On the uniform stabilizability and the margin of stabilizability of the finite dimensional approximations of distributed parameter systems

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Abstract

In this paper we consider the connection between the uniform stabilizability of the finite dimension approximate control systems and the asymptotic behavior of the margin of the stabilizability of the finite dimensional vector control systems in $\mathbb{R}^N$ as N tends toward infinity in the context of approximation of infinite dimensional control systems. By investigating the changes of the stabilizability margin of finite dimension control systems under similarity transformation, we conclude that if the finite dimensional approximate control systems are uniformly stabilizable and the uniform equivalence of the operator norm condition is satisfied by the selection of the basis for the finite dimensional subspaces, then the stabilizability margin of the resulting finite dimensional vector control system in $\mathbb{R}^N$ must be uniform bounded away from zero. Since the stabilizability margin of a finite dimensional vector control system can be computed or estimated easily in many important cases, the results presented in this paper provide a useful necessary condition for the uniform stabilizability of the approximate control systems. Several numerical examples are given to illustrate the correct application of the results in this paper.

Keywords: Approximation of distributed parameter systems, Uniform stabilizability, Stabilizability margin.

1 Introduction

Many linear quadratic control problems with an infinite dimensional state space can be abstractly stated as follows. Let H and U be Hilbert spaces in which we represent the state vector and the control vector for the system considered. Find a function $u_{opt}(\cdot) \in L^2(0,\infty;U)$ which minimizes a quadratic functional J given by

$$J(u;x_0) = \int_0^\infty \left(\|\mathcal{C}x(t)\|_H^2 + \|u(t)\|_U^2\right) dt, \tag{1.1}$$

where the function $x(\cdot)$ is a mild solution of the evolution equation in the state space H given by

$$\frac{d}{dt}x(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad t > 0, \quad x(0) = x_0. \tag{1.2}$$

For the simplicity of discussion, we only consider the cases where the linear operators $\mathcal{C} : H \mapsto H$ and $\mathcal{B} : U \mapsto H$ in the above equation are bounded. In general, the operator $\mathcal{A}$ is assumed to be an unbounded linear operator which generates a C_0-semigroup of bounded linear operators in H.

Models of the form (1.2) are widely used in engineering applications involving heat transfer in continuous media, fluid dynamics, acoustics, or vibration of flexible mechanical structures. The linear quadratic regulator (LQR) problems stated above have attracted interest from an increasingly larger control engineering community in recent years partly due to the availability of adequate computational power and the progress in computational techniques for LQR problems. A common approach to approximate the optimal control for LQR problem given by (1.1) and (1.2) is to construct a sequence of finite dimensional control problems and use the optimal controls of the finite dimensional control problems to approximate the optimal control of the infinite dimensional control problem ([G], [BK]). A widely used framework for this approach is given as follows: Let $\{H^N\}_{N=1}^\infty$ and $\{U^N\}_{N=1}^\infty$ be sequences of finite dimensional subspaces of H and U, respectively. Let $\mathcal{A}^N \in L(H^N), \mathcal{B}^N \in L(U^N, H^N)$, and $\mathcal{C}^N \in L(H^N)$ be approximations of $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ in H^N and U^N, respectively. For each index N, find a control function $u_{opt}^N(\cdot) \in L^2(0,\infty;U^N)$ which minimizes the function J^N given by

$$J^N(u^N;x_0^N) = \int_0^\infty \left(\|\mathcal{C}^N x^N(t)\|_H^2 + \|u^N\|_U^2\right) dt, \tag{1.3}$$

where the function $x^N(\cdot)$ satisfies the following equation in H^N:

$$\frac{d}{dt}x^N(t) = \mathcal{A}^N x^N(t) + \mathcal{B}^N u^N(t), \quad t > 0, \quad x^N(0) = x_0^N. \tag{1.4}$$

If a sequence $\{u^N(\cdot)\}$ converges to a function $u(\cdot)$ in $L^2(0,\infty;U)$ as N tends toward infinity, for many important applications, there exist approximation techniques that guarantee the convergence of the solution $x^N(\cdot)$ of (1.4) to the solution $x(\cdot)$ of (1.2), as N tends to infinity. This type of convergence is often referred to as the convergence of the open-loop systems (or open-loop convergence). However, we are mostly interested in the convergence of the optimal control $u^N_{opt}(\cdot)$ for LQR problems given by (1.3) and (1.4) to the optimal control $u_{opt}(\cdot)$. Moreover, in most of applications the optimal control u^N_{opt} can be written in a state feedback form

$$u^N_{opt}(t) = -(\mathcal{B}^N)^* \mathcal{P}^N x^N_{opt}(t),$$

where $(\mathcal{B}^N)^*$ is the Hilbert space adjoint operator of $\mathcal{B}^N$ and $\mathcal{P}^N \in L(H^N)$ is the unique nonnegative definite solution of the operator equation

$$(\mathcal{A}^N)^*\mathcal{P}^N + \mathcal{P}^N\mathcal{A}^N - \mathcal{P}^N\mathcal{B}^N(\mathcal{B}^N)^*\mathcal{P}^N + (\mathcal{C}^N)^*\mathcal{C}^N = 0. \tag{1.5}$$

The above equation is called the algebraic Riccati operator equation. The optimal control u_{opt} also has a similar feedback form

$$u_{opt}(t) = -\mathcal{B}^*\mathcal{P}x_{opt}(t).$$

Therefore, it is useful to find approximation techniques that can provide a sequence of the operators $\mathcal{P}^N$ which converges to the operator $\mathcal{P}$ as N tends to infinity. Thus, we will be able to approximate the optimal feedback control law for the infinite dimensional system (1.1) and (1.2). Usually, the open-loop convergence of an approximation scheme is not sufficient to guarantee the convergence of the resulting operators $\mathcal{P}^N$ to $\mathcal{P}$ (also referred to as convergence of the closed-loop system, or closed-loop convergence). It is known (see [G], [BK], etc.) that an open-loop convergent approximation technique is also closed-loop convergent if the conditions of uniform stabilizability and uniform detectability are satisfied. Since the detectability is simply the stabilizability of the pair $(\mathcal{A}^*, \mathcal{C}^*)$, we only state the definition for stabilizability.

Definition 1.1 *(Stabilizability) The control system given by* (1.2) *is said to be stabilizable, if there exists a bounded linear operator $\mathcal{K} \in L(H,U)$ such that the semigroup $\mathcal{T}$ generated by $\mathcal{A} + \mathcal{B}\mathcal{K}$ satisfies*

$$\|\mathcal{T}(t)\|_{L(H)} \le Me^{-\omega t}, \quad t \ge 0,$$

for some constants $M, \omega > 0$.

Definition 1.2 *(Uniform Stabilizability) The control systems of the form* (1.4) *generated by an approximation scheme are said to be uniformly stabilizable, if there exist constants* $M, \omega, L > 0$ *such that for every index* N*, there exists an operator* $\mathcal{K}^N \in L(H^N, U^N)$ *with* $\|\mathcal{K}^N\|_{L(H,U)} \leq L$ *and that the semigroup* $\mathcal{T}^N$ *generated by* $\mathcal{A}^N + \mathcal{B}^N\mathcal{K}^N$ *satisfies*

$$\|\mathcal{T}^N(t)\|_{L(H)} \leq Me^{-\omega t}, \quad t \geq 0.$$

For a given approximation technique, the verification of the stabilizability and the uniform stabilizability of the approximate control systems (1.4) can sometimes be difficult. On the other hand, if bases $\{\phi_k^N\}_{k=1}^{n(N)}$ and $\{\xi_k^N\}_{k=1}^{m(N)}$ are chosen for H^N and U^N, respectively, (1.4) can be represented by a linear system in $\mathbb{R}^{n(N)}$ with control input in $\mathbb{R}^{m(N)}$,

$$\frac{d}{dt}y^N(t) = A^N y^N(t) + B^N v^N(t), \quad y^N(t) \in \mathbb{R}^{n(N)}, \quad v^N(t) \in \mathbb{R}^{m(N)}, \tag{1.6}$$

where A^N and B^N are matrix representations of the operators $\mathcal{A}^N$ and $\mathcal{B}^N$ with respect to the above bases. In fact, the vectors $y^N(t)$ and $v^N(t)$ are simply the coordinate vectors of elements $x^N(t)$ and $u^N(t)$ in H^N and U^N, with respect to the specified bases. For a system of the form (1.6), the margin of the stabilizability is a useful measurement of the magnitude of control effort required for its stabilization via feedback.

