \in S\backslash S^\times$. Then there exist some $a \in K$ and $g \in fL[X]_{(f)}$ such that $h = a + g$. Assume that $a \neq 0$. Then $h \in \widehat{S}^\times$. Therefore there exist some $b \in L$ and $g' \in fL[X]_{(f)}$ such that $(a+g)(b+g') = 1$. It follows that $1 - ab = ag' + bg + gg' \in L \cap fL[X]_{(f)} = \{0\}$, hence $ab = 1$ and thus $b \in K$. This implies that $h \in S^\times$, a contradiction. "$\supseteq$": Let $h \in fL[X]_{(f)}$. Then $h \in S$. Assume that $h \in S^\times$. Then $h \in \widehat{S}^\times = L[X]_{(f)}\backslash fL[X]_{(f)}$, a contradiction.

Claim 1g: S is primary. It is sufficient to show that $P = fL[X]_{(f)}$ for every $P \in \mathrm{spec}(S)^\bullet$. Let $P \in \mathrm{spec}(S)^\bullet$. Then $P \subseteq fL[X]_{(f)}$, hence $\{0\} \neq P\widehat{S} \subseteq fL[X]_{(f)}$. This implies that $\sqrt[\widehat{S}]{P\widehat{S}} = fL[X]_{(f)}$, since $L[X]_{(f)}$ is primary. Consequently, there exists some $k \in \mathbb{N}$ such that $f^k \in P\widehat{S}$, hence $f^{k+1} \in Pf\widehat{S} \subseteq PS = P$. Since $f \in S$ this implies that $f \in P$. Therefore $(fL[X]_{(f)})^2 = f^2\widehat{S} \subseteq fS \subseteq P$ and thus $fL[X]_{(f)} \subseteq P$. Finally this implies that $P = fL[X]_{(f)}$.

Claim 1h: S is a Mori domain. By claim 1d and claim 1g we have that S is seminormal and primary. Since $\widehat{S}$ is a Krull domain, Lemma 2.6. implies that S is a Mori domain.

Let $f_1, f_2 \in K[X]$ be non-associated elements such that $\deg(f_1) = \deg(f_2) = 1$. Let $R_1 = K + f_1L[X]_{(f_1)}$, $R_2 = K + f_2L[X]_{(f_2)}$ and $R = R_1 \cap R_2$.

Claim 2a: $L(X)$ is a field of quotients of R. Let $Q \subseteq L(X)$ be the field of quotients of R. There exists some $a, b \in K^\bullet$ such that $f_1 = b + af_2$. Therefore $f_1 \in R_2$ and $f_2 = -a^{-1}b + a^{-1}f_1 \in R_1$. Since $hf_1 \in R_1$ and $hf_2 \in R_2$ for all $h \in L[X]$ it follows that $hf_1f_2 \in R$ for all $h \in L[X]$. Consequently, $L[X] \subseteq Q$ and thus $Q = L(X)$.

By claim 1 and claim 2a it is immediately clear that R is integrally closed. Lemma 2.1.1., Proposition 2.2. and claim 1 imply that R is a Mori domain. Let $N_1 = f_1L[X]_{(f_1)}$, $N_2 = f_2L[X]_{(f_2)}$, $M_1 = N_1 \cap R$ and $M_2 = N_2 \cap R$. Then $M_1, M_2 \in \mathrm{spec}(R)$.

Claim 2b: M_1 and M_2 are incomparable. Assume that $M_1 \subseteq M_2$. Then $f_1 \in N_2$, hence $f_1 \in f_2L[X]$ and thus f_1 and f_2 are associated, a contradiction. Therefore $M_1 \nsubseteq M_2$ and analogously it follows that $M_2 \nsubseteq M_1$.

Claim 2c: $R\backslash(M_1 \cup M_2) = R^\times$. We have that $R^\times \subseteq R\backslash(M_1 \cup M_2) = R\backslash(N_1 \cup N_2) = (R\backslash N_1) \cap (R\backslash N_2) \subseteq (R_1\backslash N_1) \cap (R_2\backslash N_2) = R_1^\times \cap R_2^\times = R^\times$.

Claim 2d: $\max(R) = \{M_1, M_2\}$. By claim 2c we have that $R\backslash R^\times = M_1 \cup M_2$. "$\subseteq$": Let $M \in \max(R)$. Then $M \subseteq M_1 \cup M_2$, hence $M \in \{M_1, M_2\}$. "$\supseteq$": Let I be an ideal of R such that $M_1 \subseteq I \subsetneqq R$. Then $I \subseteq M_1 \cup M_2$ and $I \nsubseteq M_2$ by claim 2b. Therefore $I = M_1$, hence $M_1 \in \max(R)$. Analogously it follows that $M_2 \in \max(R)$.

