ASYMPTOTIC ANALYSIS OF STABILIZABILITY OF A CONTROL SYSTEM FOR A DISCRETIZED BOUNDARY DAMPED WAVE EQUATION

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1. Introduction

In this paper we analyze the finite difference approximation to the boundary (weakly) damped 1–dimensional wave equation

$$\begin{aligned} u_{tt}(t,x) &= \alpha u_{xx}(t,x) + b(x)z(t) \quad x \in (0,1), t > 0 \\ u(t,0) &= 0 \qquad\qquad\qquad\qquad\quad t > 0 \\ u_x(t,1) &= -\alpha u_t(t,1) \qquad\qquad t > 0 \end{aligned} \tag{1.1}$$

where $b \in L^2(0,1)$ and $v \in L^2(0,\infty)$ is a control. Such systems arise as models for the vibration of an elastic string or the propagation of sound in a wave–duct. It is shown in ([**?**],[L])that the solution of (1.1) decays exponentially in time at a rate which is independent of the initial state . Usually, control problems such as (1.1) are discussed in a Hilbertspace setting

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}v(t); \tag{1.2}$$

where $\mathcal{A}$ is an unbounded linear operator in a Hilbertspace $\mathcal{H}$ which generates a C_0–semigroup of bounded linear operators in $\mathcal{H}$ and $\mathcal{B}$ is a bounded linear operator from a control space $\mathcal{U}$ to the state space $\mathcal{H}$. In order to implement any control law resulting from the discussion of (1.1) some approximation of (1.2) is in general necessary. A common approach is to select finite dimensional subspaces $\mathcal{H}^N \subset \mathcal{H}$ and $\mathcal{U}^N \subset \mathcal{U}$ and to formulate a sequence of finite dimensional control systems in $\mathcal{H}^N$

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) + \mathcal{B}^N v^N(t) \tag{1.3}$$

where $\mathcal{A}^N \in \mathcal{L}(\mathcal{H}^N)$, $\mathcal{B}^N \in \mathcal{L}(\mathcal{U}^N, \mathcal{H}^N)$. Any numerical realization of (1.3) refers to a specific choice of basis in $\mathcal{H}^N$ and $\mathcal{U}^N$ relative to which (1.3) is (1.3) is given by

$$\dot{y}^N(t) = A^N y^N(t) + B^N \nu^N(T) \tag{1.4}$$

where A^N and B^N are the matrix representations of the operators $\mathcal{A}^N$ and $\mathcal{B}^N$ respectively.

Naturally, a good approximation scheme should preserve the essential qualitative properties of 1.2. More specifically, it is known that $\mathcal{A}$ describing the dynamics of (1.1) generates an exponentially stable semigroup. Consequently, the choice of an approximation scheme should be determined by whether it results in finite

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dimensional approximations which either are exponentially stable uniformly in N, or are such that at least $(\mathcal{A}^N, \mathcal{B}^N)$ are uniformly stabilizable (which is sufficient for many control purposes).

Definition 1.1. *(Stabilizability) The control system (1.2) is called stabilizable, if there exists a bounded linear operator $\mathcal{K} \in L(H, U)$ such that the semigroup $\mathcal{T}$ generated by $\mathcal{A} + \mathcal{B}K$ satisfies*

$$\|\mathcal{T}(t)\|_{L(H)} \leq Me^{-\omega t}, \quad t \geq 0,$$

for some constants $M, \omega > 0$.

Definition 1.2. *(Uniform Stabilizability) The approximating control systems (1.3) are called uniformly stabilizable, if there exist constants $M, \omega, L > 0$ such that for every index N, there exists an operator $\mathcal{K}^N \in L(H^N, U^N)$ with $\|\mathcal{K}^N\|_{L(H,U)} \leq L$ and that the semigroup $\mathcal{T}^N$ generated by $\mathcal{A}^N + \mathcal{B}^N\mathcal{K}^N$ satisfies*

$$\|\mathcal{T}^N(t)\|_{L(H)} \leq Me^{-\omega t}, \quad t \geq 0.$$

We emphasize that uniform stabilizability is a property which is independent of the choice of basis in $\mathcal{H}^N$. Unfortunately it is sometimes difficult to check (it is trivially satisfied if the systems (1.3) are uniformly exponentially stable.) It is known ([PW]) that a necessary condition for uniform stabilizability of $(\mathcal{A}^N, \mathcal{B}^N)$ requires that the stabilizability radii $\gamma_s(A^N, B^N)$ satisfy $\gamma_s(A^N, B^N) \geq \varepsilon > 0$ uniformly in N and that a certain condition of uniform equivalence of operator norms in $\mathcal{H}^N$ and $\mathbb{R}^{n(N)}$, $n(N) = \dim \mathcal{H}^N$ holds. We recall the concept of stabilizability margin below.

