

VALUATION IDEAL FACTORIZATION DOMAINS

GYU WHAN CHANG AND ANDREAS REINHART

ABSTRACT. An integral domain D is a *valuation ideal factorization domain* (VIFD) if each nonzero principal ideal of D can be written as a finite product of valuation ideals. Clearly, π -domains are VIFDs. In this paper, we study the ring-theoretic properties of VIFDs and the $*$ -operation analogs of VIFDs. Among them, we show that if D is treed (resp., $*$ -treed), then D is a VIFD (resp., $*$ -VIFD) if and only if D is an h -local Prüfer domain (resp., a $*$ - h -local P*MD) if and only if every nonzero prime ideal of D contains an invertible (resp., a $*$ -invertible) valuation ideal. We also study integral domains D such that for each nonzero nonunit $a \in D$, there is a positive integer n such that a^n can be written as a finite product of valuation elements.

1. INTRODUCTION

All rings considered in this paper are commutative with identity. Let D be an integral domain with quotient field K . An *overring* of D means a subring of K containing D . As in [32, Appendix 3], we say that an ideal I of D is a *valuation ideal* if there is a valuation overring V of D such that $IV \cap D = I$. Clearly, each ideal of a valuation domain is a valuation ideal. Conversely, in [18, Corollary 2.4], Gilmer and Ohm showed that if every principal ideal of D is a valuation ideal, then D is a valuation domain. Following [9], we say that a nonzero nonunit $a \in D$ is a *valuation element* if aD is a valuation ideal, i.e., there is a valuation overring V of D such that $aV \cap D = aD$. In [9], we studied some ring-theoretic properties of *valuation factorization domains* (VFDs), which are integral domains whose nonzero nonunits can be written as a finite product of valuation elements. It is clear that valuation domains and UFDs are VFDs. In this paper, we continue our work on ideal factorization properties of integral domains. It is well-known that D is a Dedekind domain (resp., π -domain) if and only if every nonzero ideal (resp., nonzero principal ideal) of D can be written as a finite product of prime ideals; in particular, a Dedekind domain is a π -domain. Recall that a prime ideal of D is a valuation ideal [32, page 341], so a Dedekind domain (resp., π -domain) D has the property that every nonzero principal ideal of D can be written as a finite product of valuation ideals, which will be called a *valuation ideal factorization domain* (VIFD). In this paper, we study some ideal-theoretic properties of VFDs and VIFDs, and the $*$ -operation analog of a VIFD, which is called a $*$ -VIFD.

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This paper consists of five sections including the introduction where we review the definitions related with the t - and w -operations. First, in Section 2, we investigate valuation ideals, valuation elements and their connections to star operations. Let I be a proper valuation ideal of D . Among others, we show that if I is $*$ -invertible, then I_* is a valuation ideal of D and it is contained in a unique maximal t -ideal of D . In Section 3, we study the $*$ -VIFDs. We introduce the notion of $*$ -VIFDs and we show that all of these rings are integrally closed weakly Matlis domains. Furthermore, we prove that D is a VFD if and only if D is a VIFD with $\text{Pic}(D) = \{0\}$, if and only if D is a $*$ -VIFD with $\text{Cl}_*(D) = \{0\}$. It is also shown that D is a $*$ -h-local P $*$ MD if and only if every nonzero ($*$ -finite) $*$ -ideal of D can be written as a finite $*$ -product of $*$ -comaximal valuation ideals of D . We further study the $*$ -VIFDs that are $*$ -treed. In Section 4, we consider VIFDs and t -VIFDs. Finally, in Section 5, we investigate AVFDs, i.e., integral domains for which every nonzero nonunit has a power that is a finite product of valuation elements. Among other things, we show that if D is a $*$ -VIFD such that $\text{Cl}_*(D)$ is a torsion group, then D is an AVFD.

Throughout this paper, let \mathbb{N} , \mathbb{N}_0 and \mathbb{Z} denote the set of all positive integers, the set of all nonnegative integers and the set of all integers, respectively. For $x, y \in \mathbb{Z}$ with $x \leq y$, let $[x, y] = \{z \in \mathbb{Z} \mid x \leq z \leq y\}$.

1.1. Definitions related to star operations. We first review some definitions related to the t -operation which are needed for fully understanding this paper. Let D be an integral domain with quotient field K . A D -submodule A of K is called a *fractional ideal* of D if $dA \subseteq D$ for some nonzero $d \in D$. An (integral) ideal of D is a fractional ideal of D that is contained in D . Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of D . For $A, B \in F(D)$, let $(A : B) = \{x \in K \mid xB \subseteq A\}$ and $A^{-1} = (D : A)$. Observe that $(A : B) \in F(D)$ and $A^{-1} \in F(D)$.

Recall that a map $*$: $F(D) \rightarrow F(D), I \mapsto I_*$ is called a *star operation* on D if the following conditions are satisfied for all $A, B \in F(D)$ and nonzero $c \in K$.

- $A \subseteq A_* = (A_*)_*$,
- if $A \subseteq B$, then $A_* \subseteq B_*$,
- $(cA)_* = cA_*$ and
- $D_* = D$.

Let $*$ be a star operation on D . We say that $*$ is of *finite type* if for each $A \in F(D)$, $A_* = \bigcup_{C \in f(D), C \subseteq A} C_*$. Besides that, $*$ is said to be *stable* if $(A \cap B)_* = A_* \cap B_*$ for all $A, B \in F(D)$. If $*_1$ and $*_2$ are star operations on D , we mean by $*_1 \leq *_2$ that $I_{*1} \subseteq I_{*2}$ for any $I \in F(D)$.

Let $*_f : F(D) \rightarrow F(D)$ be defined by $A_{*f} = \bigcup_{C \in f(D), C \subseteq A} C_*$ for each $A \in F(D)$. Then $*_f$ is a star operation of finite type on D . Let $\tilde{*} : F(D) \rightarrow F(D)$ be defined by $A_{\tilde{*}} = \{x \in K \mid xJ \subseteq A \text{ for some } J \in f(D) \text{ with } J_* = D\}$ for each $A \in F(D)$. Then $\tilde{*}$ is a stable star operation of finite type on D [16, Example 2.1 and Remark 2.3]. If we set

- $A_v = (A^{-1})^{-1}$,
- $A_t = \bigcup \{I_v \mid I \subseteq A \text{ and } I \in f(D)\}$,
- $A_w = \{x \in K \mid xJ \subseteq A \text{ for some } J \in f(D) \text{ with } J_v = D\}$, and
- $A_d = A$

for each $A \in F(D)$, then v is a star operation on D , $t = v_f$, $w = \tilde{v}$, and d is a star operation on D such that $d = d_f = \tilde{d}$. It is known that $d \leq * \leq v$, $d \leq \tilde{*} \leq *_f \leq t \leq v$, and $\tilde{*} \leq w$ for any star operation $*$ on D .

An $I \in F(D)$ is called a *fractional $*$ -ideal* of D if $I_* = I$ and a fractional $*$ -ideal I of D is called a *$*$ -ideal* if $I \subseteq D$. A $*$ -ideal is a *maximal $*$ -ideal* if it is maximal among the proper $*$ -ideals. Let $*\text{-Max}(D)$ be the set of maximal $*$ -ideals of D and let $*\text{-Spec}(D)$ be the set of prime $*$ -ideals of D . It may happen that $*\text{-Max}(D) = \emptyset$ even though D is not a field; for example, if D is a rank-one nondiscrete valuation domain and $* = v$, then $*\text{-Max}(D) = \emptyset$. However, $*_f\text{-Max}(D) \neq \emptyset$ if and only if D is not a field; each maximal $*_f$ -ideal of D is a prime ideal; $*_f\text{-Max}(D) \subseteq *_f\text{-Spec}(D)$; each proper $*_f$ -ideal of D is contained in a maximal $*_f$ -ideal. Each prime ideal of D minimal over a $*_f$ -ideal is a $*_f$ -ideal, whence each height-one prime ideal is a $*_f$ -ideal; $D = \bigcap_{P \in *_f\text{-Max}(D)} D_P$; $I_{\tilde{*}} = \bigcap_{M \in *_f\text{-Max}(D)} I_P$ for all $I \in F(D)$; and $*_f\text{-Max}(D) = \tilde{*}\text{-Max}(D)$ (see, for example, [24, Lemma 2.1(2)]).

An integral domain D is said to be of *finite $*$ -character* if each nonzero nonunit of D is contained in only finitely many maximal $*$ -ideals. Recall from [26] that D is a *$*$ -h-local domain* if D is of finite $*$ -character and every nonzero prime $*$ -ideal of D is contained in a unique maximal $*$ -ideal of D . Note that D is *t -h-local* if and only if D is weakly Matlis [5, page 8] and D is *d -h-local* if and only if D is h-local [5, page 8]. The $*$ -dimension of D is defined by $*\text{-dim}(D) = \sup\{n \in \mathbb{N}_0 \mid P_1 \subsetneq \cdots \subsetneq P_n \text{ for some prime } *\text{-ideals } P_i \text{ of } D\}$. Hence, $*\text{-dim}(D) = 1$ if and only if D is not a field and each prime $*$ -ideal of D is a maximal $*$ -ideal.

An $I \in F(D)$ is said to be *$*$ -invertible* if $(II^{-1})_* = D$ and a fractional $*$ -ideal I of D is said to be *$*$ -finite* if $I = J_*$ for some $J \in f(D)$. We say that D is a *Prüfer $*$ -multiplication domain* (P^*MD) if each nonzero finitely generated ideal of D is $*_f$ -invertible. It is known that D is a P^*MD if and only if D_P is a valuation domain for all maximal $*_f$ -ideals P of D [16, Theorem 3.1]. A Prüfer domain is a $PvMD$ whose maximal ideals are t -ideals. Let $T_*(D)$ be the set of $*$ -invertible fractional $*$ -ideals. Then $T_*(D)$ is an abelian group under $I * J = (IJ)_*$. Let $\text{Inv}(D)$ (resp., $\text{Prin}(D)$) be the subgroup of $T(D)$ of invertible (resp., nonzero principal) fractional ideals of D . The factor group $\text{Cl}_*(D) = T_*(D)/\text{Prin}(D)$, called the *$*$ -class group* of D , is an abelian group and $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$, called the *Picard group* of D , is a subgroup of $\text{Cl}_*(D)$.

Let S be a subset of D . Then S is called *multiplicatively closed* if $1 \in S$ and $xy \in S$ for all $x, y \in S$. It is clear that if S is a multiplicatively closed set with $0 \in S$, then $D_S \simeq \{0\}$, so we always assume that $0 \notin S$. Furthermore, S is said to be *divisor-closed* or *saturated* if for all $x, y \in D$ with $xy \in S$, it follows that $x, y \in S$. If $a, b \in D$, then we write $a \mid_D b$ if there is some $c \in D$ such that $b = ac$. Hence, S is divisor-closed if and only if $a \mid_D b$ implies $a \in S$ for any $b \in S$ and $a \in D$.

Let $D[X]$ be the polynomial ring over D . For a polynomial $f \in D[X]$, let $c(f)$ denote the ideal of D generated by the coefficients of f . Let $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$. Then, by the Dedekind-Mertens lemma, it can be shown that N_v is a multiplicatively closed and divisor-closed subset of $D[X]$.

1.2. Importance of star operations. Next we give a short explanation for considering arbitrary star operations (of finite type) rather than the d -operation, the t -operation and the w -operation. First of all, star operations enable us to unify results that hold for all of the aforementioned operations. Besides that, we will see later that certain results are specific to the t -operation/ w -operation, while other

results hold for arbitrary star operations of finite type. We also emphasize that similar types of concepts have been investigated for star operations [5, 26, 30].

If not stated otherwise, then from now on D is always an integral domain with quotient field K and $$ is a star operation of finite type on D .*

2. VALUATION IDEALS AND VALUATION ELEMENTS

We first discuss briefly how to extend the star operation $*$ on D to arbitrary nonzero D -submodules of K . For a nonzero D -submodule A of K , let $A_* = \bigcup_{J \in f(D), J \subseteq A} J_*$, then $A_* = \bigcup_{J \in f(D), J \subseteq A} J_*$ because $*$ is of finite type, and hence A_* is well-defined. Next we present a few simple properties of the $*$ -closure of D -submodules of K that we will use without further mention.

Lemma 2.1. *Let D be an integral domain with quotient field K , let $*$ be a star operation of finite type on D , let $c \in K$ be nonzero and let A and B be nonzero D -submodules of K . Then A_* is a D -submodule of K , $A \subseteq A_*$, $cA_* = (cA)_*$, if $A \subseteq B_*$, then $A_* \subseteq B_*$, and $(AB)_* = (A_*B)_*$. Moreover, if $*$ is stable, then $(A \cap B)_* = A_* \cap B_*$.*

Proof. It is clear that A_* is a D -submodule of K , since A_* is an upper directed union of D -submodules of K . Observe that $A = \bigcup_{J \in f(D), J \subseteq A} J \subseteq \bigcup_{J \in f(D), J \subseteq A} J_* = A_*$. Moreover, $cA_* = c \bigcup_{J \in f(D), J \subseteq A} J_* = \bigcup_{J \in f(D), J \subseteq A} cJ_* = \bigcup_{J \in f(D), cJ \subseteq cA} (cJ)_* = \bigcup_{J \in f(D), J \subseteq cA} J_* = (cA)_*$.

Let $A \subseteq B_*$. To show that $A_* \subseteq B_*$, it is sufficient to show that $J_* \subseteq B_*$ for each $J \in f(D)$ with $J \subseteq A$. Let $J \in f(D)$ with $J \subseteq A$. Then $J \subseteq B_*$, and hence there is some $L \in f(D)$ such that $J \subseteq L_*$ and $L \subseteq B$. Thus, $J_* \subseteq (L_*)_* = L_* \subseteq B_*$.

Next we show that $(A_*B)_* = (AB)_*$ (for arbitrary nonzero D -submodules A and B of K). Clearly, $AB \subseteq A_*B \subseteq (A_*B)_*$, and hence $(AB)_* \subseteq (A_*B)_*$. Now let $x \in (A_*B)_*$. Then $x \in J_*$ for some $J \in f(D)$ with $J \subseteq A_*B$. Observe that $J \subseteq L_*L'$ for some $L, L' \in f(D)$ such that $L \subseteq A$ and $L' \subseteq B$. This implies that $x \in J_* \subseteq (L_*L')_* = (LL')_* \subseteq (AB)_*$.