Claim 2e: If T is a primary integral domain with field of quotients $L(X)$, $S \subseteq T^\bullet$ is multiplicatively closed and $S \nsubseteq T^\times$, then $S^{-1}T = L(X)$. Let $S \nsubseteq T^\times$. Assume that $S^{-1}T \neq L(X)$. Then there exists some $P \in \mathrm{spec}(S^{-1}T)^\bullet$. We have that $P \cap T = T\backslash T^\times$. There exists some $x \in S\backslash T^\times$. It follows that $1 = x^{-1}x \in S^{-1}TP = P$, a contradiction.

Claim 2f: $R_1 = R_{M_1}$ and $R_2 = R_{M_2}$. Since $R\backslash M_1 \subseteq R_1\backslash N_1 = R_1^\times$ it follows that $(R_1)_{M_1} = R_1$. Assume that $R\backslash M_1 \subseteq R_2^\times$. Then $R\backslash M_1 \subseteq R_2\backslash N_2 \cap R = R\backslash M_2$, hence $M_2 \subseteq M_1$, a contradiction. Therefore $R\backslash M_1 \nsubseteq R_2^\times$ and thus claim 2e implies that $(R_2)_{M_1} = L(X)$. Consequently, $R_{M_1} = (R_1)_{M_1} \cap (R_2)_{M_1} =$

$R_1 \cap L(X) = R_1$. Analogously it follows that $R_2 = R_{M_2}$.
By claim 2b and claim 2d it follows that $|\max(R)| = 2$. By claim 2f we have that $ht(M_1) = ht(M_2) = 1$ and hence R is 1-dimensional. Assume that R is noetherian. Then R_1 is noetherian by claim 2f. This is a contradiction to claim 1e. By Lemma 2.5.2. and Theorem 4.2. it follows that $(\widehat{R}^\times : R^\times) = \infty$. Therefore Corollary 3.3. implies that R is not a C-monoid. □

If H is a monoid and $A \subseteq H$, then let $[A]$ denote the smallest submonoid of H that contains A. A monoid H is called quasi finitely generated if there exists some finite $E \subseteq H$ such that $H = [E \cup H^\times]$. By [20, Theorem 2.7.14.] every factorial monoid H where $\mathfrak{X}(H)$ is finite is quasi finitely generated. Note that if R is a domain, then R is a quasi finitely generated monoid if and only if R is a Cohen-Kaplansky domain. For a survey on Cohen-Kaplansky domains see [2].

Proposition 5.2. *Let H be a primary Mori monoid and $M = H \setminus H^\times$. If $\mathcal{R}(M)$ is quasi finitely generated and $\widehat{\mathcal{R}(M)}$ is factorial, then H is a finitely primary C-monoid.*

Proof. Let $\mathcal{R}(M)$ be quasi finitely generated such that $\widehat{\mathcal{R}(M)}$ is factorial. By [20, Theorem 2.7.13.] it follows that $\mathcal{F}_{\widehat{\mathcal{R}(M)}/\mathcal{R}(M)} \neq \{0\}$. Since $\mathcal{F}_{\mathcal{R}(M)/H} \neq \{0\}$ and $\widehat{H} \subseteq \widehat{\mathcal{R}(M)}$ it follows that $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$. Therefore $\widehat{H}$ is completely integrally closed and since $\mathcal{R}(M) \subseteq \widehat{H}$ we have that $\widehat{H} = \widehat{\mathcal{R}(M)}$ is factorial. Consequently, H is finitely primary.
Next we show that for every prime element $p \in \widehat{H}$ there exists some $n \in \mathbb{N}$ such that $p^{2n} \sim_H p^n$. Let $p \in \widehat{H}$ be a prime element. By [20, Proposition 2.7.11.] it follows that $\widetilde{\mathcal{R}(M)} = \widehat{H}$, hence there exists some $k \in \mathbb{N}$ such that $p^k \in \mathcal{R}(M)$. Since $M = p\widehat{H} \cap H$ we have that $p^k(p\widehat{H} \cap H) \subseteq H$. Consequently, $p^{-1}H \cap \widehat{H} \subseteq p^{-(k+1)}H \cap \widehat{H}$ and thus $p^{-(lk+1)}H \cap \widehat{H} \subseteq p^{-((l+1)k+1)}H \cap \widehat{H}$ for all $l \in \mathbb{N}_0$. This is an ascending sequence of divisorial fractional ideals of H (since $\mathcal{F}_{\widehat{H}/H} \neq \{0\}$). There is some $r \in \mathbb{N}_0$ such that $p^{-(lk+1)}H \cap \widehat{H} = p^{-(rk+1)}H \cap \widehat{H}$ for all $l \in \mathbb{N}_{\geq r}$. Hence $p^{2(r+1)k+1} \sim_H p^{rk+1}$. If $n = (r+2)k$, then $p^{2n} \sim_H p^n$.
By [20, Proposition 2.7.4.1.] and [20, Theorem 2.7.13.] it follows that $\widehat{H}^\times/\mathcal{R}(M)^\times$ is finitely generated. $\widehat{H}^\times/\mathcal{R}(M)^\times$ is a torsion group (since $\widetilde{\mathcal{R}(M)} = \widehat{H}$) and thus $(\widehat{H}^\times : \mathcal{R}(M)^\times) < \infty$. Moreover we have that $\mathcal{R}(M)^\times M \subseteq H$. It is an easy consequence of [20, Corollary 2.9.8.] that H is a C-monoid. □

