UNIQUE FACTORIZATION PROPERTY OF NON-UNIQUE FACTORIZATION DOMAINS II

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ABSTRACT. Let D be an integral domain. A nonzero nonunit a of D is called a *valuation element* if there is a valuation overring V of D such that $aV \cap D = aD$. We say that D is a *valuation factorization domain* (VFD) if each nonzero nonunit of D can be written as a finite product of valuation elements. In this paper, we study some ring-theoretic properties of VFDs. Among other things, we show that (i) a VFD D is Schreier, and hence $\mathrm{Cl}_t(D) = \{0\}$, (ii) if D is a PvMD, then D is a VFD if and only if D is a weakly Matlis GCD-domain, if and only if $D[X]$, the polynomial ring over D, is a VFD and (iii) a VFD D is a weakly factorial GCD-domain if and only if D is archimedean. We also study a unique factorization property of VFDs.

0. Introduction

Let D be an integral domain with quotient field K. An overring of D means a subring of K containing D. A nonzero nonunit $x \in D$ is said to be *homogeneous* if x is contained in a unique maximal t-ideal of D. As in [5], we say that D is a *homogeneous factorization domain* (HoFD) if each nonzero nonunit of D can be written as a finite product of homogeneous elements. Let D be an HoFD, let $x \in D$ be a nonzero nonunit and let $x = \prod_{i=1}^{n} a_i = \prod_{j=1}^{m} b_j$ be two finite products of t-comaximal homogeneous elements of D. Then $n = m$ and $a_iD = b_iD$ for $i \in [1, n]$ by reordering if necessary [5, Remark 2.1]. Hence, an HoFD has a unique factorization property even though it is not a unique factorization domain (UFD). In [5], Chang studied several properties of HoFDs and constructed examples of HoFDs. In this paper, we continue to study the unique factorization property of non-unique factorization domains.

As in [15, Appendix 3], we say that an ideal I of D is a *valuation ideal* if there is a valuation overring V of D such that $IV \cap D = I$. Clearly, each ideal of a valuation domain is a valuation ideal. Conversely, in [8, Corollary 2.4], Gilmer and Ohm showed that if every principal ideal of D is a valuation ideal, then D is a valuation domain. In this paper, we will say that a nonzero nonunit $a \in D$ is a *valuation element* if aD is a valuation ideal, i.e., there is a valuation overring V of D such that $aV \cap D = aD$. It is well known that a prime ideal of D is a valuation ideal [15, page 341]. Hence, every prime element is a valuation element. Thus, every nonzero nonunit of a UFD can be written as a finite product of valuation elements. We will say that D is a *valuation factorization domain* (VFD) if each

Date: May 20, 2020.
2010 *Mathematics Subject Classification.* 13A15, 13F05, 13G05.
Key words and phrases. valuation element, VFD, PvMD, HoFD, weakly Matlis GCD-domain.

nonzero nonunit of D can be written as a finite product of valuation elements. Clearly, valuation domains and UFDs are VFDs. The purpose of this paper is to study some factorization properties of VFDs.

0.1. **Definitions related to the t-operation.** We first review some definitions related to the t-operation which are needed for fully understanding this paper. Let D be an integral domain with quotient field K. A D-submodule A of K is called a fractional ideal of D if $dA \subseteq D$ for some nonzero $d \in D$. Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of D. For $A \in F(D)$, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$; then $A^{-1} \in F(D)$. Hence, if we set

- $A_v = (A^{-1})^{-1}$ and
- $A_t = \bigcup\{I_v \mid I \subseteq A \text{ and } I \in f(D)\}$,

then the v- and t-operations are well defined. It is easy to see that $I \subseteq I_t \subseteq I_v$ for all $I \in F(D)$ and $I_t = I_v$ if I is finitely generated. Let $* = v$ or t. An $I \in F(D)$ is called a $*$-*ideal* if $I_* = I$. A $*$-ideal is a *maximal* $*$-*ideal* if it is maximal among the proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. It may happen that v-Max$(D) = \emptyset$ even though D is not a field as in the case of a rank-one nondiscrete valuation domain D. However, t-Max$(D) \neq \emptyset$ if and only if D is not a field; each maximal t-ideal of D is a prime ideal; each proper t-ideal of D is contained in a maximal t-ideal; each prime ideal of D minimal over a t-ideal is a t-ideal, whence each height-one prime ideal is a t-ideal; and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. An integral domain D is said to be of *finite* (t-)*character* if each nonzero nonunit of D is contained in only finitely many maximal (t-)ideals. Let Spec(D) (resp., t-Spec(D)) be the set of prime ideals (resp., prime t-ideals) of D; so $t\text{-Max}(D) \subseteq t\text{-Spec}(D) \subseteq \text{Spec}(D) \setminus \{(0)\}$. The t-dimension of D is defined by $t\text{-dim}(D) = \sup\{n \mid P_1 \subsetneq \cdots \subsetneq P_n$ for some prime t-ideals P_i of $D\}$. Hence, $t\text{-dim}(D) = 1$ if and only if D is not a field and $t\text{-Max}(D) = t\text{-Spec}(D)$, and if $\dim(D) = 1$, then $t\text{-Max}(D) = t\text{-Spec}(D) = \text{Spec}(D) \setminus \{(0)\}$.

An $I \in F(D)$ is said to be *invertible* (resp., t-*invertible*) if $II^{-1} = D$ (resp., $(II^{-1})_t = D$). It is easy to see that invertible ideals are t-invertible t-ideals. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is known that D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D, if and only if $D[X]$, the polynomial ring over D, is a PvMD [11, Theorems 3.2 and 3.7]; and a Prüfer domain is a PvMD whose maximal ideals are t-ideals. Let $\text{T}(D)$ be the set of t-invertible fractional t-ideals. Then $\text{T}(D)$ is an abelian group under $I * J = (IJ)_t$. Let Inv(D) (resp., Prin(D)) be the subgroup of $\text{T}(D)$ of invertible (resp., nonzero principal) fractional ideals of D. The factor group $\text{Cl}_t(D) = \text{T}(D)/\text{Prin}(D)$, called the *$t$-class group* of D, is an abelian group and $\text{Pic}(D) = \text{Inv}(D)/\text{Prin}(D)$, called the Picard group of D, is a subgroup of $\text{Cl}_t(D)$. A GCD-domain is just a PvMD with trivial t-class group.

0.2. **Results.** This paper consists of five sections including the introduction. Let D be an integral domain. In Section 1, we study basic properties of valuation elements and VFDs. Among other things, we show that (i) a VFD is integrally closed, (ii) if D is not a field, then D is a VFD with $t\text{-dim}(D) = 1$ if and only if

D is a weakly factorial GCD-domain and (iii) every nonzero nonunit of a VFD can be written as a finite product of incomparable valuation elements. In Section 2, we show that (i) a VFD is a Schreier domain, and hence it has a trivial t-class group and (ii) a UMT-domain D is a VFD if and only if $D[X]$ is a VFD. In Section 3, we study VFDs that are HoFDs. We show that if t-Spec(D) is treed, then (i) every valuation element is a homogeneous element and (ii) D is a VFD if and only if D is a weakly Matlis GCD-domain, if and only if $D[X]$ is a VFD. Finally, in Section 4, we introduce the notion of UVFDs and show that the UVFDs are precisely the weakly Matlis GCD-domains. We also characterize when a VFD is a UVFD.

1. Valuation elements and VFDs

Let D be an integral domain with quotient field K. Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0$ be the set of non-negative integers. For elements $a, b \in D$, we say that a divides b (denoted by $a \mid_D b$) if $b = ac$ for some $c \in D$. In this section, we study basic properties of valuation elements and VFDs. Our first result is very simple, but it plays a key role in the study of VFDs.

Proposition 1.1. *Let D be an integral domain, let D' be an overring of D and let $a, b \in D$ be such that $a \neq 0$ and $aD' \cap D = aD$.*

1. *If $bD' \cap D = bD$, then $abD' \cap D = abD$.*
2. *If $b \mid_D a$, then $bD' \cap D = bD$.*
3. *If $\sqrt{aD} \subseteq \sqrt{bD}$, then $bD' \cap D = bD$. In particular, if a is a valuation element of D and $\sqrt{aD} \subseteq \sqrt{bD} \subsetneq D$, then b is a valuation element of D.*

Proof. (1) Let $bD' \cap D = bD$. Observe that $abD = a(bD) = a(bD' \cap D) = abD' \cap aD = abD' \cap aD' \cap D = abD' \cap D$.

(2) Let $b \mid_D a$. There is some $c \in D$ such that $a = bc$. We infer that $bD' \cap D = c^{-1}aD' \cap D = c^{-1}(aD' \cap cD) = c^{-1}(aD' \cap D \cap cD) = c^{-1}(aD \cap cD) = bD \cap D = bD$.