Definition 1.3 *(Margin of Stabilizability) For any* $N \times N$ *matrix* A^N *and any* $N \times M$ *matrix* B^N*, the margin of stabilizability of a pair of matrices* (A^N, B^N) *is defined by*

$$\gamma_s(A^N, B^N) = \min\{\|\delta A^N, \delta B^N\|_2, (A^N + \delta A^N, B^N + \delta B^N) \textit{ is unstabilizable}\},$$

where the norm $\|\cdot\|_2$ *is the matrix norm induced by the usual Euclidean vector norm in* $\mathbb{R}^{n(N)+m(N)}$ *and* $\mathbb{R}^{n(N)}$.

It can be shown that if there exists a $m(N) \times n(N)$ matrix K^N such that $\|K^N\|_2 \leq L$ and

$$\|e^{(A^N + B^N K^N)t}\|_2 \leq Me^{-\omega t}, \quad t \geq 0,$$

for some constants $L, M, \omega > 0$, then the margin of stabilizability $\gamma_s(A^N, B^N)$ is bounded away from zero by a constant $\alpha > 0$ which only depends upon the constants L, M, and ω. Therefore, recent efforts have been made to investigate the relationship between the uniform stabilizability of systems (1.4) and the margin of stabilizability of systems (1.6) (see [BP]).

The uniform stabilizability of the approximating systems has been investigated by many researchers. The condition is established in [BK] for a large class of approximation techniques when equation (1.2) corresponds to a parabolic evolution equations. However, for more general evolution equations, in particular, for the hyperbolic evolution equations, the question of uniform stabilizability is still open. In the special cases where the system (1.2) is uniformly exponentially stable with $u(t) \equiv 0$ and the solution of (1.4) preserves the uniform exponential stability in the infinite dimensional system, then the uniform stabilizability condition holds trivially by taking $\mathcal{K}^N = 0$. But it is reported in [BIW] that the uniform exponential stability is not preserved in the solution of (1.4) for most of existing approximation techniques. This does not, however, imply that the approximate systems (1.4) derived from these approximation techniques are not uniformly *stabilizable*. The analysis of the asymptotic behavior of the margin of stabilizability of (A^N, B^N) as N tends to infinity may offer an alternative approach for the verification of the uniform stabilizability of the approximation schemes.

It should be noted that the open-loop convergence and the uniform stabilizability condition are concepts independent of the choice for the bases $\{\phi_k^N\}_{k=1}^{n(N)}$ and $\{\xi_k^N\}_{k=1}^{m(N)}$ for subspaces H^N and U^N, respectively. A change of basis in H^N is equivalent to a similarity transformation $\tilde{A}^N = S^N A^N (S^N)^{-1}$ and $\tilde{B}^N = S^N B^N$ in (1.6). It is well known that the stabilizability margin is not preserved under similarity transformation. Thus, if it is always possible to choose basis for H^N such that the margin of stabilizability of the vector systems (1.6) tends to zero (or towards infinity) as N tends to infinity, which will not affect the convergence and the uniform stabilizability of systems (1.4), it would suggest that the margin of stabilizability of the vector equation (1.6) is only related to the uniform stabilizability of the control systems (1.4) under other additional conditions.

The possibility of changing basis for H^N or U^N to modify the margin of stabilizability of vector systems (1.6) has another interesting application. It is known (see [L], [KL]) that the numerical condition number of the algebraic matrix Riccati equation

$$(A^N)^* P^N + P^N A^N - P^N B^N (B^N)^* P^N + (C^N)^* C^N = 0 \tag{1.7}$$

depends upon the margin of stabilizability of the pairs (A^N, B^N) and $((A^N)^*, (C^N)^*)$. For any closed-loop convergent approximation scheme, the ability of change the basis provides the possibility to find a numerically better conditioned matrix Riccati equation.

The main objective of our current investigation is as follows: establish conditions under which the uniform stabilizability of the approximate control systems (1.4) will imply the margin of stabilizability for the vector systems (1.6) being bounded away from zero uniformly as N tends to infinity. This would provide a necessary condition for the uniform stabilizability in terms of the behavior of the margin of stabilizability of the pairs of matrices (A^N, B^N).

The remainder of this paper can be outlined as follows. In Section 2, we investigate the range of the variation of the margin of stabilizability of (1.6) under similarity transformations. In Section 3, we discuss the relationship between the asymptotic behavior of the margin of stabilizability as the dimension of the approximation system N tends towards infinity and the uniform stabilizability of the approximate control systems. In Section 4, examples and numerical experimental results are presented to illustrate the asymptotic behavior of the margin of stabilizability of several commonly used approximation techniques and the accurate interpretation of the information provided by the results of the computations. In Section 5, we will comment on the findings presented in this paper and the directions of future investigations.

2 Margin of stabilizability under similarity transformations

In this section, we consider a control system in $\mathbb{R}^n$ with $\mathbb{R}^m$ as control space

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $m \leq n$. Henceforth we shall identify the system (2.1) with the pair of matrices (A, B). Our main objective is to demonstrate that the stabilizability margin defined in Section 1 strongly depends on the particular basis chosen. In fact, we shall show that by changing the basis, γ_s may in general be made arbitrarily small. On the other hand, for controllable systems, γ_s may be assigned arbitrarily large values by an appropriate choice of similarity transformation.

Parts of our arguments are based on the following characterization of the stabilizability margin (see [E1])

$$\gamma_s(A, B) = \min_{\mathrm{Re}\lambda\geq 0} s_n([\lambda I - A, B]) = \min_{\mathrm{Re}\lambda\geq 0} \min_{\substack{\|x\|_2=1 \\ x\in\mathbb{C}^n}} \|x^*[\lambda I - A, B]\|_2.$$

For any matrix $M^{n\times(n+m)}$, $s_n(M)$ denotes the smallest singular value of M. We use $\sigma(M)$ to denote the spectrum of a square matrix $M \in \mathbb{C}^{n\times n}$. Let G be the set of regular matrices in $\mathbb{C}^{n\times n}$. We are interested in the subset $\mathcal{I}$ of $\mathbb{R}_+$ defined by

$$\mathcal{I} = \{\alpha, \alpha = \gamma_s(TAT^{-1}, TB), T \in G\}. \tag{2.2}$$

Henceforth we shall assume that the system determined by the pair (A, B) is stabilizable. The discussion of the properties of $\mathcal{I}$ will be facilitated by the following notion.

Definition 2.1 *A subset $E \subset G$ is called an unitary generator for G if every $T \in G$ admits a factorization of the form $T = US$ with $S \in E$ and U unitary.*

As a consequence of the above definition and the invariance of γ_s under unitary transformations, we obtain the following results.

Lemma 2.1 *Let E be an unitary generator for G. Then the set $\mathcal{I}$ defined in (2.2) is also given by*

$$\mathcal{I} = \{\alpha : \alpha = \gamma_s(TAT^{-1}, TB), T \in E\}.$$

Lemma 2.2 *The set $\mathcal{I}$ is an interval.*

Proof: Define $G_+ = \{T \in G : \det(T) > 0\}$. It can be shown that G_+ is a connected unitary generator for G. Hence the conclusion follows from Lemma 2.1 and the observation that γ_s depends continuously on matrices A and B. □

We recall the concept of the stability margin σ_s of a stability matrix A which was introduced in [HP1]:

$$\sigma_s(A) = \inf\{\|\delta A\|_2 : A + \delta A \text{ is not stable}\} = \min_{\mathrm{Re}\lambda \geq 0} s_n(\lambda I - A).$$

In the case where A is not stable, one defines $\sigma_s(A) = 0$.

Lemma 2.3 *For any pair of matrices (A, B), the following inequalities hold*

$$\sigma_s(A) \leq \gamma_s(A, B) \leq \sigma_s(A) + \|B\|_2.$$

Proof: The first inequality is immediate from the definitions of σ_s and γ_s by observing that if $(A + \delta A, B + \delta B)$ is unstabilizable, $A + \delta A$ must be unstable (see [HP1]). As to the second inequality, let δA be a perturbation of A such that $\|\delta A\|_2 = \sigma_s(A)$ and $A + \delta A$ is not a stability matrix, we consider the perturbation $(\delta A, -B)$. It is clear that the perturbed pair $(A + \delta A, 0)$ is not stabilizable. This implies

$$\gamma_s(A, B) \leq \|[\delta A, -B]\|_2 \leq \sigma_s(A) + \|B\|_2.$$

□

Lemma 2.4 *For any matrix $A \in \mathbb{C}^{n \times n}$, the set $\Gamma = \{\|TAT^{-1}\|_2 : T \in G, \|T\|_2 \leq 1\}$ is bounded if and only if $A = \mu I$ for some $\mu \in \mathbb{C}$.*

