EMBEDDING DOMAIN TECHNIQUE REPRESENTATION OF THE GRADIENT FOR SOME SHAPE OPTIMIZATION PROBLEMS

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Abstract. An embedding domain technique for a class of shape optimization problems for elliptic equations with inhomogeneous boundary conditions is derived. Differentiability of the cost is shown and a representation of the gradient in terms of primal and dual variables emanable for numerical realization is obtained.

1. Introduction

This paper is devoted to the study of a class of shape optimization problems for partial differential equations by an embedding domain technique. One of the main difficulties of treating such problems numerically rests in the fact that the partial differential equation must be solved repeatedly on domains that change their geometry. The basic idea of the embedding domain technique consists in embedding the varying domains Ω_γ, with γ in a class of admissible parametrizations $\mathcal{S}$, on which the partial differential equation is posed in a larger reference domain $\hat{\Omega}$ with a simple geometry. Simultaneously the original equation must be extended appropriately to a system on $\hat{\Omega}$. Embedding domain techniques are well studied for the direct problem of solving the partial differential equation on complicated domains. Variants of the technique differ in the way of extending the equation. A comparison of different methods is given in [9]. There are numerous applications of the method to elliptic problems [5],[6] and [4], flow problems [7] and [8], and scattering problems [1], among others. For shape optimization the embedding domain method is much less developed. Besides

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the obvious advantage of facilitating the repeated solution of the partial differential on the varying domains Ω_γ, we shall be able to show that by employing an adequate duality framework and extension of the equation from Ω_γ to $\hat{\Omega}$ a simple and numerically useful representation of the gradient of the cost with respect to the domain can be obtained in terms of primal and dual variables. More precisely the outer unit normal derivative which typically appears in the characterization of the gradient ([3]) and for which it is cumbersome to obtain a sufficiently precise numerical approximation, is replaced by an independent dual variable which appears as a Lagrange multiplier in the optimality system of the shape optimization problem.

The analysis of the proposed representation of the gradient is based on uniform (w.r.t.γ) H^2-regularity of the state and adjoint equation and on the continuous dependence of their first order traces on γ along the boundary of Ω_γ.

This investigation is a continuation of our research on the use of embedding domain techniques in shape optimization. In [13] we considered a similar problem as in this paper but with homogenous boundary conditions on the moving part of the boundary. Here we show that the technique of [13] covers inhomogenous boundary conditions as well. Also we provide improved regularity and continuous dependence results which lead to an simplified proof for the differentiability of the shape optimization problem.

Before outlining the contents of the paper let us mention related research on the use of the embedding techniques in [10], [11] and [2]. A broad overview on shape optimization problems is given in the monographs of [12], [15] and [17].

The paper is organized as follows: In Section 2 we formulate the problem on Ω_γ and (equivalently) on $\hat{\Omega}$. Continuous dependence results analogous to those in [13] are recalled in Section 3. In Section 4 we address some regularity issues. Although some of the estimates may appear to be standard one has to verify that they hold uniformly with respect to the admissible shapes. Furthermore we establish the continuous dependence in H^2 of the solution of the primal and dual equation on γ. The differentiability of the cost functional is established in Section 5. In the Appendix we construct explicitly a family of uniformly bounded extension operators which ensures that the traces on Γ_γ of certain Sobolev spaces of functions defined on Ω_γ and $\hat{\Omega}$ respectively define the same spaces.

2. Problem formulation

We consider the class of Lipschitzian domains

$$\Omega_\gamma = \{(x, y) \colon 0 < x < \gamma(y), \quad y \in (0,1)\}$$

for $\gamma \in \mathcal{S}$ where $\mathcal{S}$ is the set of admissible shapes

$$\mathcal{S} = \{\gamma \in H^3(0,1) \colon \mu \leq \gamma(y) \leq \nu, y \in [0,1], |\gamma|_{H^3} \leq \rho\},$$

with $0 < \mu < \nu < 1$ and $\nu < \rho$. The curved part of $\partial\Omega_\gamma$, which is denoted by Γ_γ, is considered to be unknown:

$$\Gamma_\gamma = \{(\gamma(y), y) \colon y \in (0,1)\}.$$

Given $f \in L^2_{loc}(\mathbb{R}^2)$ and $\varphi \in C^2(\Gamma_\gamma)$ such that φ admits an extension $\phi \in C^2(\bar{\Omega}_\gamma)$ with $\phi|_{\partial\Omega_\gamma \setminus \Gamma_\gamma} = 0$ the following system

$$
\begin{aligned}
-\Delta u = &\ f \quad \text{on } \Omega_\gamma \\
u = &\ \varphi \quad \text{on } \Gamma_\gamma \\
u = &\ 0 \quad \text{on } \partial\Omega_\gamma \setminus \Gamma_\gamma
\end{aligned}
\tag{2.1}
$$

will be considered. Further let z be a measurement defined on a fixed curve $\mathcal{C} \subset \Omega_\gamma$. In this note we are concerned with the problem of recovering Ω_γ from the data z, f and φ. Alternatively z may represent a desired profile of the state u along $\mathcal{S}$. In this case the goal is to find an admissible shape $\gamma^* \in \mathcal{S}$ such that the solution of (2.1) on Ω_{γ^*} provides a "best" fit to z along $\mathcal{C}$. Both problems are covered by the following shape optimization problem

$$
\begin{aligned}
&\min J(u_\gamma) \quad \text{over } \gamma \in \mathcal{S} \\
&\text{subject to} \quad u_\gamma \text{ solves (2.1).}
\end{aligned}
\tag{P}
$$

Above the cost functional J is given by

$$J(u) = \int_{\mathcal{C}} |\tau_{\mathcal{C}} u - z|^2 ds$$

where $\mathcal{C}$ is a smooth curve in $(0, \bar{\nu}) \times (0,1)$, for some $\bar{\nu} \in (0, \mu)$, $\tau_{\mathcal{C}}$ denotes the trace operator on $\mathcal{C}$ and $z \in L^2(\mathcal{C})$.

Next we turn to an equivalent formulation of (2.1). We consider the problem

$$(\nabla w, \nabla v) = (f + \Delta\phi, v), \quad v \in H^1_0(\Omega_\gamma). \tag{2.2}$$

It is well known that (2.2) admits a unique solution $w_\gamma \in H^1_0(\Omega_\gamma)$, which by

$$u_\gamma = w_\gamma + \phi \tag{2.3}$$

yields the unique distributional solution of (2.1). Conversely, if $u_\gamma \in H^1(\Omega_\gamma)$ satisfies (2.1) then w_γ defined by (2.3) is the unique solution of (2.2).

Problem (2.2) may be equivalently reformulated as an equation on $\hat{\Omega} = (0,1) \times (0,1)$. The following spaces provide the correct abstract framework:

$$
\begin{aligned}
V_\gamma = &\ \{v \in H^1(\Omega_\gamma) \colon v|_{\partial\Omega_\gamma \setminus \Gamma_\gamma} = 0\} \\
\hat{V} = &\ \{\hat{v} \in H^1(\hat{\Omega}) \colon \hat{v}|_{\partial\hat{\Omega} \setminus \{1\} \times (0,1)} = 0\} \\
H^{1/2}_{00}(\Gamma_\gamma) = &\ \{\chi \in L^2(\Gamma_\gamma) \colon \exists v \in V_\gamma (\chi = v|_{\Gamma_\gamma})\}.
\end{aligned}
$$

We shall use

$$\|u\|_{V_\gamma} = \sqrt{(\nabla u, \nabla u)_{\Omega_\gamma}}$$

as norm in Ω_γ, the norm on $\hat{V}$ is defined analogously. The space $H^{1/2}_{00}(\Gamma_\gamma)$ is endowed with the norm

$$\|\chi\|_{H^{1/2}_{00}(\Gamma_\gamma)} = \inf\{\|v\|_{V_\gamma} : v \in V_\gamma, v|_{\Gamma_\gamma} = \chi\}.$$

Furthermore we shall utilize the family of operators $\tau_\gamma\colon V_\gamma \to H_{00}^{1/2}(\Gamma_\gamma)$ for $\gamma \in \mathcal{S}$ defined by:

$$\tau_\gamma v = v|_{\Gamma_\gamma}.$$

Since $H_{00}^{1/2}(\Gamma_\gamma)$ can be identified with the trace space of V_γ surjectivity of τ_γ follows. The inequality

$$\|\tau_\gamma v\|_{H_{00}^{1/2}(\Gamma_\gamma)} \leq \|v\|_{V_\gamma}, \quad v \in V_\gamma$$

shows that $\{\tau_\gamma\colon \gamma \in \mathcal{S}\} \subset \mathcal{L}(V_\gamma, H_{00}^{1/2}(\Gamma_\gamma))$ is uniformly bounded. We shall also consider "interior") traces of functions $\hat{v} \in \hat{V}$ on Γ_γ. It will be shown in the Appendix that replacing V_γ by $\hat{V}$ in the definition of $H_{00}^{1/2}(\Gamma_\gamma)$ defines a space which is set theoretically identical and topologically equivalent to $H_{00}^{1/2}(\Gamma_\gamma)$. This justifies to denote the mapping $\hat{v} \to \hat{v}|_{\Gamma_\gamma}$ with the same symbol τ_γ.