Definition 1.3. *(Margin of Stabilizability) For any $N \times N$ matrix A^N and any $N \times M$ matrix B^N, the margin of stabilizability of a pair of matrices (A^N, B^N) is defined by*

$$\gamma_s(A^N, B^N) = \min\{\|\delta A^N, \delta B^N\|_2, (A^N + \delta A^N, B^N + \delta B^N) \text{ is unstabilizable}\},$$

where the norm $\|\cdot\|_2$ is the matrix norm induced by the usual Euclidean vector norm in $\mathbb{R}^{N+M}$ and $\mathbb{R}^N$.

It has been shown in ([BIW])that certain mixed finite element approximations and polynomial based Galerkin approximations of the weakly damped wave equation (1.1) preserve exponential stability uniformly in N. Also there is numerical evidence in (**??**) that this is not true for commonly used schemes such as finite differences.

In this paper we analyze the spectrum of approximating systems (1.3) generated by a central difference approximation to (1.1). In particular we derive bounds for the eigenvalues of $\mathcal{A}^N$ which show that some eigenvalues of $\mathcal{A}^N$ approach the imaginary axis at a rate $\mathcal{O}(\frac{1}{n})$.Consequently, the systems (1.3) are not uniformly exponentially stable. Although the norm condition alluded to above is not satisfied for the standard choice of basis in $\mathcal{H}^N$ it can be verified for a scaled hierarchical basis. This together with a theoretical upper bound for the stabilizability margins and the knowledge of the asymptotic behavior of the eigenvalues and eigenvectors of A^N allows to prove that $\gamma_s(A^N, B^N)$ tend to zero as $N \to \infty$. Therefore, the finite dimensional systems arising from a central difference approximation to (1.1) are not uniformly stabilizable.

The outline of the paper is as follows: In Section 2 we formulate the appropriate control systems and demonstrate the advantage of using a scaled hierarchical basis in $\mathcal{H}^N$. In Section 3 we show that the eigenvalues and eigenvectors of A^N are determined by the roots of a polynomial p^N of degree $4N+2$ (which of course differs form the characteristic polynomial). We show that the roots of p^N are given by the fixed points of a certain map $\mathcal{T}$ and verify that the contraction mapping principle can be applied to certain narrow sectors in $\mathbb{C}$. Consequently, the roots of p^N can be calculated by iterating the fixed point map $\mathcal{T}$. We emphasize that this algorithm allows to calculate the roots of p^N (and consequently the eigenvactors of A^N) individually and that there is virtually no limit on the order of approximation N. Furthermore it is possible to show $\lim_{N\to\infty} \lambda_k^N = \lambda_k$, where λ_k^N and λ_k denote the k–th eigenvalue of $\mathcal{A}^N$ and $\mathcal{A}$ respectively ($\operatorname{Im}\lambda_k > 0$, $\operatorname{Im}\lambda_k^N > 0$). We think that the results of Section 3 are interesting by themselves and may provide a useful test example in numerical linear algebra.

Finally we use the precise information on the eigenvalues and eigenvectors of A^N to analyze the deterioration of the stabilizability margin $\gamma_s(A^N, B^N)$ as $N \to \infty$. We point out that the technique developped in Section 4 can be aplied to other control problems as well for a quick numerical check whether uniform stabilizability is possible.

References

[BIW] H.T. Banks, K. Ito and C. Wang: *Exponentially stable approximations of weakly damped wave equations*, CAMS 91-12

[L] J. Lagnese: *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Diff. Eqns. **50**, 1983, pp. 163–182.

[HP] D. Hinrichsen and A.J. Pritchard: *Stability radii of linear systems*, Systems & Control Letters **7**, 1986, pp. 1-10.

[PW] G. Peichl and C. Wang: *On uniform stabilizability and the margin of stabilizability* to appear in JMSEC