(3) Let $\sqrt{aD} \subseteq \sqrt{bD}$. Then there is some $k \in \mathbb{N}$ such that $a^k \in bD$. By (1), $a^k D' \cap D = a^k D$, and since $b \mid_D a^k$, we infer by (2) that $bD' \cap D = bD$. $\square$

Corollary 1.2. *Let D be an integral domain and let $a \in D$ be a valuation element.*

1. *If $b \in D$ is such that $\sqrt{aD} \subseteq \sqrt{bD}$, then aD and bD are comparable.*
2. *Each two principal ideals of D that contain a are comparable.*
3. *$\bigcap_{n \in \mathbb{N}} a^n D \in \mathrm{Spec}(D)$.*

Proof. There is some valuation overring V of D such that $aV \cap D = aD$.

(1) Let $b \in D$ be such that $\sqrt{aD} \subseteq \sqrt{bD}$. It follows from Proposition 1.1(3) that $bV \cap D = bD$. Since V is a valuation domain, we have that aV and bV are comparable, and hence aD and bD are comparable.

(2) Let $b, c \in D$ be such that $a \in bD \cap cD$. Then $bV \cap D = bD$ and $cV \cap D = cD$ by Proposition 1.1(2). Since bV and cV are comparable, we infer that bD and cD are comparable.

(3) It follows from Proposition 1.1(1) that $a^n V \cap D = a^n D$ for each $n \in \mathbb{N}$. Therefore, $(\bigcap_{n \in \mathbb{N}} a^n V) \cap D = \bigcap_{n \in \mathbb{N}} a^n D$. Since a is not a unit of V, we have that $\bigcap_{n \in \mathbb{N}} a^n V \in \mathrm{Spec}(V)$, and thus $\bigcap_{n \in \mathbb{N}} a^n D \in \mathrm{Spec}(D)$. $\square$

Remark 1.3. Let D be a VFD, let $a \in D$ be a valuation element and let $Q \in \mathrm{Spec}(D)$ be such that $Q \subsetneq \sqrt{aD}$. Then $Q \subseteq \bigcap_{n \in \mathbb{N}} a^n D$.

Proof. Let $x \in Q \setminus \{0\}$. Then $x \in bD \subseteq Q$ for some valuation element $b \in D$. We have that $\sqrt{bD} \subsetneq \sqrt{aD} = \sqrt{a^n D}$ for each $n \in \mathbb{N}$, and hence $x \in bD \subseteq \bigcap_{n\in\mathbb{N}} a^n D$ by Corollary 1.2(1). Consequently, $Q \subseteq \bigcap_{n\in\mathbb{N}} a^n D$. □

Corollary 1.4. [8, Corollary 2.4] *Let D be an integral domain. If every nonzero nonunit of D is a valuation element, then D is a valuation domain.*

Proof. Let $a, b \in D$ be nonzero nonunits. Then ab is a valuation element by assumption. Note that aD and bD are principal ideals of D that contain ab. Consequently, aD and bD are comparable by Corollary 1.2(2). Therefore, D is a valuation domain. □

Corollary 1.5. *A VFD is integrally closed.*

Proof. Let D be a VFD, let $\overline{D}$ be the integral closure of D and let $a \in D$ be a valuation element. There is some valuation overring V of D such that $aV \cap D = aD$. Since V is integrally closed, it follows that $\overline{D} \subseteq V$, and hence $a\overline{D} \cap D = aD$. It is an immediate consequence of Proposition 1.1(1) that $x\overline{D} \cap D = xD$ for each $x \in D$. If $y \in \overline{D}$, then $yz \in D$ for some nonzero $z \in D$, and thus $yz \in z\overline{D} \cap D = zD$ and $y \in D$. Consequently, $D = \overline{D}$. □

Corollary 1.6. *Let D be a quasi-local domain of dimension one. The following statements are equivalent.*

(1) *D is a valuation domain.*
(2) *D has at least one valuation element.*
(3) *D is a VFD.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Let $a \in D$ be a valuation element and let $b \in D$ be a nonzero nonunit. Then $\sqrt{aD} = \sqrt{bD}$, and hence b is a valuation element of D by Proposition 1.1(3). Therefore, D is a valuation domain by Corollary 1.4. □

A nonzero nonunit x of D is said to be *primary* if xD is a primary ideal. Clearly, prime elements are primary but not vice versa.

Proposition 1.7. *Let D be an integral domain, let $a \in D$ be a valuation element and let S be a multiplicatively closed subset of D.*

(1) *$\sqrt{aD}$ is a prime t-ideal.*
(2) *a is a primary element if and only if $\sqrt{aD}$ is a maximal t-ideal.*
(3) *If t-$\dim(D) = 1$, then every valuation element of D is a primary element.*
(4) *If $aS^{-1}D \subsetneq S^{-1}D$, then a is a valuation element of $S^{-1}D$.*

Proof. (1) Let V be a valuation overring of D such that $aD = aV \cap D$. Then $\sqrt{aD} = \sqrt{aV} \cap D$, and since $\sqrt{aV}$ is a prime ideal, $\sqrt{aD}$ is a prime ideal. Clearly, $\sqrt{aD}$ is minimal over aD and aD is a t-ideal. Thus, $\sqrt{aD}$ is a prime t-ideal.

(2) This follows from [1, Lemma 2.1].

(3) Let t-$\dim(D) = 1$ and let $b \in D$ be a valuation element. Then $\sqrt{bD}$ is a maximal t-ideal by (1) and assumption. Thus, by (2), b is a primary element.

(4) Let $aS^{-1}D \subsetneq S^{-1}D$. There is some valuation overring V of D such that $aV \cap D = aD$. Observe that S is a multiplicatively closed subset of V, and hence $S^{-1}V$ is an overring of V. Since V is a valuation domain, we have that $S^{-1}V$ is a

valuation domain. Note that $aS^{-1}D = S^{-1}(aD) = S^{-1}(aV \cap D) = aS^{-1}V \cap S^{-1}D$. Thus, a is a valuation element of $S^{-1}D$. □

Corollary 1.8. *Let D be a VFD and let S be a multiplicatively closed subset of D.*

(1) *$S^{-1}D$ is a VFD.*
(2) *If P is a height-one prime ideal of D, then D_P is a valuation domain.*

Proof. (1) This follows directly from Proposition 1.7(4).

(2) This is an immediate consequence of (1) and Corollary 1.6. □

An integral domain D is a *weakly factorial domain* (WFD) if every nonzero nonunit of D can be written as a finite product of primary elements. Let $X^1(D)$ be the set of height-one prime ideals of D. It is known that D is a WFD if and only if $D = \bigcap_{P \in X^1(D)} D_P$, where the intersection is locally finite (i.e., for each nonzero $x \in D$, x is a unit of D_P for all but finitely many $P \in X^1(D)$) and $\mathrm{Cl}_t(D) = \{0\}$ [3, Theorem]; in this case, t-$\dim(D) = 1$ (cf. Proposition 1.7(2)).

Corollary 1.9. *Let D be an integral domain that is not a field. Then D is a VFD with t-$\dim(D) = 1$ if and only if D is a weakly factorial GCD-domain.*

Proof. ($\Rightarrow$) Let D be a VFD of t-dimension one. It follows from Proposition 1.7(3) that D is a weakly factorial domain. Thus, it is an immediate consequence of Corollary 1.8(2) and [2, Theorem 18] that D is a GCD-domain.

($\Leftarrow$) Now let D be a weakly factorial GCD-domain. Then t-$\dim(D) = 1$. We next show that every primary element is a valuation element. Let $a \in D$ be a primary element and let $P = \sqrt{aD}$. Then P is a height-one prime ideal of D, and since D is a GCD-domain, D_P is a valuation domain. Note that $aD_P \cap D = aD$, and hence a is a valuation element. Thus, D is a VFD. □

Note that if D is a (one-dimensional) Bézout domain which is not of finite character (e.g., let D be the example in [13, Theorem 3.4] or let D be the ring of entire functions), then D is a GCD-domain and yet D is not a VFD (by Theorem 3.4). For more details concerning this example, we refer to [12, Example 4.2].

For $n \in \mathbb{N}$, let $[1, n] = \{k \in \mathbb{N} \mid 1 \leq k \leq n\}$. Two elements x and y of an integral domain D are said to be *incomparable* if xD and yD are incomparable under inclusion. We next show that each nonzero nonunit a of a VFD D can be written as a finite product of incomparable valuation elements, say, $a = \prod_{i=1}^{n} a_i$, and in this case, n is the number of minimal prime ideals of aD by a series of lemmas.