Finally, let $*$ be stable. Then

$$\begin{aligned} A_* \cap B_* &= \left(\bigcup_{J \in f(D), J \subseteq A} J_* \right) \cap \left(\bigcup_{L \in f(D), L \subseteq B} L_* \right) \\ &= \bigcup_{J \in f(D), J \subseteq A} \bigcup_{L \in f(D), L \subseteq B} (J_* \cap L_*) \\ &= \bigcup_{J \in f(D), J \subseteq A} \bigcup_{L \in f(D), L \subseteq B} (J \cap L)_* \\ &\subseteq \bigcup_{J \in f(D), J \subseteq A \cap B} J_* = (A \cap B)_* \subseteq A_* \cap B_*, \end{aligned}$$

where the third equality follows because $*$ is stable. Thus, $(A \cap B)_* = A_* \cap B_*$. \square

Let $*$ be a star operation of finite type on D . Following [15], we say that an overring D' of D is $*$ -closed if $(D')_* = D'$.

Lemma 2.2. *Let D be an integral domain, let $*$ be a stable star operation of finite type on D and let I be a valuation $*$ -ideal of D . Then there exists a $*$ -closed valuation overring W of D such that $IW \cap D = I$.*

Proof. There exists a valuation overring V of D such that $IV \cap D = I$. Set $W = V_*$. Since $V^2 \subseteq V$, we infer that $W^2 \subseteq (W^2)_* = (V^2)_* \subseteq W$. Observe that W is a $*$ -closed overring of V (since $V \subseteq W$, $W_* = W$ and $x+y, xy \in W$ for all $x, y \in W$). Consequently, W is a valuation overring of D (since W is an overring of a valuation domain) and $I \subseteq IW \cap D \subseteq (IW)_* \cap D = (IV)_* \cap D_* = (IV \cap D)_* = I_* = I$ by Lemma 2.1. Thus, $IW \cap D = I$. \square

Let D^\times denote the *unit group* of D and let $\text{Spec}(D)$ denote the *set of prime ideals* of D . For each ideal I of D , let $\mathcal{P}(I)$ be the *set of minimal prime ideals* of I (i.e., the set of prime ideals of D that are minimal over I) and let $\sqrt{I} = \sqrt[\vee]{I} = \bigcap_{P \in \text{Spec}(D), I \subseteq P} P = \{x \in D \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$ be the *radical* of I .

Proposition 2.3. *Let D be an integral domain that is not a field, let $*$ be a star operation of finite type on D and let I be a valuation $*$ -ideal of D .*

- (1) *There exists some $M \in *-\text{Max}(D)$ such that $I_M \cap D = I$.*
- (2) *If I is proper, then \sqrt{I} is a prime $*$ -ideal of D .*

Proof. Observe that $\tilde{*}$ is a stable star operation of finite type such that $J_{\tilde{*}} \subseteq J_*$ for each $J \in F(D)$ and $*-\text{Max}(D) = \tilde{*}-\text{Max}(D)$. Hence, I is a valuation $\tilde{*}$ -ideal of D . Thus, $IW \cap D = I$ for some $\tilde{*}$ -closed valuation overring W of D by Lemma 2.2.

(1) Set $P = W \setminus W^\times \cap D$. We show that P is a prime $\tilde{*}$ -ideal of D . Clearly, P is a prime ideal of D (since $W \setminus W^\times$ is a prime ideal of W). Let $x \in (W \setminus W^\times)_{\tilde{*}}$. Since $\tilde{*}$ is of finite type, there exists some $J \in f(D)$ with $x \in J_{\tilde{*}}$ and $J \subseteq W \setminus W^\times$. Observe that $JW = yW$ for some $y \in J$ (since $J \in f(D)$ and W is a valuation domain). This implies that $x \in J_{\tilde{*}} \subseteq J_{\tilde{*}}W \subseteq (J_{\tilde{*}}W)_{\tilde{*}} = (JW)_{\tilde{*}} = (yW)_{\tilde{*}} = yW_{\tilde{*}} = yW \subseteq W \setminus W^\times$. It follows that $W \setminus W^\times$ is $\tilde{*}$ -closed, and hence P is a $\tilde{*}$ -ideal of D .

Since P is a prime $\tilde{*}$ -ideal of D and $\tilde{*}$ is of finite type, there exists some $M \in \tilde{*}-\text{Max}(D)$ such that $P \subseteq M$. Observe that $M \in *-\text{Max}(D)$. Since $D \setminus M \subseteq D \setminus P = D \setminus (W \setminus W^\times) = W^\times \cap D \subseteq W^\times$, we have that $D_M \subseteq D_P \subseteq W$. Consequently, $I \subseteq I_M \cap D \subseteq IW \cap D = I$, and hence $I_M \cap D = I$.

(2) Let I be proper. Since $*$ is of finite type, it is clear that \sqrt{I} is a $*$ -ideal of D . Since W is a valuation domain, we have that \sqrt{IW} is a prime ideal of W . Since $\sqrt{I} = \sqrt{IW} \cap D = \sqrt{IW} \cap D$, we infer that \sqrt{I} is a prime ideal of D . \square

Corollary 2.4. *Let D be an integral domain and let I be a proper t -invertible valuation t -ideal of D . Then there exists a unique $M \in t-\text{Max}(D)$ such that $I \subseteq M$.*

Proof. This is an immediate consequence of Proposition 2.3(1) and [4, Lemma 4.2]. \square

The next example shows that the t -operation in Corollary 2.4 cannot be replaced by an arbitrary star operation of finite type.

Example 2.5. Let $D = \mathbb{Z}[X]$ be the polynomial ring over \mathbb{Z} and let \mathbb{P} be the set of prime numbers. Then D is a two-dimensional Noetherian UFD and X is a valuation element of D (since X is a prime element of D). Moreover, the prime ideal XD is contained in infinitely many maximal ideals of D (since $(pD + XD)_{p \in \mathbb{P}}$ is a sequence of distinct maximal ideals of D that contain XD). In particular, D is not of finite character and there exists a nonzero prime ideal of D that is contained in more than one maximal ideal of D .

A nonzero nonunit $a \in D$ is said to be *homogeneous* if it is contained in a unique maximal t -ideal of D . Following [7], we say that D is a *homogeneous factorization*

domain (HoFD) if every nonzero nonunit of D can be written as a finite product of homogeneous elements.

Corollary 2.6. *Let D be an integral domain.*

- (1) *If $a \in D$ is a valuation element, then a is homogeneous.*
- (2) *If D is a VFD, then D is an integrally closed HoFD.*

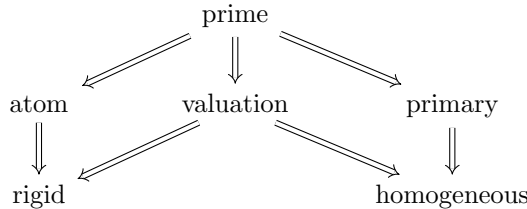
Proof. (1) Observe that aD is both a valuation ideal and a t -invertible t -ideal. Hence, by Corollary 2.4, aD is contained in a unique maximal t -ideal of D , and thus a is homogeneous.

(2) A VFD is integrally closed [9, Corollary 1.5]. Moreover, a valuation element is homogeneous by (1). Thus, D is an integrally closed HoFD. \square

Let D be a quasi-local domain of dimension one that is not a valuation domain. Then every nonzero element of D is homogeneous, while D is not a VFD and D has no valuation element [9, Corollary 1.6]. Therefore, a homogeneous element need not be a valuation element and an HoFD need not be a VFD in general.

Remark 2.7. We want to emphasize that valuation elements are not just homogeneous elements, but they also satisfy another interesting property that was studied by Zafrullah. Following [27], we say that a nonzero nonunit $a \in D$ is *rigid* if for all $b, c \in D$ with $b \mid_D a$ and $c \mid_D a$, it follows that $b \mid_D c$ or $c \mid_D b$. Moreover, D is called *semirigid* if every nonzero nonunit of D is a finite product of rigid elements. Note that every valuation element is rigid [9, Corollary 1.2(2)], and hence every VFD is semirigid. For more information on rigid elements and semirigid domains, we refer to [27, 28, 31].

In what follows we provide connections to well-known types of elements. Let $u \in D$ be a nonzero nonunit. Then u is called an *atom* of D if for all $a, b \in D$ with $u = ab$, either a is a unit of D or b is a unit of D . We say that u is *primary* if uD is a primary ideal of D . Furthermore, D is called *atomic* if every nonzero nonunit of D is a finite product of atoms of D . Observe that every atom is rigid, so every atomic domain is a semirigid domain. (Let $u \in D$ be an atom of D and let $a, b \in D$ be such that $a \mid_D u$ and $b \mid_D u$. If $a \in D^\times$, then clearly $a \mid_D b$. Now let $a \notin D^\times$. There is some $c \in D$ such that $u = ac$. We infer that $c \in D^\times$, and hence $b \mid_D u \mid_D uc^{-1} = a$.) It is known that D is a UFD if and only if D is an atomic VFD [9, Corollary 2.4]. Also note that every primary element is homogeneous [6, Lemma 2.1]. For the sake of clarity, we provide the following diagram to visualize the relations between the various types of elements.



In general, a primary atom need not be a valuation element. (Let D be an atomic quasi-local one-dimensional domain that is not a valuation domain and let $v \in D$ be an atom. Then v is a primary element of D but not a valuation element of D [9, Corollary 1.6].) We also want to mention that a valuation element is in general neither primary nor an atom. (Let V be a two-dimensional valuation domain, let

P be the unique height-one prime ideal of V and let $x \in P$ be nonzero. Then x is a valuation element of V but x is neither primary nor an atom.) Note that an atom does not have to be homogeneous. (Let $D = \mathbb{Z}[\sqrt{10}]$. Then 3 is an atom of D and 3 is not homogeneous.) Finally, a primary element need not be rigid. (Let D be a quasi-local one-dimensional domain that is not a valuation domain. Then there is some nonzero nonunit $y \in D$ such that y is not rigid (e.g. see [31]) and yet y is primary.) In particular, we obtain that none of the implications in the diagram above can be reversed.

Let I be a $*$ -ideal of D . We say that I is *$*$ -locally principal* if I_M is a principal ideal of D_M for each $M \in *-\text{Max}(D)$. It is easy to see that a $*$ -invertible $*$ -ideal of D is a $*$ -locally principal ideal for any star operation $*$ of finite type on D [20, 12.3 Theorem]. We are going to give a $*$ -locally principal ideal analog of [9, Proposition 1.1] that, among other things, shows that if $aR \cap D = aD$ and $bR \cap D = bD$, then $abR \cap D = abD$ for any $a, b \in D$ and an overring R of D . We first need a lemma which is also a natural generalization of [9, Proposition 1.1].

Lemma 2.8. *Let D be an integral domain, let R be an overring of D , let I be a nonzero principal ideal of D and let J be an ideal of D .*

- (1) *If $IR \cap D = I$ and $JR \cap D = J$, then $(IJ)R \cap D = IJ$.*
- (2) *If $(IJ)R \cap D = IJ$, then $JR \cap D = J$.*

Proof. (1) Let $IR \cap D = I$ and $JR \cap D = J$. It suffices to show that $(IJ)R \cap D \subseteq IJ$. Let $x \in (IJ)R \cap D$. Since $IJ \subseteq I$, we infer that $x \in IR \cap D = I$, and hence $xI^{-1} \subseteq II^{-1} = D$. Observe that $xI^{-1} \subseteq (IJ)RI^{-1} = ((II^{-1})J)R = JR$. This implies that $xI^{-1} \subseteq JR \cap D = J$. Therefore, $x \in xD = x(II^{-1}) = I(xI^{-1}) \subseteq IJ$.

(2) Let $(IJ)R \cap D = IJ$. It suffices to show that $JR \cap D \subseteq J$. Let $x \in JR \cap D$. Then $xI \subseteq (IJ)R$. Since $I \subseteq D$, it follows that $xI \subseteq (IJ)R \cap D = IJ$, and hence $x \in xD = x(II^{-1}) \subseteq (IJ)I^{-1} = (II^{-1})J = J$. Thus, $JR \cap D \subseteq J$. \square

Proposition 2.9. *Let D be an integral domain, let R be an overring of D , let $*$ be a star operation of finite type on D , let I be a nonzero $*$ -locally principal $*$ -ideal of D and let J be a nonzero $*$ -ideal of D .*

- (1) *If $IR \cap D = I$ and $JR \cap D = J$, then $(IJ)_*R \cap D = (IJ)_*$.*
- (2) *If $(IJ)_*R \cap D = (IJ)_*$, then $JR \cap D = J$.*
- (3) *If $IR \cap D = I$, $\sqrt{I} \subseteq \sqrt{J}$ and I and J are $*$ -invertible, then $JR \cap D = J$.*

*In particular, $IR \cap D = I$ if and only if $(I^n)_*R \cap D = (I^n)_*$ for some $n \in \mathbb{N}$ if and only if $(I^n)_*R \cap D = (I^n)_*$ for each $n \in \mathbb{N}$.*

Proof. First we show that $((IJ)_*)_M = I_M J_M$ for each $M \in *-\text{Max}(D)$. Let $M \in *-\text{Max}(D)$ and let $x \in (IJ)_*$. There is some nonzero $y \in I$ such that $I_M = yD_M$. Observe that $x \in A_*$ for some nonzero finitely generated ideal A of D with $A \subseteq IJ \subseteq I_M J_M = yJ_M$. We have that $bA \subseteq yJ$ for some $b \in D \setminus M$, and thus $x \in A_* = b^{-1}(bA)_* \subseteq b^{-1}(yJ)_* = b^{-1}yJ \subseteq yJ_M = I_M J_M$. This implies that $(IJ)_* \subseteq I_M J_M$, and hence $((IJ)_*)_M = I_M J_M$.

(1) Let $IR \cap D = I$, let $JR \cap D = J$ and let $x \in (IJ)_*R \cap D$. Since $*$ is of finite type, it is sufficient to show that $x \in ((IJ)_*)_M$ for each $M \in *-\text{Max}(D)$. Let $M \in *-\text{Max}(D)$. Then $x \in ((IJ)_*R \cap D)_M = ((IJ)_*)_M R_M \cap D_M = I_M J_M R_M \cap D_M = I_M J_M = ((IJ)_*)_M$, where the third equality holds by Lemma 2.8(1) (since I_M is a principal ideal of D_M , $I_M = (IR \cap D)_M = I_M R_M \cap D_M$, $J_M = (JR \cap D)_M = J_M R_M \cap D_M$ and R_M is an overring of D_M).

