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PUMPING IN MODELS OF FLOW IN A LOOP OF RIGID PIPES

Georg Propst [1]

Abstract. This article studies ordinary differential equations modeling incompressible flow in rigid pipes that connect two distensible vessels, one of which is periodically forced. The forcing controls either the pressure or the volume of the excited vessel and - in part of the period - can be replaced by free relaxation. The pressure losses at the junctions of the pipes and vessels are quadratic with or without switches according to the direction of the flow. Stability and net flow of the equilibria of the unforced systems is investigated. Pumping solutions are defined and proven to exist in case of nonlinear pressure losses at the junctions. In contrast to often-quoted literature, it is shown that 'impedance defined' piecewise linear models can not produce net flow with continuous solutions. For partial forcing models numerical simulations are reported.

Keywords: valveless pumping, impedance defined flow, periodic solutions, circular net flow

1. Introduction

In flow configurations with elastic tubes and/or rigid pipes that are free of valves, pumping effects such as average flow can be generated by periodic excitation. This was observed experimentally, eg. [2, 5, 9, 11, 15, 16], as well as in numerical simulations with mathematical models, eg. [6, 7, 11, 15, 16]. All these references and the present article deal with closed loop configurations so that average flow is possible for periodic motion. This is not possible in non-closed-loop configurations such as in [1, 13, 14].

However, many of the present-day engineering applications are non-closed-loop configurations with elastic tubes, see eg. [17] and the references therein. The mathematical models of configurations with elastic tubes are nonlinear partial differential equations with dynamic boundary conditions. The proof of well-posedness of such systems involves rather advanced functional analytic methods, see [12] for an overwiew and the references therein for the proofs. Numerical simulations for a non-closed-loop thick-wall elastic tube are used in [4]. Physical measurements with a closed loop consisting of a rigid and an elastic tube are described in [8].

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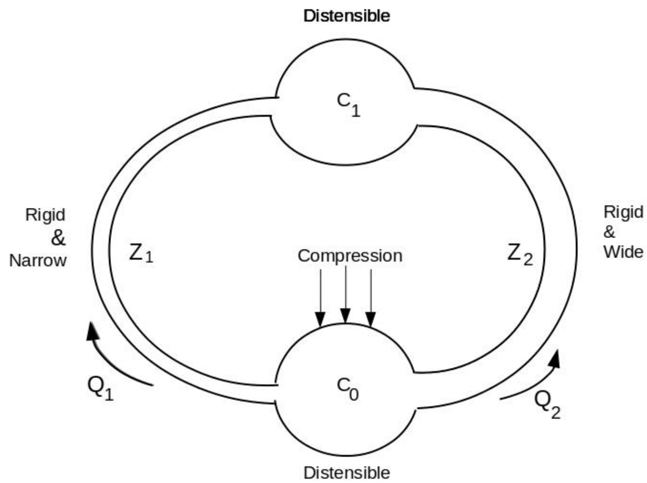

FIGURE 1. Reproduction of Figure 3 (a) in [9]

One of the origins of research on valveless pumping is in cardiovascular science [5, 9], where one is interested primarily in closed loop configurations. In the present article we consider two distensible vessels that are connected by two rigid pipes all of which are filled with an incompressible fluid such as water. This is the configuration of [9]. The momentum equations for rigid pipes are ordinary differential equations. [9] is quoted in [1, 6, 10, 11, 13] and many other published papers, even in [8, 17]. Yet, the outdated International Journal of Cardiovascular Medicine and Science is not easy to get. For ease of reference, transparency and reproducibility Figure 3 (a) and equations (1a),(2),(3),(4),(1b) of [9] are shown in Figure 1 and equations (1),(2),(3),(4),(5). We also reproduce the sentence between (4) and (5).

In [9] (and iterated in [10]) a piecewise linear model for the flows Q_1, Q_2 in two rigid pipes that connect two distensible reservoirs is considered; see Figure 1. The time interval $[0, \infty)$ is covered by intervals of length $T > 0$, $[kT, (k+1)T), k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and each of these intervals is split into $[kT, kT + T_0)$ (Phase a) and $[kT+T_0, (k+1)T)$ (Phase b), where $0 < T_0 < T$. With the nomenclature of [9] the model equations are

$$Q_0(t) = Q_1(t) + Q_2(t) \tag{1}$$

$$p_0(t) - p_1(t) = R_1 Q_1(t) \tag{2}$$

$$p_0(t) - p_1(t) = L_2\, d\, Q_2(t)/dt \tag{3}$$

$$Q_1(t) + Q_2(t) = C_1\, d\, p_1(t)/dt \tag{4}$$

When reservoir C_0 is not compressed, eq. (1) is replaced by

$$Q_1(t) + Q_2(t) = -C_0\, d\, p_0(t)/dt \tag{5}$$

The reservoir C_0 is periodically compressed (Phase a) modeled by the enforcement of Q_1+Q_2 in equation (1) and released (Phase b) according to equation (5). For the (numerical) computation of an (asymptotically) periodic solution of this model, one needs to track the volumes V_0, V_1 whose derivatives are $\mp(Q_1+Q_2)$ and this also yields the pressures $p_i = C_i^{-1}V_i$. In [9] the continuity of the solutions is not discussed, however the graphical simulation results seem to show discontinuities at $kT, kT+T_0$. In fact, for $R_1, L_2 \neq 0$, suppose that $Q_1(t), Q_2(t)$ are T-periodic solutions of (2)-(3); then they are continuous during Phase a and Phase b. So, by the fundamental theorem of calculus, we integrate (2)-(3) over a full period and get for the average $\overline{Q_1}$ of Q_1