Lemma 1.10. *Let D be an integral domain. If $v \in D$ is a finite product of valuation elements of D such that $\sqrt{vD}$ is a prime ideal, then v is a valuation element.*

Proof. Let $k \in \mathbb{N}$ and let $v \in D$ be such that $\sqrt{vD}$ is a prime ideal of D and $v = \prod_{i=1}^{k} v_i$ for some valuation elements $v_i \in D$. We have that $\sqrt{vD} = \bigcap_{i=1}^{k} \sqrt{v_i D}$. Since $\sqrt{vD}$ is a prime ideal of D, it follows that $\sqrt{vD} = \sqrt{v_j D}$ for some $j \in [1, k]$. It is an immediate consequence of Proposition 1.1(3) that v is a valuation element of D. □

Corollary 1.11. *Let D be a VFD. Then the valuation elements of D are precisely the elements $a \in D$ for which $\sqrt{aD}$ is a nonzero prime ideal of D.*

Proof. This is an immediate consequence of Proposition 1.7(1) and Lemma 1.10. □

Let I be an ideal of an integral domain D. Let $\mathcal{P}(I)$ denote the set of minimal prime ideals of I.

Lemma 1.12. *Let D be a VFD and let $a \in D$ be a nonzero nonunit. Then*

$$\min\{k \in \mathbb{N} \mid a \text{ is a product of } k \text{ valuation elements of } D\} = |\mathcal{P}(aD)|.$$

Proof. Let $n = \min\{k \in \mathbb{N} \mid a$ is a product of k valuation elements of $D\}$. We have that $a = \prod_{i=1}^n a_i$ for some valuation elements a_i of D. Let $P \in \mathcal{P}(aD)$. Set $\Sigma_P = \{i \in [1,n] \mid P \subseteq \sqrt{a_iD}\}$. Observe that $\sqrt{a_jD} = P$ for some $j \in \Sigma_P$. This implies that $\sqrt{\prod_{i\in\Sigma_P} a_iD} = \bigcap_{i\in\Sigma_P} \sqrt{a_iD} = P$, and thus $\prod_{i\in\Sigma_P} a_i$ is a valuation element of D by Corollary 1.11. Since n is minimal and Σ_P is nonempty, we infer that $|\Sigma_P| = 1$. Note that $[1,n] = \bigcup_{Q\in\mathcal{P}(aD)} \Sigma_Q$. Consequently, there is a bijection $\varphi : \mathcal{P}(aD) \to [1,n]$ such that $\Sigma_Q = \{\varphi(Q)\}$ for each $Q \in \mathcal{P}(aD)$. □

Proposition 1.13. *Let D be a VFD, let $n \in \mathbb{N}$, let $(a_i)_{i=1}^n$ be a sequence of valuation elements of D and let $a = \prod_{i=1}^n a_i$. The following statements are equivalent.*

1. *a_i and a_j are incomparable for all distinct $i, j \in [1,n]$.*
2. *$\sqrt{a_iD}$ and $\sqrt{a_jD}$ are incomparable for all distinct $i, j \in [1,n]$.*
3. *A map $f : [1,n] \to \mathcal{P}(aD)$ given by $f(i) = \sqrt{a_iD}$ is a well-defined bijection.*
4. *$n = |\mathcal{P}(aD)|$.*

Hence, every nonzero nonunit of D can be written as a finite product of incomparable valuation elements.

Proof. (1) ⇒ (2) This follows from Corollary 1.2(1).

(2) ⇒ (3) Note that if $P \in \mathcal{P}(aD)$, then $a_i \in P$ for some $i \in [1,n]$, and hence $P = \sqrt{a_iD}$. Moreover, if $j \in [1,n]$, then $a \in \sqrt{a_jD}$, and thus $Q \subseteq \sqrt{a_jD}$ for some $Q \in \mathcal{P}(aD)$. As shown before, $Q = \sqrt{a_kD}$ for some $k \in [1,n]$. It follows that $k = j$, and hence $\sqrt{a_jD} = Q \in \mathcal{P}(aD)$. Thus, f is a well-defined bijection.

(3) ⇒ (4) This is obvious.

(4) ⇒ (1) If there are distinct $i, j \in [1,n]$ such that a_iD and a_jD are comparable, then a_ia_j is a valuation element, which contradicts Lemma 1.12.

Moreover, by Lemma 1.12 again, every nonzero nonunit of D can be written as a finite product of incomparable valuation elements. □

Now let D be a VFD. It is an easy consequence of Proposition 1.13 that if $n, m \in \mathbb{N}$ and $(a_i)_{i=1}^n$ and $(b_j)_{j=1}^m$ are two sequences of incomparable valuation elements of D with $\prod_{i=1}^n a_i = \prod_{j=1}^m b_j$, then $n = m$ and $\sqrt{a_iD} = \sqrt{b_iD}$ for each $i \in [1,n]$ by reordering if necessary.

Corollary 1.14. *Let D be a VFD and let $\Omega = \{\sqrt{xD} \mid x \in D \setminus \{0\}, \sqrt{xD} \in \mathrm{Spec}(D)\}$.*

1. *The valuation elements of D are precisely the nonzero nonunits $a \in D$ for which each two principal ideals of D that contain a are comparable.*
2. *If $a \in D$ is a valuation element and $P, Q \in \Omega$ are such that $a \in P \cap Q$, then P and Q are comparable.*
3. *$\Omega = \bigcup_{a\in D\setminus\{0\}} \mathcal{P}(aD) = \{\sqrt{xD} \mid x \in D$ is a valuation element$\}$.*

Proof. (1) This is an easy consequence of Corollary 1.2(2) and Proposition 1.13.

(2) Let $a \in D$ be a valuation element and let $P, Q \in \Omega$ be such that $a \in P \cap Q$. Then $\sqrt{aD} \in \Omega$ and $\sqrt{aD} \subseteq P \cap Q$. Moreover $P = \sqrt{pD}$ and $Q = \sqrt{qD}$ for some $p, q \in D$. Without restriction let $\sqrt{aD} \subsetneq P$ and $\sqrt{aD} \subsetneq Q$. Therefore, $aD \subseteq pD \cap qD$ by Corollary 1.2(1), and thus pD and qD are comparable by (1). Consequently, P and Q are comparable.

(3) ($\subseteq$) First let $P \in \Omega$. There is some nonzero $x \in D$ such that $P = \sqrt{xD}$. Observe that $P \in \mathcal{P}(xD)$. ($\subseteq$) Next let $a \in D$ be a nonzero nonunit and let $Q \in \mathcal{P}(aD)$. Then $a \in yD \subseteq Q$ for some valuation element $y \in D$. It follows from Corollary 1.11 that $\sqrt{yD}$ is a prime ideal of D, and hence $Q = \sqrt{yD}$. ($\subseteq$) Finally, let $z \in D$ be a valuation element of D. Set $A = \sqrt{zD}$. It follows from Corollary 1.11 that $A \in \Omega$. □

2. Schreier domains

Let D be an integral domain. Then D is called a *pre-Schreier domain* if for all nonzero $x, y, z \in D$ with $x \mid_D yz$, there are some $a, b \in D$ such that $x = ab$, $a \mid_D y$ and $b \mid_D z$. Moreover, D is called a *Schreier domain* if D is an integrally closed pre-Schreier domain. Clearly, GCD-domains are Schreier domains. Schreier domains were introduced by Cohn [6], and later, in [14], Zafrullah introduced the notion of pre-Schreier domains.

(Pre-)Schreier domains are rather "nice" integral domains. Let $D[X]$ be the polynomial ring over D. Recall that a polynomial $f \in D[X]$ is called *primitive* if each common divisor of the coefficients of f is a unit of D. We say that D satisfies *Gauß' Lemma* if the product of each two primitive polynomials over D is primitive. Clearly, UFDs satisfy Gauß' Lemma, and we use this fact to show that if D is a UFD, then $D[X]$ is also a UFD. It is well known (cf. [4, Propositions 3.2 and 3.3]) that every (pre-)Schreier domain satisfies Gauß' Lemma.

Proposition 2.1. *A VFD is a Schreier domain.*

Proof. Let D be a VFD. It follows from Corollary 1.5 that D is integrally closed. Next we show by induction that for each $k \in \mathbb{N}$, for each valuation element $x \in D$ and for all nonzero $y, z \in D$ such that $x \mid_D yz$ and $|\mathcal{P}(yD)| + |\mathcal{P}(zD)| = k$, there are some $a, b \in D$ such that $x = ab$, $a \mid_D y$ and $b \mid_D z$.