Returning to the discussion of (2.2) we set

$$F = f + \Delta\phi \quad \in L^2(\Omega_\gamma)$$

and let $\tilde{F}_\gamma$ denote the extension by zero

$$\tilde{F}_\gamma = \begin{cases} F & \text{on } \Omega_\gamma \\ 0 & \text{on } \hat{\Omega} \setminus \Omega_\gamma. \end{cases}$$

Subsequently," ˆ " will stand for the extension by zero and " ˆ " will indicate that the corresponding entity refers to $\hat{\Omega}$.

Next we formulate a "global" problem on $\hat{\Omega}$:

Find a pair $(\hat{w}, \hat{g}) \in \hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ such that

$$\begin{cases} (\nabla\hat{w}, \nabla\hat{v})_{\hat{\Omega}} - (\tilde{F}_\gamma, \hat{v})_{\hat{\Omega}} - \langle \hat{g}, \tau_\gamma \hat{v}\rangle = 0, & \hat{v} \in \hat{V} \\ \tau_\gamma \hat{w} = 0, \end{cases} \tag{2.4}$$

($\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_{00}^{1/2}(\Gamma_\gamma)$ and $H_{00}^{1/2}(\Gamma_\gamma)^*$). Surjectivity of τ_γ implies the existence of a unique solution $(\hat{w}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ satisfying the following a priori bounds

$$\begin{aligned} \|\hat{w}_\gamma\| &\leq \|\tilde{F}_\gamma\|_{\hat{V}^*} \\ \|\hat{g}_\gamma\| &\leq 2\|\tilde{F}_\gamma\|_{\hat{V}^*} \leq 2\|f + \Delta\phi\|_{L^2(\hat{\Omega})}. \end{aligned} \tag{2.5}$$

In a fashion similar to Theorem 3.1 in [13] one establishes the following result which explains the equivalence of problems (2.2) and (2.4):

Theorem 2.1. *(i) Let $(\hat{w}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H^{1/2}_{00}(\Gamma_\gamma)^*$ denote the solution to (2.4) and define $w_\gamma = \hat{w}_\gamma|_{\Omega_\gamma} \in V_\gamma$. Then w_γ is the unique solution to (2.2), $\hat{w}_\gamma|_{\hat{\Omega}\setminus\Omega_\gamma} \equiv 0$ and*

$$\langle \tau_1 w_\gamma, \tau_0 v\rangle_{\partial\Omega_\gamma} = \langle \hat{g}_\gamma, \tau_\gamma v\rangle \tag{2.6}$$

holds for all $v \in V_\gamma$.

(ii) Conversely, let $w_\gamma \in V_\gamma$ be the solution of (2.2). Then (2.6) uniquely defines an element $\hat{g}_\gamma$ in $H^{1/2}_{00}(\Gamma_\gamma)^*$ and $(\tilde{w}_\gamma, \hat{g}_\gamma)$ is the solution of (2.4).

Above τ_1 denotes the first order trace operator on $\partial\Omega_\gamma$. Next we turn to a global formulation of the inhomogenous equation (2.1). Let $(\hat{w}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H^{1/2}_{00}(\Gamma_\gamma)^*$ be the solution of (2.4), $w_\gamma = \hat{w}_\gamma|_{\Omega_\gamma}$ and let $\hat{\phi}_\gamma$ denote a fixed extension of $\phi \in C^2(\bar{\Omega}_\gamma)$ to the closure of $\hat{\Omega}$ such that $\hat{\phi}_\gamma \in C^2(\bar{\hat{\Omega}})$.

Furthermore define $\hat{f}_\gamma \in L^2(\hat{\Omega})$ by

$$\hat{f}_\gamma = \begin{cases} f & \text{on } \Omega_\gamma \\ -\Delta\hat{\phi}_\gamma & \text{on } \hat{\Omega}\setminus\Omega_\gamma. \end{cases} \tag{2.7}$$

and set

$$\hat{u}_\gamma = \hat{w}_\gamma + \hat{\phi}_\gamma. \tag{2.8}$$

One verifies that $\hat{u}_\gamma$ extends (2.3) from Ω_γ to $\hat{\Omega}$ and satisfies

$$\tau_\gamma \hat{u}_\gamma = \varphi.$$

Moreover, for $\hat{v} \in \hat{V}$ one obtains

$$\begin{aligned}
(\nabla\hat{u}_\gamma, \nabla\hat{v})_{\hat{\Omega}} &= (\nabla\hat{w}_\gamma, \nabla\hat{v})_{\hat{\Omega}} + (\nabla\hat{\phi}_\gamma, \nabla\hat{v})_{\hat{\Omega}} \\
&= (f + \Delta\phi, \hat{v})_{\Omega_\gamma} + \langle \hat{g}_\gamma, \tau_\gamma\hat{v}\rangle + (\nabla\phi, \nabla\hat{v})_{\Omega_\gamma} + (\nabla\hat{\phi}_\gamma, \nabla\hat{v})_{\hat{\Omega}\setminus\Omega_\gamma} \\
&= (f, \hat{v})_{\Omega_\gamma} + \int_{\partial\Omega_\gamma} \frac{\partial\phi}{\partial n}\tau_0\ v\ ds + \langle \hat{g}_\gamma, \tau_\gamma\hat{v}\rangle + (\nabla\hat{\phi}_\gamma, \nabla\hat{v})_{\hat{\Omega}\setminus\Omega_\gamma} \\
&= (\hat{f}_\gamma, \hat{v})_{\hat{\Omega}} + \langle \hat{g}_\gamma, \tau_\gamma\hat{v}\rangle.
\end{aligned}$$

Above, $\frac{\partial}{\partial n}$ denotes the exterior normal derivative to Ω_γ.

One concludes that $(\hat{u}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H^{\frac{1}{2}}_{00}(\Gamma_\gamma)^*$ solves the following system on $\hat{\Omega}$:

Find $(\hat{u}, \hat{g}_\gamma) \in \hat{V} \times H^{1/2}_{00}(\Gamma_\gamma)^*$ such that

$$\begin{cases} (\nabla\hat{u}, \nabla\hat{v})_{\hat{\Omega}} - (\hat{f}_\gamma, \hat{v})_{\hat{\Omega}} - \langle \hat{g}, \tau_\gamma\hat{v}\rangle = 0, \\ \tau_\gamma\hat{u} = \varphi. \end{cases} \tag{2.9}$$

Conversely, one verifies that any solution $(\hat{u}_\gamma, \hat{g}_\gamma)$ of (2.9) gives rise to a solution $(\hat{w}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H^{1/2}_{00}(\Gamma_\gamma)^*$ of (2.4) where

$$\hat{w}_\gamma = \hat{u}_\gamma - \hat{\phi}_\gamma.$$

Uniqueness of the solution of (2.4) therefore implies uniqueness of the solution of (2.9). Summarizing the preceding discussion one obtains

Theorem 2.2. *(i) Let $\hat{\phi}_\gamma$ be any extension of ϕ to $\hat{\Omega}$ such that $\hat{\phi}_\gamma \in C^2(\bar{\hat{\Omega}})$ and let $(\hat{u}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ denote the solution of (2.9). Then $u_\gamma = \hat{u}_\gamma|_{\Omega_\gamma}$ is the unique solution of (2.1) and*

$$\langle \tau_1 u_\gamma, \tau_0 v\rangle_{\partial\Omega_\gamma} = \langle \hat{g}_\gamma, \tau_\gamma v\rangle + \int_{\Gamma_\gamma} \frac{\partial \hat{\phi}_\gamma}{\partial n}\, \tau_\gamma\, v\, ds \tag{2.10}$$

holds for all $v \in V_\gamma$.

(ii) Conversely, let $u_\gamma \in V_\gamma$ be the solution of (2.1) and $\hat{w}_\gamma$ be the corresponding solution of (2.4). Then (2.10) uniquely defines an element $\hat{g}_\gamma \in H_{00}^{1/2}(\Gamma_\gamma)^*$ and $(\hat{u}_\gamma, \hat{g}_\gamma)$ with

$$\hat{u}_\gamma = \hat{w}_\gamma + \hat{\phi}_\gamma$$

is the solution of (2.9).

Proof. It remains to show (2.10). This will be a consequence of (2.6) and (2.8):

$$\begin{aligned}\langle \tau_1 u_\gamma, \tau_0 v\rangle_{\partial\Omega_\gamma} &= \langle \tau_1 w_\gamma, \tau_0 v\rangle_{\partial\Omega_\gamma} + \langle \tau_1 \phi, \tau_0 v\rangle_{\partial\Omega_\gamma}\\ &= \langle \hat{g}_\gamma, \tau_0 v\rangle + \int_{\Gamma_\gamma} \frac{\partial \phi_\gamma}{\partial n}\, \tau_\gamma\, v\, ds.\end{aligned}$$

□

It will be advantageous to use the following extensions of φ: Choose $\chi \in C^2(0,1)$ such that

$$\begin{array}{ll}\chi(0) = 0, \quad \chi(x) \geq 0, & x \in (0,1)\\ \chi(x) = 1 & x \in (\mu, 1)\end{array}$$

and define:

$$\begin{aligned}\hat{\phi}_\gamma(x,y) &= \varphi(\gamma(y), y)\, \chi(x) \quad \text{for } (x,y) \in \hat{\Omega}\\ \phi_\gamma &= \hat{\phi}_\gamma|_{\Omega_\gamma}.\end{aligned}$$

Consequently $\hat{\phi}_\gamma$ does not depend on γ and we shall therefore write ϕ instead. By assumption $\varphi \in C^2(0,1)$ and hence $\hat{\phi} \in C^2(\bar{\hat{\Omega}}) \cap \hat{V}$, $\phi_\gamma \in C^2(\bar{\Omega}_\gamma) \cap V_\gamma$.