Proof: For $i = 1, \cdots, n$ and $\epsilon > 0$ define $T_{\epsilon,i} = \mathrm{diag}(t_1, \cdots, t_n)$ with $t_k = 1$ for all $k \neq i$ and $t_i = \epsilon$. Hence we obtain

$$\|T_{\epsilon,i}AT_{\epsilon,i}^{-1}\|_2 \geq \frac{1}{\sqrt{n}}\|T_{\epsilon,i}AT_{\epsilon,i}^{-1}\|_1 \geq \frac{1}{\sqrt{n}}\left[|a_{i,i}| + \frac{1}{\epsilon}\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{j,i}|\right].$$

since the term on the right hand side of the above inequalities tends toward infinity when ϵ tends toward zero, this implies that Γ can be bounded only if $a_{j,i} = 0$ for all $j \neq i$, i.e., $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Assume that A is diagonal and has at least two distinct diagonal elements, for example, $\lambda_1 \neq \lambda_2$. Define

$$T_\epsilon = \begin{bmatrix} S_\epsilon & 0 \\ 0 & I_{n-2} \end{bmatrix}, \quad S_\epsilon = \begin{bmatrix} \epsilon & 1 \\ 0 & \epsilon \end{bmatrix},$$

where I_{n-2} is a $(n-2)\times(n-2)$ identity matrix. Note that $[T_\epsilon AT_\epsilon^{-1}]_{1,2} = \epsilon^{-1}(\lambda_2 - \lambda_1)$, therefore,

$$\|T_\epsilon AT_\epsilon^{-1}\|_2 \geq \frac{1}{\sqrt{n}}\|T_\epsilon AT_\epsilon^{-1}\|_\infty \geq \frac{1}{\sqrt{n}}\left[|\lambda_1| + \frac{1}{\epsilon}|\lambda_2 - \lambda_1|\right],$$

which implies that Γ is unbounded. □

As a consequence of the above lemma, we infer the existence of a sequence of regular matrices T_k with $\|T_k\|_2 = 1$ such that

$$\lim_{k\to\infty}\|T_kA^{-1}T_k^{-1}\|_2 = +\infty,$$

which implies (see [HP1, p.6, Proposition 4.4]) that

$$\lim_{k\to\infty}\sigma_s(T_kAT_k^{-1}) \leq \lim_{k\to\infty}s_n(T_kA^{-1}T_k^{-1}) \leq \lim_{k\to\infty}\frac{1}{\|T_kA^{-1}T_k^{-1}\|_2} = 0, \tag{2.3}$$

for all $A \neq \mu I$. Combining the above results, we obtain the following.

Proposition 2.1 (i) *For any stabilizable pair (A, B) with $A \neq \mu I$ for any $\mu \in \mathbb{C}$, the lower bound of the interval $\mathcal{I}$ is zero.*

(ii) *For any stabilizable pair (A, B) with $A = \mu I$, the lower bound of the interval $\mathcal{I}$ is given by $|\min\{\mathrm{Re}\mu, 0\}|$.*

Proof: Let $\{T_k\}_{k=1}^{\infty}$ be a sequence of matrices for which (2.3) holds and define $S_k = \alpha_k T_k$ with $\alpha_k \in \mathbb{R}\setminus\{0\}, \lim_{k\to\infty}\alpha_k = 0$. ¿From Lemma 2.3, we obtain

$$\begin{aligned}\inf\{\alpha : \alpha \in \mathcal{I}\} &= \inf\{\gamma_s(TAT^{-1}, TB), T \in G\} \\ &\leq \lim_{k\to\infty}(\sigma_s(T_kAT_k^{-1}) + |\alpha_k|\|T_kB\|_2) = 0.\end{aligned}$$

The second part of the statement follows from

$$\gamma_s^2(S_k A S_k^{-1}, S_k B) = \min_{\lambda \in \mathbb{C}_+} \min_{\|x\|=1} (|\lambda - \mu|^2 + |\alpha_k|^2 \|T_k B x\|_2^2).$$

$\square$

Now we turn to establish the upper bounds for $\mathcal{I}$.

Proposition 2.2 *For any uncontrollable pair (A, B), the interval $\mathcal{I}$ is bounded.*

Proof: We assume without lost of generality that the system (A, B) is in the following control canonical form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{2.4}$$

We may also assume that $A_{2,2} \in \mathbb{C}^{p \times p}$ is a stability matrix. Define a subset $G(p)$ of G such that any element $T \in G(p)$ has the form

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{bmatrix}, \quad T_{2,2} \in \mathbb{C}^{p \times p}.$$

It is easy to verify that $G(p)$ is an unitary generator for G. Hence, by Lemma 2.1, $\mathcal{I} = \{\gamma_s(TAT^{-1}, TB), T \in G(p)\}$. Observe that for $T \in G(p)$

$$TAT^{-1} = \begin{bmatrix} T_{1,1} A_{1,1} T_{1,1}^{-1} & * \\ 0 & T_{2,2} A_{2,2} T_{2,2}^{-1} \end{bmatrix}.$$

On the other hand, for every pair of matrices (A, B) in the control canonical form (2.4), $\gamma_s(A, B) \leq \sigma_s(A_{2,2})$ holds. In fact, let $\delta A_{2,2} \in \mathbb{C}^{p \times p}$ such that $A_{2,2} + \delta A_{2,2}$ is not a stability matrix and $\|\delta A_{2,2}\|_2 = \sigma_s(A_{2,2})$. Then the perturbation

$$\delta A = \begin{bmatrix} 0 & 0 \\ 0 & \delta A_{2,2} \end{bmatrix}, \quad \delta B = 0$$

has the properties that $(A + \delta A, B + \delta B)$ is not stabilizable and $\|[\delta A, \delta B]\|_2 = \sigma_s(A_{2,2})$. Therefore, using the results in [HP1, p.6, Proposition 4.4], we have

$$\gamma_s(TAT^{-1}, TB) \leq \sigma_s(T_{2,2} A_{2,2} T_{2,2}^{-1}) \leq \alpha,$$

where α is given by $\alpha = \min\{|\mathrm{Re}\lambda|, \lambda \in \sigma(A_{2,2})\}$. $\square$

Next, we consider the upper bound of $\mathcal{I}$ in the case of a controllable system (A, B). First, we recall the concept of the margin of controllability (see [E1], [E2], [M])

$$\gamma_c(A, B) = \inf\{\|[\delta A, \delta B]\|_2 : (A + \delta A, B + \delta B) \text{ is not controllable}\}.$$

It has been shown in [E1], [WD] that the margin of controllability can be characterized by

(2.5) $$\gamma_c(A,B) = \min_{\lambda\in\mathbb{C}} s_n([\lambda I - A, B]) = \min_{\substack{\|x\|_2=1 \\ x\in\mathbb{C}^n}} \|x^*[x^*AxI - A, B]\|_2.$$

¿From the first equality above, it is apparent that $\gamma_c(A,B) \le \gamma_s(A,B)$.

Lemma 2.5 *Let (A,B) be controllable. Then*

$$\sup_{T\in G} \gamma_c^2(TAT^{-1}, TB) \;=\; \sup_{T\in G} \min_{\substack{\|x\|_2=1 \\ T^*x\in \mathrm{Ker}(B^*)}} \|x^*[x^*TAT^{-1}xI - TAT^{-1}]\|_2^2.$$

Proof: Let constants α, β be defined as follows

$$\alpha = \sup_{T\in G} \gamma_c^2(TAT^{-1}, TB), \quad \beta = \sup_{T\in G} \min_{\substack{\|x\|_2=1 \\ T^*x\in \mathrm{Ker}(B^*)}} \|x^*[x^*TAT^{-1}xI - TAT^{-1}]\|_2^2.$$

Then, it is easy to see from (2.5) that $\alpha \le \beta$. For any $S \in G$, consider the sequence of matrices $T_k = kS$. From (2.5) we obtain

$$\gamma_c(T_kAT_k^{-1}, T_kB) = \|x_k^*[x_k^*SAS^{-1}x_kI - SAS^{-1}]\|_2^2 + k^2\|x_k^*SB\|_2^2,$$

for some unit vector x_k. By the compactness of the unit sphere in $\mathbb{R}^k$, there exists a subsequence (again denoted by x_k) that converges to a unit vector z. If α is finite, then $S^*z \in \mathrm{Ker}(B^*)$. As a consequence, we have

$$\alpha \ge \|z^*[z^*SAS^{-1}zI - SAS^{-1}]\|_2^2 \ge \min_{\substack{\|x\|_2=1 \\ S^*x\in \mathrm{Ker}(B^*)}} \|x^*[x^*SAS^{-1}xI - SAS^{-1}]\|_2^2.$$

The above inequalities hold for all $S \in G$, this implies $\alpha \ge \beta$. Thus, $\alpha = \beta$. $\square$

Proposition 2.3 *For a controllable system (A,B),*

$$\sup_{T\in G} \gamma_c(TAT^{-1}, TB) = +\infty.$$

Thus, the upper bound of the interval $\mathcal{I}$ is $+\infty$.