$$\begin{aligned}
\overline{Q_1} &= \frac{1}{T}\int_{kT}^{(k+1)T} Q_1(t)\,dt \\
&= \frac{L_2}{TR_1}\left(\int_{kT}^{kT+T_0} \frac{d}{dt}Q_2(t)\,dt + \int_{kT+T_0}^{(k+1)T} \frac{d}{dt}Q_2(t)\,dt\right) \\
&= \frac{L_2}{TR_1}\left(\lim_{t\uparrow kT+T_0} Q_2(t) - \lim_{t\downarrow kT} Q_2(t) + \lim_{t\uparrow (k+1)T} Q_2(t) - \lim_{t\downarrow kT+T_0} Q_2(t)\right).
\end{aligned}$$

Periodicity of Q_2 implies $\lim_{t\uparrow(k+1)T} Q_2(t) = \lim_{t\uparrow kT} Q_2(t)$. It follows that $\overline{Q_1} \neq 0$ if and only if either $Q_2(t)$ is discontinuous at kT or at $kT+T_0$ or at both such that the jumps do not cancel each other. In particular, a continuous periodic solution of this model has average flow $\overline{Q_1} = 0$. So the main result of [9], namely nonzero average flow, must have been produced by simulations with discontinuous flows – without stating this aspect explicitely; [10] reports on avoiding a discontinuity (in p_0), but Q_1 in Figure 2.11 of [10] again looks very discontinuous.

However, discontinuous (in time) flows are not physically realistic, in particular not for valveless models that are of concern here.

It happens in mathematical modeling that physically unrealistic features are accepted, just for simplicity and as long as the features are not important for the aspect of interest. But in [9, 10] discontinuous (in time) flow is crucial for the main theme of the model, namely average flow in a valveless circular configuration. The discontinuities are not discussed in these works, nor in the many publications citing [9]. In [9] the discontinuities are missed, uncommented simulation artefacts. This should be repaired.

In the present article we generalize the model of [9] (a full momentum equation for either pipe) and introduce more flexibility (lengths and cross sections of the pipes, etc). But as long as the model is (piecewise) linear and the solutions are continuous, there will be neither pumping nor average flow. Thus, we introduce nonlinear loss terms of the form $\zeta\frac{\varrho}{2}Q_i^2$ and - for some versions - prove existence of continuous periodic and pumping solutions with average flow. For the two phase models, we present simulation results that are continuous and pumping.

That the models are simple is advantageous, since thereby the role of the various parameters and in particular of the non-linearity becomes very clear. Some of the quoted references were (co)authored by physiologists, so they understand and accept that some aspects of the cardiovascular system are represented by a configuration as simple as in Figure 1.

2. The models

Let p^0, p^1 be the time varying pressures in two fluid filled vessels that are connected by a rigid pipe of length $\ell > 0$. The idealized momentum equation for the motion of the fluid in the pipe is

$$\rho \ell u'(t) = p^0(t) - p^1(t). \tag{6}$$

$\rho > 0$ is the density of the incompressible fluid and u is the fluid velocity in direction from vessel 0 toward vessel 1. $'$ stands for the derivative with respect to time t. We will frequently suppress writing the explicit t-dependence. We model pipe friction by subtracting the pressure $R\ell u$ from the right hand side of (6). $R \geq 0$ is Poiseuille's friction coefficient of the pipe. According to Bernoulli's law along streamlines, the pressure drop due to acceleration at the junctions of the pipe and the vessels, is modeled by subtracting $\zeta \frac{\rho}{2} u^2$ from the pressure in the vessel, where $\zeta \geq 1$ is a dimensionless friction coefficient depending on the geometry and smoothness of the junction. Thereby it is assumed that the fluid in the vessel is at rest. This is in accordance with flow out of the vessel into the pipe when the cross section of the vessel is large in comparison to the cross section of the pipe. But it is less in accordance with flow in the direction from the pipe into the vessel. To take this asymmetry into account, the quadratic term is turned off for the latter direction leading to $\frac{1}{2}(1 + (-1)^j \mathrm{sgn} u)\zeta \frac{\rho}{2} u^2$ for vessel $j = 0, 1$ (cf. [15, 16]). Modeling a vessel to be distensible like an elastic balloon, we let $p^j = C^j(V^j - V^j_*)$, where $C^j > 0, V^j_*, j = 0, 1$ are given model parameters. The variation of the hydrostatic pressure at the bottom of a vertical rigid cylindrical open tank due to the variation of the level height of the fluid in the tank yields a linear model of the same structure (eg. eq. (7) in [13]). We will consider closed loop configurations with two pipes connecting the two vessels as in [9]. The subindices i are used for the two pipes. We combine three types of forcing in vessel 0 and two versions of the quadratic terms: with or without signum switches.