Let $k \in \mathbb{N}$, let $x \in D$ be a valuation element and let $y, z \in D$ be nonzero such that $x \mid_D yz$ and $|\mathcal{P}(yD)| + |\mathcal{P}(zD)| = k$. Observe that $yz \in xD \subseteq \sqrt{xD} \in \text{Spec}(D)$, and hence $y \in \sqrt{xD}$ or $z \in \sqrt{xD}$. Without restriction let $y \in \sqrt{xD}$. Note that $y = \prod_{P \in \mathcal{P}(yD)} y_P$ for some incomparable valuation elements $y_P \in D$ with $\sqrt{y_P D} = P$ for each $P \in \mathcal{P}(yD)$ by Proposition 1.13. Consequently, there is some $P \in \mathcal{P}(yD)$ such that $\sqrt{y_P D} \subseteq \sqrt{xD}$. It follows from Corollary 1.2(1) that $y_P D$ and xD are comparable.

Case 1: $y_P D \subseteq xD$. Then $y \in xD$. Set $a = x$ and $b = 1$. Then $x = ab$, $a \mid_D y$ and $b \mid_D z$.

Case 2: $xD \subsetneq y_P D$. There is some nonunit $w \in D$ such that $x = y_P w$. We infer by Proposition 1.1(2) that w is a valuation element of D. There is some $y' \in D$ such that $y = y_P y'$. Note that $\mathcal{P}(y'D) = \mathcal{P}(yD) \setminus \{P\}$, and hence $|\mathcal{P}(y'D)| + |\mathcal{P}(zD)| < k$.

Moreover, $y_P w = x \mid_D yz = y_P y'z$, and thus $w \mid_D y'z$. It follows by the induction hypothesis that $w = a'b$, $a' \mid_D y'$ and $b \mid_D z$ for some $a', b \in D$. Set $a = y_P a'$. Then $x = ab$ and $a \mid_D y$.

We infer that for each valuation element $x \in D$ and for all nonzero $y, z \in D$ with $x \mid_D yz$ there are some $a, b \in D$ such that $x = ab$, $a \mid_D y$ and $b \mid_D z$. Now it is straightforward to show by induction that for each $n \in \mathbb{N}$, for each $x \in D$ which is a product of n valuation elements of D and for each nonzero $y, z \in D$ with $x \mid_D yz$, there are some $a, b \in D$ such that $x = ab$, $a \mid_D y$ and $b \mid_D z$. This implies that D is a Schreier domain. □

However, Schreier domains need not be VFDs. For example, it is known that a Prüfer domain is a Schreier domain if and only if it is a Bézout domain [7, Proposition 2], while a VFD that is a Prüfer domain is an h-local Prüfer domain (by Corollary 3.7). Hence, if $D = \mathbb{Z}_{\mathbb{Z}\setminus(2)\cup(3)} + X\mathbb{Q}[\![X]\!]$, then D is a Schreier domain but not a VFD.

Corollary 2.2. *Let D be an integral domain. The following statements are equivalent.*

1. *D is a VFD.*
2. *D is a Schreier domain and every nonzero prime t-ideal of D contains a valuation element of D.*
3. *D is a pre-Schreier domain and every nonzero prime t-ideal of D contains a valuation element of D.*

Proof. (1) ⇒ (2) It follows from Proposition 2.1 that D is a Schreier domain. It is obvious that every nonzero prime t-ideal of D contains a valuation element of D.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) Let Σ be the set of finite products of units and valuation elements of D. Observe that Σ is a multiplicatively closed subset of D. Next we show that Σ is divisor-closed. Let $x \in D$ be such that $x \mid_D y$ for some $y \in \Sigma$. There are some $n \in \mathbb{N}$ and some elements $y_i \in D$ for each $i \in [1, n]$ which are either units or valuation elements of D such that $y = \prod_{i=1}^n y_i$. Therefore, $x = \prod_{i=1}^n x_i$ for some elements $x_i \in D$ such that $x_i \mid_D y_i$ for each $i \in [1, n]$. It follows from Proposition 1.1(2) that x_i is a unit or a valuation element of D for each $i \in [1, n]$. Therefore, $x \in \Sigma$.

It is sufficient to show that $D \setminus \{0\} \subseteq \Sigma$. Assume that there is some $z \in D \setminus (\Sigma \cup \{0\})$. Then $zD \cap \Sigma = \emptyset$, because Σ is divisor-closed by the previous paragraph. Consequently, there is some prime t-ideal P of D such that $zD \subseteq P$ and $P \cap \Sigma = \emptyset$. On the other hand, P contains a valuation element of D, and hence $P \cap \Sigma \neq \emptyset$, a contradiction. □

Let I be a t-ideal of an integral domain D. Then I is said to be *t-finite* if $I = J_t$ for some $J \in f(D)$. It is known that I is t-invertible if and only if I is t-finite and I_P is principal for all $P \in t\text{-Max}(D)$ [11, Corollary 2.7].

Corollary 2.3. *Let D be a VFD.*

1. $\mathrm{Cl}_t(D) = \{0\}$.
2. *Every atom of D is a prime element.*
3. *If D is a t-finite conductor domain, i.e., the intersection of each two principal ideals of D is t-finite, then D is a GCD-domain.*

Proof. This follows directly from Proposition 2.1 and [7, Proposition 2 and Corollary 6]. □

A *Mori domain* is an integral domain which satisfies the ascending chain condition on integral t-ideals, equivalently, each t-ideal is of finite type. Mori domains include Noetherian domains, Krull domains and UFDs.

Corollary 2.4. *Let D be a VFD. The following statements are equivalent.*

(1) *D is atomic.*
(2) *D is a Mori domain.*
(3) *D is a Krull domain.*
(4) *D is a UFD.*

Proof. Clearly, every UFD is a Krull domain and every Krull domain is a Mori domain. By definition, a Mori domain satisfies the ascending chain condition on principal ideals and is, therefore, atomic. If D is atomic, then every nonzero nonunit of D is a finite product of prime elements by Corollary 2.3(2), and thus D is a UFD. □

Observe that if D is a Krull domain which is not a UFD (e.g., $D = \mathbb{Z}[\sqrt{-5}]$), then D and $D[X]$ are both examples of Krull domains that fail to be VFDs. Furthermore, if V is a nondiscrete valuation domain (e.g., $V = \overline{\mathbb{Z}}_M$, where $\overline{\mathbb{Z}}$ is the ring of algebraic integers and M is a maximal ideal of $\overline{\mathbb{Z}}$), then V is a VFD and yet V is not atomic. We next study when $D[X]$ is a VFD.

Lemma 2.5. *Let $D[X]$ be the polynomial ring over an integral domain D and let $a \in D$ be a nonzero nonunit. Then a is a valuation element of D if and only if a is a valuation element of $D[X]$.*

Proof. Let K be the quotient field of D.

($\Rightarrow$) By assumption, $aV \cap D = aD$ for some valuation overring V of D. Note that if M is a maximal ideal of V, then $V(X) := V[X]_{M[X]}$ is a valuation overring of $D[X]$ and $V(X) \cap K[X] = V[X]$; hence if $u \in aV(X) \cap D[X]$, then $u = af$ for some $f \in V[X]$. Hence, $aV \cap D = aD$ implies $af \in aD[X]$, and thus $f \in D[X]$. Therefore, $aV(X) \cap D[X] = aD[X]$.

($\Leftarrow$) Let W be a valuation overring of $D[X]$ such that $aW \cap D[X] = aD[X]$ and let $V = W \cap K$. Then V is a valuation overring of D and

$$\begin{aligned} aD &= aD[X] \cap K = aW \cap D[X] \cap K \\ &= (aW \cap aK) \cap (D[X] \cap K) = a(W \cap K) \cap D \\ &= aV \cap D. \end{aligned}$$

Thus, a is a valuation element of D. □

Let $D[X]$ be the polynomial ring over D. For $f \in D[X]$, let $c(f)$ denote the ideal of D generated by the coefficients of f. A nonzero prime ideal Q of $D[X]$ is called an *upper to zero* in $D[X]$ if $Q \cap D = (0)$. Following [10], we say that D is a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal. Then D is an integrally closed UMT-domain if and only if D is a PvMD [10, Proposition 3.2].

Proposition 2.6. *Let $D[X]$ be the polynomial ring over an integral domain D. Then $D[X]$ is a VFD if and only if D is a VFD and every upper to zero in $D[X]$ contains a valuation element of $D[X]$.*

Proof. ($\Rightarrow$) Let $D[X]$ be a VFD. Let $a \in D$ be a nonzero nonunit. Then a is a nonzero nonunit of $D[X]$, and hence a is a finite product of valuation elements of $D[X]$. Clearly, each of these valuation elements is contained in D, and thus a is a finite product of valuation elements of D by Lemma 2.5. Therefore, D is a VFD. It is clear that every upper to zero in $D[X]$ contains a valuation element of $D[X]$.