The proof of this proposition is quite long and technical, for the clarity of the presentation, we feel that it may be more appropriate to place it in an appendix. See Appendix A for details of the proof of the above proposition.

3 Uniform stabilizability and margin of stabilizability of finite dimensional approximations

Consider an infinite dimensional linear control system of the form (1.2) in a Hilbert space H with a Hilbert space U as the space for control input. An approximation technique uses a sequence of finite dimensional linear control systems of the form (1.4) in the subspaces H^N of H with the subspaces U^N of U as the space for control input. In order to carry out numerical computations, a basis $\{\phi_k^N\}_{k=1}^{n(N)}$ for H^N and a basis $\{\xi_k^N\}_{k=1}^{m(N)}$ for U^N are chosen for each finite dimensional system. As a consequence, the coordinate vectors of the state and the input functions of (1.4), with respect to the above bases satisfy (1.6). In this section, we consider the uniform stabilizability of systems (1.4) as it is defined in Definition 1.2 for N sufficiently large and the asymptotic behavior of the margin of stabilizability for the pairs of matrices (A^N, B^N) as N tends to infinity.

Suppose that the approximation technique provides a sequence of finite dimensional control systems (1.4) that satisfies the uniform stabilizability condition. This property, as well as the property of open-loop convergence, is independent of the choice of the particular bases for H^N and U^N. In general, the finite dimensional operator $\mathcal{A}^N$ is not a multiple of the identity map in H^N (thus its matrix representation is not a multiple of the identity matrix), by Proposition 2.1, it is possible to choose bases in H^N and U^N so that the margin of stabilizability of the resulting pair of matrices (A^N, B^N) is less than N^{-1}. As a consequence, we have

$$\lim_{N\to\infty} \gamma_s(A^N, B^N) = 0.$$

Conversely, suppose the sequence of finite dimensional systems (1.4) given by a particular approximation scheme is not uniformly stabilizable, however, each of the finite dimensional systems is controllable (This could be the case when every sequence of feedback operators $\mathcal{K}^N$ that stabilizes the systems (1.4) up to a specified exponential decay rate has the property that the norms $\|\mathcal{K}^N\|_{L(U^N,H^N)}$ tends to infinity as N tends to infinity). Then, by Proposition 2.3, it is possible to choose bases in H^N and U^N so that the margin of stabilizability of the resulting pair of matrices (A^N, B^N) is equal to 1 for all N. The above observations indicate that the connection between the uniform stabilizability of the systems (1.4) and asymptotic behavior of the margin of stabilizability of its Euclidean space representations (1.6) is influenced by the specific choice of bases for the subspaces H^N and U^N.

In fact, consider linear operators $\delta\mathcal{A}^N \in L(H^N)$ and $\delta\mathcal{B}^N \in L(U^N, H^N)$ such that the pair of the perturbed operators $(\mathcal{A}^N + \delta\mathcal{A}^N, \mathcal{B}^N + \delta\mathcal{B}^N)$, is unstabilizable. As a consequence, the resulting pair of matrices $(A^N + \delta A^N, B^N + \delta B^N)$ which corresponds to the matrix represenation of the operators $\mathcal{A}^N + \delta\mathcal{A}^N$ and $\mathcal{B}^N + \delta\mathcal{B}^N$ with respect to any choice of bases for H^N and U^N is also unstabilizable. When the magnitude of the perturbation $\|[\delta A^N, \delta B^N]\|_2$ is measured in the matrix norm induced by the usual Euclidean norm, the same perturbations to the operators $\mathcal{A}^N$ and $\mathcal{B}^N$ could correspond to different magnitudes of perturbations to the matrix representations A^N and B^N if the bases of H^N and U^N are chosen differently. Hence, it seems useful to define a measure of the margin of stabilizability for (1.4) that is independent of the choice of bases.

For any pair of operators $\mathcal{A}^N \in L(H^N), \mathcal{B}^N \in L(U^N, H^N)$, let $[\mathcal{A}^N, \mathcal{B}^N]$ be a linear operator in $L(H^N \times U^N, H^N)$ defined by

$$[\mathcal{A}^N, \mathcal{B}^N](x^N, u^N) = \mathcal{A}^N x^N + \mathcal{B}^N u^N, \quad (x^N, u^N) \in H^N \times U^N.$$

Definition 3.1 *For a given pair of operators $(\mathcal{A}^N, \mathcal{B}^N)$ with $\mathcal{A}^N \in L(H^N)$ and $\mathcal{B}^N \in L(U^N, H^N)$, the margin of stabilizability is defined as*

$$\begin{aligned} & \gamma_s(\mathcal{A}^N, \mathcal{B}^N) \\ = \; & \min\left\{\|[\delta\mathcal{A}^N, \delta\mathcal{B}^N]\|_{L(H^N\times U^N, H^N)}, (\mathcal{A}^N + \delta\mathcal{A}^N, \mathcal{B}^N + \delta\mathcal{B}^N) \text{ is not stabilizable}\right\}. \end{aligned}$$

The following theorem was essentially proved in [H]. However, the results were stated in terms of the margin of stabilizability of the matrix representations of the operators which, as we indicated above, are only valid when the bases for the approximation subspaces are chosen appropriately.

Theorem 3.1 *If the finite dimensional systems* (1.4) *are uniformly stabilizable for all N, then there exists a constant $\epsilon > 0$ such that $\gamma_s(\mathcal{A}^N, \mathcal{B}^N) \geq \epsilon$ for all N.*

Proof: For any perturbation $(\mathcal{A}^N, \mathcal{B}^N)$, consider the following system

$$\dot{x}^N(t) = (\mathcal{A}^N + \delta\mathcal{A}^N + (\mathcal{B}^N + \delta\mathcal{B}^N)\mathcal{K}^N)x^N(t), \tag{3.1}$$

where $\mathcal{K}^N$ is the stabilizing feedback operator in Definition 1.2. Therefore, using the semigroup $\mathcal{T}^N$ in Definition 1.2, we can write

$$x^N(t) = \mathcal{T}^N(t)x^N(0) + \int_0^t \mathcal{T}^N(t-\tau)(\delta\mathcal{A}^N + \delta\mathcal{B}^N\mathcal{K}^N)x^N(\tau)d\tau.$$

By Gronwall's inequality, we obtain

$$\|x^N(t)\|_H \leq Me^{-\omega t}e^{tM\|\delta\mathcal{A}^N + \delta\mathcal{B}^N\mathcal{K}^N\|_{L(H)}}\|x^N(0)\|_H.$$

Hence (3.1) is stable for all perturbations $(\delta\mathcal{A}^N, \delta\mathcal{B}^N)$ such that

$$M\|\delta\mathcal{A}^N + \delta\mathcal{B}^N\mathcal{K}^N\|_{L(H)} < \omega.$$

The above inequality holds automatically if

$$M\sqrt{1+L^2}\|[\delta\mathcal{A}^N, \delta\mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} < \omega.$$

It follows from

$$\begin{aligned}
\sup_{x\neq 0}\frac{\|\delta\mathcal{A}^N x + \delta\mathcal{B}^N\mathcal{K}^N x\|_H^2}{\|x\|_H^2} &\leq (1+L^2)\sup_{x\neq 0}\frac{\|\delta\mathcal{A}^N x + \delta\mathcal{B}^N\mathcal{K}^N x\|_H^2}{\|x\|_H^2 + \|\mathcal{K}^N x\|_U^2} \\
&\leq (1+L^2)\sup_{\substack{x\neq 0 \\ u\in U^N}}\frac{\|\delta\mathcal{A}^N x + \delta\mathcal{B}^N u\|_H^2}{\|x\|_H^2 + \|u\|_U^2} \\
&\leq (1+L^2)\sup_{(x,u)\neq 0}\frac{\|\delta\mathcal{A}^N x + \delta\mathcal{B}^N u\|_H^2}{\|x\|_H^2 + \|u\|_U^2}.
\end{aligned}$$

This implies $\gamma_s(\mathcal{A}^N, \mathcal{B}^N) \geq \omega(M\sqrt{1+L^2})^{-1}$. □

The margin of stabilizability of the pair of operators is not easily computable. But if certain equivalence between the norm of the operators and the norm of its matrix representation is given, the above results will also hold for the systems (1.6).