2.1. Pressure forcing (`prefor`).

Let $p(t)$ be a controllable pressure in addition to the elasticity or to the hydrostatic pressure in vessel 0. Denoting the constant cross section of pipe i by $A_i > 0$ we get for $i = 1, 2$ and $j = 0, 1$

$$\begin{aligned} \rho \ell_i u_i' &= p + p^0 - p^1 - R_i \ell_i u_i - \left(\frac{1 + \mathrm{sgn} u_i}{2} \zeta_i^0 - \frac{1 - \mathrm{sgn} u_i}{2} \zeta_i^1\right) \frac{\rho}{2} u_i^2 && (7) \\ V^{j\prime} &= -(-1)^j (A_1 u_1 + A_2 u_2) && (8) \end{aligned}$$

where $p^j = C^j(V^j - V_*^j)$. It is reasonable to assume that the volume V_* outside of the always filled and undeformed pipes is constant, i.e. $V^0(t) + V^1(t) \equiv V_*$. Then (7),(8) can be reduced to an equivalent model with state vector $(u_1(t), u_2(t), V^0(t)) \in \mathbb{R}^3$: We omit the $j = 1$ equation in (8) and call the remaining V^0-equation (3'); and we replace $p^0 - p^1$ in (7) by $-P_* + CV^0$ and call the result (2'). The constants are $P_* = C^0V_*^0 + C^1(V_* - V_*^1)$ and $C = C^0 + C^1$. If (u_1, u_2, V^0) is a solution of (2'),(3'), then $(u_1, u_2, V^0, V_* - V^0)$ is a solution of (7),(8) and the first three components of a solution to (7),(8) are a solution to (2'),(3'). We refer to (7),(8) or (2'),(3') as model `preforsgn`; when the signum switches $\frac{1}{2}(1 \pm \mathrm{sgn} u_i)$ are replaced by 1, we call it `prefor`.

2.2. **Flow forcing (`flofor`).** Similar as in [15],[16],[13](model (d)) instead of the pressure p let the flow in and out of vessel 0 be controlled by a forcing function $f(t)$: $V^{0\prime} = -f$. A possible physical realization is given by a rigid tank whose time varying volume is enforced by a piston (cf. [15],[16]). Of course, $V^{1\prime} = +f$ or $V^1 = V_* - V^0$. Subtracting $(7)_1$ from $(7)_2$ in `prefor`, and using $A_1u_1 + A_2u_2 = f$ and its derivative, gives

$$\begin{aligned}
\rho(A_1\ell_2 + A_2\ell_1)u_2' \;=\;& (\rho\delta_1\frac{A_2}{A_1}f - A_1R_2\ell_2 - A_2R_1\ell_1)u_2 \\
&+\frac{\rho}{2A_1}(A_1^2\delta_2 - A_2^2\delta_1)u_2^2 \\
&+R_1\ell_1 f - \frac{\rho}{2A_1}\delta_1 f^2 + \rho\ell_1 f'
\end{aligned} \tag{9}$$

where $\delta_i = \zeta_i^1 - \zeta_i^0$. We refer to this model as `flofor`; when each ζ_i^j is multiplied by the switch $\frac{1}{2}(1+(-1)^j\mathrm{sgn} u_i)$, we call it `floforsgn`. These are inhomogeneous ODEs with a time dependent coefficient of u_2.

For some of the analysis it is more convenient not to eliminate u_1^2 in the right hand side of $(7)_1-$ $(7)_2$, which, by $u_1 = (f - A_2u_2)/A_1$, gives

$$\rho(A_1\ell_2 + A_2\ell_1)u_2' = -(A_1R_2\ell_2 + A_2R_1\ell_1)u_2 + \frac{\rho}{2}A_1(\delta_2u_2^2 - \delta_1u_1^2) + R_1\ell_1 f + \rho\ell_1 f'. \tag{10}$$

2.3. **Partial forcing (`parflo`).** Inspired by [9] we look at a model wherein each time-period is split into two phases.. Let $T > 0$ and $T > T_0 \geq 0$ be fixed. For $k \in \mathbb{Z}$, the time interval $[kT, kT + T_0)$ is called phase a and $[kT + T_0, (k+1)T)$ is called phase b. During phase a the model for the flow is `flofor` (or `floforsgn`) with some appropriate forcing $f(t)$, and during phase b the model is `prefor` (or `preforsgn` respectively) with $p \equiv 0$, which means 'relaxation' for vessel 0. Initial conditions (i.c.) are given at $t = 0$, for notational uniformity we denote them by $u_2(0-), V^0(0-)$. In case of periodicity the solution may be extended to $t < 0$.

The two phase model has the four state components u_1, u_2, V^0, V^1 and, for $k = 0, 1, 2, \ldots$, is defined as follows:

Phase a, $t \in [kT, kT + T_0)$:

$$
\begin{aligned}
u_2 &\quad \text{satisfies (9) with i.c. } u_2(kT) = u_2(kT-) \\
u_1 &= \frac{1}{A_1}(f - A_2 u_2) \\
V^{0\prime} &= -f \text{ with i.c. } V^0(kT) = V^0(kT-) \\
V^1 &= V_* - V^0 \\
f &\quad \text{is continuously differentiable and satisfies} \\
&\quad \text{the continuity condition } f(kT) = A_1 u_1(kT-) + A_2 u_2(kT-)
\end{aligned}
$$

Phase b, $t \in [kT + T_0, (k+1)T)$:

$$
\begin{aligned}
u_i &\quad \text{satisfy (2') with } p \equiv 0 \text{ and i.c. } u_i(kT + T_0) = u_i(kT + T_0-) \\
V^{0\prime} &= -(A_1 u_1 + A_2 u_2), \text{ with i.c. } V^0(kT + T_0) = V^0(kT + T_0-) \\
V^1 &= V_* - V^0
\end{aligned}
$$