(2) Let $(IJ)_*R \cap D = (IJ)_*$ and let $x \in JR \cap D$. Since $*$ is of finite type, it remains to show that $x \in J_M$ for each $M \in *\text{-Max}(D)$. Let $M \in *\text{-Max}(D)$. Then $x \in JR \cap D \subseteq (JR \cap D)_M = J_MR_M \cap D_M$ and $I_MJ_MR_M \cap D_M = ((IJ)_*)_M R_M \cap D_M = ((IJ)_*R \cap D)_M = ((IJ)_*)_M = I_MJ_M$. Consequently, $x \in J_MR_M \cap D_M = J_M$ by Lemma 2.8(2) (since I_M is a principal ideal and R_M is an overring of D_M).

(3) Let $IR \cap D = I$, let $\sqrt{I} \subseteq \sqrt{J}$ and let I and J be $*$ -invertible. Since $*$ is of finite type and I is a $*$ -invertible $*$ -ideal, we have that I is $*$ -finite. Since $I \subseteq \sqrt{J}$ and I is $*$ -finite, we infer that $(I^n)_* \subseteq J$ for some $n \in \mathbb{N}$. It follows that $(I^n)_* = (JL)_*$ for some $*$ -invertible $*$ -ideal L of D (since J is a $*$ -invertible $*$ -ideal of D). It follows from (1) above that $(I^n)_*R \cap D = (I^n)_*$, and hence $(JL)_*R \cap D = (JL)_*$. Therefore, $JR \cap D = J$ by (2) above. \square

Corollary 2.10. (cf. [9, Corollary 1.2]) *Let D be an integral domain, let $*$ be a star operation of finite type on D , let I be a proper $*$ -invertible valuation $*$ -ideal of D and let J be a $*$ -invertible $*$ -ideal of D .*

- (1) *If L is a $*$ -invertible $*$ -ideal of D with $I \subseteq J \cap L$, then J and L are comparable.*
- (2) *If $\sqrt{I} \subseteq \sqrt{J}$, then I and J are comparable.*
- (3) *$\bigcap_{n \in \mathbb{N}} (I^n)_*$ is a prime $*$ -ideal of D .*
- (4) *If $\sqrt{I} \subsetneq \sqrt{J}$, then $I \subseteq \bigcap_{n \in \mathbb{N}} (J^n)_*$.*

Proof. Note that the $*$ -invertible $*$ -ideals of D are precisely the $\tilde{*}$ -invertible $\tilde{*}$ -ideals of D [24, Lemma 2.1(3)]. By Lemma 2.2, there is a $\tilde{*}$ -closed valuation overring V of D such that $IV \cap D = I$.

(1) Let L be a $*$ -invertible $*$ -ideal of D such that $I \subseteq J \cap L$. Then $JV \cap D = J$ and $LV \cap D = L$ by Proposition 2.9(3). Moreover, since V is a valuation domain, JV and LV are comparable. Thus, J and L are comparable.

(2) Let $\sqrt{I} \subseteq \sqrt{J}$. Then $JV \cap D = J$ by Proposition 2.9(3). Since IV and JV are comparable, we have that I and J are comparable.

(3) Clearly, $\bigcap_{n \in \mathbb{N}} (I^n)_*$ is a $*$ -ideal of D . Note that $(I^n)_* = (I^n)_{\tilde{*}}$ for each $n \in \mathbb{N}$ by [24, Lemma 2.1(3)]. There is a finitely generated ideal L of D with $I = L_{\tilde{*}}$. Consequently, $LV = aV$ for some $a \in L$. Note that $aV = LV \subseteq IV \subseteq (IV)_{\tilde{*}} = (LV)_{\tilde{*}} = (aV)_{\tilde{*}} = aV$, and hence $IV = aV$. Along similar lines, one can prove that $(I^n)_*V = a^nV$ for each $n \in \mathbb{N}$. It follows by Proposition 2.9(1) that $(I^n)_*V \cap D = (I^n)_*$ for each $n \in \mathbb{N}$. Note that $\bigcap_{n \in \mathbb{N}} a^nV$ is a prime ideal of V . Consequently, $\bigcap_{n \in \mathbb{N}} (I^n)_* = \bigcap_{n \in \mathbb{N}} ((I^n)_*V \cap D) = (\bigcap_{n \in \mathbb{N}} (I^n)_*V) \cap D$ is a prime ideal of D .

(4) Let $\sqrt{I} \subsetneq \sqrt{J}$ and let $n \in \mathbb{N}$. Then $\sqrt{I} \subseteq \sqrt{J} = \sqrt{(J^n)_*}$, and hence I and $(J^n)_*$ are comparable by (2). If $(J^n)_* \subseteq I$, then $\sqrt{J} = \sqrt{(J^n)_*} \subseteq \sqrt{I}$, a contradiction. Therefore, $I \subseteq (J^n)_*$. \square

In this paper we study integral domains D in which each nonzero principal ideal can be written as a finite $*$ -product of valuation $*$ -ideals for a given star operation $*$ on D , and in this case, all of the valuation ideals in question must be $*$ -invertible. The next result shows that such a finite $*$ -product of $*$ -invertible valuation $*$ -ideals can be written in a specific form. For example, if I is a finite $*$ -product of $*$ -invertible valuation $*$ -ideals of D , say, $I = (\prod_{i=1}^n I_i)_*$ and each I_i is a proper $*$ -ideal of D , then $\mathcal{P}(I) \subseteq \{\sqrt{I_k} \mid k \in [1, n]\}$ and $n \geq |\mathcal{P}(I)|$.

Proposition 2.11. *Let D be an integral domain, let $*$ be a star operation of finite type on D and let I be a finite $*$ -product of $*$ -invertible valuation $*$ -ideals of D . Then $|\mathcal{P}(I)| = \min\{m \in \mathbb{N}_0 \mid I \text{ is a } * \text{-product of } m \text{ } * \text{-invertible valuation } * \text{-ideals}\}$.*

of D and there are $*$ -invertible valuation $*$ -ideals $(I(P))_{P \in \mathcal{P}(I)}$ of D such that $I = (\prod_{P \in \mathcal{P}(I)} I(P))_*$ and $\sqrt{I(Q)} = Q$ for each $Q \in \mathcal{P}(I)$.

Proof. Without restriction we can assume that I is proper. Let n be the smallest positive integer such that I is the $*$ -product of n $*$ -invertible valuation $*$ -ideals of D . Then there are $*$ -invertible valuation $*$ -ideals I_i of D such that $I = (\prod_{i=1}^n I_i)_*$. First we show that $\mathcal{P}(I) \subseteq \{\sqrt{I_i} \mid i \in [1, n]\}$. Let $P \in \mathcal{P}(I)$. Then $\prod_{i=1}^n I_i \subseteq I \subseteq P$, and hence there is some $j \in [1, n]$ such that $I_j \subseteq P$. We infer that $I \subseteq I_j \subseteq \sqrt{I_j} \subseteq P$. Since $\sqrt{I_j}$ is a prime ideal of D by Proposition 2.3(2), it follows that $P = \sqrt{I_j}$.

This implies that $|\mathcal{P}(I)| \leq n$. Next we show that for each $P \in \mathcal{P}(I)$, $|\{i \in [1, n] \mid P \subseteq \sqrt{I_i}\}| = 1$. Let $P \in \mathcal{P}(I)$. Then $P = \sqrt{I_j}$ for some $j \in [1, n]$, and thus $\{i \in [1, n] \mid P \subseteq \sqrt{I_i}\} \supseteq \{i \in [1, n] \mid P = \sqrt{I_i}\} \neq \emptyset$. Set $\mathcal{I} = \{i \in [1, n] \mid P \subseteq \sqrt{I_i}\}$ and $J = (\prod_{i=1, P \subseteq \sqrt{I_i}}^n I_i)_*$. It is sufficient to show that \mathcal{I} is a singleton. Note that $\sqrt{J} = \bigcap_{i=1, P \subseteq \sqrt{I_i}}^n \sqrt{I_i} = P$ (since $\{i \in [1, n] \mid P = \sqrt{I_i}\} \neq \emptyset$). Clearly, J is a $*$ -invertible $*$ -ideal of D . Since $\sqrt{I_j} = P = \sqrt{J}$, we infer by Proposition 2.9(3) that J is a valuation ideal of D . Because of the minimality of n , we have that \mathcal{I} is a singleton.

Since $[1, n] \subseteq \bigcup_{Q \in \mathcal{P}(I)} \{i \in [1, n] \mid Q \subseteq \sqrt{I_i}\}$, it follows that $n \leq \sum_{Q \in \mathcal{P}(I)} |\{i \in [1, n] \mid Q \subseteq \sqrt{I_i}\}| = |\mathcal{P}(I)|$, and hence $|\mathcal{P}(I)| = n$. Let $f : [1, n] \rightarrow \mathcal{P}(I)$ be defined by $f(i) = \sqrt{I_i}$ for each $i \in [1, n]$. Then f is a well-defined bijection. For each $Q \in \mathcal{P}(I)$, set $I(Q) = I_{f^{-1}(Q)}$. Then $I = (\prod_{Q \in \mathcal{P}(I)} I(Q))_*$ and $\sqrt{I(Q)} = Q$ for each $Q \in \mathcal{P}(I)$. \square

We continue our investigation of ideals that are finite $*$ -products of $*$ -invertible valuation $*$ -ideals. The next result serves as a preparatory result for Theorem 3.7, but it is more generally applicable, since it holds for arbitrary integral domains.

Proposition 2.12. *Let D be an integral domain, let $*$ be a star operation of finite type on D and let Ω be the set of finite $*$ -products of $*$ -invertible valuation $*$ -ideals of D . Then for all $I, J, L \in \Omega$ with $(JL)_* \subseteq I$, there are some $J', J'', L', L'' \in \Omega$ such that $I = (J'L')_*$, $J = (J'J'')_*$ and $L = (L'L'')_*$.*

Proof. First we show that for each $*$ -invertible valuation $*$ -ideal I of D and all $J, L \in \Omega$ with $(JL)_* \subseteq I$, there are some $J', J'', L', L'' \in \Omega$ such that $J = (J'J'')_*$, $L = (L'L'')_*$ and $I = (J'L')_*$. By Proposition 2.11, it is sufficient to show by induction that for each $n \in \mathbb{N}_0$, each $*$ -invertible valuation $*$ -ideal I of D and all $J, L \in \Omega$ with $|\mathcal{P}(J)| + |\mathcal{P}(L)| = n$ and $(JL)_* \subseteq I$, there are some $J', J'', L', L'' \in \Omega$ such that $J = (J'J'')_*$, $L = (L'L'')_*$ and $I = (J'L')_*$. Let $n \in \mathbb{N}_0$, let I be a $*$ -invertible valuation $*$ -ideal of D and let $J, L \in \Omega$ be such that $|\mathcal{P}(J)| + |\mathcal{P}(L)| = n$ and $(JL)_* \subseteq I$. Without restriction let I be proper. Since I is a valuation ideal of D , \sqrt{I} is a prime ideal of D by Proposition 2.3(2), and thus $J \subseteq \sqrt{I}$ or $L \subseteq \sqrt{I}$. Without restriction let $J \subseteq \sqrt{I}$. By Proposition 2.11, there are $*$ -invertible valuation $*$ -ideals $(J(P))_{P \in \mathcal{P}(J)}$ of D such that $J = (\prod_{P \in \mathcal{P}(J)} J(P))_*$. Consequently, $J(Q) \subseteq \sqrt{I}$ for some $Q \in \mathcal{P}(J)$. This implies that $\sqrt{J(Q)} \subseteq \sqrt{I}$. It follows from Corollary 2.10(2) that $J(Q)$ and I are comparable.

CASE 1: $J(Q) \subseteq I$. Then $J(Q) = (IA)_*$ for some $*$ -invertible $*$ -ideal A of D . It follows from Proposition 2.9(2) that A is a valuation ideal of D . Set $J' = I$, $J'' = (A \prod_{P \in \mathcal{P}(J) \setminus \{Q\}} J(P))_*$, $L' = D$ and $L'' = L$. Then $J', J'', L', L'' \in \Omega$, $J = (J'J'')_*$, $L = (L'L'')_*$ and $I = (J'L')_*$.

CASE 2: $I \subsetneq J(Q)$. Then $I = (J(Q)C)_*$ for some $*$ -invertible $*$ -ideal C of D . We infer by Proposition 2.9(2) that C is a valuation ideal of D . Set $B =$

$(\prod_{P \in \mathcal{P}(J) \setminus \{Q\}} J(P))_*$. Then $B \in \Omega$ and $(J(Q)BL)_* = (JL)_* \subseteq I = (J(Q)C)_*$. Since $J(Q)$ is $*$ -invertible, it follows that $(BL)_* \subseteq C$. Moreover, $\mathcal{P}(B) = \mathcal{P}(J) \setminus \{Q\}$, and hence $|\mathcal{P}(B)| + |\mathcal{P}(L)| < n$. Therefore, there are $B', B'', L', L'' \in \Omega$ such that $B = (B'B'')_*$, $L = (L'L'')_*$ and $C = (B'L')_*$ by the induction hypothesis. Set $J' = (J(Q)B')_*$ and $J'' = B''$. Then $J', J'' \in \Omega$, $J = (J(Q)B)_* = (J'J'')_*$ and $I = (J(Q)C)_* = (J'L')_*$.

It remains to show by induction that for each $m \in \mathbb{N}_0$ and all $I, J, L \in \Omega$ such that I is a $*$ -product of m $*$ -invertible valuation $*$ -ideals of D and $(JL)_* \subseteq I$, there are some $J', J'', L', L'' \in \Omega$ such that $J = (J'J'')_*$, $L = (L'L'')_*$ and $I = (J'L')_*$. The statement is clearly true for $m = 0$. Let $m \in \mathbb{N}_0$ and $I, J, L \in \Omega$ be such that I is a $*$ -product of $m + 1$ $*$ -invertible valuation $*$ -ideals of D and $(JL)_* \subseteq I$. Clearly, there are some $A, B \in \Omega$ such that A is a $*$ -product of m $*$ -invertible valuation $*$ -ideals of D and B is a $*$ -invertible valuation $*$ -ideal of D such that $I = (AB)_*$. Then $(JL)_* \subseteq I \subseteq B$. As shown before, there are $M', M'', N', N'' \in \Omega$ such that $J = (M'M'')_*$, $L = (N'N'')_*$ and $B = (M'N')_*$. We have that $(M''N''B)_* = (M'M''N'N'')_* = (JL)_* \subseteq I = (AB)_*$, and hence $(M''N'')_* \subseteq A$. It follows by the induction hypothesis that there are $C', C'', D', D'' \in \Omega$ such that $M'' = (C'C'')_*$, $N'' = (D'D'')_*$ and $A = (C'D')_*$. Set $J' = (M'C')_*$, $J'' = C''$, $L' = (N'D')_*$ and $L'' = D''$. Then $J', J'', L', L'' \in \Omega$, $J = (M'M'')_* = (J'J'')_*$, $L = (N'N'')_* = (L'L'')_*$ and $I = (AB)_* = (C'D'M'N')_* = (J'L')_*$. \square

3. $*$ -VALUATION IDEAL FACTORIZATION DOMAINS

A π -domain is an integral domain whose nonzero principal ideals can be written as a finite product of prime ideals [2]. Hence, each nonzero principal ideal of a π -domain can be written as a finite product of valuation ideals, because a prime ideal is a valuation ideal. In this section, we study such type of integral domains in the more general setting of star operations. We begin this section with the definition of $*$ -VIFDs for which we note that an ideal I of an integral domain D is a fractional ideal of D with $I \subseteq D$, so D is also a $*$ -ideal of D .