Condition 3.1 *(Uniform Equivalence of the Operator Norms) A given choice of bases for the subspaces H^N and U^N is said to satisfy the uniform equivalence of the operator norms condition, if there exist positive constants c_1 and c_2 indepedent of N such that the matrix representation A^N, B^N of any operators $\mathcal{A}^N \in L(H^N)$, $\mathcal{B}^N \in L(U^N, H^N)$ satisfy*

$$c_1\|[A^N, B^N]\|_2 \leq \|[\mathcal{A}^N, \mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} \leq c_2\|[A^N, B^N]\|_2. \tag{3.2}$$

Corollary 3.1 *If the finite dimensional systems* (1.4) *are uniformly stabilizable for all N and if the choice of bases for H^N and U^N satisfies the uniform equivalence of the operator norm condition, then there exists a constant $\epsilon > 0$ such that for all N, $\gamma_s(A^N, B^N) \geq \epsilon$.*

Proof: For any perturbation $(\delta A^N, \delta B^N)$ such that

$$\|[\delta A^N, \delta B^N]\|_2 < \frac{\omega}{c_2 M\sqrt{1+L^2}},$$

(3.2) implies that

$$\|[\delta\mathcal{A}^N, \delta\mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} < \frac{\omega}{M\sqrt{1+L^2}}.$$

By the proof of Theorem 3.1, this implies (3.1) is stable. Therefore, the pair $(A^N + \delta A^N, B^N + \delta B^N)$ is stabilizable. $\square$

Naturally we would like to know how restrictive is Condition 3.1. More practically, how can one check whether or not a given choice of bases satisfies Condition 3.1. Consider a basis $\{\phi_k^N\}_{k=1}^{n(N)}$ for H^N and $\{\xi_k^N\}_{k=1}^{m(N)}$ for U^N. For vectors $x^N \in H^N$ and $u^N \in U^N$, let $y^N \in \mathbb{R}^{n(N)}$ and $v^N \in \mathbb{R}^{m(N)}$ be coordinate vectors of x^N and u^N, respectively. That is,

$$x^N = \sum_{k=1}^{n(N)} y_k^N \phi_k^N, \quad u^N = \sum_{k=1}^{m(N)} v_k^N \xi_k^N.$$

Then it is easy to verify that

$$\|x^N\|_H^2 = (y^N)^* M^N y^N,$$

where M^N is given by $[M^N]_{k,j} = \langle \phi_k^N, \phi_j^N \rangle_H, k, j = 1, \cdots, n(N)$, and

$$\|(x^N, u^N)\|_{H\times U} = ((y^N)^*, (v^N)^*) Q^N \begin{pmatrix} y^N \\ v^N \end{pmatrix}, \text{ where } Q^N \text{ is given by } \begin{bmatrix} M^N & 0 \\ 0 & V^N \end{bmatrix},$$

and $[V^N]_{k,j} = \langle \xi_k^N, \xi_j^N \rangle_U$, for $k, j = 1, \cdots, m(N)$. The matrices M^N and Q^N are positive definite. Let $\{\lambda_k^N\}_{k=1}^{n(N)}$ and $\{\mu_k^N\}_{k=1}^{n(N)+m(N)}$ be eigenvalues of M^N and Q^N such that

$$\lambda_1^N \geq \lambda_2^N \geq \cdots \geq \lambda_{n(N)}^N, \quad \mu_1^N \geq \mu_2^N \geq \cdots \geq \mu_{n(N)+m(N)}^N.$$

Then the following inequalities hold:

$$\lambda_{n(N)}^N \|y^N\|_2^2 \leq \|x^N\|_H^2 \leq \lambda_1^N \|y^N\|_2^2, \tag{3.3}$$

$$\mu_{n(N)+m(N)}^N \|((y^N)^*, (v^N)^*)\|_2^2 \leq \|(x^N, u^N)\|_{H\times U}^2 \leq \mu_1^N \|((y^N)^*, (v^N)^*)\|_2^2. \tag{3.4}$$

Since the operator norms are given by

$$\|[\mathcal{A}^N, \mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} = \max_{(x^N,u^N)\neq 0} \frac{\|\mathcal{A}^N x^N + \mathcal{B}^N u^N\|_H}{\|(x^N, u^N)\|_{H\times U}},$$

and

$$\|[A^N, B^N]\|_2 = \max_{((y^N)^*,(v^N)^*)\neq 0} \frac{\|A^N y^N + B^N v^N\|_2}{\|((y^N)^*, (v^N)^*)\|_2},$$

we obtain

$$\|[\mathcal{A}^N, \mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} \geq \sqrt{\frac{\lambda_{n(N)}^N}{\mu_1^N}} \|[A^N, B^N]\|_2, \tag{3.5}$$

$$\|[\mathcal{A}^N, \mathcal{B}^N]\|_{L(H^N\times U^N, H^N)} \leq \sqrt{\frac{\lambda_1^N}{\mu_{n(N)+m(N)}^N}} \|[A^N, B^N]\|_2. \tag{3.6}$$

On the other hand, it is possible to find vectors (x^N, u^N) such that equality holds for any inequalities in (3.3) and (3.4). Therefore, it is possible to find operators $\mathcal{A}^N$ and $\mathcal{B}^N$ such that the equalities in (3.5) or (3.6) hold. Hence, we have the following results.

Proposition 3.1 *Condition 3.1 holds if and only if there exist positive constants τ_1 and τ_2 such that*

$$\sqrt{\frac{\lambda_1^N}{\mu_{n(N)+m(N)}^N}} \leq \tau_1, \qquad \sqrt{\frac{\lambda_{n(N)}^N}{\mu_1^N}} \geq \tau_2,$$

for all N.

Theorem 3.1 and its corollary establish a necessary condition for the uniform stabilizability of the approximate control systems (1.4). In the context of verification the closed-loop convergence of an approximation technique, it is more useful to have a sufficient condition for the uniform stabilizability. The discussions at the beginning of this section indicate that if systems (1.4) are controllable, Condition 3.1 is still necessary in order to obtain a sufficient condition for the uniform stabilizability of (1.4) in terms of the asymptotic behavior of the margin of stabilizability of the pairs of matrices (A^N, B^N).

4 Examples and numerical experiments

In this section, we consider two examples of distributed parameter control systems and approximation techniques commonly used for these systems. In particular, we will check the uniform equivalence of operator norms condition for these approximation methods.

4.1 Heat equation

We consider the following one-dimensional diffusion equation

$$\partial_t x(t,s) = a\partial_s^2 x(t,s) + b(s)u(t), \quad s \in (0,1), t > 0, \tag{4.1}$$

with boundary condition

$$x(t,0) = x(t,1) = 0, \quad t > 0. \tag{4.2}$$

The function $x(t,\cdot) \in L^2(0,1) = H$ is the state vector which represents the temperature distribution on an one-dimensional rod. The function $u(\cdot) \in L^2(0,\infty)$ is the control input that is a heating/cooling source and the function b is taken to be the characteristic function of a subinterval $[\epsilon_1, \epsilon_2] \subset [0,1]$. Let H^N be the space of linear B-spline functions which satisfy the Dirichlet boundary conditions (see [S]). A Galerkin approximation scheme is defined as follows: let $x^N(t,\cdot) \in H^N$ which satisfies

$$\frac{d}{dt}\langle x^N(t,\cdot), z^N\rangle_H = -a\int_0^1 \partial_s x^N(t,s) z^N_s(s)ds + u^N(t)\int_0^1 b(s)z^N(s)ds, \tag{4.3}$$

for all $z^N \in H^N$. The most commonly used basis $\{B^N_k\}_{k=1}^N$ for H^N is defined by

$$B^N_k(s) = \begin{cases} 1-(N+1)|s-s^N_k|, & s \in [s^N_{k-1}, s^N_{k+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{4.4}$$

where $s^N_k = k/(N+1), k = 0, \cdots, N+1$. The control space in this example is $\mathbb{R}^1$, the basis for $\mathbb{R}^1$ can be any non-zero real value. Let $\theta^N > 0$ be a basis for $U^N = \mathbb{R}^1$. By using the notation defined in Section 3, the mass matrix M^N is defined by

$$[M^N]_{k,j} = \langle B^N_k, B^N_j\rangle_H, \quad k,j = 1, \cdots, N,$$

and the matrix V^N is equal to $(\theta^N)^2$. It can be shown that the eigenvalues λ^N_k of M^N are given by

$$\lambda^N_k = \frac{1}{6(N+1)}\left(4 + 2\cos(\frac{2k\pi}{N+1})\right), \quad k = 1, \cdots, N.$$

As a consequence, the eigenvalues of matrix Q^N are $\lambda^N_1, \cdots, \lambda^N_N$ and $(\theta^N)^2$. Thus, in order to satisfy the uniform equivalence of operator norm condition, we need to take $\theta^N = c/\sqrt{N+1}$ for some constant $c > 0$. Indeed for such a value of θ^N, the following inequalities hold

$$\sqrt{\frac{\lambda^N_1}{\mu^N_{N+1}}} \le \frac{1}{\sqrt{\min\{3, c\}}}, \quad \sqrt{\frac{\lambda^N_N}{\mu^N_1}} \ge \frac{1}{\sqrt{3\max\{1, c\}}},$$

because

$$\frac{1}{3(N+1)} \le \lambda_k \le \frac{1}{N+1}.$$

It is known (see for example, [BK]) that the linear spline based Galerkin approximation schemes for equations (4.1)–(4.2) satisfy the uniform stabilizability condition. Therefore, by Corollary 3.1, if $\theta^N = 1/\sqrt{N+1}$, the stabilizability margin of the matrix pair (A^N, B^N) which corresponds to the matrix representations of the operators $\mathcal{A}^N$ and $\mathcal{B}^N$ for the finite dimensional control system in H^N should be