$(t-)$ denotes the left-hand limit at t. The two phase model is called `parflo` (`parflosgn` respectively). A solution of a two phase model is continuously differentiable on $[kT, kT+T_0), [kT+T_0, (k+1)T)$ and continuous at $kT, kT+T_0, k = 0, 1, 2, \dots$. The continuity of u_2, V^0, V^1 is a consequence of the initial conditions and the continuity of u_1 follows from the continuity condition on f. Because of the continuity condition, the forcing f is not independent of the state, nevertheless in $(kT, kT + T_0)$ it may be any continuously differentiable function. For example, similar to [9] (but not as rough), it could imitate a beating heart by $f(t) = f(kT) + a(t - kT)$. If we choose $a = 2(V^0(kT) - T_0 f(kT) - V_0^0)/T_0^2$, then f enforces $V^0(kT + T_0-) = V_0^0$. V_0^0 is a model parameter (in [9] $V_0^0 = 0$).

Since $f \equiv 0$ enforces constant volumes in phase a, there is no unforced `parflo`, `parflosgn` unless $T_0 = 0$ so that it coincides with `prefor`, `preforsgn` with $p \equiv 0$.

3. Equilibria, net flow and stability

To shorten the presentation we exclude the case $R_1 = R_2 = 0$.

An equilibrium of an unforced dynamic system is a solution such that all time derivatives vanish.

A (constant or nonconstant) periodic solution to one of the above models is called a *net flow* (or *average flow*) if the means of its velocity components are not zero.

In the non-loop models as in [13, 14], net flow is impossible, because the volumes in two tanks connected by one pipe must be periodic too.

First we show that equilibria of `preforsgn` do not exhibit net flow. We write ζ_i for the factor in front of $\frac{\varrho}{2}u_i^2$ in (7), i.e. $\zeta_i = -\zeta_i^1$ if $u_i < 0$ and $\zeta_i = +\zeta_i^0$ if $u_i > 0$. Let (u_1, u_2, V^0, V^1) be an equilibrium of `preforsgn` with $p \equiv 0$. Because of (8) $u_1 > 0$ implies $u_2 < 0$ and $u_1 < 0$ implies $u_2 > 0$. In case $p^0 - p^1 > 0$, (7) implies

$R_i\ell_i u_i + \zeta_i \frac{\rho}{2} u_i^2 > 0$. So if $u_1 < 0$ we have the contradiction $R_1\ell_1 u_1 - \zeta_1^1 \frac{\rho}{2} u_1^2 < 0$ and if $u_1 > 0$, then $u_2 < 0$ and we have the contradiction $R_2\ell_2 u_2 - \zeta_2^1 \frac{\rho}{2} u_2^2 < 0$. Thus $u_1 = u_2 = 0$ which implies $p^0 - p^1 = 0$. An analogous argument holds in case $p^0 - p^1 < 0$. Therefore, in equilibrium only $p^0 - p^1 = 0$ is possible, which implies $R_i\ell_i u_i + \zeta_i \frac{\rho}{2} u_i^2 = 0$. So either $u_1 = u_2 = 0$ or $\zeta_i \rho u_i = -2R_i\ell_i$ with $R_1, R_2 > 0$. But in both the cases $u_i > 0$ and $u_i < 0$ the product $\zeta_i u_i$ is positive, and we have a contradiction. Therefore in the unforced `preforsgn` model every equilibrium has zero flow, $u_1 = u_2 = 0$. V^0 and V^1 must be such that $C^0(V^0 - V_*^0) = C^1(V^1 - V_*^1)$ or $V^0 = P_*/C, V^1 = V_* - V^0$.

Note that the nonlinear term in (7) or (2') is continuously differentiable: for $u_i \geq 0$ its partial derivative wrt u_i is $-\zeta_i^0 \rho u_i$ and for $u_i \leq 0$ it is $+\zeta_i^1 \rho u_i$ which coincide at $u_i = 0$. The Jacobian of the homogeneous right hand side of (2'),(3') in an equilibrium (u_1, u_2, V^0) is

$$\begin{pmatrix} -R_1/\rho - \zeta_1^0 u_1/\ell_1 & 0 & C/\rho\ell_1 \\ 0 & -R_2/\rho + \zeta_2^1 u_2/\ell_2 & C/\rho\ell_2 \\ -A_1 & -A_2 & 0 \end{pmatrix} \tag{11}$$

for $u_1 \geq 0$, since then $u_2 \leq 0$. For $u_1 \leq 0$ and $u_2 \geq 0$, $-\zeta_1^0$ is replaced by $+\zeta_1^1$ and $+\zeta_2^1$ is replaced by $-\zeta_2^0$. Thus, the characteristic polynomial of the Jacobian at $(0, 0, V^0)$ is

$$\lambda(\lambda + \frac{R_1}{\rho})(\lambda + \frac{R_2}{\rho}) + \frac{CA_1}{\rho\ell_1}(\lambda + \frac{R_2}{\rho}) + \frac{CA_2}{\rho\ell_2}(\lambda + \frac{R_1}{\rho}).$$

In the simple case $R_1 = R_2 =: R > 0$ the eigenvalues are

$$\lambda_1 = -\frac{R}{\rho} \text{ and } \lambda_{2,3} = -\frac{R}{2\rho} \pm \sqrt{\frac{R^2}{4\rho^2} - \frac{C}{\rho}(\frac{A_1}{\ell_1} + \frac{A_2}{\ell_2})},$$

the real parts of which are all negative and the equilibrium $(0, 0, V_0)$ of `preforsgn` (2'),(3') is locally exponentially stable. The decay rate increases with R and decreases with ρ.