3. Continuous dependence

We provide a framework which allows to compare traces of functions along Γ_γ for different $\gamma \in \mathcal{S}$. For this purpose define

$$H_{00}^{1/2}(0,1) = \{\varphi \in L^2(0,1)\colon \varphi = \hat{v}(1,\cdot) \text{ for some } \hat{v} \in \hat{V}\},$$

and endow $H_{00}^{1/2}(0,1)$ with the natural quotient norm. Below we shall argue the existence of isomorphisms $\mathcal{J}_\gamma \in \mathcal{L}\left(H_{00}^{1/2}(\Gamma_\gamma), H_{00}^{1/2}(0,1)\right)$, which together with their inverses are uniformly bounded, i.e. there exists constants κ_1 and κ_2 such that

$$\|\mathcal{J}_\gamma\| \le \kappa_1, \quad \|\mathcal{J}_\gamma^{-1}\| \le \kappa_2 \quad \text{for all} \quad \gamma \in \mathcal{S}.$$

In addition we introduce $\mathfrak{T}_\gamma \in \mathcal{L}(\hat{V}, H_{00}^{1/2}(0,1))$ by

$$\mathfrak{T}_\gamma = \mathcal{J}_\gamma \tau_\gamma.$$

Note that the family $\{\mathfrak{T}_\gamma : \gamma \in \mathcal{S}\}$ is uniformly bounded.

Lemma 3.1. *(i) Let $\gamma_n, \bar{\gamma} \in \mathcal{S}$ satisfy $\lim_{n\to\infty} \gamma_n = \bar{\gamma}$ in $W^{1,\infty}(0,1)$. Then $\mathfrak{T}_{\gamma_n}$ converges strongly to $\mathfrak{T}_{\bar{\gamma}}$.*

(ii) Assume that $\gamma_n, \bar{\gamma}$ satisfy $\lim_{n\to\infty} \gamma_n = \bar{\gamma}$ in $C([0,1]), 0 < \mu \le \gamma_n,\ \bar{\gamma} < \nu < 1$ and let $\hat{v}_n \in \hat{V}$ converge weakly to $\hat{v} \in \hat{V}$. Then $\mathfrak{T}_{\gamma_n} \hat{v}_n = \varphi$ for all $n \in \mathbb{N}$ implies $\mathfrak{T}_{\bar{\gamma}} \hat{v} = \varphi$.

The proof can be given in the same manner as in [13].

Theorem 3.3. *Let $\gamma_n, \bar{\gamma} \in \mathcal{S}$, $f_{\gamma_n} \in L^2(\Omega_{\gamma_n})$, $f_{\bar{\gamma}} \in L^2(\Omega_{\bar{\gamma}})$ satisfy $\lim_{n\to\infty} \gamma_n = \bar{\gamma}$ in $W^{1,\infty}(0,1)$ and $\lim_{n\to\infty} \tilde{f}_{\gamma_n} = \tilde{f}_{\bar{\gamma}}$ in $L^2(\hat{\Omega})$. Then the solutions $(\hat{u}_{\gamma_n}, \hat{g}_{\gamma_n}) \in \hat{V} \times H_{00}^{1/2}(\Gamma_{\gamma_n})^*$ of (2.9) satisfy*

$$\lim_{n\to\infty} \hat{u}_{\gamma_n} = \hat{u}_{\bar{\gamma}} \quad \textit{in } V$$

$$w^* - \lim_{n\to\infty} (\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n} = (\mathcal{J}_{\bar{\gamma}}^{-1})^* \hat{g}_{\bar{\gamma}},$$

where $(\hat{u}_{\bar{\gamma}}, \hat{g}_{\bar{\gamma}})$ is the solution of (2.9) with $(\gamma, \tilde{f}_\gamma)$ replaced by $(\bar{\gamma}, \tilde{f}_{\bar{\gamma}})$.

Sketch of proof. We start with the description of the isomorphisms $\mathcal{J}_\gamma \in \mathcal{L}\big(H_{00}^{1/2}(\Gamma_\gamma), H_{00}^{1/2}(0,1)\big)$. Consider the family of C^1-diffeomorphisms $T_\gamma \colon \Omega_\gamma \to \hat{\Omega}$ given by

$$T_\gamma(x,y) = \big(\frac{x}{\gamma(y)}, y\big) \quad \text{for} \quad (x,y) \in \Omega_\gamma \text{ and } \gamma \in \mathcal{S},$$

which enduce via $v \to \hat{v} = v \circ T_\gamma^{-1}$ an equicontinuous family of surjections $V_\gamma \to \hat{V}$ with respect to $\gamma \in \mathcal{S}$. For $\gamma \in \mathcal{S}$ define the operator $\mathcal{J}_\gamma \in \mathcal{L}\left(H_{00}^{1/2}(\Gamma_\gamma), H_{00}^{1/2}(0,1)\right)$ by

$$(\mathcal{J}_\gamma \varphi)(y) = (v \circ T_\gamma^{-1})(1, y),$$

where $v \in V_\gamma$ is chosen such that $\varphi = \tau_\gamma v$. One checks that the definition is independent of the choice of v and that for a constant κ_1 independent of $\gamma \in \mathcal{S}$

$$\begin{aligned} \|\mathcal{J}_\gamma \varphi\|_{H_{00}^{1/2}(0,1)} &= \inf\{\|\hat{v}\|_{\hat{V}} \colon \hat{v} \in \hat{V}, \hat{v}(1,\cdot) = \mathcal{J}_\gamma \varphi\} \\ &\le \|v \circ T_\gamma^{-1}\|_{\hat{V}} \le \kappa_1 \|v\|_{V_\gamma}, \end{aligned}$$

which implies that $\|\mathcal{J}_\gamma\| \le \kappa_1$, for all $\gamma \in \mathcal{S}$. Conversely, if $\chi \in H_{00}^{1/2}(0,1)$ then $\chi = \hat{v}(1,\cdot)$ for some $\hat{v} \in \hat{V}$. For $v_\gamma = \hat{v} \circ T_\gamma \in V_\gamma$ we have

$$\mathcal{J}_\gamma(\tau_\gamma v_\gamma) = (v_\gamma \circ T_\gamma^{-1})(1,\cdot) = \hat{v}(1,\cdot) = \chi$$

and consequently $\mathcal{J}_\gamma$ is surjective and $\mathcal{J}_\gamma^{-1}\chi = \tau_\gamma v_\gamma$. This implies that $\|\mathcal{J}_\gamma^{-1}\| \le \kappa_2$ for a constant κ_2 independent of $\gamma \in \mathcal{S}$.

Let $(\hat{w}_{\gamma_n}, \hat{g}_{\gamma_n})$ stand for the unique solution of (2.4) with γ replaced by γ_n. Using the extension of φ described in Remark 2.3 one verifies that

$$\lim_{n\to\infty} \widetilde{\Delta\phi}_{\gamma_n} = \widetilde{\Delta\phi}_{\bar{\gamma}} \quad \text{in } L^2(\hat{\Omega})$$

and consequently

$$\lim_{n\to\infty} \tilde{F}_{\gamma_n} = \tilde{F}_{\bar{\gamma}} \quad \text{in } L^2(\hat{\Omega}).$$

Exactly as in the proof of Theorem 4.1 in [13] one concludes that the assertion of Theorem 2.2 holds for $(\hat{w}_{\gamma_n}, \hat{g}_{\gamma_n})$:

$$\hat{w}_{\gamma_n} \to \hat{w}_{\bar{\gamma}} \quad \text{in } \hat{V}$$

$$\left(\mathcal{J}_{\gamma_n}^{-1}\right)^* \hat{g}_{\gamma_n} \rightharpoonup^* \left(\mathcal{J}_{\gamma_n}^{-1}\right)^* \hat{g}_{\bar{\gamma}} \quad \text{in} \quad H_{00}^{1/2}(0,1)^*$$

and $(\hat{w}_{\bar{\gamma}}, \hat{g}_{\bar{\gamma}}) \in \hat{V} \times H_{00}^{1/2}(\Gamma_{\bar{\gamma}})^*$ is the unique solution of (2.4) with $(\gamma_n, \tilde{F}_{\gamma_n})$ replaced by $(\bar{\gamma}, \tilde{F}_{\bar{\gamma}})$. Since $\lim_{n\to\infty} \hat{f}_{\gamma_n} = \hat{f}_{\bar{\gamma}}$ Theorem 2.2 follows from (2.9) and (2.8). $\square$

4. Regularity

In this section we discuss the regularity of the restrictions of $\hat{u}$ of (2.9) to Ω_γ and to $\hat{\Omega}\setminus\bar{\Omega}_\gamma$. In view of the representation (2.8) and Theorem 2.1 it suffices to study the regularity of the solution w_γ of (2.2). Some of the following results may appear to be standard, however, they differ from readily available descriptions of the properties of elliptic equations in that they are uniform with respect to the parametrization $\gamma \in \mathcal{S}$ of the boundary of Ω_γ.