Corollary 5.3. *Let H be a primary monoid such that $\widehat{H}$ is factorial. Then H' and $\widetilde{H}$ are finitely primary C-monoids.*

Proof. By Proposition 2.7.1. it follows that H' and $\widetilde{H}$ are primary. Obviously, $\widehat{H} = \widehat{H'} = \widehat{\widetilde{H}}$. Therefore Lemma 2.5.2. implies that $\mathcal{F}_{\widehat{H}/\widetilde{H}} \supseteq \mathcal{F}_{\widehat{H}/H'} \neq \{0\}$. Consequently, H' and $\widetilde{H}$ are finitely primary. It follows by Theorem 4.1.2. that H' and $\widetilde{H}$ are Mori monoids. Without restriction we can assume that $\widetilde{H} \neq \widehat{H}$. By Lemma 2.5.3. we have that $\mathcal{F}_{\widehat{H}/H'} = H' \setminus H'^\times$ and $\mathcal{F}_{\widehat{H}/\widetilde{H}} = \widetilde{H} \setminus \widetilde{H}^\times$. Therefore $\mathcal{R}(H' \setminus H'^\times) = \mathcal{R}(\widetilde{H} \setminus \widetilde{H}^\times) = \widehat{H}$. Clearly $\widehat{H}$ is quasi finitely generated, hence H' and $\widetilde{H}$ are C-monoids by Proposition 5.2.. □

In Theorem 4.3. we showed that if (R, M) is a primary integral domain and $\mathcal{R}(M)$ is a QCK-domain, then R is a C-monoid. Proposition 5.2. is some sort of monoid theoretical pendant to this result. Note that the complete integral closure of a Cohen-Kaplansky domain is always factorial. A simple consequence of Proposition 5.2. is that if (R, M) is a primary Mori domain such that $\mathcal{R}(M)$ is a Cohen-Kaplansky domain, then R is a C-monoid. The following question arises: How strong are the assumptions in Proposition 5.2.? To give a partial answer to this question we need a result about root closed integral domains which has been proved in [6, Theorem 1.]. Next we show that this result remains true under slightly weaker conditions.

Proposition 5.4. *Let R be a 1-dimensional, root closed integral domain such that $\widehat{R}$ is a Krull domain and such that $2 < |\widehat{R}/P|$ and $\widehat{R}/P$ is algebraic over a finite field for all $P \in \max(\widehat{R})$. Then $R = \widehat{R}$.*