For `prefor` with $p \equiv 0$ in equilibrium we have $0 = p^0 - p^1 - R_i\ell_i u_i + \delta_i \frac{\rho}{2} u_i^2$ and $0 = A_1 u_1 + A_2 u_2$. $u_i = 0$ is possible only if $p^0 - p^1 = 0$ with the volumina as in `preforsgn`. For $u_1, u_2 \neq 0$ we use the u_1-equation to replace $p^0 - p^1$ in the u_2-equation, use $u_1 = -A_2 u_2/A_1$, multiply by A_1 and divide by u_2 to get the contradiction $A_1 R_2 \ell_2 + A_2 R_1 \ell_1 = 0$ in case $A_1^2\delta_2 - A_2^2\delta_1 = 0$ (in particular in the linear model when $\delta_1 = \delta_2 = 0$) and

$$u_2 = \frac{2A_1}{\rho} \frac{A_1 R_2 \ell_2 + A_2 R_1 \ell_1}{A_1^2\delta_2 - A_2^2\delta_1} \neq 0, \text{ if } A_1^2\delta_2 - A_2^2\delta_1 \neq 0. \tag{12}$$

From u_2 we get $p^0 - p^1$ which yields $V^0 = (p^0 - p^1 + P_*)/C$ and V^1.

This formula can be interpreted as follows: $\delta_2 = \zeta_2^1 - \zeta_2^0 > 0$ means that in pipe 2 the pressure loss at the junction of vessel 1 is larger than at vessel 0 resulting in a force from 0 toward 1 and a positive u_2. Similarly a negative δ_1 favors a negative u_1 and thus a positive u_2.

The Jacobian of `prefor` (2'),(3') (with $\delta_i = \zeta_i^1 - \zeta_i^0$) is as in (11) except that $-\zeta_1^0$ is replaced by $+\delta_1$ and $+\zeta_2^1$ is replaced by $+\delta_2$. For $R_1 = R_2 > 0$ the stability of the equilibrium $(0, 0, V^0)$ is established as for `preforsgn` above. The characteristic polynomial of (2'),(3') of `prefor` at the netflow equilibrium (u_1, u_2, V^0) in (12) is

$$\lambda(\lambda+\frac{R_1}{\rho}-\frac{\delta_1}{\ell_1}u_1)(\lambda+\frac{R_2}{\rho}-\frac{\delta_2}{\ell_2}u_2)+\frac{CA_1}{\rho\ell_1}(\lambda+\frac{R_2}{\rho}-\frac{\delta_2}{\ell_2}u_2)+\frac{CA_2}{\rho\ell_2}(\lambda+\frac{R_1}{\rho}-\frac{\delta_1}{\ell_1}u_1).$$

Let us consider the case

$$\frac{R_1}{\rho} - \frac{\delta_1}{\ell_1}u_1 = \frac{R_2}{\rho} - \frac{\delta_2}{\ell_2}u_2 =: a. \tag{13}$$

In this case the eigenvalues of the Jacobian are as those for $(0, 0, V^0)$ with R/ρ replaced by a; so the net flow equilibrium is unstable if $a < 0$. When $R_1 = R_2 =: R > 0$, the first equality in (13) holds iff $\delta_1 A_2 \ell_2 = -\delta_2 A_1 \ell_1$. And $a < 0$ iff $R\ell_2 - \delta_2 \rho u_2 < 0$ which holds, for example, if $A_1 = A_2, \ell_1 = \ell_2, \delta_1 = -\delta_2 \neq 0$.

The model `flofor` with $f \equiv 0$ in case $A_1^2\delta_2 - A_2^2\delta_1 = 0$ has the only equilibrium $u_2 = 0$, otherwise there is the additional netflow equilibrium $u_2 \neq 0$ of (12). The derivative with respect to u_2 of the right hand side of (9) is $-A_1R_2\ell_2 - A_2R_1\ell_1 + \frac{\rho}{A_1}(A_1^2\delta_2 - A_2^2\delta_1)u_2$. So $u_2 = 0$ is stable and the netflow equilibrium is unstable.

Finally, consider `floforsgn` with $f \equiv 0$. The equilibrium $u_2 = 0$ is stable, because the derivative of (9) is negative at $u_2 = 0$. If $u_2 > 0$ then $u_1 < 0$ since $A_1u_1 + A_2u_2 \equiv 0$ and in (9) δ_1 is replaced by $+\zeta_1^1$, δ_2 is replaced by $-\zeta_2^0$. So (9) implies $0 = -(A_1R_2\ell_2 + A_2R_1\ell_1)u_2 + \frac{\rho}{2A_1}(A_1^2(-\zeta_2^0) - A_2^2\zeta_1^1)u_2^2$ which is a contradiction since $\zeta_i^j \geq 1$. Analogously, $u_2 < 0$ leads to a contradiction. Thus `floforsgn` does not have netflow equilibria.