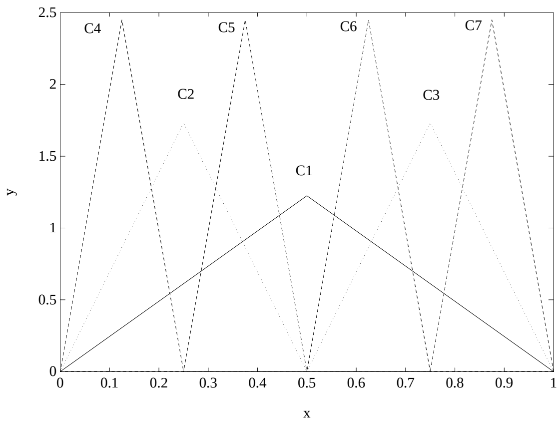

Figure 4.1: Basis elements in E_3

uniformly bounded away from zero as N tends toward infinity. From (4.3), we find that

$$A^N = (M^N)^{-1}K^N, \quad B^N = (M^N)^{-1}b^N,$$

where $[K^N]_{k,j} = \langle (B_k^N)', (B_j^N)' \rangle_H$ and $b_k^N = \theta^N \langle b, B_k^N \rangle_H$. We note that the matrix A^N is a symmetric stability matrix (i.e., the eigenvalues of A^N are negative real numbers). In this case, using a result in [BP], we have $\gamma_s(A^N, B^N) = s_N(A^N, B^N)$ where as in Section 2, $s_N(A^N, B^N)$ corresponds to the minimal singular value of the rectangular matrix $[A^N, B^N]$ which can be easily computed. The stabilizability margin of the matrix-pairs (A^N, B^N) are listed in Table 4.2.

Another useful basis for H^N when $N = 2^p - 1$ is the so-called hierarchical basis $E_p = \{C_k^N\}_{k=1}^N$ which arises naturally in the context of approximation theory based on the wavelet functions. Using the notation defined in (4.4), the sets E_p can be described recursively as follows

$$\begin{aligned} E_1 &= \left\{ C_1^N = \frac{B_1^1}{\|B_1^1\|_{L^2(0,1)}} \right\}, \\ E_p &= \left\{ C_{2^{(p-1)}-1+l}^N = \frac{B_{2j-1}^{2^p-1}}{\|B_{2l-1}^{2^p-1}\|_{L^2(0,1)}}, l = 1, \cdots, 2^{p-1} \right\} \bigcup E_{p-1}. \end{aligned}$$

Thus, all the basis elements in E_p have norm 1. In Figure 4.1, we depict the basis elements in E_3. For any $N = 2^p - 1$, the basis elements C_k^N can be easily written as

Table 4.1: Ratio between largest and smallest eigenvalues of mass matrices.

Dimension	Regular	Hierarchical
3	2.0938	7.00
7	2.7170	18.26
15	2.9245	48.94
31	2.9808	120.91
63	2.9951	287.80
127	2.9987	667.29
255	2.9996	1517.38

linear combinations of B_k^N's as follows

$$C_k^N(s) = \sum_{j=1}^{N} \alpha_{k,j}^N B_j^N(s), \quad \alpha_{k,j}^N = C_k^N(s_j).$$

If we define the matrix T^N by $[T^N]_{k,j} = \alpha_{k,j}^N$, then the mass matrix M_H^N corresponding to the new set of basis elements is given by $T^N M_B^N (T^N)^*$ where M_B^N is the mass matrix corresponding to the basis $\{B_k^N\}_{k=1}^N$. The spectral structure of the new mass matrix is more complicate to analyze (Note that the transformation from M_B^N to M_H^N is not a similarity transformation). However, the computational results show that the ratio $|\lambda_N^N/\lambda_1|$ tends toward infinity as N tends toward infinity (see Table 4.1). This indicates that the uniform equivalence of the operator norms condition is not satisfied. The matrices A^N and B^N for the hierarchical basis are defined by (4.3). Although A^N is still a stability matrix, it is no longer symmetric. However, as it is indicated in Section 2, the minimal singular value $s_N([A^N, B^N])$ provides an upper bound for $\gamma_s(A^N, B^N)$. The minimal singular values for the matrices $[A^N, B^N]$ are listed in Table 4.2, we observe a substantial decrease of the stabilizability margin as N increases. The difference in the stabilizability margin for the two systems is more clearly illustrated in Figure 4.2. The above example illustrates that although the finite dimensional control systems considered above are equivalent in terms of approximation of the infinite dimensional control system, the asymptotic behavior of the stabilizability margin for the finite dimensional vector systems can be substantially different. In fact, even though the approximation scheme is uniformly stabilizable in the sense defined in Section 1, without satisfying the uniform equivalence of the operator norms condition, the stabilizability margin of the finite dimensional vector control systems obtained from the hierarchical basis apparently tends toward zero as the dimension of the approximation systems tends toward infinity.

Figure 4.2: Minimum singular values of matrices $[A^N, B^N]$

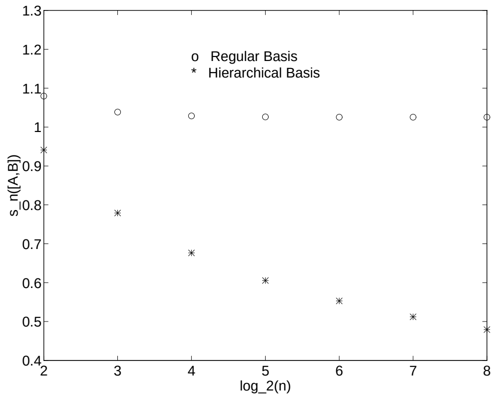

Table 4.2: Minimum singular values of matrices $[A^N, B^N]$

Dimension	Regular Basis	Hierarchical Basis
3	1.0797	0.9410
7	1.0385	0.7791
15	1.0287	0.6765
31	1.0262	0.6055
63	1.0256	0.5530
127	1.0254	0.5121
255	1.0254	0.4792

4.2 Weakly damped wave equation

We consider an one-dimensional version of the weakly damped wave equation discussed in [BIW]:

$$\partial_t^2 x(t,s) = \partial_s^2 x(t,s) + b(s)u(t), \quad s \in (0,1), t > 0, \tag{4.5}$$

with boundary condition

$$\partial_s x(t,0) = \alpha \partial_t x(t,0) + \beta x(t,0), \tag{4.6}$$
$$\partial_s x(t,1) = -\alpha \partial_t x(t,1) - \beta x(t,1), \tag{4.7}$$

where α, β are positive constants. The function $(x(t,\cdot), x_t(t,\cdot)) \in H^1(0,1) \times L^2(0,1) = H$ is considered as the state of the system defined by (4.5)–(4.7). In practice, the components of the above state vector represent the transversal displacement and the velocity of a string or a membrane. The function $b(\cdot)$ is taken to be the characteristic function $\chi[\delta_1, \delta_2]$ of the sub-interval $[\delta_1, \delta_2]$ of $(0,1)$.

A finite difference approximation scheme is defined by

$$\frac{d}{dt}\begin{pmatrix} x^N(t) \\ y^N(t) \end{pmatrix} = \begin{bmatrix} 0 & I \\ -K^N & -D^N \end{bmatrix} \begin{pmatrix} x^N(t) \\ y^N(t) \end{pmatrix} + \begin{pmatrix} 0 \\ b^N \end{pmatrix} u(t), \tag{4.8}$$

where the matrices K^N and D^N are defined as

$$K^N = (N-1)^2 \begin{bmatrix} \omega^N & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & \omega^N \end{bmatrix},$$

and $[D^N]_{1,1} = \alpha(N-1)$, $[D^N]_{N,N} = \alpha(N-1)$ and $[D^N]_{i,j} = 0$ for all other entries, the coefficient $\omega^N = 1 + \beta/(N-1)$. The vector b^N is given by $b_j^N = 1$ if $s_j^N \in [\delta_1, \delta_2]$, otherwise, $b_j^N = 0$. Since the finite difference approximation scheme is derived from approximation of the derivatives of the solution of (4.5)–(4.7) by finite difference quotients of the approximate solution on the grid points, an interpolation map needs to be specified explicitly in order to fit the framework given in Section 3.