Theorem 4.4. *Let w_γ denote the variational solution of (2.2). Then $w_\gamma \in H_0^1(\Omega_\gamma) \cap H^2(\Omega_\gamma)$ and*

$$\|w_\gamma\|_{H^2(\Omega_\gamma)} \;\le\; c\|F_\gamma\|_{L^2(\Omega_\gamma)}$$

holds with a constant $c > 0$ independent of $\gamma \in \mathcal{S}$.

Proof. The transformation $T_\gamma\colon \Omega_\gamma \to \hat{\Omega}$

$$T_\gamma(x,y) = \left(\frac{x}{\gamma(y)}, y\right), \ \gamma \in \mathcal{S} \tag{4.1}$$

defines a family of uniformly bounded $C^{1,1}$ - diffeomorphisms, i.e.

$$\max\{\|T_\gamma\|_{1,1}, \ \|T^{-1}\|_{1,1}\} \ \le k_1.$$

Here and below k_i denote constants which are independent of $\gamma \in \mathcal{S}$. It follows that the mappings

$$v \to v \circ T_\gamma^{-1}$$

determine isomorphisms of $H^2(\Omega_\gamma)$ onto $H^2(\hat{\Omega})$ (cf. [18]). Moreover there exist constants k_2 and k_3 such that

$$k_2\|v\|_{H^2(\Omega_\gamma)} \ \le \ \|v \circ T_\gamma^{-1}\|_{H^2(\hat{\Omega})} \ \le \ k_3\|v\|_{H^2(\Omega_\gamma)} \tag{4.2}$$

for all $v \in H^2(\Omega_\gamma)$. Henceforth we shall use the notation

$$V = v \circ T_\gamma^{-1}$$

for $v \in H^1(\Omega_\gamma)$. Based on T_γ the system (2.2) is transformed into

$$(\nabla W, \mathcal{A}_\gamma \nabla V)_{\hat{\Omega}} = (\mathfrak{F}_\gamma, V)_{\hat{\Omega}}, \quad V \in H_0^1(\hat{\Omega}) \tag{4.3}$$

where $\mathfrak{F}_\gamma = \gamma F_\gamma \circ T_\gamma^{-1}$ and

$$\mathcal{A}_\gamma(\xi,\eta) = \begin{pmatrix} \frac{1}{\gamma(\eta)} + \xi^2\frac{\dot{\gamma}(\eta)^2}{\gamma(\eta)} & -\xi\dot{\gamma}(\eta) \\ -\xi\dot{\gamma}(\eta) & \gamma(\eta) \end{pmatrix}, \ (\xi,\eta) \in \hat{\Omega}, \ \gamma \in \mathcal{S}. \tag{4.4}$$

It will be shown in the Appendix that $\mathcal{A}_\gamma$ is uniformly elliptic on $\hat{\Omega}$ uniformly in $\gamma \in \mathcal{S}$. The assumptions on $\mathcal{S}$ ensure that the hypotheses of Section II-6 in [La] are satisfied and hence the second fundamental inequality

$$\|U\|_{H^2(\hat{\Omega})} \le k_4\left(\|\mathfrak{L}U\|_{L^2(\hat{\Omega})} + \|U\|_{H^1(\hat{\Omega})}\right),$$

holds for all $U \in H^2(\hat{\Omega}) \cap H_0^1(\hat{\Omega})$. Above $\mathfrak{L}$ stands for the elliptic operator which is determined by the left hand side of (4.3). This inequality is the basis for a homotopy argument to establish the existence of a unique solution W_γ of

$$\mathfrak{L}W = \mathfrak{F}_\gamma$$

in $H^2(\hat{\Omega})$ which consequently satisfies

$$\|W_\gamma\|_{H^2(\hat{\Omega})} \le k_4\left(|\mathfrak{F}_\gamma|_{L^2(\hat{\Omega})} + \|W_\gamma\|_{H^1(\hat{\Omega})}\right).$$

For details we refer to the proof of Theorem 7.1 in [La]. The uniform ellipticity of $\mathcal{A}_\gamma$ and (4.3) leads to

$$\|W_\gamma\|_{H^2(\hat{\Omega})} \le 2k_4|\mathfrak{F}_\gamma|_{L^2(\hat{\Omega})}, \tag{4.5}$$

which together with (4.2) implies the desired bound. $\square$

Next we establish a continuous dependence result for the solution of the transformed equation (4.3) with respect to γ.

Theorem 4.5. *Let $W_\gamma, \gamma \in \mathcal{S}$, denote the solution of (4.3). Then the mapping $\gamma \to W_\gamma$ is continuous on $\mathcal{S} \to H^2(\hat{\Omega})$.*

Proof. Consider a fixed $\bar{\gamma} \in \mathcal{S}$ and choose $\gamma \in \mathcal{S}$ arbitrarily. Let $W_{\bar{\gamma}}$ and W_γ denote the corresponding solutions of (4.3). The difference $D = W_\gamma - W_{\bar{\gamma}}$ satisfies

$$(\nabla D, \mathcal{A}_{\bar{\gamma}} \nabla V) = \big(\text{ div } (\nabla W_\gamma(\mathcal{A}_\gamma - \mathcal{A}_{\bar{\gamma}})) + \mathcal{F}_\gamma - \mathcal{F}_{\bar{\gamma}}, V\big), \tag{4.6}$$

for $V \in H_0^1(\hat{\Omega})$, which is of the type (4.3). Consequently, $D \in H^2(\hat{\Omega})$ satisfies

$$\begin{aligned} \|D\|_{H^2(\hat{\Omega})} &= \|W_\gamma - W_{\bar{\gamma}}\|_{H^2(\hat{\Omega})} \\ &\le k_5 \Big(|\text{ div } \big(\nabla W_\gamma(\mathcal{A}_\gamma - \mathcal{A}_{\bar{\gamma}})\big)|_{L^2(\hat{\Omega})} + |\mathcal{F} - \mathcal{F}_{\bar{\gamma}}|_{L^2(\hat{\Omega})}\Big) . \end{aligned}$$

One can verify the estimate

$$|\text{ div } \big(\nabla W_\gamma(\mathcal{A}_\gamma - \mathcal{A}_{\bar{\gamma}})\big)|_{L^2(\hat{\Omega})} \ \le k_6 \ |\gamma - \bar{\gamma}|_{W^{2,\infty}}.$$

The transformed inhomogeneity $\mathcal{F}_\gamma$ is determined from

$$\mathcal{F}_\gamma = \gamma \, f \circ T_\gamma^{-1} + \text{ div } ((\nabla \psi_\gamma) \mathcal{A}_\gamma)$$

where $\psi_\gamma = \phi \circ T_\gamma^{-1}$. A lengthy calculation establishes the bound

$$|\text{ div } \big((\nabla \psi_\gamma) \mathcal{A}_\gamma\big) - \text{ div } \big((\nabla \psi_{\bar{\gamma}}) \mathcal{A}_{\bar{\gamma}}\big)|_{L^2(\hat{\Omega})} \le k_6 \ \|\gamma - \bar{\gamma}\|_{W^{2,\infty}}.$$

Since

$$\lim_{\gamma \to \bar{\gamma}} \int_{\hat{\Omega}} |\gamma \, f \circ T_\gamma^{-1} - \bar{\gamma} \, f \circ T_{\bar{\gamma}}^{-1}|^2 = 0$$

holds for $f \in C(\bar{\hat{\Omega}})$ it extends by density to $f \in L^2(\bar{\hat{\Omega}})$. This completes the proof of the theorem. ☐

Theorem 4.1 implies the existence and uniform boundedness of the trace $\nabla W_\gamma(\cdot, \gamma(\cdot))$ and Theorem 4.2 ensures its continuous dependence on γ. We have

Corollary 4.1. *Let u_γ denote the variational solution of (2.1). Then $|\nabla u_\gamma(\gamma(\cdot), \cdot)|_{L^2(0,1)} \le k_7$ for a constant k_7 independent of $\gamma \in \mathcal{S}$. Furthermore*

$$\lim \int_0^1 |\nabla u_\gamma(\gamma(y), y) - \nabla u_{\bar{\gamma}}(\bar{\gamma}(y), y)|^2 \, dy = 0$$

as γ tends to $\bar{\gamma}$ in $\mathcal{S}$.