($\Leftarrow$) Let D be a VFD and let every upper to zero in $D[X]$ contain a valuation element of $D[X]$. Then D is a Schreier domain by Proposition 2.1. It follows from [6, Theorem 2.7] (or from [4, Theorem 4.8]) that $D[X]$ is a Schreier domain. Let P be a nonzero prime t-ideal of $D[X]$. If $P \cap D = (0)$, then P contains a valuation element of $D[X]$ by assumption. Now let $P \cap D \neq (0)$. It is clear that $P \cap D$ contains a valuation element of D. It follows from Lemma 2.5 that P contains a valuation element of $D[X]$. Consequently, $D[X]$ is a VFD by Corollary 2.2. $\square$

Corollary 2.7. *Let $D[X]$ be the polynomial ring over a UMT-domain D. Then D is a VFD if and only if $D[X]$ is a VFD.*

Proof. By Proposition 2.6 it is sufficient to show that if D is a VFD, then every upper to zero in $D[X]$ contains a valuation element of $D[X]$. Let D be a VFD. Then D is integrally closed by Corollary 1.5, and hence D is a PvMD because D is a UMT-domain. Hence, D is a GCD-domain by Corollary 2.3(1). Let P be an upper to zero in $D[X]$. Since D is a GCD-domain, it follows that $P = fD[X]$ for some prime element $f \in D[X]$. Since every prime element of $D[X]$ is a valuation element, it follows that P contains a valuation element of $D[X]$. $\square$

3. VFDs which are HoFDs

An integral domain D is called a *weakly Matlis domain* if (i) D is of finite t-character and (ii) D is independent, i.e., no two distinct maximal t-ideals of D contain a common nonzero prime ideal. It is easy to see that if D is not a field, then D is a weakly factorial GCD-domain if and only if D is a weakly Matlis GCD-domain with t-$\dim(D) = 1$.

Let D be an integral domain. We say that $a, b \in D$ are t-comaximal if $(a, b)_v = D$. Two elements $a, b \in D$ are said to be *coprime* if for each $c \in D$ with $aD \cup bD \subseteq cD$, it follows that c is a unit of D. Hence, if $a, b \in D$ are t-comaximal, then a, b are coprime. Note that if $a, b \in D$ are two homogeneous elements that are not t-comaximal, then ab is also a homogeneous element of D. Thus, every nonzero nonunit of an HoFD can be written as a finite product of t-comaximal homogeneous elements. It is known that D is an HoFD if and only if D is a weakly Matlis domain with trivial t-class group [5, Theorem 2.2]. We first study when VFDs are HoFDs.

Proposition 3.1. *Let D be a VFD. The following statements are equivalent.*

(1) *D is an HoFD.*
(2) *D is a weakly Matlis domain.*
(3) *Every nonzero prime t-ideal of D is contained in a unique maximal t-ideal.*
(4) *Every valuation element of D is homogeneous.*

(5) *D is of finite t-character.*
(6) *Every valuation element of D is contained in only finitely many maximal t-ideals.*

If every maximal t-ideal of D is the radical of a principal ideal, then these equivalent conditions are satisfied.

Proof. (1) $\Rightarrow$ (2) This follows from [5, Theorem 2.2].

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) Let $a \in D$ be a valuation element of D. Then $\sqrt{aD}$ is a nonzero prime t-ideal of D by Proposition 1.7(1). Consequently, $|\{M \in t\text{-Max}(D) \mid a \in M\}| = |\{M \in t\text{-Max}(D) \mid \sqrt{aD} \subseteq M\}| = 1$, and hence a is homogeneous.

(4) $\Rightarrow$ (1) This is obvious.

(2) $\Rightarrow$ (5) $\Rightarrow$ (6) This is clear.

(6) $\Rightarrow$ (4) Let $a \in D$ be a valuation element. Set $\Sigma = \{Q \in t\text{-Max}(D) \mid a \in Q\}$. Assume that $|\Sigma| \geq 2$. Then there are some distinct $M, N \in \Sigma$. Since Σ is finite, there are some $b \in M \setminus \bigcup_{Q \in \Sigma \setminus \{M\}} Q$ and $c \in N \setminus \bigcup_{Q \in \Sigma \setminus \{N\}} Q$. Note that a and b are not t-comaximal. We infer by Proposition 2.1 and [4, Proposition 3.3] that $aD \cup bD \subseteq dD$ for some nonunit $d \in D$. It follows by analogy that $aD \cup cD \subseteq eD$ for some nonunit $e \in D$. Since $a \in dD \cap eD$, we have that dD and eD are comparable by Corollary 1.2(2). Without restriction let $eD \subseteq dD$. There is some $A \in t\text{-Max}(D)$ such that $dD \subseteq A$, and hence $aD \cup bD \cup cD \subseteq A$. This implies that $M = A = N$, a contradiction.

Now let every maximal t-ideal of D be the radical of a principal ideal. We infer by Corollary 1.14(2) that every valuation element of D is homogeneous. □

We say that D is a *t-treed domain* if the set of prime t-ideals of D is treed under inclusion. The class of t-treed domains includes PvMDs and integral domains of t-dimension one. We next study VFDs that are t-treed. We first need a lemma.

Lemma 3.2. *Let D be a t-treed domain. Then every valuation element of D is homogeneous.*

Proof. Let $a \in D$ be a valuation element. Assume to the contrary that a is not homogeneous. Hence, there are at least two distinct maximal t-ideals M_1 and M_2 of D containing a. Let $S = D \setminus (M_1 \cup M_2)$, and note that $a \in S^{-1}D$ is a valuation element by Proposition 1.7(4). Hence, by replacing D with $S^{-1}D$, we assume that D is a treed domain with two maximal ideals M_1 and M_2.

Since a is a valuation element, there is a valuation overring V of D such that $aV \cap D = aD$. Note that if M is the maximal ideal of V, then $M \cap D$ is a proper prime ideal of D, and hence, without loss of generality, we may assume that $M \cap D \subseteq M_1$. Thus, $aD \subseteq aD_{M_1} \cap D \subseteq aV \cap D = aD$, whence $aD_{M_1} \cap D = aD$. Choose $b \in M_2 \setminus M_1$. Then $\sqrt{aD} \subsetneq \sqrt{bD}$ because Spec(D) is treed. Thus, by Proposition 1.1(3), $bD = bD_{M_1} \cap D = D_{M_1} \cap D = D$, a contradiction. Therefore, a is contained in a unique maximal t-ideal of D. □

Next we want to point out that a weaker form of Lemma 3.2 can be proved without relying on prime avoidance.

Remark 3.3. Let D be a VFD. If each two valuation elements of D that are incomparable are t-comaximal, then every valuation element of D is homogeneous.

Proof. Let $a \in D$ be a valuation element. Assume to the contrary that a is not homogeneous. Then there are distinct maximal t-ideals M and Q of D containing a. Since D is a VFD, there are valuation elements $b, c \in D$ such that $b \in M \setminus Q$ and $c \in Q \setminus M$. Since aD and bD are contained in M, we infer that aD and bD are comparable, and hence $aD \subseteq bD$. It follows by analogy that $aD \subseteq cD$. Therefore, bD and cD are comparable by Corollary 1.2(2), a contradiction. □

A PvMD is a ring of Krull type if it is of finite t-character, and a PvMD is an independent ring of Krull type if it is weakly Matlis. Recall that a PvMD D is an HoFD if and only if $D[X]$ is an HoFD, if and only if D is an independent ring of Krull type with $\mathrm{Cl}_t(D) = \{0\}$ [5, Corollary 2.6].

Theorem 3.4. *The following statements are equivalent for a t-treed domain D.*

1. *D is a VFD.*
2. *D is an HoFD and a PvMD.*
3. *D is an independent ring of Krull type and* $\mathrm{Cl}_t(D) = \{0\}$.
4. *$D[X]$ is a VFD.*
5. *D is a weakly Matlis GCD-domain.*

Proof. (1) ⇒ (2) By Lemma 3.2, it suffices to show that D is a PvMD. Let M be a maximal t-ideal of D. Then D_M is a VFD by Corollary 1.8(1) and $\mathrm{Spec}(D_M)$ is linearly ordered by assumption. Let $a \in D_M$ be a nonzero nonunit. Then, since $\mathrm{Spec}(D_M)$ is linearly ordered, $\sqrt{aD_M}$ is a prime ideal, and hence a is a valuation element of D_M by Corollary 1.11. Thus, D_M is a valuation domain by Corollary 1.4.

(2) ⇒ (1) Let $a \in D$ be a homogeneous element. Then there is a unique maximal t-ideal M of D with $a \in M$. Hence,

$$aD = \bigcap_{P \in t\text{-}\mathrm{Max}(D)} aD_P = aD_M \cap \Big(\bigcap_{P \in t\text{-}\mathrm{Max}(D) \setminus \{M\}} D_P \Big) = aD_M \cap D,$$

and since D_M is a valuation domain, a is a valuation element of D. Thus, D is a VFD.