The most common interpolation map consists of linear interpolation of the solution of (4.8), i.e., for each vector $(x^N, y^N) \in \mathbb{R}^N \times \mathbb{R}^N$, the corresponding function in H is given by

$$(x^N(s), y^N(s)) = \left(\sum_{k=1}^{N} x_k^N L_k^N(s), \sum_{k=1}^{N} y_k^N L_k^N(s) \right),$$

where L_k^N is defined by

$$L_k^N(s) = \begin{cases} 1-(N-1)|s-s_{k-1}^N|, & s \in [s_{k-2}^N, s_k^N] \bigcap [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

where $s_k^N = k/(N-1)$. Let $H_1^N = H_2^N = span\{L_k^N\}_{k=1}^N$, then the approximation subspace H^N is given by $H^N = H_1^N \times H_2^N$. The inner product in H can be defined as follows:

$$\langle (x,y), (\hat{x}, \hat{y}) \rangle_H = \langle x', x' \rangle_{L^2(0,1)} + \beta(x(0)\hat{x}(0) + x(1)\hat{x}(1)) + \langle y, \hat{y} \rangle_{L^2(0,1)}.$$

Then, the mass matrix M^N is given by

$$M^N = \begin{bmatrix} K^N/(N-1) & 0 \\ 0 & W^N \end{bmatrix},$$

where $[W^N]_{k,j} = \langle B_k^N, B_j^N \rangle_{L^2(0,1)}$. An eigenvalue of the matrix M^N is either an eigenvalue of the matrix $K^N/(N-1)$ or an eigenvalue of the matrix W^N. Therefore, if the uniform equivalence of the operator norms condition is satisfied, the ratio between the largest eigenvalue and the smallest eigenvalue of K^N (or W^N) must be bounded uniformly. We observe that the eigenvectors of both K^N and W^N have the form

$$z^N = a \begin{pmatrix} 1 \\ p \\ \vdots \\ p^{N-1} \end{pmatrix} + b \begin{pmatrix} 1 \\ p^{-1} \\ \vdots \\ p^{-(N-1)} \end{pmatrix}. \tag{4.9}$$

In the case of matrix K^N, the value p is a root of the polynomial equation

$$p^{2N} = \frac{((\omega^N - 2)p + 2)^2}{(\omega^N - 2 + p)^2} = \frac{(p - 1 - \beta p/(N-1))^2}{(p - 1 + \beta p/(N-1))^2}, \tag{4.10}$$

and in the case of matrix W^N, p is a root of the polynomial equation

$$p^{2N+2} = 1. \tag{4.11}$$

In fact, every root of (4.10) (or (4.11)) gives an eigenvector of the matrix K^N (or W^N, respectively) of the form (4.9) except when $p = \pm 1$. Moreover, all roots p of these equations have the property $|p| = 1$. Let $\{p_k^N\}_{k=1}^{2N}$ be roots of equation (4.10) (or (4.11)), the following equalities also hold:

$$\lim_{N\to\infty} \min_{1\le k\le 2N} |p_k^N - 1| = 0, \quad \lim_{N\to\infty} \min_{1\le k\le 2N} |p_k^N + 1| = 0.$$

For a root $p \ne \pm 1$ of (4.10) the corresponding eigenvalue of K^N is given by $(N-1)^2(2-p-p^{-1})$ and for a root $p \ne \pm 1$ of (4.11) the corresponding eigenvalue of W^N is given by $6(4+p+p^{-1})/(N-1)$. Therefore, if $\{\lambda_k^N\}_{k=1}^N$ and $\{\nu_k^N\}_{k=1}^N$ are eigenvalues of the matrix K^N and the matrix W^N, respectively, then we have

$$\lim_{N\to\infty} \frac{\max_{1\le k\le N} |\lambda_k^N|}{\min_{1\le k\le N} |\lambda_k^N|} = +\infty, \quad 1 \le \lim_{N\to\infty} \frac{\max_{1\le k\le N} |\nu_k^N|}{\min_{1\le k\le N} |\nu_k^N|} \le 3.$$

The ratio between the largest eigenvalue and the smallest eigenvalue of the mass matrix are given in Table 4.3. Thus, the uniform equivalence of the operator norms condition can not be satisfied for this selection of the basis. Indeed, the basis of H_1^N for the first component of the state vector should be changed.

An alternative choice of the space H_1^N for $N = 2^p + 1$ is interestingly a hierarchical basis $F_p = \{e_k\}_{k=1}^N$ defined recursively as follows:

$$\begin{aligned} F_0 &= \left\{ e_1 = \frac{L_1^2}{\|(L_1^2)'\|_{L^2(0,1)}}, e_2 = \frac{L_2^2}{\|(L_2^2)'\|_{L^2(0,1)}} \right\}, \\ F_{p+1} &= F_p \bigcup \left\{ e_{2^p+1+l} = \frac{L_{2l-1}^{2^{p+1}}}{\|(L_{2l-1}^{2^{p+1}})'\|_{L^2(0,1)}}, l = 1, \cdots, 2^p \right\}. \end{aligned}$$

In Figure 4.3 we depict the basis functions in F_3. We observe that the vector $L_{2l-1}^{2^{p+1}}$ is orthogonal to the space spanned by the collection of vectors F_p with respect to the inner product

$$\langle x, y\rangle_1 = \langle x', y'\rangle_{L^2(0,1)} + \beta(x(0)y(0) + x(1)y(1)).$$

We also make a slight change of the basis for H_2^N by taking $\{\sqrt{N-1}L_k^N\}_{k=1}^N$, under the new selection of the bases for H_1^N and H_2^N, respectively, the new mass matrix has the form

$$M_H^N = \begin{bmatrix} V^N & 0 \\ 0 & (N-1)W^N \end{bmatrix}, \quad \text{where} \quad V^N = \begin{bmatrix} 1+\beta & -1 & 0 & \cdots & 0 \\ -1 & 1+\beta & & \ddots & \vdots \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

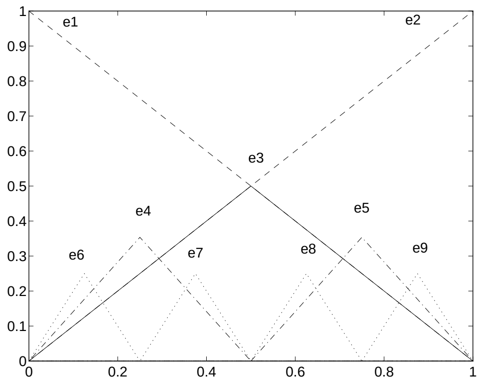

Figure 4.3: Basis elements in F_3

It can be easily observed that the uniform equivalence of the operator norms condition is satisfied by this selection of the basis. In fact, the ratio between the smallest eigenvalue of the mass matrix and the largest eigenvalues of the mass matrix are given in Table 4.3. Let $(\hat{x}^N, \hat{y}^N) \in \mathbb{R}^N \times \mathbb{R}^N$ be such that

$$(\sum_{k=1}^{N} \hat{x}_k^N \hat{L}_k^N(s), \sum_{k=1}^{N} \hat{y}_k^N \hat{L}_k^N(s)) = (\sum_{k=1}^{N} x_k^N L_k^N(s), \sum_{k=1}^{N} y_k^N L_k^N(s)),$$

Table 4.3: Ratio between the smallest and the largest eigenvalue of the mass matrix.

Index N	Regular	Hierarchical
5	184.7	27.807
9	716.6	29.349
17	2776.8	29.822
33	10868.4	29.953
65	42928.2	29.988
129	170549.7	29.997
257	679798.5	29.999

Table 4.4: Stabilizability margins for approximation of the wave equation.

Index N	Regular Basis	Hierarchical Basis
5	0.0574179	0.1243204
9	0.0177448	0.0739211
17	0.0030037	0.0197859
33	0.0004082	0.0048777
65	0.0000485	0.0012843
129	0.0000129	0.0003293

where $F_p = \{\hat{L}_k^N\}_{k=1}^N$. Then, we have

$$\begin{pmatrix} x^N \\ y^N \end{pmatrix} = T^N \begin{pmatrix} \hat{x}^N \\ \hat{y}^N \end{pmatrix},$$

where the matrix T^N is given by

$$T^N = \begin{bmatrix} S^N & 0 \\ 0 & \sqrt{(N-1)I} \end{bmatrix},$$

the matrix S^N is defined by $[S^N]_{k,j} = \hat{L}_k^N(s_{j-1}^N)$, for $k, j = 1, \cdots, N$. Under the new basis, the state equation that is equivalent to (4.8) is given by

$$\frac{d}{dt}\begin{pmatrix} \hat{x}^N(t) \\ \hat{y}^N(t) \end{pmatrix} = (T^N)^{-1}\begin{bmatrix} 0 & I \\ -K^N & -D^N \end{bmatrix} T^N \begin{pmatrix} \hat{x}^N(t) \\ \hat{y}^N(t) \end{pmatrix} + (T^N)^{-1}\begin{pmatrix} 0 \\ b^N \end{pmatrix} u(t).$$

From our computation results, we observed in both above cases that if the eigenvalues of the matrix A^N are given by $\sigma_k + i\mu_k$ for $k = 1, \cdots, 2N$, then the minimum of the function $s_{2N}([\lambda I - A^N, B^N])$ always locates at (or very close to) $\lambda = \mu_k$ for some k. As a result, we are able to compute the stabilizability margin of the pairs of matrices $[A^N, B^N]$ and $[(T^N)^{-1}AT^N, (T^N)^{-1}B^N]$. The computed margin of the stabilizability are given in Table 4.4. Since the margin of the stabilizability of the finite dimensional systems corresponding to the hierarchical basis tends to zero as N tends toward infinity, using Corollary 3.1, we can conclude based on the numerical results that the finite difference approximation of the wave equation is not uniformly exponentially stabilizable.