Proof. In view of (2.3) it suffices to consider the variational solution w_γ of (2.2). The first assertion is a consequence of (4.5), the trace theorem in $H^2(\hat{\Omega})$ and (4.2). The continuity of the trace follows from the following estimate where D_{xy} denotes the Jacobian

$$
\begin{aligned}
&\int_0^1 |\nabla w_\gamma(\gamma(y),y) - \nabla w_{\bar\gamma}(\bar\gamma(y),y)|^2 dy \\
&= \int_0^1 |\nabla W_\gamma(1,y) D_{xy} T_\gamma(\gamma(y),y) - \nabla W_{\bar\gamma}(1,y) D_{xy} T_{\bar\gamma}(\bar\gamma(y),y)|^2 dy \\
&\le 2\int_0^1 |\nabla W_\gamma(1,y)|^2 \, |D_{xy} T_\gamma(\gamma(y),y) - D_{xy} T_{\bar\gamma}(\bar\gamma(y),y)|^2 dy \\
&+ 2\int_0^1 |\nabla W_\gamma(1,y) - \nabla W_{\bar\gamma}(1,y)|^2 \, |D_{xy} T_{\bar\gamma}(\bar\gamma(y),y)|^2 dy,
\end{aligned}
\tag{4.7}
$$

and the first part of the corollary and Theorem 4.2. $\square$

To characterize the derivative of J with respect to γ we also need to discuss the regularity of the solutions to the adjoint equation associated to (P) which is of the form

$$
(\nabla\lambda, \nabla v)_{\Omega_\gamma} = \langle \delta_\gamma, v\rangle_{H^{-1}(\Omega_\gamma), H_0^1(\Omega_\gamma)} \tag{4.8}
$$

where $\delta_\gamma \in H^{-1}(\Omega_\gamma)$ is a distribution with support contained in $(0,\bar\nu)\times(0,1)$. For (P) it is given by

$$
\langle \delta_\gamma, v\rangle = -2\int_{\mathcal{C}} (\tau_{\mathcal{C}} u_\gamma - z)\tau_{\mathcal{C}} v \, ds, \qquad \text{for } v \in H_0^1(\Omega_\gamma).
$$

The Lax-Milgram theorem implies the existence of a unique solution $\lambda_\gamma \in H_0^1(\Omega_\gamma)$ of (4.8). Let us define

$$
\begin{aligned}
R_\gamma &= \{(x,y):\ \mu < x < \gamma(y),\ y \in (0,1)\} \\
S_\gamma &= \{(x,y):\ \bar\nu < x < \gamma(y),\ y \in (0,1)\},
\end{aligned}
$$

and a C^∞ - function ψ

$$
\psi(x) \begin{cases} = 0 & \text{for } \ x < \bar\nu \\ \in [0,1] & \text{for } \ \bar\nu \le x \le \mu \\ = 1 & \text{for } \ x > \mu. \end{cases}
$$

Furthermore decompose λ_γ as

$$
\lambda_\gamma = \psi_\gamma \lambda_\gamma + (1-\psi_\gamma)\lambda_\gamma =: \lambda_\gamma^1 + \lambda_\gamma^2,
$$

and note that $\lambda_\gamma = \lambda_\gamma^1$ on R_γ and $\lambda_\gamma = \lambda_\gamma^2$ on $(0,\bar\nu)\times(0,1)$. We shall exploit the fact that $\lambda_\gamma^1 \in H_0^1(S_\gamma)$ satisfies

$$
\Delta\lambda = 2\lambda_{\gamma,x}\psi_{\gamma,x} + \lambda_\gamma \psi_{\gamma,xx} =: h_\gamma
$$

in the distributional sense on S_γ. By Theorem 4.1 with Ω_γ replaced by S_γ we have $\lambda_\gamma^1 \in H^2(S_\gamma)$ and

$$
\|\lambda_\gamma\|_{H^2(R_\gamma)} \ \le \ \|\lambda_\gamma^1\|_{H^2(S_\gamma)} \le c\|\lambda_\gamma\|_{H^1(S_\gamma)} \le C
$$

with constants c and C independent of $\gamma \in \mathcal{S}$. Next we argue that the analogs of Theorem 4.2 and consequently of Corollary 4.1 hold as well. Let $\tilde{T}_\gamma \colon \mathcal{S}_\gamma \to \hat{\Omega}$ denote the transformation analogous to T_γ which is defined in (4.1) with Ω_γ replaced by S_γ and define, for $\gamma \in \mathcal{S}$,

$$L^1_\gamma = \lambda^1_\gamma \circ \tilde{T}^{-1}_\gamma \in H^1_0(\hat{\Omega}) \cap H^2(\hat{\Omega}).$$

Theorem 4.6. *There exists a constant C such that*

$$\|\lambda_\gamma\|_{H^2(R_\gamma)} \le C \text{ for all } \gamma \in \mathcal{S}.$$

Moreover $\lim L^1_\gamma = L^1_{\bar{\gamma}}$ *in* $H^2(\hat{\Omega})$ *whenever* $\gamma \to \bar{\gamma}$ *in* $\mathcal{S}$.

Proof. The first assertion has already been proven. To prove the second, note that $\tilde{T}_\gamma$ induces an isomorphism $i_\gamma \colon H^1_0(\mathcal{S}_\gamma) \to H^1_0(\hat{\Omega})$ determined by

$$i_\gamma v = v \circ \tilde{T}^{-1}_\gamma.$$

One verifies that L^1_γ is the variational solution of

$$(\nabla L^1_\gamma, \mathcal{A}_\gamma \nabla V)_{L^2(\hat{\Omega})} = (\mathcal{H}_\gamma, V)_{L^2(\hat{\Omega})},\ V \in H^1_0(\hat{\Omega}),$$

where $\mathcal{H}_\gamma = \gamma\, h_\gamma \circ \tilde{T}^{-1}_\gamma$. Consequently for $\gamma, \bar{\gamma} \in \mathcal{S}$ we find

$$\big(\nabla(L^1_\gamma - L^1_{\bar{\gamma}}), \mathcal{A}_{\bar{\gamma}} \nabla V\big) = -\big(\nabla L^1_\gamma, (\mathcal{A}_\gamma - \mathcal{A}_{\bar{\gamma}})\nabla V\big) + (\mathcal{H}_\gamma - \mathcal{H}_{\bar{\gamma}}, V),$$

for $V \in H^1_0(\hat{\Omega})$, which corresponds to (4.6) with $\mathcal{F}_\gamma - \mathcal{F}_{\bar{\gamma}}$ replaced by $\mathcal{H}_\gamma - \mathcal{H}_{\bar{\gamma}}$. From the above equality it follows that

$$\lim L^1_\gamma = L^1_{\bar{\gamma}} \ \text{ in } \ H^2(\hat{\Omega})$$

as $\gamma \to \bar{\gamma}$ in $\mathcal{S}$, once it is verified that

$$\lim L_\gamma = L_{\bar{\gamma}} \ \text{ in } \ H^1(\hat{\Omega}) \tag{4.9}$$

as $\gamma \to \bar{\gamma}$ in $\mathcal{S}$. Here L_γ is the solution of the adjoint equation transformed by T_γ

$$(\nabla L_\gamma, \mathcal{A}_\gamma \nabla V) = \langle (i^{-1}_\gamma)^* \delta_\gamma, V \rangle_{H^{-1}(\hat{\Omega}), H^1_0(\hat{\Omega})}.$$

We find

$$\begin{aligned}(\nabla(L_\gamma - L_{\bar{\gamma}}), \mathcal{A}_{\bar{\gamma}} \nabla V) = &-(\nabla L_\gamma, (\mathcal{A}_\gamma - \mathcal{A}_{\bar{\gamma}})\nabla V) \\ &+ \langle (i^{-1}_\gamma)^* \delta_\gamma - (i^{-1}_{\bar{\gamma}})^* \delta_{\bar{\gamma}}, V \rangle_{H^{-1}(\hat{\Omega}), H^1_0(\hat{\Omega})},\end{aligned}$$

and since

$$\begin{aligned}&|\langle (i^{-1}_\gamma)^* \delta_\gamma - (i^{-1}_{\bar{\gamma}})^* \delta_{\bar{\gamma}}, V \rangle_{H^{-1}(\hat{\Omega}), H^1_0(\hat{\Omega})}| = |\langle \delta_\gamma, (i^{-1}_\gamma) V\rangle - \langle \delta_{\bar{\gamma}}, i^{-1}_{\bar{\gamma}} V\rangle| \\ &= |\int_{\mathcal{C}} (\tau_{\mathcal{C}} \hat{u}_\gamma - z) \tau_{\mathcal{C}} i^{-1}_\gamma V ds - \int_{\mathcal{C}} (\tau_{\mathcal{C}} \hat{u}_{\bar{\gamma}} - z) \tau_{\mathcal{C}} i^{-1}_{\bar{\gamma}} V ds| \\ &\le k_7 \big(|\hat{u}_\gamma - \hat{u}_{\bar{\gamma}}|_{H^1(\hat{\Omega})} + |\tau_{\mathcal{C}} u_{\bar{\gamma}} - z|_{L^2(\mathcal{C})} |\tau_{\mathcal{C}} (i^{-1}_\gamma V - i^{-1}_{\bar{\gamma}} V)|_{L^2(\mathcal{C})} \big) \\ &\le k_8 \big(|\hat{u}_\gamma - \hat{u}_{\bar{\gamma}}|_{H^1(\hat{\Omega})} + |\gamma - \bar{\gamma}|^{1/2}_\infty \big) |V|_{H^1_0(\hat{\Omega})},\end{aligned}$$