In summary, the unforced sgn models have zero flow equilibria only. The explanation is that the sgn switches at each end of the pipes together resist flow in any direction. But in the no-sgn models the difference of the friction coefficients $\delta_i = \zeta_i^1 - \zeta_i^0$ may be nonzero and if $A_1^2\delta_2 - A_2^2\delta_1 \neq 0$, then there is a netflow with u_2 in the direction of the sign of this quantity. However, these net flow equilibria are unstable and thus are not physically realistic.

4. Existence of periodic solutions

The models `prefor` and `flofor` can be written in the form

$$x' = Ax + b(t) + \varepsilon h(t, x, \varepsilon), \tag{14}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix, the continuous function b is T-periodic, ε is a scalar and h is $\mathbb{R}^n$ valued. We deduce the existence of T-periodic solutions from the theory in [3] chapter IV, and check the sufficient properties of A, b, h, listed after [3, IV (2.4)], namely

- $x' = Ax$ is noncritical with respect to periodic functions,
- h is continuous and locally Lipschitz in x uniformly in t and uniformly in ε in a bounded interval that contains 0.

In `prefor` we let $R_1 = R_2 > 0$. Then the eigenvalues of the constant matrix A in (2'),(3'), which is (11) with $\zeta_i^j = 0$, all have negative real part. The same is true for the 1×1 matrix

$$A = \frac{1}{\rho(A_1\ell_2 + A_2\ell_1)}(-A_1R_2\ell_2 - A_2R_1\ell_1)$$

in (9). Therefore, $x' = Ax$ is noncritical with respect to periodic functions

$$b(t) = \begin{pmatrix} (p(t) - P_*)/\rho\ell_1 \\ (p(t) - P_*)/\rho\ell_2 \\ 0 \end{pmatrix}$$

in `prefor` and

$$b(t) = \frac{1}{\rho(A_1\ell_2 + A_2\ell_1)}(R_1\ell_1 f - \frac{\rho}{2A_1}\delta_1 f^2 + \rho\ell_1 f')$$

in `flofor`. Let ε be that number among δ_1, δ_2 that has the largest absolute value. For `prefor` we define the nonlinear term

$$h(t, \begin{pmatrix} u_1 \\ u_2 \\ V_0 \end{pmatrix}, \varepsilon) = \frac{\rho}{2}\begin{pmatrix} \delta_1 u_1^2/2\varepsilon\ell_1 \\ \delta_2 u_2^2/2\varepsilon\ell_2 \\ 0 \end{pmatrix}.$$

Because $|\delta_i/\varepsilon| \leq 1$ and $|u^2 - w^2| \leq |u + w||u - w|$, h is locally Lipschitz in the second argument with the required uniformity. The same is true for h in `flofor`,

$$h(t, u_2, \varepsilon) = \frac{1}{A_1\ell_2 + A_2\ell_1}(\frac{\delta_1}{\varepsilon}\frac{A_2}{A_1}fu_2 + \frac{1}{2A_1}(A_1^2\frac{\delta_2}{\varepsilon} - A_2^2\frac{\delta_1}{\varepsilon})u_2^2),$$

if f is continuous and periodic. Therefore, [3, IV Theorem 2.1] yields

Theorem 4.1. *Suppose p is continuous and T-periodic. There are positive constants ε_1, ρ_1 such that for $|\delta_i| = |\zeta_i^1 - \zeta_i^0| < \varepsilon_1$ there exists a T-periodic solution of* `prefor` *and this is the only such solution in a ρ_1 neighborhood of the periodic solution of the linear system with $\delta_i = 0$. If f is continuous and T-periodic, the same statement (with appropriate ε_1, ρ_1) holds for* `flofor`.

Remark. The existence proof with the help of (14) is not applicable for the sgn models. The difficulty in `preforsgn` is not lack of Lipschitz continuity, as $\mathrm{sgn} u_i u_i^2$ is continuously differentiable at $u_i = 0$. But

$$-\delta_1 = \frac{1 + \mathrm{sgn} u_i}{2}\zeta_i^0 - \frac{1 - \mathrm{sgn} u_i}{2}\zeta_i^1 = \frac{1}{2}(\zeta_i^0 - \zeta_i^1) + \frac{1}{2}\mathrm{sgn} u_i(\zeta_i^0 + \zeta_i^1)$$

where $\zeta_i^0 + \zeta_i^1 \geq 2$ can not be made small. Moreover in `floforsgn`, the terms involving δ_1 depend on $u_1 = (f - A_2u_2)/A_1$ and are discontinuous in time. So we have no proof of existence of periodic solutions to the sgn models.

In our formulation of `parflo` we tried to mimic [9] and included the continuity condition on the forcing f in order to guarantee continuity of u_1. But this makes f depend on the state variables u_i and so this model is not a standard ODE problem. Also, splitting the period into two parts before and after $kT + T_0$ is not covered by existence proofs in the literature on periodic solutions. Another, more standard version would be to prescribe the pressure p in vessel 0 during phase a and set it to zero during phase b. Of course, continuity of the state variables u_i, V^j is to be brought about by adequate initial conditions at kT and $kT + T_0$. This renders a forcing that is independent of the state variables, but possibly discontinuous at $kT, kT + T_0$. Instead of analyzing the existence of periodic solutions to such a model (call it `parpre`), we propose to use the above results on `prefor` and approximate a discontinuous forcing p by continuous functions.

5. Pumping solutions

In [13] a dynamic system that has a solution with certain properties was defined to be a pump. The following definition is a bit more specific, less precise and more handy.