Proof. Case 1: R is local: Let $M = R \backslash R^\times$. Assume that $R \neq \widehat{R}$. By Lemma 2.5. it follows that $\{0\} \neq \mathcal{F}_{\widehat{R}/R} = M$ is a radical ideal of $\widehat{R}$. By Lemma 3.1. we have that $\widehat{R}$ is a semilocal principal ideal domain. Since $M \subseteq P$ for all $P \in \max(\widehat{R})$ it follows that $M = \mathcal{J}(\widehat{R})$. Let $N \in \max(\widehat{R})$. There exists some $b \in \widehat{R}$ such that $b + N \neq N, 1 + N$, hence $b - 1 \notin N$. Consequently, there exists some finite subfield L of $\widehat{R}/N$ such that $b + N \in L^\times$. Let $n = |L^\times|$. Then $b^n - 1 \in N$.
Next we show that $\mathcal{J}(\widehat{R}) \in \max(\widehat{R})$. Assume to the contrary that $\mathcal{J}(\widehat{R}) \notin \max(\widehat{R})$. Let $N' = \bigcap_{P \in \max(\widehat{R}), P \neq N} P$. Then $N + N' = R$ and hence there exist some $x \in N$ and $y \in N'$ such that $x + y = 1$. Let $a = x + by$. Then $a \in 1 + N'$ and $a \in b + N$, hence $a^n \in 1 + N'$ and $a^n \in b^n + N = 1 + N$. Therefore $a^n - 1 \in N \cap N' = \mathcal{J}(\widehat{R}) \subseteq R$ and hence $a \in R$. This implies that $a - 1 \in N' \cap R = M \subseteq N$. Consequently, $b - 1 = (a-1) - (a-b) \in N$, a contradiction.
It follows that $\widehat{R} \backslash \widehat{R}^\times = N = \mathcal{J}(\widehat{R})$. It remains to show that $\widehat{R} \subseteq R$, since then $R = \widehat{R}$, a contradiction. Let $x \in \widehat{R}$. If $x \notin \widehat{R}^\times$, then $x \in \mathcal{J}(\widehat{R}) \subseteq R$. If $x \in \widehat{R}^\times$, then there exists some finite subfield K of $\widehat{R}/N$ such that $x + N \in K^\times$. Let $m = |K^\times|$. It follows that $x^m - 1 \in N = R \backslash R^\times$. Consequently, $x \in R$.
Case 2: R is not necessarily local: Let $M \in \max(R)$. Then R_M is a local, 1-dimensional, root closed integral domain. Since $\widehat{R}$ is a Krull domain we have that $\widehat{R_M} = \widehat{R}_M$ is a Krull domain. Let $N \in \max(\widehat{R_M}) = \max(\widehat{R}_M)$. There exists some $P \in \operatorname{spec}(\widehat{R})$ such that $P \cap R = M$ and $N = P_M$. Let $Q \in \max(\widehat{R})$ be such that $P \subseteq Q$. Then $Q \cap R = M$ and thus $\widehat{R}_M \supsetneq Q_M \supseteq P_M$. Consequently, $P = P_M \cap \widehat{R} = Q_M \cap \widehat{R} = Q \in \max(\widehat{R})$. It follows that $\widehat{R_M}/N = \widehat{R}_M/P_M \cong \widehat{R}/P$. Therefore $2 < |\widehat{R_M}/N|$ and $\widehat{R_M}/N$ is algebraic over a finite field. By case 1 it follows that $R_M = \widehat{R_M} = \widehat{R}_M$. Consequently, $R = \bigcap_{A \in \max(R)} R_A = \bigcap_{A \in \max(R)} \widehat{R}_A = \widehat{R}$. □

Proposition 5.5. *Let R be a primary integral domain with field of quotients K and $M = R \backslash R^\times$ such that $\mathcal{R}(M)$ is a QCK-domain. If $|R/M| > 2$ and $\widetilde{\mathcal{R}(M)}$ is a ring, then $\mathcal{R}(M)$ is a Cohen-Kaplansky domain.*

Proof. Let $|R/M| > 2$, let $\widetilde{\mathcal{R}(M)}$ be a ring and set $S = \mathcal{R}(M)$. By [2, Theorem 4.3.] it remains to show that $|\max(\overline{S})| = |\max(S)|$. It is sufficient to show that $\overline{S}_P$ is local for every $P \in \max(S)$. Let $P \in \max(S)$ and $\mathfrak{f} = \mathcal{F}_{\overline{S}/S}$. Case 1: $P + \mathfrak{f} = S$: Observe that $P\overline{S} \cap S = P$ and $\overline{S}/P\overline{S} \cong S/P$. Therefore $P\overline{S} \in \max(\overline{S})$ and hence $\max(\overline{S}_P) = \{(P\overline{S})_P\}$. Case 2: $\mathfrak{f} \subseteq P$: Of course $\widetilde{S_P} = \widetilde{S}_P$ is a ring. Since S is 1-dimensional by Theorem 4.2. it follows that $\widetilde{S_P}$ is noetherian and 1-dimensional by the Theorem of Krull-Akizuki. Obviously, $\widehat{\widetilde{S_P}} = \overline{\widetilde{S_P}} = \overline{S_P}$ is a Krull domain. Next we show that $2 < |\overline{S_P}/Q| < \infty$ for all $Q \in \max(\overline{S_P})$. Let $Q \in \max(\overline{S_P}) = \max(\overline{S}_P)$. Then there exists some $Q' \in \max(\overline{S})$ such that $Q' \cap S = P$ and $Q = Q'_P$. Therefore $\overline{S}_P/Q \cong \overline{S}/Q'$ and since $\mathfrak{f} \subseteq Q'$ it follows by Lemma 3.4.2. that $|\overline{S_P}/Q| < \infty$. Since $Q \cap R = M$ we have that R/M can be embedded into $\overline{S_P}/Q$, hence $2 < |\overline{S_P}/Q|$. By Proposition 5.4. it follows that $\widetilde{S_P} = \overline{S_P}$. Since S_P is noetherian and $K^\times/S_P^\times$ is finitely generated we have that $|\max(\overline{S}_P)| = |\max(\overline{S_P})| = |\max(S_P)| = 1$ by [2, Theorem 4.3.]. □

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