(2) ⇔ (3) This follows from [5, Corollary 2.6].

(1) ⇒ (4) If D is a VFD, then D is a PvMD by the proof of (1) ⇒ (2). Thus, $D[X]$ is a VFD by Corollary 2.7.

(4) ⇒ (1) This is an immediate consequence of Proposition 2.6.

(3) ⇔ (5) This follows because a GCD-domain is a PvMD with trivial t-class group. □

Next we want to point out that even a Schreier domain with a unique maximal t-ideal need not be a VFD. In particular, weakly Matlis Schreier domains and Schreier domains which are HoFDs need not be VFDs. Recall that a quasi-local integral domain D with maximal ideal M is a *pseudo-valuation domain* (*PVD*) if for all ideals A and B of D, it follows that $A \subseteq B$ or $BM \subseteq AM$ [9, Theorem 1.4].

Example 3.5. [4, Example 2.10] Let $T = \mathbb{C}[X]$, let K be a quotient field of T and let S be the integral closure of T in an algebraic closure $\overline{K}$ of K. Let Q be a maximal ideal of S, let $\overline{\mathbb{Q}} \subseteq \overline{K}$ be the algebraic closure of $\mathbb{Q}$ and let $D = \overline{\mathbb{Q}} + Q_Q$. Then D is a Schreier domain with a unique maximal t-ideal and yet D is not a VFD.

Proof. It follows from [4, Example 2.10] and its proof that D is a Schreier domain and a PVD, but not a Bézout domain. Since D is a PVD, we have that $\mathrm{Spec}(D)$ is linearly ordered, and thus D is t-treed. In particular, D has a unique maximal t-ideal. Since D is not a Bézout domain, it follows that D is not a valuation domain, and thus D is not a GCD-domain. Therefore, D is not a VFD by Theorem 3.4. □

Corollary 3.6. *A PvMD D is a VFD if and only if D is an HoFD.*

Proof. It is well known that a PvMD is a t-treed domain. Thus, the result follows directly from Theorem 3.4. □

In [5, Section 4], Chang studied HoFDs that are PvMDs and he also constructed several examples of such kind of integral domains. An integral domain D has *finite character* if each nonzero element of D is contained in at most finitely many maximal ideals of D. The domain D is said to be *h-local* if D has finite character and each nonzero prime ideal of D is contained in a unique maximal ideal. Hence, D is an h-local domain if D is a weakly Matlis domain whose maximal ideals are t-ideals.

Corollary 3.7. *A Prüfer domain D is a VFD if and only if D is an h-local Prüfer domain with* $\mathrm{Pic}(D) = \{0\}$.

Proof. It is clear that a Prüfer domain is an independent ring of Krull type if and only if it is an h-local Prüfer domain. Thus, the result follows directly from Theorem 3.4. □

Let D be a UMT-domain or a t-treed domain. Then D is a VFD if and only if $D[X]$, the polynomial ring over D, is a VFD by Corollary 2.7 and Theorem 3.4. However, we don't know if this is true in general.

Question 3.8. Let $D[X]$ be the polynomial ring over a VFD D. Is $D[X]$ a VFD?

4. Unique valuation factorization domains

For $n \in \mathbb{N}$ let $\mathcal{S}_n$ be the symmetric group on n letters.

Definition 4.1. Let D be an integral domain. We say that D is a *unique VFD* (*UVFD*) if the following two conditions are satisfied.

1. (1) Every nonzero nonunit of D is a finite product of incomparable valuation elements of D.
2. (2) If $n, m \in \mathbb{N}$ and $(a_i)_{i=1}^n$ and $(b_j)_{j=1}^m$ are two sequences of incomparable valuation elements of D with $\prod_{i=1}^n a_i = \prod_{j=1}^m b_j$, then $n = m$ and there is some $\sigma \in \mathcal{S}_n$ such that $a_i D = b_{\sigma(i)} D$ for each $i \in [1, n]$.

Clearly, UVFDs are VFDs by Proposition 1.13. Moreover, by the remark after Proposition 1.13, it follows that a VFD D is a UVFD if and only if for all $n \in \mathbb{N}$ and all sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ of incomparable valuation elements of D with $\prod_{i=1}^n a_i = \prod_{i=1}^n b_i$ and $\sqrt{a_j D} = \sqrt{b_j D}$ for each $j \in [1, n]$, it follows that $a_j D = b_j D$ for each $j \in [1, n]$.

Theorem 4.2. *Let D be a VFD and let $\Omega = \{\sqrt{xD} \mid x \in D \backslash \{0\}, \sqrt{xD} \in \mathrm{Spec}(D)\}$. The following statements are equivalent.*

(1) *D is a UVFD.*
(2) *Each two incomparable valuation elements of D are coprime.*
(3) *D is a PvMD.*
(4) *D_P is a valuation domain for each $P \in \Omega$.*
(5) *For all $A, B, C \in \Omega$ with $A \cup B \subseteq C$, A and B are comparable.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in D$ be incomparable valuation elements of D and let $c \in D$ be such that $aD \cup bD \subseteq cD$. Set $v = ac$ and $w = bc$. Then $\sqrt{vD} = \sqrt{aD}$, $\sqrt{wD} = \sqrt{bD}$ and $vb = aw$. It follows from Corollary 1.11 that v and w are valuation elements of D. Note that $\sqrt{aD}$ and $\sqrt{bD}$ are incomparable by Corollary 1.2(1). Therefore, v and b are incomparable and a and w are incomparable. We infer that $vD = aD$ by assumption, and hence c is a unit of D.

(2) $\Rightarrow$ (3) By Theorem 3.4, it is sufficient to show that D is t-treed. Assume to the contrary that there is some $M \in t\text{-Max}(D)$ and some incomparable prime t-ideals P and Q of D that are contained in M. Observe that there are some valuation elements $a, b \in D$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. It is clear that a and b are incomparable. Therefore, a and b are coprime. It follows from Proposition 2.1 and [4, Proposition 3.3] that a and b are t-comaximal, a contradiction.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (5) Let $A, B, P \in \Omega$ be such that $A \cup B \subseteq P$. Then D_P is a valuation domain and A_P and B_P are prime ideals of D_P. Hence, A_P and B_P are comparable, and thus $A = A_P \cap D$ and $B = B_P \cap D$ are comparable.

(5) $\Rightarrow$ (1) Let $n \in \mathbb{N}$ and let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences of incomparable valuation elements of D such that $\prod_{i=1}^n a_i = \prod_{i=1}^n b_i$ and $\sqrt{a_iD} = \sqrt{b_iD}$ for each $i \in [1, n]$. Let $i \in [1, n]$. Since $a_i \mid_D \prod_{j=1}^n b_j$ we infer by Proposition 2.1 that $a_i = \prod_{j=1}^n b'_j$ for some elements b'_j of D such that $b'_j \mid_D b_j$ for each $j \in [1, n]$. If $j \in [1, n] \setminus \{i\}$, then $\sqrt{a_iD}$ and $\sqrt{b_jD}$ are incomparable, and since $\sqrt{a_iD} \cup \sqrt{b_jD} \subseteq \sqrt{b'_jD}$, we infer that b'_j is a unit of D. This implies that $a_iD = b'_iD \supseteq b_iD$. It follows by analogy that $b_iD \supseteq a_iD$, and hence $a_iD = b_iD$. Thus, D is a UVFD. $\square$

Corollary 4.3. *Let D be a UVFD and let S be a multiplicatively closed subset of D. Then $S^{-1}D$ is a UVFD.*

Proof. It follows from Theorem 4.2 that D is a VFD and a PvMD. It is clear that $S^{-1}D$ is a PvMD. Moreover, $S^{-1}D$ is a VFD by Corollary 1.8(1). Therefore, $S^{-1}D$ is a UFVD by Theorem 4.2. $\square$

Corollary 4.4. *Let $D[X]$ be the polynomial ring over an integral domain D. Then D is a UVFD if and only if $D[X]$ is a UVFD.*

Proof. Note that D is a PvMD if and only if $D[X]$ is a PvMD [11, Theorem 3.7]. Thus, the result is an immediate consequence of Corollary 2.7 and Theorem 4.2. $\square$

Corollary 4.5. *An integral domain is a UVFD if and only if it is a weakly Matlis GCD-domain.*

Proof. This is an immediate consequence of Theorems 3.4 and 4.2. $\square$

Let D be an integral domain with quotient field K. We say that D satisfies the *Principal Ideal Theorem* if each minimal prime ideal of each nonzero principal ideal of D is of height one. It is well known that Noetherian domains satisfy the Principal Ideal Theorem. Recall that an element $x \in K$ is said to be *almost integral* over D if there exists some nonzero $c \in D$ such that $cx^n \in D$ for each $n \in \mathbb{N}$. We say that D is *completely integrally closed* if for all $x \in K$ such that x is almost integral over D, it follows that $x \in D$. Moreover, D is said to be *archimedean* if for each nonunit $x \in D$, $\bigcap_{n\in\mathbb{N}} x^n D = (0)$. Observe that if t-dim$(D) = 1$, then D satisfies the Principal Ideal Theorem. Moreover, if D is completely integrally closed or D satisfies the Principal Ideal Theorem, then D is archimedean.