5 Conclusion

The analysis and the examples in this paper have shown that the relationship between the uniform stabilizability and the stabilizability margin of the matrix representation for approximate control systems depends upon the norm used in different approximation subspaces relative to the norm induced by the oringinal infinite dimensional state space. The uniform norm equivalence condition is not satisfied in most of the commonly used approximation schemes. However, a simple change of variable can be helpful in establishing this condition in many cases. Since the margin of the stabilizability (or an upper bound of the margin of stabilizability) can often be computed easily, if the margin of the stabilizability of the finite dimensional systems tends toward zero as the dimension of the approximate control system tends toward infinity, using the theoretical results presented in this paper, we can conclude that the uniform stabilizability condition is not satisfied.

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A Proof of Proposition 2.3

Proposition: *For a controllable system* (A, B),

$$\sup_{T \in G} \gamma_c(TAT^{-1}, TB) = +\infty.$$

Thus, the upper bound of the interval $\mathcal{I}$ *is* $+\infty$.

Proof: If $\text{Rank}(B) = n$, from (2.5) it is immediate that by taking $T_k = kI$ the above result holds. Now consider the case $\text{Rank}(B) < n$. For any integer $k = n-2, \cdots, 1, 0$, we define subspaces $\mathcal{S}_k$ of $\mathbb{C}^n$ as follows

$$\mathcal{S}_k = \{\text{Range}([B, AB, A^2B, \cdots A^kB])\}^{\perp}.$$

It is easy to see that

$$\{0\} = \mathcal{S}_{n-1} \subset \mathcal{S}_{n-2} \subset \cdots \subset \mathcal{S}_1 \subset \mathcal{S}_0 = \ker(B^*),$$

because the set $\mathrm{Range}([B, AB, \cdots, A^{k+1}B])$ contains the set $\mathrm{Range}([B, AB, \cdots, A^k B])$. Moreover, if for some k, $\mathcal{S}_{k+1} = \mathcal{S}_k$, this implies

$$\mathrm{Range}([B, AB, \cdots, A^k B]) = \mathrm{Range}([B, AB, \cdots, A^{n-1}B]).$$

By the controllability of the pair (A, B), $\mathcal{S}_k = \{0\}$. Let $\mathcal{S}_i$ be the smallest non-trivial subspace, then from i to 0, all the inclusions are strict inclusions. Thus, we can define subspaces $\mathcal{E}_k$ for $k = i, \cdots, 0$ as follows

$$\mathcal{E}_i = \mathcal{S}_i, \quad \mathcal{E}_k = \mathcal{S}_{k+1}^{\perp} \bigcap \mathcal{S}_k, \quad \text{for} \quad k = i-1, i-2, \cdots, 0, \quad \mathcal{E}_{-1} = \mathrm{Range}(B).$$

We note that the family of subspaces $\{\mathcal{E}_k\}_{k=0}^{i}$ forms an orthogonal partition of the subspace $\mathrm{Ker}(B^*)$ and the family of subspaces $\{\mathcal{E}_k\}_{k=-1}^{i}$ forms an orthogonal partition of $\mathbb{C}^n$. From the definition of $\mathcal{E}_k$, if $z \in S_k$, $A^* z \in S_{k-1}$. Thus, z is in the orthogonal complement of $\mathcal{E}_j$ for any $j < k-1$. On the other hand, it is not difficult to see that for every vector $z \in \mathcal{E}_k$, $A^* z \notin \mathcal{S}_k$, otherwise,

$$(A^* z)^* [B, AB, \cdots, A^k B] = z^* [AB, \cdots, A^{k+1} B] = 0, \quad \text{and} \quad z^* B = 0.$$

which would mean that $z \in \mathcal{S}_{k+1}$. Combine the above remarks, we obtain the following important conclusion: if P_k is the orthogonal projection operator from $\mathbb{C}^n$ onto $\mathcal{E}_k$, then we have $P_{k-1} A^* z \neq 0$ for all $z \in \mathcal{E}_k$ and $z \neq 0$. This also holds for $k = 0$. Now we define a sequence of linear transformations T_l as follows: $T_l^* z = z$ for $z \in \mathrm{Range}(B)$ and $T_l^* z = l^k z$ if $z \in \mathcal{E}_k$. Consider any vector $z \in \mathrm{Ker}(B^*)$, then there exist vectors $z_i, \cdots, z_0$ such that $z_k \in \mathcal{E}_k$ and $z = \sum_{k=0}^{i} z_k$. Let us define $w_{i,j} = P_j A^* z_i$, then we have

$$(T_l^{-1})^* A^* T_l^* z = \sum_{k=0}^{i} l w_{k,k-1} + \sum_{k=0}^{i} \sum_{j=k}^{i} l^{k-j} w_{k,j}.$$

Therefore, we obtain

$$[z^* (T_l^{-1})^* A^* T_l^* z I - (T_l^{-1})^* A^* T_l^*] z = (l\tau_1 + \tau_2(l)) \sum_{k=0}^{i} z_k - \sum_{k=0}^{i} l w_{k,k-1} - \sum_{k=0}^{i} \sum_{j=k}^{i} l^{k-j} w_{k,j},$$

where

$$\tau_1 = \sum_{k=1}^{i} z_{k-1}^* w_{k,k-1}, \quad \tau_2(l) = \sum_{j=0}^{i} z_j^* (\sum_{k=0}^{j} l^{k-j} w_{k,j}).$$

Thus, for any $z \in \operatorname{Ker}(B^*)$, we have

$$\|z^*[z^*T_lA(T_l)^{-1}zI - T_lA(T_l)^{-1}]\|_2 = l\|\phi_1(z) + l^{-1}\phi_2(z,l)\|_2,$$

where

$$\phi_1(z) = -w_{0,-1} + \sum_{k=1}^{i}(\tau_1 z_{k-1} - w_{k,k-1}) + \tau_1 z_i, \quad \phi_2(z,l) = \sum_{j=0}^{i}\left(\tau_2(l)z_j - \sum_{k=0}^{j} l^{k-j}w_{k,j}\right).$$

Therefore, we have

$$\begin{aligned} &\min_{\substack{\|z\|_2=1 \\ z\in\operatorname{Ker}(B^*)}} \|z^*[z^*T_lA(T_l)^{-1}zI - T_lA(T_l)^{-1}]\|_2 \\ \geq &\min_{\substack{\|z\|_2=1 \\ z\in\operatorname{Ker}(B^*)}} l\|\phi_1(z)\|_2 - \sup_{\substack{\|z\|_2=1 \\ z\in\operatorname{Ker}(B^*)}} \|\phi_2(z,l)\|_2. \end{aligned}$$

It is easy to see that there exists a constant K such that $\|\phi_2(z,l)\|_2^2 \leq K$ for all unit vector $z \in \operatorname{Ker}(B^*)$ and $l > 0$. On the other hand, we can show that $\phi_1(z) \neq 0$ for all unit vectors $z \in \operatorname{Ker}(B^*)$. In fact, if $\phi_1(z) = 0$, for some unit vector $z \in \operatorname{Ker}(B^*)$, then $w_{0,-1}$ must be equal to zero which implies $z_0 = 0$. If $z_0 = 0$, this implies $w_{1,0} = 0$ which forces $z_1 = 0$. Applying this argument for $k = 2, \cdots, i$, we obtain $z_k = 0$ for all $k = 0, \cdots, i$. This contradicts the assumption that $\|z\|_2 = 1$ and $z \in \operatorname{Ker}(B^*)$. By the compactness argument, there must exist a constant $\epsilon > 0$ such that

$$\min_{\substack{\|z\|_2=1 \\ z\in\operatorname{Ker}(B^*)}} \|\phi_1(z)\|_2 \geq \epsilon.$$

Combining with the previous lemma, we conclude that

$$\sup_{T\in G} \gamma_c(TAT^{-1}, TB) = +\infty.$$

□