Definition 3.1. Let D be an integral domain and let $*$ be a star operation on D . Then D is called a $*$ -valuation ideal factorization domain ($*$ -VIFD) if each nonzero principal ideal I of D can be written as a finite $*$ -product of valuation ideals of D , i.e., there are some $n \in \mathbb{N}$ and valuation ideals I_i of D such that $I = (\prod_{i=1}^n I_i)_*$. We say that D is a VIFD if D is a d -VIFD.

Let $*_1$ and $*_2$ be two star operations of finite type on D such that $*_1 \leq *_2$. It is easy to see that $(I_{*_1})_{*_2} = (I_{*_2})_{*_1} = I_{*_2}$ for all $I \in F(D)$. Hence, by definition, a $*_1$ -VIFD is a $*_2$ -VIFD. In particular,

$$\text{VIFD} \Rightarrow *\text{-VIFD} \Rightarrow t\text{-VIFD}$$

for any star operation $*$ of finite type.

Lemma 3.2. Let D be an integral domain, let $*$ be a star operation of finite type on D and let I be a $*$ -invertible valuation ideal of D . Then I_* is a $*$ -invertible valuation $*$ -ideal of D .

Proof. Since $*\text{-Max}(D) = \tilde{*}\text{-Max}(D)$, a $*$ -invertible ideal of D is $\tilde{*}$ -invertible, so I is a $\tilde{*}$ -invertible ideal of D . There exists a valuation overring V of D such that $IV \cap D = I$. Set $W = V_{\tilde{*}}$ and note that W is a valuation overring of D by the proof of Lemma 2.2. Since $\tilde{*}$ is stable, we have that $I_{\tilde{*}} \subseteq I_{\tilde{*}}W \cap D \subseteq (I_{\tilde{*}}W)_{\tilde{*}} \cap D = (IV)_{\tilde{*}} \cap D_{\tilde{*}} = (IV \cap D)_{\tilde{*}} = I_{\tilde{*}}$, and hence $I_{\tilde{*}}W \cap D = I_{\tilde{*}}$. We infer that $I_{\tilde{*}}$ is a

$\tilde{*}$ -invertible valuation $\tilde{*}$ -ideal of D . Therefore, $I_{\tilde{*}}$ is a $\tilde{*}$ -invertible valuation $\tilde{*}$ -ideal of D . Finally, note that $I_{\tilde{*}} = (I_{\tilde{*}})_{\tilde{*}} = I_{*}$. \square

We next give a first elementary characterization of $\tilde{*}$ -VIFDs. Note that these characterizations view the concepts of $\tilde{*}$ -VIFDs from three different angles. First, we can replace finite $\tilde{*}$ -products of valuation ideals by finite $\tilde{*}$ -products of valuation $\tilde{*}$ -ideals (and vice versa). Second, we can lift the existence of representations of arbitrary nonzero principal ideals as finite $\tilde{*}$ -products of valuation ideals to arbitrary $\tilde{*}$ -invertible $\tilde{*}$ -ideals. Finally, we prove the interchangeability of the star operations $\tilde{*}$ and $*$ in this characterization.

Proposition 3.3. *Let D be an integral domain and let $*$ be a star operation of finite type on D . The following statements are equivalent.*

- (1) D is a $\tilde{*}$ -VIFD.
- (2) D is a $*$ -VIFD.
- (3) Each nonzero principal ideal of D is a finite $\tilde{*}$ -product of valuation $\tilde{*}$ -ideals.
- (4) Each $\tilde{*}$ -invertible $\tilde{*}$ -ideal of D is a finite $\tilde{*}$ -product of valuation $\tilde{*}$ -ideals.
- (5) Each $\tilde{*}$ -invertible $\tilde{*}$ -ideal of D is a finite $\tilde{*}$ -product of valuation ideals.

Proof. (1) \Rightarrow (3) This is an immediate consequence of Lemma 3.2.

(2) \Rightarrow (1) This follows from the fact that $\tilde{*} \leq *$.

(3) \Rightarrow (2) This is an immediate consequence of the following observation: If I and J are $\tilde{*}$ -invertible $\tilde{*}$ -ideals of D , then I and J are $*$ -invertible $*$ -ideals of D and $(IJ)_{\tilde{*}} = ((IJ)_{\tilde{*}})_{\tilde{*}} = (IJ)_{*}$ (since $(IJ)_{\tilde{*}}$ is a $\tilde{*}$ -invertible $\tilde{*}$ -ideal of D).

(3) \Rightarrow (4) Let I be a proper $\tilde{*}$ -invertible $\tilde{*}$ -ideal of D . Choose a nonzero $a \in I$. Clearly, $aD = (\prod_{j=1}^m J_j)_{\tilde{*}}$ with $m \in \mathbb{N}$ and proper $\tilde{*}$ -invertible valuation $\tilde{*}$ -ideals J_j . Observe that J_j is a t -invertible t -ideal for each $j \in [1, m]$. Consequently, for each $j \in [1, m]$, J_j is contained in a unique maximal t -ideal by Corollary 2.4. We infer that aD is contained in only finitely many maximal t -ideals of D , and hence I is contained in only finitely many maximal t -ideals of D . Since I is a t -invertible t -ideal of D , we have that I_M is a principal ideal of D_M for each $M \in t\text{-Max}(D)$. Let $N \in t\text{-Max}(D)$ be such that $I \subseteq N$. Then $I_N = bD_N$ for some $b \in I$. Note that $bD = (\prod_{i=1}^n I_i)_{\tilde{*}}$ for $n \in \mathbb{N}$ and $\tilde{*}$ -invertible valuation $\tilde{*}$ -ideals I_i of D . Without restriction we can assume that there is some $r \in [1, n]$ such that $I_i \subseteq N$ for each $i \in [1, r]$ and $I_i \not\subseteq N$ for each $i \in [r+1, n]$. Observe that $I_N = ((\prod_{i=1}^r I_i)_{\tilde{*}})_N$. Let $\ell \in \mathbb{N}$ and let $(N_i)_{i=1}^{\ell}$ be the distinct maximal t -ideals of D that contain I . Then for each $i \in [1, \ell]$ there is some $m_i \in \mathbb{N}$ and some finite $\tilde{*}$ -product $(\prod_{j=1}^{m_i} J_{i,j})_{\tilde{*}}$ of $\tilde{*}$ -invertible valuation $\tilde{*}$ -ideals of D such that $J_{i,j} \subseteq N_i$ for each $j \in [1, m_i]$ and such that $I_{N_i} = ((\prod_{j=1}^{m_i} J_{i,j})_{\tilde{*}})_{N_i}$. Note that for each $N \in t\text{-Max}(D)$, we have that $I_N = ((\prod_{i=1}^{\ell} \prod_{j=1}^{m_i} J_{i,j})_{\tilde{*}})_N$ (since every proper $\tilde{*}$ -invertible valuation $\tilde{*}$ -ideal of D is contained in a unique maximal t -ideal of D). Since I and $(\prod_{i=1}^{\ell} \prod_{j=1}^{m_i} J_{i,j})_{\tilde{*}}$ are $\tilde{*}$ -invertible $\tilde{*}$ -ideals of D (and hence t -ideals of D), this implies that $I = (\prod_{i=1}^{\ell} \prod_{j=1}^{m_i} J_{i,j})_{\tilde{*}}$ is a finite $\tilde{*}$ -product of valuation $\tilde{*}$ -ideals of D .

(4) \Rightarrow (5) \Rightarrow (1) This is obvious. \square

Corollary 3.4. *Let D be an integral domain and let $*$ be a star operation of finite type on D . The following statements are equivalent.*

- (1) D is a VFD.
- (2) $\text{Cl}_*(D) = \{0\}$ and every $\tilde{*}$ -invertible $\tilde{*}$ -ideal of D is a finite $\tilde{*}$ -product of valuation ideals.
- (3) D is a $\tilde{*}$ -VIFD and $\text{Cl}_*(D) = \{0\}$.

Proof. (1) \Rightarrow (2) Recall from [9, Corollary 2.3(1)] that $\text{Cl}_t(D) = \{0\}$, so $\text{Cl}_*(D) = \{0\}$. Now let J be a proper $*$ -invertible $*$ -ideal of D . Then $J = aD$ for some nonzero nonunit $a \in D$. Since D is a VFD, J is a finite product of principal valuation ideals of D . Thus, J is a finite $*$ -product of principal valuation ideals of D .

(2) \Rightarrow (3) This follows directly from the fact that a nonzero principal ideal is a $*$ -invertible $*$ -ideal.

(3) \Rightarrow (1) Let $a \in D$ be a nonzero nonunit. Then $aD = (\prod_{i=1}^n I_i)_*$ for some $*$ -invertible valuation $*$ -ideals I_i of D by Proposition 3.3. Clearly, each I_i is principal, and hence aD is a finite $*$ -product of principal valuation ideals of D . Consequently, aD is a finite product of principal valuation ideals of D . Thus, a is a finite product of valuation elements of D . \square

Corollary 3.5. *Let D be an integral domain and let $*_1$ and $*_2$ be star operations of finite type on D such that $*_1 \leq *_2$. Then D is a VFD if and only if D is a $*_1$ -VIFD and $\text{Cl}_{*_2}(D) = \{0\}$.*

Proof. If D is a VFD, then it is an immediate consequence of Corollary 3.4 that D is a $*_1$ -VIFD and $\text{Cl}_{*_2}(D) = \{0\}$. Now let D be a $*_1$ -VIFD and $\text{Cl}_{*_2}(D) = \{0\}$. Then $\text{Cl}_{*_1}(D) = \{0\}$, and hence D is a VFD by Corollary 3.4. \square

A π -domain is a VIFD, because a prime ideal is a valuation ideal. We next study the relationship between a VFD and a VIFD, which is an analog of the fact that a UFD is a π -domain with trivial Picard group.

Corollary 3.6. *Let D be a Krull domain.*

- (1) D is a t -VIFD.
- (2) D is a VFD if and only if D is a UFD.
- (3) D is a VIFD if and only if D is a π -domain.

Proof. (1) This is clear.

(2) A Krull domain is a UFD if and only if its t -class group is trivial. Consequently, the result follows from (1) and Corollary 3.4.

(3) Observe that $t\text{-dim}(D) \leq 1$. In general, a π -domain is a VIFD. Conversely, suppose that D is a VIFD. It suffices to show that each height-one prime ideal of D is invertible by [2, Theorem 1]. Let P be a height-one prime ideal. Choose $a \in P$ such that $aD_P = P_P$. Then aD is a finite product of valuation ideals, so P contains an invertible valuation ideal, say Q , containing a . By Proposition 2.3, $Q_P \cap D = Q$, and since $P_P = aD_P \subseteq Q_P$, it follows that $Q = P$. Thus, P is invertible. \square

Following [1, 12], we say that D is a $*$ -Schreier domain if for all $*$ -invertible $*$ -ideals I, J and L of D such that $(JL)_* \subseteq I$, there are some $*$ -invertible $*$ -ideals J' and L' of D such that $J \subseteq J', L \subseteq L'$ and $I = (J'L')_*$. Recall from [1, Corollary 3.3] that D is a Schreier domain if and only if D is an integrally closed d -Schreier domain with $\text{Pic}(D) = \{0\}$, so there is a clear distinction between the concepts of Schreier and d -Schreier domains (cf. [12, Proposition 2]).

Theorem 3.7. *Let D be an integral domain and let $*$ be a star operation of finite type on D . Then D is a $*$ -VIFD if and only if D is a $*$ -Schreier domain and each nonzero prime $*$ -ideal of D contains a $*$ -invertible valuation ($*$ -)ideal of D .*

Proof. (\Rightarrow) First let D be a $*$ -VIFD. Then every $*$ -invertible $*$ -ideal of D is a finite $*$ -product of $*$ -invertible valuation $*$ -ideals of D by Proposition 3.3. In particular, the set of $*$ -invertible $*$ -ideals is the set of finite $*$ -products of $*$ -invertible valuation $*$ -ideals of D . We infer by Proposition 2.12 that D is a $*$ -Schreier domain. Now let

P be a nonzero prime $*$ -ideal of D . Then there is a nonzero nonunit $a \in D$ such that $aD \subseteq P$. Since aD is a finite $*$ -product of $*$ -invertible valuation $*$ -ideals of D , we have that P contains a $*$ -invertible valuation $*$ -ideal of D .