Definition 5.1. *A periodic solution x with mean $\overline{x}$ to a dynamic system $x'(t) = F(x(t), f(t))$ where f is a periodic forcing with mean $\overline{f}$ is called a* pumping solution *if $F(\overline{x}, \overline{f}) \neq 0$, i.e. $\overline{x}$ is not an equilibrium of the $\overline{f}$-forced system.*

Here and throughout we write $\overline{x}$ for the mean $\frac{1}{T}\int_t^{t+T} x\,d\tau$ of a periodic function x on $\mathbb{R}$ with period $T > 0$.

Suppose u_1, u_2, V^0, V^1 is a periodic solution of `prefor` with $\overline{p} = 0$. Taking the mean in (7), (8) gives

$$\begin{aligned} 0 &= -R_i\ell_i\overline{u_i} + \delta_i\frac{\rho}{2}\overline{u_i^2} + \overline{p^0 - p^1} \\ 0 &= A_1\overline{u_1} + A_2\overline{u_2}. \end{aligned} \tag{15}$$

Extracting $\overline{p^0 - p^1}$ from the u_1-equation and inserting it into the u_2-equation, we get

$$R_2\ell_2\overline{u_2} - \delta_2\frac{\rho}{2}\overline{u_2^2} = -R_1\ell_1\frac{A_2}{A_1}\overline{u_2} - \delta_1\frac{\rho}{2}\overline{u_1^2}$$

or, if $R_1 + R_2 > 0$,

$$\overline{u_2} = A_1\frac{\rho}{2}\frac{\delta_2\overline{u_2^2} - \delta_1\overline{u_1^2}}{A_1R_2\ell_2 + A_2R_1\ell_1}, \tag{16}$$

which is nonzero iff $\delta_2\overline{u_2^2} \neq \delta_1\overline{u_1^2}$, e.g. if $\delta_1\delta_2 < 0$. So we have a pumping solution (unless u_2 in (12) equals $\overline{u_2}$ in (16)).

The same formula for $\overline{u_2}$ holds for a periodic solution of `flofor` with $\overline{f} = 0$. This can be seen by taking the average in (10).

In `parflo` we need to split the integrals into the part over $[kT, kT+T_0)$ denoted by a superscript a, e.g. $u_1^a = \int_{kT}^{kT+T_0} u_1\, d\tau$, and the part over $[kT+T_0, (k+1)T)$ denoted by a superscript b, e.g. $u_1^b = \int_{kT+T_0}^{(k+1)T} u_1\, d\tau$. Let u_1, u_2, V^0, V^1 be a T-periodic solution of `parflo`. Then

$$0 = \int_{kT}^{(k+1)T} V_0'\, d\tau = (V_0')^a + (V_0')^b = -f^a - (A_1 u_1 + A_2 u_2)^b$$

or

$$u_1^b = \frac{1}{A_1}(-f^a - A_2 u_2^b). \tag{17}$$

From (10) and $u_1 = \frac{1}{A_1}(f - A_2 u_2)$ we get in Phase a

$$\rho(\ell_2 u_2' - \ell_1 u_1') = -R_2 \ell_2 u_2 + R_1 \ell_1 u_1 + \frac{\rho}{2}(\delta_2 u_2^2 - \delta_1 u_1^2).$$

Subtracting $(7)_1$ from $(7)_2$, the same equation holds in Phase b. Integration yields

$$\begin{aligned}
\rho(\ell_2 u_2' - \ell_1 u_1')^a &= -R_2 \ell_2 u_2^a + R_1 \ell_1 \frac{1}{A_1}(f^a - A_2 u_2^a) + \frac{\rho}{2}(\delta_2 u_2^2 - \delta_1 u_1^2)^a \\
\rho(\ell_2 u_2' - \ell_1 u_1')^b &= -R_2 \ell_2 u_2^b + R_1 \ell_1 \frac{1}{A_1}(-f^a - A_2 u_2^b) + \frac{\rho}{2}(\delta_2 u_2^2 - \delta_1 u_1^2)^b
\end{aligned}$$

where we used (17). Taking the sum of these two equations, the left hand sides cancel because of periodicity and continuity at $kT + T_0$. In the sum of the right hand sides f^a and $-f^a$ cancel, so again we arrive at formula (16) for $\overline{u_2}$.

A physical interpretation of (16) is that a positive $\delta_i = \zeta_i^1 - \zeta_i^0$ has the effect of a valve in positive u_i-direction by reducing the mean of the pressure at vessel 1 more than at vessel 0; this supports positive flow from 0 to 1. The $-\delta_1$ in the $\overline{u_2}$ formula represents the inhibition of positive flow in pipe 2 by a pressure difference due to $\delta_1 > 0$. For example if $\delta_2 > 0$ (< 0) and $\delta_1 < 0$ (> 0), then both pipes contribute to a positive (negative) average velocity $\overline{u_2}$.

Remark. Similar computations for the sgn models lead to terms of the form $\widehat{u_i^2} := \frac{1}{T}\int_0^T \mathrm{sgn} u_i(t)\, u_i(t)^2\, dt$ in addition to $\overline{u_i^2} > 0$. But the sign of $\widehat{u_i^2}$ depends of the shape of $u_i(t)$, and so the prediction of pumping in the sgn models is not as straightforward as in the non-sgn models.