Proposition 4.6. *Let D be a VFD that is not a field. The following statements are equivalent.*

(1) *D is a weakly factorial GCD-domain.*
(2) t-dim$(D) = 1$.
(3) *D is completely integrally closed.*
(4) *D is archimedean.*
(5) *D satisfies the Principal Ideal Theorem.*

Proof. Set $\Omega = \{\sqrt{xD} \mid x \in D \setminus \{0\}, \sqrt{xD} \in \mathrm{Spec}(D)\}$. By Corollary 1.14(3), we have that $\Omega = \bigcup_{a\in D\setminus\{0\}} \mathcal{P}(aD) = \{\sqrt{xD} \mid x \in D$ is a valuation element$\}$.

(1) $\Leftrightarrow$ (2) This follows from Corollary 1.9.

(1) $\Rightarrow$ (3) Note that D is an intersection of one-dimensional valuation overrings of D, and hence D is an intersection of completely integrally closed overrings of D. Therefore, D is completely integrally closed.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (5) Let $P \in \Omega$. Then $P = \sqrt{pD}$ for some valuation element $p \in D$. It remains to show that P is of height one. Let Q be a prime ideal of D such that $Q \subsetneq P$. We infer by Remark 1.3 that $Q \subseteq \bigcap_{n\in\mathbb{N}} p^n D = (0)$, and thus $Q = (0)$.

(5) $\Rightarrow$ (2) Note that Ω is the set of height-one prime ideals of D. It remains to show that each maximal t-ideal of D is an element of Ω. It follows by Corollary 1.8(2) that D_P is a valuation domain for each $P \in \Omega$. Thus, D is a PvMD by Theorem 4.2. Let $M \in t$-Max(D). Then $M = \bigcup_{P\in\Omega, P\subseteq M} P$. Observe that D is t-treed, and hence $M \in \Omega$. □

Note that if V is a two-dimensional valuation domain (e.g., $V = \mathrm{Int}(\mathbb{Z})_M$, where $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ is the ring of integer-valued polynomials and M is a height-two prime ideal of $\mathrm{Int}(\mathbb{Z})$), then V and $V[X]$ are both examples of non-archimedean VFDs.

Let D be an integral domain. We say that a nonzero element $a \in D$ has *prime radical* if $\sqrt{aD}$ is a prime ideal of D. Next we study VFDs in which each minimal prime ideal of a nonzero t-finite t-ideal is minimal over a t-invertible t-ideal (i.e., for each nonzero t-finite t-ideal I of D and every $P \in \mathcal{P}(I)$ there is some t-invertible t-ideal J of D such that $P \in \mathcal{P}(J)$). In other words, we study VFDs D for which

$$\begin{aligned} & \bigcup \{\mathcal{P}(I) \mid I \text{ is a nonzero } t\text{-finite } t\text{-ideal of } D\} \\ = & \bigcup \{\mathcal{P}(I) \mid I \text{ is a } t\text{-invertible } t\text{-ideal of } D\}. \end{aligned}$$

Suppose that D satisfies one of the following conditions.

(1) D is a PvMD.
(2) D is of t-dimension one.

(3) D has finitely many prime ideals.
(4) For each t-finite t-ideal I of D there is some $a \in D$ such that $\sqrt{I} = \sqrt{aD}$.

Then every minimal prime ideal of a nonzero t-finite t-ideal of D is minimal over a t-invertible t-ideal of D.

Lemma 4.7. *Let D be an integral domain in which every nonzero nonunit is a finite product of elements with prime radical and let I be a t-invertible t-ideal of D. Then $\mathcal{P}(I)$ is finite and for each $\emptyset \neq \mathcal{L} \subseteq \mathcal{P}(I)$, $\bigcap_{Q \in \mathcal{L}} Q$ is the radical of a principal ideal of D.*

Proof. First we show that every prime ideal in $\mathcal{P}(I)$ is the radical of a principal ideal. Let $P \in \mathcal{P}(I)$. Then P is a prime t-ideal, and hence I_P is a principal ideal, i.e., $I_P = aD_P$ for some $a \in P$. There is some $b \in P$ such that $a \in bD$ and $\sqrt{bD} \in \operatorname{Spec}(D)$. We have that $P_P = \sqrt{I_P} = \sqrt{aD_P} \subseteq \sqrt{bD_P} \subseteq P_P$. Consequently, $P_P = \sqrt{bD_P} = (\sqrt{bD})_P$, and hence $P = \sqrt{bD}$.

Thus, by [12, Lemma 2.5], $\mathcal{P}(I)$ is finite. Let $\emptyset \neq \mathcal{L} \subseteq \mathcal{P}(I)$. If $Q \in \mathcal{L}$, then $Q = \sqrt{a_Q D}$ for some $a_Q \in D$. This implies that $\bigcap_{Q \in \mathcal{L}} Q = \bigcap_{Q \in \mathcal{L}} \sqrt{a_Q D} = \sqrt{(\prod_{Q \in \mathcal{L}} a_Q) D}$. □

Remark 4.8. Let D be a VFD in which each minimal prime ideal of a nonzero t-finite t-ideal is minimal over a t-invertible t-ideal and let I be a nonzero t-finite t-ideal of D. Then $\mathcal{P}(I)$ is finite and for each $\emptyset \neq \mathcal{L} \subseteq \mathcal{P}(I)$, $\bigcap_{Q \in \mathcal{L}} Q$ is the radical of a principal ideal of D.

Proof. First we show that every minimal prime ideal of I is the radical of a principal ideal of D. Let $P \in \mathcal{P}(I)$. There is some t-invertible t-ideal J of D such that $P \in \mathcal{P}(J)$. Consequently, P is the radical of a principal ideal of D by Lemma 4.7. Thus, by the proof of Lemma 4.7, $\mathcal{P}(I)$ is finite and $\bigcap_{Q \in \mathcal{L}} Q$ is the radical of a principal ideal of D for all $\emptyset \neq \mathcal{L} \subseteq \mathcal{P}(I)$. □

Proposition 4.9. *Let D be a VFD in which each minimal prime ideal of a nonzero t-finite t-ideal is minimal over a t-invertible t-ideal. Then every t-finite t-ideal I of D with $\sqrt{I} \in \operatorname{Spec}(D)$ is principal.*

Proof. It is sufficient to show by induction that for every $m \in \mathbb{N}$ and every finite $E \subseteq D \setminus \{0\}$ such that $\sum_{e \in E} |\mathcal{P}(eD)| = m$ and $\sqrt{(E)_t} \in \operatorname{Spec}(D)$, it follows that $(E)_t$ is principal.

Let $m \in \mathbb{N}$ and $E \subseteq D \setminus \{0\}$ be such that E is finite, $\sum_{e \in E} |\mathcal{P}(eD)| = m$ and $\sqrt{(E)_t} \in \operatorname{Spec}(D)$. Set $I = (E)_t$, $Q = \sqrt{I}$ and $\Sigma = \{e \in E \mid Q \in \mathcal{P}(eD)\}$. By Remark 4.8, there is some $a \in D$ such that $Q = \sqrt{aD}$. Without restriction we can assume that $a \in I$. It follows from Proposition 1.13 that for each $e \in E$, $e = \prod_{P \in \mathcal{P}(eD)} e_P$, where e_A is a valuation element of D with $\sqrt{e_A D} = A$ for each $A \in \mathcal{P}(eD)$.

Case 1: $\Sigma = \emptyset$. Let $e \in E$. Since $e \in Q$, there is some $P \in \mathcal{P}(eD)$ such that $P \subsetneq Q$. This implies that $\sqrt{e_P D} \subsetneq \sqrt{aD}$, and hence $e \in e_P D \subseteq aD$ by Corollary 1.2(1). Consequently, $E \subseteq aD$, and thus $I = aD$.

Case 2: $\Sigma \neq \emptyset$. Observe that if $g, h \in \Sigma$, then $\sqrt{g_Q D} = \sqrt{h_Q D} = Q$, and thus $g_Q D$ and $h_Q D$ are comparable by Corollary 1.2(1). Since Σ is finite and nonempty, there is some $f \in \Sigma$ such that $e_Q D \subseteq f_Q D$ for all $e \in \Sigma$. Set $c = f_Q$.