(4.9) follows from Theorem 3.1. ∎

5. Differentiability

In this section we derive a convenient form of the Fréchet derivative of the cost functional

$$J(u_\gamma) = \int_{\mathcal{C}} |\tau_{\mathcal{C}} u_\gamma - z|^2 ds, \tag{5.1}$$

where u_γ is the solution to (2.1). Due to Theorem 2.2

$$J(u_\gamma) = J(\hat{u}_\gamma) \quad \text{for } \gamma \in \mathcal{S},$$

with $(\hat{u}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ the solution to (2.9). We shall utilize the adjoint system in $\hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ given by

$$\left\{ \begin{array}{l} (\nabla \hat{\lambda}_\gamma, \nabla \hat{v}) + 2(\tau_{\mathcal{C}} \hat{u}_\gamma - z, \tau_{\mathcal{C}} \hat{v})_{\mathcal{C}} - \langle \hat{\mu}_\gamma, \tau_\gamma \hat{v} \rangle = 0, \quad \hat{v} \in \hat{V}, \\ \tau_\gamma \hat{\lambda}_\gamma = 0. \end{array} \right. \tag{5.2}$$

Based on Theorem 2.1 equation (5.2) can be related to

$$\begin{array}{rll} \Delta \lambda = & 2(\tau_{\mathcal{C}} u_\gamma - z) & \text{on } \Omega_\gamma \\ \lambda = & 0 & \text{on } \partial \Omega_\gamma, \end{array} \tag{5.3}$$

just as (2.1) is related to (2.9). By Theorem 4.3 the solution λ_γ of (5.3) satisfies $\lambda|_{R_\gamma} \in H^2(R_\gamma)$ and $\left\{ \|\lambda_\gamma|_{R_\gamma}\|_{H^2(R_\gamma)} \colon \gamma \in \mathcal{S} \right\}$ is uniformly bounded. An analogous statement holds for u_γ. As a consequence the Langrange multipliers $\hat{g}_\gamma, \hat{\mu}_\gamma$ admit the following interpretation in $L^2(\Gamma_\gamma)$

$$\begin{array}{rl} \hat{g}_\gamma & = \frac{\partial \hat{u}_\gamma^-}{\partial n} - \frac{\partial \hat{u}_\gamma^+}{\partial n} \\ \hat{\mu}_\gamma & = \frac{\partial \lambda_\gamma}{\partial n}, \end{array}$$

where we have set

$$\hat{u}_\gamma^- = \hat{u}_\gamma|_{\Omega_\gamma} \quad \text{and} \quad \hat{u}_\gamma^+ = \hat{u}_\gamma|_{\hat{\Omega} \setminus \bar{\Omega}_\gamma}$$

and where n denotes the outer unit normal of $\Omega_\gamma \cap \Gamma_\gamma$. Furthermore we have

$$\begin{array}{l} \langle \hat{g}_\gamma, \varphi \rangle_{\Gamma_\gamma} = \langle (\mathcal{J}_\gamma^{-1})^* \hat{g}_\gamma, \mathcal{J}_\gamma \varphi \rangle = \displaystyle\int_0^1 g_\gamma(y)(\mathcal{J}_\gamma \varphi)(y) \sqrt{1 + \dot{\gamma}^2(y)} dy \\ \langle \hat{\mu}_\gamma, \varphi \rangle_{\Gamma_\gamma} = \langle (\mathcal{J}_\gamma^{-1})^* \hat{\mu}_\gamma, \mathcal{J}_\gamma \varphi \rangle = \displaystyle\int_0^1 \mu_\gamma(y)(\mathcal{J}_\gamma \varphi)(y) \sqrt{1 + \dot{\gamma}^2(y)} dy \end{array} \tag{5.4}$$

for $\varphi \in H_{00}^{1/2}(\Gamma_\gamma)$, g_γ and $\mu_\gamma \in L^2(0,1)$.

Lemma 5.2. *The solutions $(\hat{u}_\gamma, \hat{g}_\gamma)$ and $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma) \in \hat{V} \times H_{00}^{1/2}(\Gamma_\gamma)^*$ of (2.9) and (5.2) satisfy*

$$\begin{array}{rl} g_\gamma & = \sqrt{1 + \dot{\gamma}^2}\, \tau_\gamma \hat{u}_{\gamma,x}^- \\ \mu_\gamma & = \sqrt{1 + \dot{\gamma}^2}\, \tau_\gamma \hat{\lambda}_{\gamma,x}^-. \end{array}$$

The proof of this lemma will be given at the end of the section. Next we characterize the Fréchet derivative of the cost functional with respect to γ. For this purpose we need to consider $J(u_\gamma)$ with γ in a neighborhood of $\mathcal{S}$ and we define

$$\tilde{\mathcal{S}} = \left\{\gamma \in H^3(0,1)\colon \tilde{\mu} \le \gamma \le \tilde{\nu}, |\gamma|_{H^3} \le \tilde{\rho}\right\},$$

where $0 < \tilde{\mu} < \mu < \nu < \tilde{\nu} < \min(1, \tilde{\rho})$ and $\rho < \tilde{\rho}$. Without loss of generality we can assume that the bounds derived for u_γ, λ_γ in Section 4 are uniform with respect to $\gamma \in \tilde{\mathcal{S}}$.

Theorem 5.7. *Let $f \in L^\infty(\Omega)$. Then $\gamma \to J(u_\gamma)$ is Fréchet differentiable in $\mathcal{S}$ and its derivative is characterized by*

$$D_\gamma J(u_\gamma)\bar{\gamma} = -\int_0^1 \mu_\gamma(y) g_\gamma(y) \bar{\gamma}(y) dy,$$

for every $\bar{\gamma} \in H^3(0,1)$.

Proof. We give a proof that is considerably simpler than that of an analogous result in [13]. It is based on Corollary 4.1. The directional derivative will be calculated first. Fix $\gamma \in \mathcal{S}, \bar{\gamma} \in H^3(0,1)$ and let $t \neq 0$ be such that $\gamma + t\bar{\gamma} \in \tilde{\mathcal{S}}$. We start by splitting

$$\begin{aligned}
&\tfrac{1}{t}\big(J(u_{\gamma+t\bar{\gamma}}) - J(u_\gamma)\big) = \tfrac{1}{t}\big(|\tau_{\mathcal{C}}\hat{u}_{\gamma+t\bar{\gamma}} - z|^2_{L^2(\mathcal{C})} - |\tau_{\mathcal{C}}\hat{u}_\gamma - z|^2_{L^2(\mathcal{C})}\big) \\
&= \tfrac{1}{t}\big(\tau_{\mathcal{C}}(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma, \tau_{\mathcal{C}}\hat{u}_{\gamma+t\bar{\gamma}} - z\big)_{\mathcal{C}} + \tfrac{1}{t}(\tau_{\mathcal{C}}\hat{u}_{\gamma+t\bar{\gamma}} - \tau_{\mathcal{C}}\hat{u}_\gamma, \tau_{\mathcal{C}}\hat{u}_\gamma - z)_{\mathcal{C}} \\
&= \tfrac{1}{2t}\big[-\big(\nabla\hat{\lambda}_{\gamma+t\bar{\gamma}}, \nabla(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\big)_{\hat{\Omega}} + \langle\hat{\mu}_{\gamma+t\bar{\gamma}}, \tau_{\gamma+t\bar{\gamma}}(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\rangle_{\Gamma_{\gamma+t\bar{\gamma}}}\big] \\
&+\tfrac{1}{2t}\big[-\big(\nabla\hat{\lambda}_\gamma, \nabla(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\big)_{\hat{\Omega}} + \langle\hat{\mu}_\gamma, \tau_\gamma(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\rangle_{\Gamma_\gamma}\big] =: A + B.
\end{aligned}$$

The above equality follows from (5.2) with $\hat{v} = \hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma$. Since the estimation of B is similar to that of A we focus on A. Replacing the first term of A by inserting the primal equation (2.9) one obtains

$$\begin{aligned}
2A &= \frac{1}{t}(\hat{f}_\gamma - \hat{f}_{\gamma+t\bar{\gamma}}, \hat{\lambda}_{\gamma+t\bar{\gamma}})_{\hat{\Omega}} + \frac{1}{t}\langle\hat{g}_\gamma, \tau_\gamma\hat{\lambda}_{\gamma+t\bar{\gamma}}\rangle_{\Gamma_\gamma} \\
&+ \frac{1}{t}\langle\hat{\mu}_{\gamma+t\bar{\gamma}}, \tau_{\gamma+t\bar{\gamma}}(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\rangle_{\Gamma_{\gamma+t\bar{\gamma}}} = A1 + A2 + A3.
\end{aligned}$$

It will be advantageous to use the disjoint sets

$$E^+ = \{y \in (0,1)\colon \bar{\gamma}(y) \ge 0\} \text{ and } E^- = \{y \in (0,1)\colon \bar{\gamma}(y) < 0\}.$$