(\Leftarrow) Now let D be a $*$ -Schreier domain such that each nonzero prime $*$ -ideal of D contains a $*$ -invertible valuation $*$ -ideal of D . Let Ω be the set of all finite $*$ -products of $*$ -invertible valuation $*$ -ideals of D . Assume that D is not a $*$ -VIFD. Then there exists a $*$ -invertible $*$ -ideal I of D such that $I \notin \Omega$. Let $\Sigma = \{J \mid J \text{ is a } * \text{-ideal of } D \text{ such that } I \subseteq J \text{ and } A \not\subseteq J \text{ for each } A \in \Omega\}$. Assume that $I \notin \Sigma$. Then there are $n \in \mathbb{N}$ and $*$ -invertible valuation $*$ -ideals I_i of D such that $(\prod_{i=1}^n I_i)_* \subseteq I$. Since D is a $*$ -Schreier domain, it follows (by induction) that there are some $*$ -invertible $*$ -ideals J_i of D such that $I = (\prod_{i=1}^n J_i)_*$ and $I_j \subseteq J_j$ for each $j \in [1, n]$. We infer by Proposition 2.9(3) that J_j is a valuation ideal of D for each $j \in [1, n]$. Therefore, $I \in \Omega$, a contradiction. Hence, $I \in \Sigma$. Then $\Sigma \neq \emptyset$, and since $*$ is of finite type and each element of Ω is $*$ -finite, we have that Σ is ordered inductively (under inclusion). Consequently, there is a maximal element $P \in \Sigma$ by Zorn's lemma. We show that P is a prime $*$ -ideal of D . Clearly, P is a proper $*$ -ideal of D . Assume that P is not a prime ideal of D , then there are $a, b \in D$ such that $ab \in P$ and $a, b \notin P$. We have that $P \subsetneq (P + aD)_*$ and $P \subsetneq (P + bD)_*$, and hence $(P + aD)_* \notin \Sigma$ and $(P + bD)_* \notin \Sigma$. Consequently, there are $A, B \in \Omega$ such that $A \subseteq (P + aD)_*$ and $B \subseteq (P + bD)_*$. This implies that $(AB)_* \subseteq (P^2 + aP + bP + abD)_* \subseteq P$ and $(AB)_* \in \Omega$, a contradiction. Hence, P is a prime $*$ -ideal of D . Now since $I \subseteq P$, we have that P is nonzero, and thus P contains a $*$ -invertible valuation $*$ -ideal J of D . Note that $J \in \Omega$, a contradiction. \square

The next result is a valuation ideal analog of [9, Proposition 1.7(4)] that if $a \in D$ is a valuation element, then either a is a unit of D_S or a is a valuation element of D_S for any multiplicatively closed subset S of D .

Lemma 3.8. *Let D be an integral domain, let S be a multiplicatively closed subset of D and let I be a valuation ideal of D . Then I_S is a valuation ideal of D_S .*

Proof. Let V be a valuation overring of D such that $IV \cap D = I$. Then $I_S = (IV \cap D)_S = I_S V_S \cap D_S$ and V_S is a valuation overring of D_S . Thus, I_S is a valuation ideal of D_S . \square

Let $*$ be a star operation of finite type on D and let $*_S : F(D_S) \rightarrow F(D_S)$ be defined by $(I_S)_{*_S} = (I_*)_S$ for each $I \in F(D)$. Then $*_S$ is a star operation of finite type on D_S [20, 4.4 Theorem]. If P is a prime ideal of D such that $S = D \setminus P$, then we write $*_P$ instead of $*_S$.

Proposition 3.9. *Let D be an integral domain, let $*$ be a star operation of finite type on D and let P be a prime $*$ -ideal of D . If D is a $*$ -VIFD, then D_P is a VFD.*

Proof. Let $a \in D$ be a nonzero nonunit of D_P . Then $aD = (\prod_{i=1}^n I_i)_*$ for some $n \in \mathbb{N}$ and valuation $*$ -ideals I_i of D . If $j \in [1, n]$, then I_j is $*$ -invertible, and hence $(I_j)_P = a_j D_P$ for some $a_j \in I_j$. Therefore,

$$\begin{aligned} aD_P &= \left(\left(\prod_{i=1}^n I_i \right)_* \right)_P = \left(\left(\prod_{i=1}^n I_i \right)_P \right)_{*_P} \\ &= \left(\prod_{i=1}^n (I_i)_P \right)_{*_P} \end{aligned}$$

$$\begin{aligned}
&= \left(\prod_{i=1}^n (a_i D_P) \right)_{*P} \\
&= \left(\prod_{i=1}^n a_i \right) D_P.
\end{aligned}$$

We infer by Lemma 3.8 that a_i is either a unit of D_P or a valuation element of D_P for each $i \in [1, n]$. Consequently, D_P is a VFD. \square

An integral domain is called a *Mori domain* if it satisfies the ACC on t -ideals. Moreover, we say that D is *completely integrally closed* if for each $x \in K$ for which there is some nonzero $c \in D$ such that $cx^n \in D$ for all $n \in \mathbb{N}$, it follows that $x \in D$. Observe that D is a *Krull domain* if and only if D is a completely integrally closed Mori domain [17, Theorem 2.3.11]. The purpose of the next result is to generalize [9, Corollary 2.4] that characterizes when a VFD is a Mori domain.

Corollary 3.10. *Let D be a t -VIFD. The following statements are equivalent.*

- (1) D is a Mori domain.
- (2) D_M is a UFD for each $M \in t\text{-Max}(D)$.
- (3) D is a Krull domain.

Proof. (1) \Rightarrow (2) Let $M \in t\text{-Max}(D)$. Then D_M is a Mori domain [17, Proposition 2.10.4.2]. Moreover, D_M is a VFD by Proposition 3.9. Thus, D_M is a UFD [9, Corollary 2.4].

(2) \Rightarrow (3) Clearly, D_M is a Krull domain for each $M \in t\text{-Max}(D)$. Furthermore, D is of finite t -character by Corollary 2.4. Consequently, D is a Krull domain.

(3) \Rightarrow (1) This is obvious. \square

Let $*$ be a star operation of finite type on D . Next, we are going to show that a $*$ -VIFD is an integrally closed weakly Matlis domain. However, since $*$ $\leq t$, a $*$ -VIFD is a t -VIFD, so it suffices to show that a t -VIFD is an integrally closed weakly Matlis domain.

Proposition 3.11. *Let D be an integral domain such that every nonzero prime t -ideal of D contains a t -invertible valuation ideal of D . Then D is an integrally closed weakly Matlis domain.*

Proof. It is an immediate consequence of Lemma 3.2 that every nonzero prime t -ideal of D contains a t -invertible valuation t -ideal. Let \overline{D} be the integral closure of D and let Ω be the set of t -invertible t -ideals I of D such that $I\overline{D} \cap D = I$. It follows from Proposition 2.9 that Ω is a multiplicatively closed and divisor-closed subset of the monoid of t -invertible t -ideals of D . (The notions of multiplicatively closed and divisor-closed can be defined in analogy for monoids. For instance, see [17].) Assume that Ω is not the set of all t -invertible t -ideals of D . Then there exists a t -invertible t -ideal J of D such that $J \notin \Omega$. Since Ω is divisor-closed, we infer that $L \not\subseteq J$ for each $L \in \Omega$. Let $\Sigma = \{A \mid A \text{ is a } t\text{-ideal of } D \text{ such that } J \subseteq A \text{ and } L \not\subseteq A \text{ for each } L \in \Omega\}$. It is clear that $J \in \Sigma$. Observe that Σ is ordered inductively under inclusion (since each element of Ω is t -finite). Consequently, Σ has a maximal element Q by Zorn's lemma. It is straightforward to show that Q is a nonzero prime t -ideal of D (e.g. as in the proof of Theorem 3.7). Hence, Q contains a t -invertible valuation t -ideal B of D . On the other hand, we have that $B \in \Omega$, a contradiction. We infer that Ω is the set of all t -invertible t -ideals of D . Next we show that D is integrally closed. It remains to prove that $\overline{D} \subseteq D$.

Let $x \in \overline{D}$. Then $x = \frac{a}{b}$ for some $a \in D$ and some nonzero $b \in D$. Observe that $bD \in \Omega$. It follows that $a = bx \in b\overline{D} \cap D = (bD)\overline{D} \cap D = bD$. Thus, $x \in D$.

Finally, we show that D is weakly Matlis. It follows from Proposition 2.3(1) that for each nonzero prime t -ideal P of D , there exists a t -invertible t -ideal I of D and an $M \in t\text{-Max}(D)$ such that $I \subseteq P$ and $I_M \cap D = I$. We infer by [4, Theorem 4.3] that D is weakly Matlis. \square

It is an immediate consequence of Proposition 3.11 that if every nonzero prime $*$ -ideal of D contains a $*$ -invertible valuation ideal, then D is an integrally closed weakly Matlis domain. Nevertheless, it follows from Example 2.5 that D need not be $*$ -h-local.

Corollary 3.12. *Let D be an integral domain and let $*$ be a star operation of finite type on D . If D is a $*$ -VIFD, then D is an integrally closed weakly Matlis domain.*

Proof. This is an immediate consequence of Proposition 3.11. \square

We say that D is a $*$ -treed domain if the set of prime $*$ -ideals of D is treed under inclusion. Hence, D is $*$ -treed if and only if $\text{Spec}(D_M)$ is linearly ordered under inclusion for all maximal $*$ -ideals M of D . The class of $*$ -treed domains includes P*MDs, integral domains of $*$ -dimension one, and treed domains. Moreover, D is said to be a *ring of Krull type* if D is a PvMD of finite t -character and D is called an *independent ring of Krull type* if D is a weakly Matlis PvMD.

Theorem 3.13. *Let D be an integral domain and let $*$ be a star operation of finite type on D . The following statements are equivalent.*

- (1) D is a $*$ -h-local P*MD.
- (2) Every nonzero ($*$ -finite) $*$ -ideal of D can be written as a finite $*$ -product of $*$ -comaximal valuation ideals.
- (3) Every nonzero principal ideal of D can be written as a finite $*$ -product of $*$ -comaximal valuation ideals.
- (4) D is $*$ -treed and D is a $*$ -VIFD.
- (5) D is $*$ -treed and every nonzero prime ideal of D contains a $*$ -invertible valuation ideal.
- (6) D is $*$ -treed and every nonzero $*$ -ideal of D is a finite $*$ -product of valuation ideals.

Proof. (1) \Rightarrow (2), (6) Since D is a P*MD, it is clear that D is $*$ -treed. It remains to show that every nonzero proper $*$ -ideal of D is a finite $*$ -product of $*$ -comaximal valuation ideals. Let I be a nonzero proper $*$ -ideal of D . Then $\bigcap_{M \in * \text{-Max}(D)} I_M = I$, and since D is of finite $*$ -character, there are only finitely many maximal $*$ -ideals, say, $(M_i)_{i=1}^k$, such that $I = (\bigcap_{i=1}^k I_{M_i}) \cap D$. Let $I_i = I_{M_i} \cap D$ for $i \in [1, k]$. Then, since D_{M_i} is a valuation domain, each I_i is a valuation $*$ -ideal. Note that if $i, j \in [1, k]$ are distinct, then $(D_{M_i})_{M_j}$ is the quotient field of D . Hence, $(I_i + I_j)_* = D$ for all distinct $i, j \in [1, k]$, and thus $\bigcap_{i=1}^k I_i = (\prod_{i=1}^k I_i)_*$. Therefore, $I = (\prod_{i=1}^k I_i)_*$ is a finite $*$ -product of the $*$ -comaximal valuation ideals I_i .

(2) \Rightarrow (3) This is clear.

(3) \Rightarrow (4) Let M be a maximal $*$ -ideal of D and let $a \in M$ be nonzero. Then, by assumption, $aD = (\prod_{i=1}^n Q_i)_*$ for some $n \in \mathbb{N}$ and proper $*$ -comaximal valuation ideals Q_i of D . Note that Q_i and Q_j are $*$ -comaximal for each distinct $i, j \in [1, n]$, so M contains exactly one of the Q_i 's, say Q_1 for convenience. Consequently,

$$aD_M = \left(\left(\prod_{i=1}^n Q_i \right)_* \right)_M = \left(\prod_{i=1}^n (Q_i)_M \right)_{*_M} = (Q_1)_M$$

and $(Q_1)_M$ is a valuation ideal of D_M by Lemma 3.8, which means that a is a valuation element of D_M . Thus, every nonzero nonunit of D_M is a valuation element, and hence D_M is a valuation domain [18, Corollary 2.4]. Therefore, D is $*$ -treed.

(4) \Rightarrow (5) Let P be a nonzero prime ideal of D , and choose a nonzero $a \in D$. Then $aD = \left(\prod_{i=1}^n Q_i \right)_*$ for some $n \in \mathbb{N}$ and valuation ideals Q_i of D . Clearly, for each $i \in [1, n]$, Q_i is $*$ -invertible and P contains at least one of the Q_i 's.

(5) \Rightarrow (1) It follows from Proposition 3.11 that D is a weakly Matlis domain. It suffices to show that D is a P*MD; equivalently, D_M is a valuation domain for each $M \in *-\text{Max}(D)$. Now let M be a maximal $*$ -ideal of D . Then $\text{Spec}(D_M)$ is linearly ordered under inclusion because $*-\text{Spec}(D)$ is treed, and each prime ideal of D_M contains a valuation element by assumption and Lemma 3.8. Now if $b \in D_M$ is a nonzero nonunit, then $\sqrt{bD_M}$ is a prime ideal, so there is a valuation element $c \in \sqrt{bD_M}$. Hence, $\sqrt{cD_M} \subseteq \sqrt{bD_M}$, and thus b is a valuation element of D_M [9, Proposition 1.1(3)]. Thus, D_M is a valuation domain [9, Corollary 1.4].

(6) \Rightarrow (4) This is obvious. \square

Corollary 3.14. *Let D be a t -treed domain. Then D is a VFD if and only if $\text{Cl}_t(D) = \{0\}$ and every nonzero prime ideal of D contains a valuation element.*

Proof. (\Rightarrow) It is clear that every nonzero prime ideal of D contains a valuation element. Thus, the result follows because a VFD has a trivial t -class group [9, Corollary 2.3(1)].

(\Leftarrow) If $a \in D$ is a valuation element, then aD is a t -invertible valuation ideal. Hence, D is an independent ring of Krull type by Theorem 3.13. Therefore, $\text{Cl}_t(D) = \{0\}$ implies that D is a weakly Matlis GCD-domain, so D is a VFD [9, Theorem 3.4]. \square

Corollary 3.15. *Let D be an integral domain and let $*$ be a star operation of finite type on D such that $*-\dim(D) = 1$. The following statements are equivalent.*

- (1) D is a P*MD of finite $*$ -character.
- (2) D is a $*$ -VIFD.
- (3) Every nonzero prime ideal of D contains a $*$ -invertible valuation ideal.
- (4) Each nonzero $*$ -ideal of D is a finite $*$ -product of valuation ideals.