6. Simulations

We briefly report on numerical simulations with `parflo, parflosgn` using $f(t) = A_1 u_1(kT-) + A_2 u_2(kT-) + a(t - kT)$ during Phase a as described in section 2. It happened that the numerical solution with `parflo` led to a blow up in finite time. Roughly speaking, this behavior is induced by opposite signs of the δ_i, large $|\delta_i|$, large C_i and small R_i. We never observed a blow up with `parflosgn` (which, incidently, can produce subharmonics with period $2T$, eg. with the parameters of the example below, except $T = 1, C^0 = C^1 = 5$). But in a wide range of physically reasonable parameters the observation was that both `parflo` and `parflosgn` converge to a limit cycle with period T. Figure 2 shows the last two cycles of length $T = 2$ of a simulation with `parflo` that was started at $t = 0$ and stopped at $t = 600$. For this simulation the parameters are $T = 2, T_0 = T/5, \rho = 1, \ell_1 = \ell_2 = 30, A_1 = A_2 = 5, R_1 = R_2 = 0.5, C^0 = C^1 = 25, V_*^0 = V_*^1 = 1, \zeta_1^1 = 1, \zeta_1^0 = 2, \zeta_2^1 = 2, \zeta_2^0 = 1$ so that $\delta_2 = -\delta_1 = 1$, $V^0(0) = V^1(0) = 30, u_1(0) = u_2(0) = 0, V_0^0 = V^0(0)/4$. In the centimeter-gram-second system of units these values could describe a laboratory desktop configuration with water as fluid. With the same parameters the result with `parflosgn` looks quite similar, only the vertical distance of the parallel curves for u_1 and u_2 is about half as wide and the maxima (minima) are somewhat smaller (larger). Numerical integration of the simulation results over the last period gives $\overline{u_1} = -0.8860$ (-0.2808) and $\overline{u_2} = 0.8875$ (0.2830) for `parflo` (for `parflosgn`). This means counter-clockwise net flow as predicted in (16). However, because $A_1 = A_2$ in this example, we should have $\overline{u_1} = -\overline{u_2}$, otherwise the solution could not be periodic. That discrepancy seems to be due to inexact numerical integration rather than to non-periodicity in $u_i(t)$.

In this example we have chosen identical pipes in order to demonstrate that pumping is not due to asymmetric properties (inertia or friction) of the two pipes, as deemed necessary in some of the literature, eg. [1, 5, 9, 15, 16]. Of course, if we let, in addition, $\delta_1 = \delta_2$, then in `parflo`, $u_1(t) = u_2(t), t \geq 0$ and $\overline{u_i} = 0$ as in (16). The same is true for `parflosgn` if $\zeta_1^j = \zeta_2^j, j = 0, 1$. On the other hand, the curves $u_1(t), u_2(t)$ are not parallel as in Figure 2 when the pipe parameters ℓ_i, R_i are chosen different for $i = 1$ and $i = 2$.

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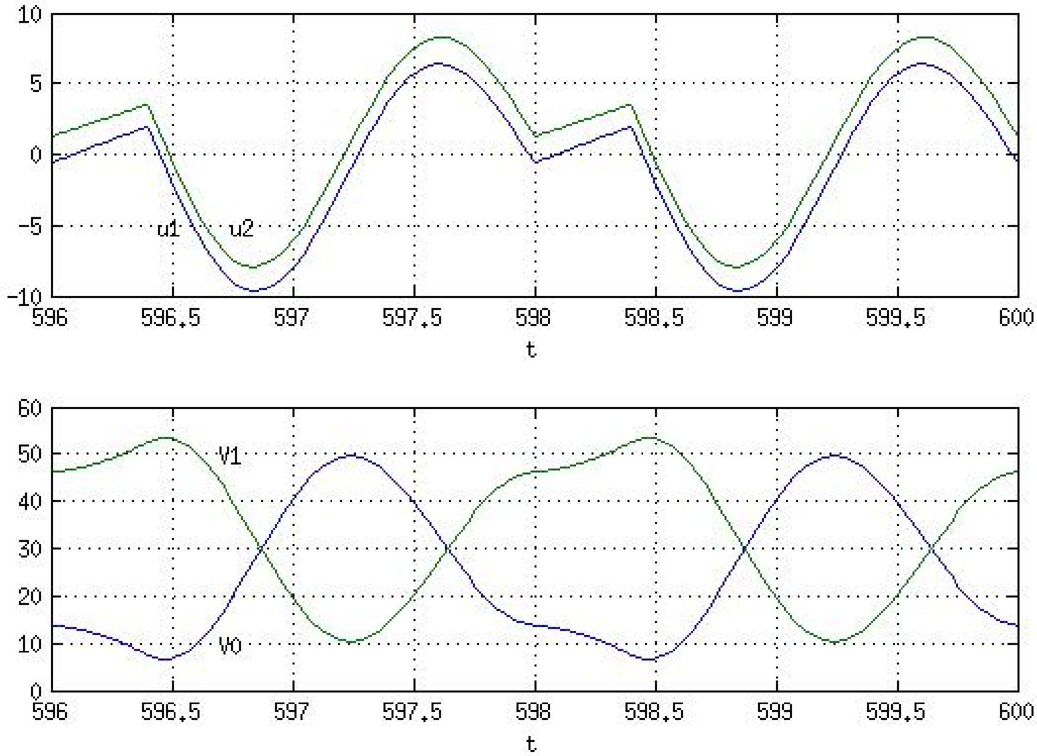

Figure 2. Simulation results

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