By the definition of $\hat{f}$ in (2.7) and since $\hat{\lambda}_{\gamma+t\bar{\gamma}}|_{\hat{\Omega}\setminus\Omega_{\gamma+t\bar{\gamma}}}$ we obtain

$$A1 = \frac{1}{t}\int_{E^+}\int_\gamma^{\gamma+t\bar{\gamma}} (-\Delta\hat{\phi} - f)\;\hat{\lambda}_{\gamma+t\bar{\gamma}}\,dxdy.$$

An argument similar to the one used in [13] establishes the estimate

$$|A1| \le (|\Delta\hat{\phi}|_\infty + |f|_\infty)\,|\bar{\gamma}|_\infty^{3/2}\,|\hat{\lambda}_{\gamma+t\bar{\gamma}}|_{H^1}\;t^{1/2}.$$

As a consequence $A1$ tends to zero for $t \to 0^+$. In view of (5.4) we obtain (see [13] for details)

$$\begin{aligned} A2 &= \tfrac{1}{t} \int_{E+} g_\gamma(y)\, \hat\lambda_{\gamma+t\bar\gamma}(\gamma(y),y) \sqrt{1+\dot\gamma^2}\, dy \\ &= \tfrac{1}{t} \int_{E+} g_\gamma(y)\, \int_\gamma^{\gamma+t\bar\gamma} \big(\tau-\gamma(y)\big) \hat\lambda_{\gamma+t\bar\gamma,xx}(\tau,y) d\tau \sqrt{1+\dot\gamma^2}\, dy \\ &\quad - \int_{E+} g_\gamma(y)\, \hat\lambda_{\gamma+t\bar\gamma,x}\big((\gamma+t\bar\gamma)(y),y\big)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy. \end{aligned}$$

Since $\|\hat\lambda_\gamma\|_{H^2(\hat\Omega)}$ is uniformly bounded for $\gamma \in \tilde{\mathcal{S}}$ the first term in $A2$ vanishes as $t \to 0$, whereas we obtain for the second term by Corollary 4.1

$$\begin{aligned} &- \lim_{t\to 0^+} \int_{E+} g_\gamma(y) \hat\lambda_{\gamma+t\bar\gamma,x}\big((\gamma+t\bar\gamma)(y),y\big)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy \\ &= - \int_{E+} g_\gamma(y) \hat\lambda^-_{\gamma,x}(\gamma(y),y)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy. \end{aligned}$$

In the discussion of $A3$ we use the fact that the trace of $\hat u_\gamma$ on Γ_γ does not depend on γ. This observation and (5.4) result in

$$\begin{aligned} A3 &= \tfrac{1}{t} \int_0^1 \mu_{\gamma+t\bar\gamma}(y) \left[\hat u_{\gamma+t\bar\gamma}\big((\gamma+t\bar\gamma)(y),y\big)\right. \\ &\quad \left. -\hat u_\gamma\big((\gamma+t\bar\gamma)(y),y\big)\right] \sqrt{1+(\dot\gamma+t\dot{\bar\gamma})^2}\, dy \\ &= \tfrac{1}{t} \int_0^1 \mu_{\gamma+t\bar\gamma}(y) \left[\hat u_\gamma(\gamma(y),y) - \hat u_\gamma\big((\gamma+t\bar\gamma)(y),y\big)\right] \sqrt{1+(\dot\gamma+t\dot{\bar\gamma})^2}\, dy \\ &= \tfrac{1}{t} \int_{E-} \mu_{\gamma+t\bar\gamma}(y) \int_{\gamma+t\bar\gamma}^\gamma \int_{\gamma(y)}^\xi \hat u_{\gamma,xx}(\tau,y) d\tau d\xi \sqrt{1+(\dot\gamma+t\dot{\bar\gamma})^2}\, dy \\ &= - \int_{E-} \mu_{\gamma+t\bar\gamma}(y) \hat u^-_{\gamma,x}(\gamma(y),y)\bar\gamma(y) \sqrt{1+(\dot\gamma+t\dot{\bar\gamma})^2}\, dy. \end{aligned}$$

Lemma 5.1 and Corollary 4.1 imply that $\mu_{\gamma+t\bar\gamma} \to \mu_\gamma$ in $L^2(0,1)$ as $t \to 0^+$ and in view of Theorem 4.1 we infer that

$$\lim_{t\to 0^+} A3 = - \int_{E-} \mu_\gamma(y)\, \hat u^-_{\gamma,x}(\gamma(y),y)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy.$$

Summarizing the previous discussion we obtain

$$\begin{aligned} \lim_{t\to 0^+} A = &\ -\tfrac{1}{2} \int_{E+} g_\gamma(y) \lambda^-_{\gamma,x}(\gamma(y),y)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy \\ &\ -\tfrac{1}{2} \int_{E-} \mu_\gamma(y) u^-_{\gamma,x}(\gamma(y),y)\bar\gamma(y)\sqrt{1+\dot\gamma^2}\, dy \\ = &\ -\tfrac{1}{2} \int_0^1 g_\gamma(y)\mu_\gamma(y)\bar\gamma(y) dy. \end{aligned}$$

The last equality is a consequence of Lemma 5.1. Replacing $\bar\gamma$ by $-\bar\gamma$ shows that the same argument can be used to obtain the above limit also for $t \to 0^-$. By an analogous argument one shows that B converges to the same limit as A. Hence the Gateaux derivative of J is given by

$$DJ(u_\gamma)\bar\gamma = - \int_0^1 g_\gamma(y)\mu_\gamma(y)\bar\gamma(y) dy.$$

To verify that the Gateaux derivative is in fact a Fréchet derivative we note first that $DJ(u_\gamma) \in H^3(0,1)^*$. Moreover the estimate

$$\begin{aligned} \|DJ(u_\gamma) - DJ(u_{\gamma^*})\| \le &\ |g_\gamma - g_{\gamma^*}|_{L^2(0,1)}\ |\mu_\gamma|_{L^2(0,1)} \\ &+ |g_{\gamma^*}|_{L^2(0,1)} |\mu_\gamma - \mu_{\gamma^*}|_{L^2(0,1)} \end{aligned}$$

together with Corollary 4.1 and Lemma 5.1 implies that

$$\lim \|DJ(u_\gamma) - DJ(u_{\gamma^*})\| = 0$$

as $\gamma \to \gamma^*$ in $H^3(0,1)$. Consequently J is also Fréchet differentiable. This concludes the proof. $\square$

Proof of Lemma 5.1. We show the statement concerning $\hat{u}_\gamma$. Using Green's formula one verifies that

$$\hat{g}_\gamma = \frac{\partial}{\partial n}(\hat{u}_\gamma^- - \hat{u}_\gamma^+).$$

Note that by Theorem 4.2 the first order traces of $\hat{u}_\gamma^- : = \hat{u}_\gamma|_{\Omega_\gamma}$ and $\hat{u}_\gamma^+ = \hat{u}_\gamma|_{\hat{\Omega}\setminus\bar{\Omega}_\gamma}$ exist in $H^{1/2}(\Gamma_\gamma)$. The outer unit normal n on Γ_γ is determined by

$$n(y) = \frac{1}{\sqrt{1+\dot{\gamma}^2(y)}} \begin{pmatrix} 1 \\ -\dot{\gamma}(y) \end{pmatrix}, \quad y \in (0,1),$$

which results in

$$\begin{aligned} \hat{g}_\gamma(y) = (1+\dot{\gamma}^2(y))^{-1/2} & \left(\hat{u}_{\gamma,x}^-(\gamma(y),y) - \hat{u}_{\gamma,x}^+(\gamma(y),y)\right) \\ & - \dot{\gamma}(y)\left(\hat{u}_{\gamma,y}^-(\gamma(y),y) - \hat{u}_{\gamma,y}^+(\gamma(y),y)\right)\Big), \end{aligned}$$

for a.a. $y \in (0,1)$. Taking tangential derivatives implies that

$$\left(\hat{u}_{\gamma,y}^- - \hat{u}_{\gamma,y}^+\right)(\gamma(y),y) = -\dot{\gamma}(y)\left(\hat{u}_{\gamma,x}^- - \hat{u}_{\gamma,x}^+\right)(\gamma(y),y),$$

which results in

$$\hat{g}_\gamma(y) = \sqrt{1+\dot{\gamma}^2(y)}\ \left(\hat{u}_{\gamma,x}^- - \hat{u}_{\gamma,x}^+\right)\ (\gamma(y),y),$$

for a.a. $y \in (0,1)$. The particular choice of $\hat{\phi}$ and (2.8) ensure that $\hat{u}_{\gamma,x}^+(\gamma(y),y) = 0$, for $y \in (0,1)$, which completes the proof of the first part of Lemma 5.1. The second part is verified in an analogous manner. $\square$

Appendix

In order to show the space V_γ can be replaced by $\hat{V}$ in the definition of $H_{00}^{1/2}(\Gamma_\gamma)$ the existence of a uniform extension operator is established.