Proof. Since $*-\dim(D) = 1$, we have that D is $*$ -treed. Moreover, D is $*$ -h-local if and only if D is of finite $*$ -character (since every nonzero prime $*$ -ideal of D is a maximal $*$ -ideal). Now the equivalence is an immediate consequence of Theorem 3.13. \square

Next we characterize when a $*$ -VIFD with $*-\dim(D) = 1$ is atomic, which is a variant of [10, Theorem 4.3], because D is a $*$ -VIFD if D is a P*MD of finite $*$ -character and $*-\dim(D) = 1$ by Corollary 3.15. We first need a definition. Assume that D is a P*MD of finite $*$ -character and $*-\dim(D) = 1$. Let $M \in *-\text{Max}(D)$. Then D_M is a one-dimensional valuation domain, and hence there exists a valuation $v_M : K \setminus \{0\} \rightarrow \mathbb{R}$ (where \mathbb{R} is the additive group of real numbers) such that $D_M \setminus \{0\} = \{x \in K \setminus \{0\} \mid v_M(x) \geq 0\}$. For each $x \in K \setminus \{0\}$, let $\|x\| = \sum_{M \in *-\text{Max}(D)} v_M(x)$ (this is well-defined, since D is of finite $*$ -character). Observe

that for each nonzero $a, b \in D$, $a \mid_D b$ if and only if $v_N(a) \leq v_N(b)$ for each $N \in *-\text{Max}(D)$. Moreover, $\|xy\| = \|x\| + \|y\|$ for all $x, y \in K \setminus \{0\}$. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers.

We say that D satisfies the ACCP if D satisfies the ACC on principal ideals of D . Moreover, D is called a *BF-domain* if D is atomic and for each nonzero nonunit $b \in D$, there is some $n \in \mathbb{N}$ such that b is not a product of more than n atoms of D . The next result can be proved along the same lines as [10, Theorem 4.3], but for the sake of completeness, we include a proof.

Proposition 3.16. *Let D be an integral domain and let $*$ be a star operation of finite type on D such that D is a P^*MD , D is of finite $*$ -character and $*-\dim(D) = 1$. The following statements are equivalent.*

- (1) *For each nonzero nonunit $a \in D$, there exists some $r \in \mathbb{R}_{>0}$ such that $\|b\| \geq r$ for each nonzero nonunit $b \in D$ such that $b \mid_D a$.*
- (2) *D is a BF-domain.*
- (3) *D satisfies the ACCP.*
- (4) *D is atomic.*

Proof. (1) \Rightarrow (2) Let $a \in D$ be a nonzero nonunit and let $t = \inf\{\|b\| \mid b \in D \text{ is a nonzero nonunit such that } b \mid_D a\}$. Note that $t > 0$. Set $k = \lfloor \frac{\|a\|}{t} \rfloor$. It is sufficient to show that if $\ell \in \mathbb{N}$ is such that a is a product of ℓ nonunits of D , then $\ell \leq k$. Let $\ell \in \mathbb{N}$ be such that a is a product of ℓ nonunits of D . Then $a = \prod_{i=1}^{\ell} a_i$ for some nonunits $a_i \in D$. Observe that $t\ell \leq \sum_{i=1}^{\ell} \|a_i\| = \|a\|$, and thus $\ell \leq k$.

(2) \Rightarrow (3) \Rightarrow (4) This follows from [17, Propositions 1.1.4 and 1.3.2].

(4) \Rightarrow (1) Let $a \in D$ be a nonzero nonunit. Note that $\mathcal{P}(aD)$ is the set of maximal $*$ -ideals of D that contain aD (since $*-\dim(D) = 1$). Set $\Omega = \{\mathcal{P}(uD) \mid u \in D \text{ is an atom such that } \mathcal{P}(uD) \subseteq \mathcal{P}(aD)\}$. Then Ω is finite (since D is of finite $*$ -character) and $\Omega \neq \emptyset$ (since D is atomic). Consequently, there exists a finite nonempty set \mathcal{A} of atoms of D such that $\Omega = \{\mathcal{P}(uD) \mid u \in \mathcal{A}\}$. Note that $\{v_P(u) \mid u \in \mathcal{A}, P \in \mathcal{P}(uD)\}$ is finite and nonempty, since \mathcal{A} is finite and nonempty and D is of finite $*$ -character. Set $r = \min\{v_P(u) \mid u \in \mathcal{A}, P \in \mathcal{P}(uD)\}$. Then $r \in \mathbb{R}_{>0}$.

Assume that there exists a nonzero nonunit $b \in D$ with $\|b\| < r$ and $b \mid_D a$. Since D is atomic, there is an atom $v \in D$ with $v \mid_D b$. We have that $\mathcal{P}(vD) \in \Omega$ (since $v \mid_D a$), and hence $\mathcal{P}(vD) = \mathcal{P}(uD)$ for some $u \in \mathcal{A}$. Let $M \in *-\text{Max}(D)$. If $v \notin M$, then $u \notin M$, and thus $v_M(v) = 0 = v_M(u)$. If $v \in M$, then $u \in M$, and hence $v_M(v) \leq \|v\| \leq \|b\| \leq r \leq v_M(u)$. In any case, we have that $v_N(v) \leq v_N(u)$ for each $N \in *-\text{Max}(D)$. Consequently, $v \mid_D u$. This implies that $u = v\varepsilon$ for some $\varepsilon \in D^\times$ (since u and v are atoms of D). There exists some $P \in \mathcal{P}(vD) = \mathcal{P}(uD)$. Note that $v_P(u) = v_P(v) \leq \|v\| \leq \|b\| < r \leq v_P(u)$, a contradiction. \square

4. VIFDS AND t -VIFDS

We begin this section with easy examples of VIFDs, which are Dedekind domains. Dedekind domains are integral domains whose nonzero ideals can be written as a finite product of prime ideals. Then Dedekind domains are π -domains, and D is a Dedekind domain if and only if D is a π -domain of Krull dimension at most one.

Example 4.1. Let D be a Dedekind domain. Then D is a VIFD because each prime ideal is a valuation ideal. Moreover, note that a Dedekind domain is a PID if and only if its ideal class group is trivial. Note also that if D is a VFD, then $\text{Cl}_t(D) = \{0\}$ [9, Corollary 2.3(1)]. Hence, D is a VFD if and only if D is a PID.

As in [23], we say that D is a *ZPUI domain* if every nonzero proper ideal I of D can be written as $I = J \prod_{i=1}^n P_i$, where J is an invertible ideal of D , $n \in \mathbb{N}$ and the $(P_i)_{i=1}^n$ are prime ideals of D . It is known that D is a ZPUI domain if and only if D is a strongly discrete h-local Prüfer domain [23, Theorem 2.3]. As a w -operation analog, we say that D is a *w-ZPUI domain* if every nonzero proper w -ideal of D can be written as $I = (J \prod_{i=1}^n P_i)_w$ for some w -invertible ideal J of D , $n \in \mathbb{N}$ and $(P_i)_{i=1}^n$ pairwise w -comaximal prime w -ideals of D [8, Definition 3.1]. It is known that D is a w -ZPUI domain if and only if D is a strongly discrete independent ring of Krull type [8, Theorem 3.5].

Following [14], we say that D is a *unique representation domain (URD)* if each t -invertible t -ideal of D can be uniquely expressed as a finite t -product of pairwise t -comaximal t -ideals with prime radical. Then D is a URD if and only if each nonzero principal ideal of D can be written as a finite t -product of pairwise t -comaximal t -ideals with prime radical, if and only if D is t -treed and each nonzero principal of D has only finitely many minimal prime ideals [14, Corollary 2.12]. Hence, we have the following corollary, while a URD need not be a w -VIFD (see, for example, [14, Corollary 2.17] and Corollary 4.6).

Proposition 4.2. *Let D be an integral domain.*

- (1) *If D is a ZPUI domain, then D is a VIFD.*
- (2) *If D is a w -ZPUI domain, then D is a w -VIFD.*
- (3) *If D is a w -VIFD that is t -treed, then D is a URD.*

Proof. (1) Let D be a ZPUI domain. Then D is an h-local Prüfer domain, and thus the result follows by Theorem 3.13.

(2) A w -ZPUI domain is an independent ring of Krull type. Thus, if D is a w -ZPUI domain, then D is a w -VIFD by Theorem 3.13.

(3) This follows from Proposition 2.3(2), Lemma 3.2 and Theorem 3.13. \square

The next result shows that the localization of a w -VIFD is also a w -VIFD as in the case of VIFD that every localization of a VIFD is a VIFD by Lemma 3.8.

Proposition 4.3. *Let D be a w -VIFD and let S be a multiplicatively closed subset of D .*

- (1) *D_S is a w -VIFD.*
- (2) *If $\text{Cl}_t(D_S) = \{0\}$, then D_S is a VFD.*
- (3) *If $S = D \setminus \bigcup_{i=1}^n M_i$ for some $n \in \mathbb{N}$ and maximal t -ideals M_i of D , then D_S is a VFD.*

Proof. (1) Let A be a nonzero principal ideal of D_S . Then $A = aD_S$ for some nonzero nonunit a of D . Hence, by assumption, $aD = (\prod_{i=1}^n I_i)_w$ for some valuation ideals I_i of D , and since $\prod_{i=1}^n I_i$ is t -invertible, we have that

$$A = \left(\left(\prod_{i=1}^n I_i \right)_w \right)_S = \left(\left(\prod_{i=1}^n I_i \right)_S \right)_{w_S} = \left(\prod_{i=1}^n (I_i)_S \right)_{w_S} = \left(\prod_{i=1}^n (I_i)_S \right)_t.$$

The last equality holds, since $(\prod_{i=1}^n (I_i)_S)_{w_S}$ is a w_S -invertible w_S -ideal of D_S , and thus it is a t -ideal of D_S . Note that $(I_i)_S$ is a valuation ideal by Lemma 3.8, and hence D_S is a w -VIFD by Proposition 3.3.

(2) This follows from (1) and Corollary 3.4.

(3) If I is a t -invertible ideal of D , then $II^{-1} \not\subseteq \bigcup_{i=1}^n M_i$, and hence I_S is invertible. Thus, I_S is principal because $\text{Pic}(D_S) = \{0\}$. Thus, by the proof of (1)

above, every nonzero principal ideal of D_S can be written as a finite (t -)product of principal valuation ideals, which implies that D_S is a VFD. \square

Let $D[X]$ be the polynomial ring over D . A nonzero prime ideal Q of $D[X]$ is called an *upper to zero* in $D[X]$ if $Q \cap D = (0)$. Following [21], we say that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t -ideal of $D[X]$. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain [21, Proposition 3.2].

Lemma 4.4. *Let D be a UMT-domain, let $D[X]$ be the polynomial ring over D and let $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$. Then D is a t -treed domain if and only if $D[X]_{N_v}$ is treed.*

Proof. The result follows directly from the fact that D is a UMT-domain if and only if each prime ideal of $D[X]_{N_v}$ is extended from D [21, Theorem 3.1]. \square

It is easy to see that if D is a Krull domain, then D is a t -VIFD, $t\text{-Spec}(D)$ is treed, and D is an independent ring of Krull type. Now we characterize when D is a t -VIFD under the assumption that D is t -treed.

Theorem 4.5. *Let D be a t -treed domain, let $D[X]$ be the polynomial ring over D and let $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$. The following statements are equivalent.*

- (1) *Every nonzero t -ideal of D is a finite t -product of valuation ideals.*
- (2) *D is a t -VIFD.*
- (3) *D is an independent ring of Krull type.*
- (4) *$D[X]$ is an independent ring of Krull type.*
- (5) *$D[X]$ is a t -VIFD and D is a UMT-domain.*
- (6) *$D[X]_{N_v}$ is a VFD and D is a UMT-domain.*
- (7) *$D[X]_{N_v}$ is an h -local Prüfer domain.*
- (8) *Every nonzero prime ideal of D contains a t -invertible valuation ideal.*

Proof. (1) \Rightarrow (2) This is clear.

(2) \Rightarrow (8) \Rightarrow (3) \Rightarrow (1) This follows from Theorem 3.13.

(3) \Leftrightarrow (4) See [3, Corollary 2.9].

(4) \Rightarrow (5) Note that $D[X]$ is a PvMD, so D is a PvMD [22, Theorem 3.7] and $D[X]$ is t -treed. Thus, the result follows by the equivalence of (2) and (3) above.

(5) \Rightarrow (6) $\text{Cl}_t(D[X]_{N_v}) = \{0\}$ by [22, Theorem 2.14, Proposition 2.1 and Corollary 2.3]. Thus, $D[X]_{N_v}$ is a VFD by Propositions 3.3 and 4.3(2).

(6) \Rightarrow (7) Since D is a UMT-domain, $D[X]_{N_v}$ is treed by Lemma 4.4. Thus, $D[X]_{N_v}$ is an independent ring of Krull type [9, Theorem 3.4]. Moreover, since each maximal ideal of $D[X]_{N_v}$ is a t -ideal, the result follows.

(7) \Rightarrow (4) See [8, Lemma 2.2]. \square

We say that D is a *generalized Krull domain* (in the sense of [25]) if D is a PvMD such that $t\text{-dim}(D) \leq 1$ and D is of finite t -character. Recall that $t\text{-dim}(D) = 1$ if and only if D is not a field and each maximal t -ideal of D is a height-one prime ideal, so an integral domain of t -dimension one is t -treed. The class of integral domains of t -dimension at most one includes Krull domains, generalized Krull domains, and one-dimensional integral domains.

Corollary 4.6. *Let D be an integral domain with $t\text{-dim}(D) = 1$. The following statements are equivalent.*

- (1) *Each nonzero t -ideal of D is a finite t -product of valuation ideals.*

- (2) D is a t -VIFD.
- (3) D is a generalized Krull domain.
- (4) D is a ring of Krull type.

Proof. It is clear that if $t\text{-dim}(D) = 1$, then D is an independent ring of Krull type if and only if D is a generalized Krull domain, if and only if D is a ring of Krull type. Thus, the result follows directly from Theorem 4.5. \square

Let $D(X) = \{\frac{f}{g} \mid f, g \in D[X] \text{ and } c(g) = D\}$. Then $D(X)$, called the *Nagata ring* of D , is a ring such that $D[X] \subseteq D(X) \subseteq D[X]_{N_v}$. It is known that each nonzero prime ideal of a treed domain is a t -ideal. Hence, a treed domain is a t -treed domain whose nonzero maximal ideals are t -ideals. We next characterize when a treed domain is a VIFD.

Corollary 4.7. *Let D be a treed domain. The following statements are equivalent.*

- (1) D is an h -local Prüfer domain.
- (2) D is a VIFD.
- (3) Each nonzero ideal of D is a finite product of valuation ideals.
- (4) Every nonzero principal ideal of D can be written as a finite product of comaximal valuation ideals.
- (5) Every nonzero principal ideal of D can be written as a finite intersection of comaximal valuation ideals.
- (6) Every nonzero prime ideal of D contains an invertible valuation ideal.
- (7) $D(X)$ is a VFD and D is a UMT-domain.

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (6) These follow directly from Theorem 3.13.