Next we show that $e \in cD$ and $|\mathcal{P}(ec^{-1}D)| \leq |\mathcal{P}(eD)|$ for each $e \in E$. Let $e \in E$. Then $P \subseteq Q$ for some $P \in \mathcal{P}(eD)$. If $e \in \Sigma$, then $P = Q$ and $e \in e_PD \subseteq cD$. If $e \notin \Sigma$, then $\sqrt{e_PD} \subsetneq \sqrt{cD}$, and thus $e \in e_PD \subseteq cD$ by Corollary 1.2(1). In any case we have that $e_P \in cD$. Set $d = e_Pc^{-1}$. Note that $d \mid_D e_P$ and e_P is a valuation element of D. It follows from Proposition 1.1(2) that d is either a unit or a valuation element of D. Note that $e = e_Pb$, where $b \in D$ is a product of $|\mathcal{P}(eD)|-1$ valuation elements of D. Consequently, $ec^{-1} = db$ is a unit or a product of at most $|\mathcal{P}(eD)|$ valuation elements of D. It follows from Lemma 1.12 that $|\mathcal{P}(ec^{-1}D)| \leq |\mathcal{P}(eD)|$.

We infer that $I \subseteq cD$. Set $F = \{ec^{-1} \mid e \in E\}$ and $J = (F)_t$. Clearly, $F \subseteq D \setminus \{0\}$ is finite, J is a t-finite t-ideal of D and $I = cJ$. Note that $\mathcal{P}(fc^{-1}D) = \mathcal{P}(fD) \setminus \{Q\}$, and thus $|\mathcal{P}(fc^{-1}D)| < |\mathcal{P}(fD)|$. Therefore, $\sum_{g\in F} |\mathcal{P}(gD)| = \sum_{e\in E} |\mathcal{P}(ec^{-1}D)| < \sum_{e\in E} |\mathcal{P}(eD)| = m$. It follows by Remark 4.8 that $\sqrt{J} = \sqrt{bD}$ for some $b \in D$. Without restriction let $J \neq D$. Since $\sqrt{cD} \subseteq \sqrt{bD}$, it follows by Proposition 1.1(3) that b is a valuation element of D, and hence $\sqrt{J} \in \mathrm{Spec}(D)$. It follows by the induction hypothesis that J is principal. Consequently, I is principal. □

Proposition 4.10. *Let D be a VFD in which each minimal prime ideal of a nonzero t-finite t-ideal is minimal over a t-invertible t-ideal. Then D is a GCD-domain.*

Proof. By Remark 4.8 it is sufficient to show by induction that for each $n \in \mathbb{N}$ and every nonzero t-finite t-ideal I of D with $|\mathcal{P}(I)| = n$, it follows that I is principal.

Let $n \in \mathbb{N}$ and let I be a nonzero t-finite t-ideal of D such that $|\mathcal{P}(I)| = n$. Without restriction let $n \geq 2$ and let $P \in \mathcal{P}(I)$. By Remark 4.8 there are some $c, d \in D$ such that $P = \sqrt{cD}$ and $\bigcap_{Q\in\mathcal{P}(I)\setminus\{P\}} Q = \sqrt{dD}$. Observe that $\sqrt{I} = \sqrt{cdD}$, and hence $c^kd^k \in I$ for some $k \in \mathbb{N}$. Set $a = c^k$ and $b = d^k$. Then $P = \sqrt{aD}$, $\bigcap_{Q\in\mathcal{P}(I)\setminus\{P\}} Q = \sqrt{bD}$ and $ab \in I$. Set $J = (I + aD)_t$. Then J is a t-finite t-ideal of D such that $\sqrt{J} = P$, and hence J is principal by Proposition 4.9. Consequently, there is some t-finite t-ideal L of D such that $I = JL$.

Next we show that $\mathcal{P}(L) = \mathcal{P}(I) \setminus \{P\}$. First let $A \in \mathcal{P}(I) \setminus \{P\}$. Then $JL = I \subseteq A$. If $J \subseteq A$, then $P \subseteq A$, and hence $P = A$, a contradiction. Therefore, $L \subseteq A$, and since $I \subseteq L$, we infer that $A \in \mathcal{P}(L)$. Now let $B \in \mathcal{P}(L)$. Since $ab \in I$, we have that $Jb \subseteq I$, and hence $b \in L \subseteq B$. Consequently, $\sqrt{bD} \subseteq B$. This implies that $C \subseteq B$ for some $C \in \mathcal{P}(I) \setminus \{P\}$. We have that $C \in \mathcal{P}(L)$ (as shown before), and thus $B = C \in \mathcal{P}(I) \setminus \{P\}$. By the induction hypothesis, L is principal. Thus, $I = JL$ is principal. □

We do not know whether every VFD is a weakly Matlis GCD-domain, but we do know this is affirmative under certain additional assumptions. In what follows, we summarize a variety of conditions that force a VFD to be a weakly Matlis GCD-domain.

Theorem 4.11. *The following statements are equivalent for a VFD D.*

(1) *D is a weakly Matlis GCD-domain.*
(2) *D is a UVFD.*
(3) *D is a PvMD.*
(4) *D is a t-treed domain.*

(5) *Each minimal prime ideal of each nonzero t-finite t-ideal of D is minimal over a t-invertible t-ideal.*
(6) *D is a t-finite conductor domain.*
(7) *D is a UMT-domain.*

Proof. (1) $\Leftrightarrow$ (2) This is an immediate consequence of Corollary 4.5.
(2) $\Leftrightarrow$ (3) This follows from Theorem 4.2.
(3) $\Rightarrow$ (4), (5), (6), (7) This is clear.
(4) $\Rightarrow$ (3) This follows from Theorem 3.4.
(5) $\Rightarrow$ (3) This is an immediate consequence of Proposition 4.10.
(6) $\Rightarrow$ (3) This follows from Corollary 2.3.
(7) $\Rightarrow$ (3) Note that every integrally closed UMT-domain is a PvMD, and thus the statement follows by Corollary 1.5. □

Proposition 4.12. *Let D be a VFD and let $\Omega = \{\sqrt{xD} \mid x \in D \setminus \{0\}, \sqrt{xD} \in \operatorname{Spec}(D)\}$. Let one of the following conditions be satisfied.*
(1) *For all $P \in \Omega$, each nonzero prime t-ideal of D contained in P is in Ω.*
(2) *For all $P \in \Omega$ there is a unique height-one prime ideal Q of D with $Q \subseteq P$.*
(3) *t-$\dim(D) \leq 2$.*

Then D is a weakly Matlis GCD-domain.

Proof. (1) By Theorems 4.2 and 4.11 it remains to show that D_P is a valuation domain for each $P \in \Omega$. Let $P \in \Omega$. Then D_P is a VFD by Corollary 1.8(1). Moreover, P_P is both the unique maximal t-ideal of D_P and the radical of a principal ideal of D_P. Next we show that every nonzero prime t-ideal of D_P is the radical of a principal ideal of D_P. Let Q be a nonzero prime t-ideal of D_P. Then $Q \cap D$ is a nonzero prime t-ideal of D contained in P. Therefore, $Q \cap D$ is the radical of a principal ideal of D, and thus $Q = (Q \cap D)_P$ is the radical of a principal ideal of D_P. Consequently, D_P satisfies (5) in Theorem 4.11, and thus D_P is a PvMD again by Theorem 4.11. We infer that D_P is a valuation domain (since P_P is a maximal t-ideal of D_P).

(2) By Theorems 4.2 and 4.11 it is sufficient to show that for all $A, B, C \in \Omega$ with $A \cup B \subseteq C$, A and B are comparable. Let $A, B, C \in \Omega$ be such that $A \cup B \subseteq C$. There are some height-one prime ideals P and Q of D such that $P \subseteq A$ and $Q \subseteq B$. Since $P \cup Q \subseteq C$, it follows that $P = Q$, and hence $P \subseteq A \cap B$. Therefore, A and B are comparable by Corollary 1.14(2).

(3) Note that Ω contains the set of height-one prime ideals of D, and thus D is a weakly Matlis GCD-domain by (1). □

A weakly Matlis GCD-domain is a VFD by Corollary 4.5. Moreover, if D is a t-treed domain (e.g., PvMD or t-$\dim(D) = 1$), then D is a VFD if and only if D is a weakly Matlis GCD-domain by Theorem 3.4. We end this paper with a question.

Question 4.13. Let D be a VFD. Is D a weakly Matlis GCD-domain?

Acknowledgements. We want to thank the referee for many helpful suggestions and comments which improved the quality of this paper. This research was completed while the second-named author visited Incheon National University during 2019. The first-named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A1B06029867). The second-named author was supported by the Austrian Science Fund FWF, Project Number J4023-N35.

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