(1) \Leftrightarrow (7) Since D is treed, every nonzero prime of D is a t -ideal. Moreover, if each maximal ideal of D is a t -ideal, then (i) D is an independent ring of Krull type if and only if D is an h -local Prüfer domain and (ii) $D(X) = D[X]_{N_v}$. Thus, the result is an immediate consequence of Theorem 4.5.

(4) \Leftrightarrow (5) This follows from the fact that if I and J are comaximal ideals of D , then $IJ = I \cap J$. \square

Next, we provide a variant of [9, Proposition 4.6]. Recall that D satisfies the Principal Ideal Theorem if every minimal prime ideal of a nonzero principal ideal of D is of height one. For example, Noetherian domains and Krull domains satisfy the Principal Ideal Theorem, while a VFD does (in general) not satisfy the Principal Ideal Theorem.

Theorem 4.8. *Let D be a t -VIFD. The following statements are equivalent.*

- (1) D is a generalized Krull domain.
- (2) $t\text{-dim}(D) \leq 1$.
- (3) D satisfies the Principal Ideal Theorem.
- (4) D is completely integrally closed.
- (5) $\bigcap_{n \in \mathbb{N}} (I^n)_t = (0)$ for each proper t -invertible t -ideal I of D .

Proof. (1) \Rightarrow (2), (4) It is obvious that $t\text{-dim}(D) \leq 1$. If $M \in t\text{-Max}(D)$, then D_M is a valuation domain of dimension ≤ 1 , and hence D_M is completely integrally closed. Therefore, $D = \bigcap_{M \in t\text{-Max}(D)} D_M$ is completely integrally closed.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (1) We have to show that D is a PvMD, D is of finite t -character and $t\text{-dim}(D) \leq 1$. It follows from Corollary 3.12 that D is of finite t -character. Let $M \in t\text{-Max}(D)$. Observe that D_M satisfies the Principal Ideal Theorem. Moreover,

D_M is a VFD by Proposition 3.9. Consequently, D_M is a PvMD by [9, Proposition 4.6]. It follows from [13, Theorem 4.1(2)] that D is a PvMD. Finally, let P be a nonzero prime t -ideal of D . Since D satisfies the Principal Ideal Theorem, P is the union of all height-one prime ideals of D that are contained in P . Since D is t -treed, there is precisely one height-one prime ideal of D that is contained in P , and hence P is a height-one prime ideal of D . Thus, $t\text{-dim}(D) \leq 1$.

(4) \Rightarrow (5) Let I be a proper t -invertible t -ideal of D and set $J = \bigcap_{n \in \mathbb{N}} (I^n)_t$. Observe that $I^{-1}J \subseteq J$, and thus $I^{-1} \subseteq (J : J)$. Assume that $J \neq (0)$. Then $(J : J) = D$ (since D is completely integrally closed), and hence $I = I_v = I^{-1} = D$, a contradiction.

(5) \Rightarrow (3) Let P be a minimal prime ideal of a nonzero principal ideal of D . Since D is a t -VIFD, there exists a t -invertible valuation t -ideal I of D such that P is a minimal prime ideal of I . Let Q be a prime ideal of D such that $Q \subsetneq P$. It suffices to show that $Q = (0)$. Assume that $Q \neq (0)$. There exists some $x \in Q \setminus \{0\}$. We infer that $xD \subseteq J \subseteq Q$ for some t -invertible valuation t -ideal J of D . Observe that $\sqrt{J} \subseteq Q \subsetneq P = \sqrt{I}$. Therefore, $x \in xD \subseteq J \subseteq \bigcap_{n \in \mathbb{N}} (I^n)_t = (0)$ by Corollary 2.10(4), a contradiction. \square

5. ALMOST VALUATION FACTORIZATION DOMAINS

We will say that D is an *almost valuation factorization domain* (AVFD) if for each nonzero nonunit $a \in D$, there is an $n \in \mathbb{N}$ such that a^n can be written as a finite product of valuation elements. Clearly, a VFD is an AVFD and a t -VIFD with torsion t -class group is an AVFD (see Proposition 5.5).

Lemma 5.1. *Let D be an integral domain and let $b \in D$ be a nonzero nonunit that can be written as a finite product of valuation elements of D .*

- (1) $\min\{k \in \mathbb{N} \mid b \text{ is a product of } k \text{ valuation elements of } D\} = |\mathcal{P}(bD)|$.
- (2) *There are valuation elements $(y_P)_{P \in \mathcal{P}(bD)}$ of D such that $b = \prod_{P \in \mathcal{P}(bD)} y_P$ and $\sqrt{y_Q D} = Q$ for each $Q \in \mathcal{P}(bD)$.*

Proof. This can be proved along the same lines as [9, Lemma 1.12 and Proposition 1.13]. \square

For the next result we mimic the proof of [9, Proposition 2.1] that a VFD is a Schreier domain.

Proposition 5.2. *Let D be an integral domain and let Ω be the set of all finite products of units and valuation elements of D . Then for each $x, y, z \in \Omega$ with $x \mid_D yz$, there are some $a, b \in \Omega$ such that $x = ab$, $a \mid_D y$, $b \mid_D z$ and $\frac{y}{a}, \frac{z}{b} \in \Omega$.*

Proof. If $r, s \in \Omega$, then we write $r \mid_\Omega s$ if $r \mid_D s$ and $\frac{s}{r} \in \Omega$.

CLAIM: For each valuation element $x \in D$ and all $y, z \in \Omega$ such that $x \mid_D yz$, there are some $a, b \in \Omega$ such that $x = ab$, $a \mid_\Omega y$ and $b \mid_\Omega z$.

By Lemma 5.1(1), it is sufficient to show by induction that for all $k \in \mathbb{N}$, for each valuation element $x \in D$ and for all $y, z \in \Omega$ such that $x \mid_D yz$ and $|\mathcal{P}(yD)| + |\mathcal{P}(zD)| = k$, there are some $a, b \in \Omega$ such that $x = ab$, $a \mid_\Omega y$ and $b \mid_\Omega z$. Let $k \in \mathbb{N}$, let $x \in D$ be a valuation element and let $y, z \in \Omega$ be such that $x \mid_D yz$ and $|\mathcal{P}(yD)| + |\mathcal{P}(zD)| = k$. Since $yz \in xD \subseteq \sqrt{xD} \in \text{Spec}(D)$ by [9, Proposition 1.7(1)], we have that $y \in \sqrt{xD}$ or $z \in \sqrt{xD}$. Without restriction let $y \in \sqrt{xD}$. By Lemma 5.1(2) there are valuation elements $(y_P)_{P \in \mathcal{P}(yD)}$ of D such that $y = \prod_{P \in \mathcal{P}(yD)} y_P$ and $\sqrt{y_Q D} = Q$ for each $Q \in \mathcal{P}(yD)$. Consequently, there

is some $Q \in \mathcal{P}(yD)$ such that $y_Q \in \sqrt{xD}$, and hence $\sqrt{y_Q D} \subseteq \sqrt{xD}$. We infer by [9, Corollary 1.2(1)] that $y_Q D \subseteq xD$ or $xD \subseteq y_Q D$. Set $y' = \prod_{P \in \mathcal{P}(yD) \setminus \{Q\}} y_P$.

CASE 1: $y_Q D \subseteq xD$. Then $y_Q = xu$ for some $u \in D$. It follows from [9, Proposition 1.1(3)] that u is a unit or a valuation element of D . Set $a = x$ and $b = 1$. Obviously, $a, b \in \Omega$, $x = ab$ and $b \mid_{\Omega} z$. Moreover, $a \mid_D y_Q \mid_D y$ and $\frac{y}{a} = uy' \in \Omega$, and thus $a \mid_{\Omega} y$.

CASE 2: $xD \subsetneq y_Q D$. Then $x = y_Q w$ for some nonunit $w \in D$. We infer by [9, Proposition 1.1(3)] that w is a valuation element of D . Observe that $y_Q w = x \mid_D yz = y_Q y' z$, and thus $w \mid_D y' z$. Furthermore, $\mathcal{P}(y' D) = \mathcal{P}(yD) \setminus \{Q\}$, and hence $|\mathcal{P}(y' D)| + |\mathcal{P}(zD)| < k$. We infer by the induction hypothesis that there are some $a', b \in \Omega$ such that $w = a'b$, $a' \mid_{\Omega} y'$ and $b \mid_{\Omega} z$. Set $a = y_Q a'$. Then $a \in \Omega$, $x = y_Q w = y_Q a' b = ab$ and $a \mid_{\Omega} y$. This proves the claim.

Note that Ω consists precisely of the units and the finite nonempty products of valuation elements of D . Therefore, it is sufficient to show by induction that for all $n \in \mathbb{N}$, $x \in D$ and $y, z \in \Omega$ such that x is a product of n valuation elements of D and $x \mid_D yz$, there are some $a, b \in \Omega$ such that $x = ab$, $a \mid_{\Omega} y$ and $b \mid_{\Omega} z$. If $n = 1$, then the statement follows from the claim. Now let $n \in \mathbb{N}$, $x \in D$ and $y, z \in \Omega$ be such that x is a product of $n+1$ valuation elements of D and $x \mid_D yz$. Then $x = \prod_{i=1}^{n+1} v_i$ for some valuation elements v_i of D . Set $x' = \prod_{i=1}^n v_i$. Then $x' \mid_D yz$. We infer by the induction hypothesis that there are some $a', b' \in \Omega$ such that $x' = a'b'$, $a' \mid_{\Omega} y$ and $b' \mid_{\Omega} z$. Since $a'b'v_{n+1} = x \mid_D yz$, we infer that $v_{n+1} \mid_D \frac{y}{a'} \frac{z}{b'}$. Since v_{n+1} is a valuation element of D and $\frac{y}{a'}, \frac{z}{b'} \in \Omega$, it follows by the claim that there are some $a'', b'' \in \Omega$ such that $v_{n+1} = a''b''$, $a'' \mid_{\Omega} \frac{y}{a'}$ and $b'' \mid_{\Omega} \frac{z}{b'}$. Set $a = a'a''$ and $b = b'b''$. Then $a, b \in \Omega$, $x = x'v_{n+1} = a'b'a''b'' = ab$, $a \mid_{\Omega} y$ and $b \mid_{\Omega} z$. \square

We say that D is an *almost Schreier domain* if for all nonzero $x, y, z \in D$ with $x \mid_D yz$ there are $n \in \mathbb{N}$ and $a, b \in D$ such that $x^n = ab$, $a \mid_D y^n$ and $b \mid_D z^n$ [11]. It is known that if D is an almost Schreier domain, then $\text{Cl}_t(D)$ is a torsion group [11, Theorem 3.1] and if D is an integrally closed almost Schreier domain, then $D[X]$ is an almost Schreier domain [11, Theorem 4.4].

Theorem 5.3. *Let D be an integral domain. Then D is an AVFD if and only if D is an almost Schreier domain and every nonzero prime t -ideal of D contains a valuation element of D .*

Proof. Let Ω be the set of all finite products of units and valuation elements of D .

(\Rightarrow) Let D be an AVFD. Let $x, y, z \in D$ be nonzero such that $x \mid_D yz$. Since D is an AVFD, there are $r, s, t \in \mathbb{N}$ such that $x^r, y^s, z^t \in \Omega$. Set $n = rst$. Then $n \in \mathbb{N}$, $x^n, y^n, z^n \in \Omega$ and $x^n \mid_D y^n z^n$. We infer by Proposition 5.2 that there are $a, b \in D$ such that $x^n = ab$, $a \mid_D y^n$ and $b \mid_D z^n$. Therefore, D is an almost Schreier domain. Now let P be a nonzero prime t -ideal of D . Choose a nonzero $a \in P$. Then there exists $n \in \mathbb{N}$ such that a^n is a finite product of valuation elements of D . Note that $a^n \in P$. Thus, P contains a valuation element of D .

(\Leftarrow) Let D be an almost Schreier domain for which every nonzero prime t -ideal contains a valuation element of D . Let $\Sigma = \{a \in D \mid a^n \in \Omega \text{ for some } n \in \mathbb{N}\}$. Clearly, Ω and Σ are multiplicatively closed subsets of D . We show that Σ is a divisor-closed subset of D . Let $a \in \Sigma$ and $b \in D$ be such that $b \mid_D a$. There are some $n, m \in \mathbb{N}$ and $v_i \in D$ such that $a^n = \prod_{i=1}^m v_i$ and v_i is a unit or a valuation element of D for each $i \in [1, m]$. We have that $b^n \mid_D \prod_{i=1}^m v_i$. Since D is an almost Schreier domain, it follows by induction that there are some $k \in \mathbb{N}$ and $w_i \in D$

such that $b^{nk} = \prod_{i=1}^m w_i$ and $w_j \mid_D v_j^k$ for each $j \in [1, m]$. Let $j \in [1, m]$. If v_j is a unit of D , then w_j is a unit of D . Now let v_j be a valuation element of D . Since $\sqrt{v_j D} \subseteq \sqrt{w_j D}$, we have that w_j is a unit or a valuation element of D by [9, Proposition 1.1(3)]. Consequently, $b^{nk} \in \Omega$, and thus $b \in \Sigma$.

It remains to show that $D \setminus \{0\} \subseteq \Sigma$. Assume that there is some $z \in D \setminus (\Sigma \cup \{0\})$. Since Σ is a divisor-closed subset of D (as shown before), we have that $zD \cap \Sigma = \emptyset$. Therefore, there exists a prime t -ideal P of D such that $zD \subseteq P$ and $P \cap \Sigma = \emptyset$. Since $z \in P$, we have that P is nonzero, and thus P contains a valuation element of D . This implies that $\emptyset \neq P \cap \Omega \subseteq P \cap \Sigma = \emptyset$, a contradiction. \square

Corollary 5.4. *Let D be an AVFD. Then D is an integrally closed weakly Matlis domain and $\text{Cl}_t(D)$ is a torsion group.*

Proof. It is an immediate consequence of Proposition 3.11 and Theorem 5.3 that D is an integrally closed weakly Matlis almost Schreier domain. We infer by [11, Theorem 3.1] that $\text{Cl}_t(D)$ is a torsion group. \square

Recall that (i) VFDs are AVFDs by definition and (ii) D is a VFD if and only if D is a VIFD with $\text{Pic}(D) = \{0\}$, if and only if D is a w -VIFD with $\text{Cl}_t(D) = \{0\}$ by Corollary 3.5. The next result also shows that a VIFD D for which $\text{Pic}(D)$ is a torsion group is an AVFD.

Proposition 5.5. *Let D be an integral domain and let $*$ be a star operation of finite type on D such that D is a $*$ -VIFD and $\text{Cl}_*(D)$ is a torsion group. Then D is an AVFD.*

Proof. Let $a \in D$ be a nonzero nonunit. Then $aD = (\prod_{i=1}^n I_i)_*$ for some $n \in \mathbb{N}$ and proper valuation $*$ -ideals I_i of D by Proposition 3.3. Clearly, each I_i is $*$ -invertible, so by assumption that $\text{Cl}_*(D)$ is a torsion group, there is an $m \in \mathbb{N}$ such that for each $i \in [1, n]$, there is some $a_i \in D$ such that $(I_i^m)_* = a_i D$. Then each a_i is a valuation element by Proposition 2.9. Thus, a^m can be written as a finite product of valuation elements. \square

We say that D is an *almost GCD domain* (AGCD domain) if for each $a, b \in D$, there is an $n \in \mathbb{N}$ such that $a^n D \cap b^n D$ is principal [29]. It is known that an integrally closed domain D is an AGCD domain if and only if D is a PvMD such that $\text{Cl}_t(D)$ is a torsion group [29, Corollary 3.8 and Theorem 3.9] and an AGCD domain is an almost Schreier domain [11, Proposition 2.2].

Corollary 5.6. *Let D be a t -treed domain. The following statements are equivalent.*

- (1) D is an AVFD.
- (2) D is an integrally closed weakly Matlis AGCD domain.
- (3) D is an independent ring of Krull type such that $\text{Cl}_t(D)$ is a torsion group.
- (4) $\text{Cl}_t(D)$ is a torsion group and each nonzero prime ideal of D contains a valuation element.
- (5) D is a t -VIFD such that $\text{Cl}_t(D)$ is a torsion group.
- (6) $D[X]$ is an AVFD.

Proof. (1) \Rightarrow (4) It is clear that each nonzero prime ideal of D contains a valuation element. Moreover, $\text{Cl}_t(D)$ is a torsion group by Corollary 5.4.

(2) \Leftrightarrow (3) See, for example, [29, Corollary 3.8 and Theorem 3.9].

(3) \Rightarrow (1) Note that D is a t -VIFD by Theorem 4.5. Thus, by Proposition 5.5, D is an AVFD.

(3) \Rightarrow (6) Observe that D is a PvMD with $\text{Cl}_t(D)$ torsion. Hence, $\text{Cl}_t(D[X])$ is a torsion group [29, Theorem 5.6]. Moreover, $D[X]$ is a t -VIFD by Theorem 4.5. Thus, $D[X]$ is an AVFD by Proposition 5.5.

(4) \Rightarrow (3), (5) This follows from Theorem 4.5.

(5) \Rightarrow (1) This is an immediate consequence of Proposition 5.5.

(6) \Rightarrow (1) Let $D[X]$ be an AVFD. Let a be a nonzero nonunit of D . Then a is a nonzero nonunit of $D[X]$, and hence there are some $k, n \in \mathbb{N}$ and valuation elements v_i of $D[X]$ such that $a^k = \prod_{i=1}^n v_i$. Observe that $v_i \in D$ for each $i \in [1, n]$, and hence v_i is a valuation element of D for each $i \in [1, n]$ by [9, Lemma 2.5]. Therefore, D is an AVFD. \square

Corollary 5.7. *Let D be an integral domain and let $D[X]$ be the polynomial ring over D . Then $D[X]$ is an AVFD if and only if D is an AVFD and every upper to zero in $D[X]$ contains a valuation element of $D[X]$.*

Proof. (\Rightarrow) Let $D[X]$ be an AVFD. Then D is an AVFD by Corollary 5.6 and every upper to zero in $D[X]$ contains a valuation element of $D[X]$ by Theorem 5.3.

(\Leftarrow) Now let D be an AVFD such that every upper to zero in $D[X]$ contains a valuation element of $D[X]$. It follows by Theorem 5.3 and Corollary 5.4 that D is an integrally closed almost Schreier domain and each nonzero prime (t -)ideal of D contains a valuation element of D . We infer by [11, Theorem 4.4] that $D[X]$ is an almost Schreier domain. Now let Q be a nonzero prime t -ideal of $D[X]$. If $Q \cap D = (0)$, then Q contains a valuation element of $D[X]$ by assumption. Now let $Q \cap D \neq (0)$ and set $P = Q \cap D$. Then P is a nonzero prime ideal of D , and thus P contains a valuation element of D . Consequently, Q contains a valuation element of D . It follows from [9, Lemma 2.5] that Q contains a valuation element of $D[X]$. Therefore, $D[X]$ is an AVFD by Theorem 5.3. \square

Example 5.8. Let L be an algebraic number field and let D be an order in L . Then D is an almost Schreier domain [11, Remark 6.4] and if D is the principal order in L (i.e., D is integrally closed), then D is an AVFD by Corollary 5.6.

It is known that an atomic VFD is a UFD [9, Corollary 2.4], and hence it satisfies the Principal Ideal Theorem. The next result shows that this is true for AVFDs.

Proposition 5.9. *Let D be an atomic domain.*

- (1) *If D is an AVFD, then D satisfies the Principal Ideal Theorem.*
- (2) *If D is a t -VIFD such that $\text{Cl}_t(D)$ is a torsion group, then D is a generalized Krull domain.*

Proof. (1) Let D be an AVFD, let $x \in D$ be nonzero and let $P \in \mathcal{P}(xD)$. There exists an $n \in \mathbb{N}$ such that x^n is a finite product of valuation elements of D . Note that $P \in \mathcal{P}(x^n D)$, and hence $P \in \mathcal{P}(vD)$ for some valuation element $v \in D$. This implies that $P = \sqrt{vD}$ by Proposition 2.3(2). Assume that P is not a height-one prime ideal of D . Then there is a nonzero prime ideal Q of D such that $Q \subsetneq P$. Note that Q contains a valuation element $a \in D$. Clearly, there is an atom $u \in Q$ such that $aD \subseteq uD$. We infer by [9, Proposition 1.1(2)] that u is a valuation element of D . Moreover, $\sqrt{uD} \subseteq Q \subsetneq P = \sqrt{vD}$, and hence $uD \subsetneq vD \subsetneq D$ [9, Corollary 1.2(1)]. This contradicts the fact that u is an atom of D .

(2) Let D be a t -VIFD such that $\text{Cl}_t(D)$ is a torsion group. Then D is an AVFD by Proposition 5.5. Consequently, D satisfies the Principal Ideal Theorem by (1), and hence D is a generalized Krull domain by Theorem 4.8. \square

The *exponent* of a group G , denoted $\exp(G)$, is defined by $\inf\{k \in \mathbb{N} \mid x^k = 1 \text{ for all } x \in G\}$. We proceed with providing a partial generalization of [9, Corollary 2.4] that characterizes when a VFD is atomic. Let $\mathbb{N}_{\geq m} = \{x \in \mathbb{N} \mid x \geq m\}$ for each $m \in \mathbb{N}$.

Theorem 5.10. *Let D be an integral domain such that $\exp(\text{Cl}_t(D))$ is finite. Then D is a Krull domain if and only if D is an atomic t -VIFD, and in this case, D is an AVFD.*

Proof. Clearly, every Krull domain is an atomic t -VIFD. Conversely, suppose that D is an atomic t -VIFD. It follows from Proposition 5.9(2) that D is a generalized Krull domain. Therefore, D satisfies the ACCP by Proposition 3.16.

Next we show that D satisfies the ACC on t -invertible t -ideals of D . Let $(I_i)_{i \in \mathbb{N}}$ be an ascending sequence of t -invertible t -ideals of D . Let n be the exponent of $\text{Cl}_t(D)$. Observe that $(I_i^n)_t$ is a principal ideal of D for each $i \in \mathbb{N}$. Since D satisfies the ACCP, there exists some $m \in \mathbb{N}$ such that $(I_m^n)_t = (I_k^n)_t$ for all $k \in \mathbb{N}_{\geq m}$. Let $k \in \mathbb{N}_{\geq m}$ and let $M \in t\text{-Max}(D)$. There are nonzero $a, b \in D$ such that $(I_m)_M = aD_M$ and $(I_k)_M = bD_M$. It follows that $a^n D_M = ((I_m^n)_t)_M = ((I_k^n)_t)_M = b^n D_M$, and thus $(\frac{a}{b})^n$ is a unit of D_M . Note that D_M is integrally closed by Corollary 3.12. Consequently, $\frac{a}{b} \in D_M$, and hence $\frac{a}{b}$ is a unit of D_M . This implies that $(I_m)_M = aD_M = bD_M = (I_k)_M$. Therefore, $I_m = I_k$.

It remains to show that every nonzero t -ideal of D is t -invertible. Let I be a nonzero t -ideal of D . Then there exists a maximal element J of the set of all t -invertible t -ideals of D that are contained in I . Assume that $J \subsetneq I$. Then there exists some $a \in I \setminus J$. Since D is a PvMD, we infer that $(J + aD)_t$ is a t -invertible t -ideal of D . Furthermore, $J \subsetneq (J + aD)_t \subseteq I$, a contradiction. Therefore, $I = J$ is a t -invertible t -ideal of D .

It is obvious that if $\exp(\text{Cl}_t(D))$ is finite, then $\text{Cl}_t(D)$ is a torsion group. Hence, in this case, D is an AVFD by Proposition 5.5. \square

We end this paper with an example which shows that the assumption that $\exp(\text{Cl}_t(D))$ is finite in Theorem 5.10 is crucial.

Example 5.11. Let D be a one-dimensional atomic Prüfer domain of finite character such that D is not a Dedekind domain. (For an example of such a domain see [19].) Then D is an atomic t -VIFD that is not a Krull domain.

REFERENCES

- [1] Z. Ahmad, T. Dumitrescu, *Star almost Schreier domains*, Comm. Algebra **42** (2014), 4898–4910.
- [2] D.D. Anderson, *π -domains, overrings, and divisorial ideals*, Glasgow Math. J. **19** (1978), 199–203.
- [3] D.D. Anderson, G.W. Chang and M. Zafrullah, *Integral domains of finite t -character*, J. Algebra **396** (2013), 169–183.
- [4] D.D. Anderson, J.L. Mott and M. Zafrullah, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. B (7), **6** (1992), 613–630.
- [5] D.D. Anderson, M. Zafrullah, *On $*$ -semi-homogeneous integral domains*, Advances in commutative algebra, 7–31, Trends Math. Birkhäuser/Springer, Singapore, 2019.
- [6] D.F. Anderson, G.W. Chang, J. Park, *Generalized weakly factorial domains*, Houst. J. Math. **29** (2003) 1–13.
- [7] G.W. Chang, *Unique factorization property of non-unique factorization domains*, J. Algebra Appl. **20** (2021), 2150038 18 pp.
- [8] G.W. Chang and H.S. Choi, *Ideal factorization in strongly discrete independent rings of Krull type*, J. Algebra Appl. **22** (2023), 2350045 21 pp.

- [9] G.W. Chang and A. Reinhart, *Unique factorization property of non-unique factorization domains II*, J. Pure App. Algebra **224** (2020), 106430 18 pp.
- [10] J. Coykendall and R.E. Hasenauer, *Factorization in Prüfer domains*, Glasg. Math. J. **60** (2018), 401–409.
- [11] T. Dumitrescu and W. Khalid, *Almost-Schreier domains*, Comm. Algebra **38** (2010), 2981–2991.
- [12] T. Dumitrescu and M. Zafrullah, *t-Schreier domains*, Comm. Algebra **39** (2011), 808–818.
- [13] S. El Baghdadi, M. Fontana, and M. Zafrullah, *Intersections of quotient rings and Prüfer v-multiplication domains*, J. Algebra Appl. **15** (2016), 1650149 18 pp.
- [14] S. El Baghdadi, S. Gabelli, and M. Zafrullah, *Unique representation domains. II*, J. Pure Appl. Algebra **212** (2008), 376–393.
- [15] J. Elliott, *Rings, modules, and closure operations*, Springer Monogr. Math., Springer, Cham, 2019.
- [16] M. Fontana, P. Jara, and E. Santos, *Prüfer *-multiplication domains and semistar operations*, J. Algebra Appl. **2** (2003), 21–50.
- [17] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure Appl. Math. 278, Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [18] R. Gilmer and J. Ohm, *Primary ideals and valuation ideals*, Trans. Amer. Math. Soc. **117** (1965), 237–250.
- [19] A. Grams, *Atomic rings and the ascending chain condition for principal ideals*, Proc. Cambridge Philos. Soc. **75** (1974), 321–329.
- [20] F. Halter-Koch, *Ideal Systems. An Introduction to Multiplicative Ideal Theory*, Marcel Dekker, New York, 1998.
- [21] E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra **17** (1989), 1955–1969.
- [22] B.G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$* , J. Algebra **123** (1989), 151–170.
- [23] B. Olberding, *Factorization into prime and invertible ideals*, J. London Math. Soc. **62** (2000), 336–344.
- [24] B. Olberding and A. Reinhart, *Radical factorization in finitary ideal systems*, Comm. Algebra **48** (2020), 228–253.
- [25] E.M. Pirtle, *The ideal transform in a generalized Krull domain*, Rocky Mountain J. Math. **2** (1972), 45–48.
- [26] S. Xing, D.D. Anderson, M. Zafrullah, **-almost super-homogeneous ideals in *-h-local domains*, J. Algebra Appl. **21** (2022), 2250136 32 pp.
- [27] M. Zafrullah, *Semirigid GCD domains*, Manuscripta Math. **17** (1975), 55–66.
- [28] M. Zafrullah, *Rigid elements in GCD domains*, J. Natur. Sci. Math. **17** (1977), 7–14.
- [29] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), 29–62.
- [30] M. Zafrullah, *On *-potent domains and *-homogeneous ideals*, Rings, monoids and module theory, 89–109, Springer Proc. Math. Stat., 382, Springer, Singapore, 2021.
- [31] M. Zafrullah, *Semirigid GCD domains II*, J. Algebra Appl. **21** (2022), 21 pp.
- [32] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, New Jersey, 1960.

DEPARTMENT OF MATHEMATICS EDUCATION, INCHEON NATIONAL UNIVERSITY, INCHEON 22012, KOREA

Email address: whan@inu.ac.kr

INSTITUT FÜR MATHEMATIK UND WISSENSCHAFTLICHES RECHNEN, KARL-FRANZENS-UNIVERSITÄT GRAZ, NAWI GRAZ, HEINRICHSTRASSE 36, 8010 GRAZ, AUSTRIA

Email address: andreas.reinhart@uni-graz.at