Lemma A.3. *For each $\gamma \in \mathcal{S}$ there exists an extension operator $\mathcal{E}_\gamma \in \mathcal{L}(V_\gamma, \hat{V})$ and a positive constant M independent of $\gamma \in \mathcal{S}$ such that*

$$\|\mathcal{E}_\gamma\| \le M \quad \textit{for all } \gamma \in \mathcal{S}.$$

Proof. Consider the transformation $T_\gamma\colon \hat{\Omega} \to (-1,1)\times(0,1)$ defined by

$$T_\gamma(x,y) = \begin{cases} \left(\frac{x}{\gamma(y)} - 1, y\right) & \text{for } (x,y)\in\Omega_\gamma \\ \left(\frac{x-\gamma(y)}{1-\gamma(y)}, y\right) & \text{for } (x,y)\in\hat{\Omega}\setminus\Omega_\gamma. \end{cases}$$

One verifies that $T_\gamma(\Omega_\gamma) = (-1,0)\times(0,1)$ and $T_\gamma(\hat{\Omega}\setminus\Omega_\gamma) = (0,1)\times(0,1)$. Moreover T_γ is a bijection which together with its inverse $T_\gamma^{-1}\colon (-1,1)\times(0,1)\to\hat{\Omega}$ given by

$$T_\gamma^{-1}(\xi,\eta) = \begin{cases} \big((1+\xi)\gamma(\eta),\eta\big) & \text{for } (\xi,\eta)\in(-1,0)\times(0,1) \\ \big((1-\gamma(\eta))\xi+\gamma(\eta),\eta\big) & \text{for } (\xi,\eta)\in[0,1)\times(0,1) \end{cases}$$

is Lipschitz continuous. For $v\in V_\gamma\cap C^1(\bar{\Omega}_\gamma)$ we define $\bar{v}\in C^1\left([-1,0]\times[0,1]\right)$ by

$$\bar{v}(\xi,\eta) = (v\circ T_\gamma^{-1})(\xi,\eta) \quad \text{for } (\xi,\eta)\in(-1,0)\times(0,1)$$

and extend $\bar{v}$ by reflection to obtain $V\in H^1\big((-1,1)\times(0,1)\big)$:

$$V(\xi,\eta) = \begin{cases} \bar{v}(\xi,\eta) & \text{for } (\xi,\eta)\in(-1,0]\times(0,1) \\ \bar{v}(-\xi,\eta & \text{for } (\xi,\eta)\in(0,1)\times(0,1). \end{cases}$$

The desired extension $\mathcal{E}_\gamma$ is determined by

$$\mathcal{E}_\gamma v = V\circ T_\gamma.$$

For $(x,y)\in\Omega_\gamma$ one obtains

$$(\mathcal{E}_\gamma v)(x,y) = V\big(T_\gamma(x,y)\big) = \bar{v}\big(T_\gamma(x,y)\big) = v(x,y),$$

and hence $\mathcal{E}_\gamma v$ extends v to $\hat{\Omega}$. Since $T_\gamma|_{\Omega_\gamma}$ and $T_\gamma|_{\hat{\Omega}\setminus\Omega_\gamma}$ are continuously differentiable $\mathcal{E}_\gamma v\in\hat{V}$ follows. The estimate

$$\begin{aligned}
\|\mathcal{E}_\gamma v\|_{\hat{V}}^2 &= \Big(\int_{\Omega_\gamma} + \int_{\hat{\Omega}\setminus\Omega_\gamma}\Big)\big|\big(\nabla_{(\xi,\eta)}V\big)\big(T_\gamma(x,y)\big)DT_\gamma(x,y)\big|^2 dx\,dy \\
&= \Big(\int_{\Omega_R} + \int_{\Omega_L}\Big)\big|\big(\nabla_{(\xi,\eta)}V\big)DT_\gamma\big(T_\gamma^{-1}(\xi,\eta)\big)\big|^2\cdot\frac{1}{|\det DT_\gamma\big(T_\gamma^{-1}(\xi,\eta)\big)|}d\xi\,d\eta \\
&\le \Big(\frac{1}{\mu}+\frac{1}{1-\nu}\Big)\ |\nabla V|_{L^2\left(-1,1)\times(0,1)\right)} = 2\Big(\frac{1}{\mu}+\frac{1}{1-\nu}\Big)|\nabla\bar{v}|_{L^2\left(\Omega_L\right)} \\
&\le 2\nu\Big(\frac{1}{\mu}+\frac{1}{1-\nu}\Big)\quad \|v\|_{V_\gamma}^2,
\end{aligned}$$

where $\Omega_L = (-1,0)\times(0,1)$ and $\Omega_R = (0,1)\times(0,1)$, extends by density of $V_\gamma\cap C^1(\bar{\Omega}_\gamma)$ to V_γ. It follows that

$$\|\mathcal{E}_\gamma\|^2 \le 2\nu\Big(\frac{1}{\mu}+\frac{1}{1-\nu}\Big), \qquad \text{for } \gamma\in\mathcal{S}.$$

□

Lemma A.4. *Let $\hat{H}_{00}(\Gamma_\gamma)$ stand for the space which is defined by replacing V_γ by $\hat{V}$ in the definition of $H_{00}^{1/2}(\Gamma_\gamma)$. Then the spaces $H_{00}^{1/2}(\Gamma_\gamma)$ and $\hat{H}_{00}^{1/2}(\Gamma_\gamma)$ coincide algebraically and topologically.*

Proof. We denote by $R_\gamma \in \mathcal{L}(\hat{V}, V_\gamma)$ the restriction operator of $\hat{V}$ to V_γ. Clearly $\|R_\gamma\| \le 1$ for all $\gamma \in \mathcal{S}$ holds. For every $\chi \in H_{00}^{1/2}(\Gamma_\gamma)$ there exists $w \in V_\gamma$ such that $w|_{\Gamma_\gamma} = \chi$. The extension $\hat{w} = \mathcal{E}_\gamma w$ satisfies $\hat{w}|_{\Gamma_\gamma} = \chi$, hence $\chi \in H_{00}^{1/2}(\Gamma_\gamma)$ and

$$\|\chi\|_{\hat{H}_{00}^{1/2}(\Gamma_\gamma)} \le \|\mathcal{E}_\gamma w\|_{\hat{V}} \le \|\mathcal{E}_\gamma\| \|w\|_{V_\gamma}$$

follows for all $w \in V_\gamma$ with $w|_{\Gamma_\gamma} = \chi$. This implies

$$\|\chi\|_{H_{00}^{1/2}(\Gamma_\gamma)} \le \|\mathcal{E}_\gamma\| \|\chi\|_{H_{00}^{1/2}(V_\gamma)}.$$

Conversely, let $\hat{\chi} \in H_{00}^{1/2}(\Gamma_\gamma)$ and $\hat{w} \in \hat{V}$ be such that $\hat{w}|_{\Gamma_\gamma} = \hat{\chi}$. Then $w = R_\gamma \hat{w} \in V_\gamma$ satisfies $w|_{\Gamma_\gamma} = \hat{\chi}$ which entails $\hat{\chi} \in H_{00}^{1/2}(\Gamma_\gamma)$. The estimate

$$\|\hat{\chi}\|_{H_{00}^{1/2}(\Gamma_\gamma)} \le \|w\|_{V_\gamma} \le \|R_\gamma\| \|\hat{w}\|_{\hat{V}} \le \|\hat{w}\|_{\hat{V}}$$

implies

$$\|\hat{\chi}\|_{H_{00}^{1/2}(\Gamma_\gamma)} \le \|\hat{\chi}\|_{H_{00}^{1/2}(\Gamma_\gamma)}.$$

The above argument shows that $H_{00}^{1/2}(\Gamma_\gamma) = \hat{H}_{00}^{1/2}(\Gamma_\gamma)$ and implies the uniform equivalence of the norms in $H_{00}^{1/2}(\Gamma_\gamma)$ and $\hat{H}_{00}^{1/2}(\Gamma_\gamma)$. □

Finally we verify that the transformed equation (4.3) satisfies a uniform ellipticity condition which is uniform in $\gamma \in \mathcal{S}$:

Lemma A.5. *Let $\mathcal{A}_\gamma$, $\gamma \in \mathcal{S}$ be defined by (4.4). There exists a constant $\kappa > 0$ such that*

$$z^T \mathcal{A}_\gamma(\xi, \eta) z \ge |z|^2$$

holds for all $(\xi, \eta) \in \hat{\Omega}$, $z \in \mathbb{R}^2$ and $\gamma \in \mathcal{S}$.

Proof. The assumptions on $\mathcal{S}$ imply

$$|\dot{\gamma}|_\infty \le \rho, \quad \text{for } \gamma \in \mathcal{S}.$$

For $z^T = (\alpha, \beta) \in \mathbb{R}^2$ one obtains

$$z^T \mathcal{A}_\gamma(\xi, \eta) z = \frac{1}{\gamma}[\alpha^2 + (\alpha\xi\dot{\gamma} - \beta\gamma)^2].$$

Adopting an argument in [16] we consider first the case $|\alpha| \ge \frac{\mu}{2\rho}|\beta|$ which leads to

$$|\alpha\xi\dot{\gamma} - \beta\gamma| \ge |\beta|\mu - |\alpha|\rho \ge \frac{\mu}{2}|\beta|.$$

If $|\alpha| \ge \frac{\mu}{2\rho}|\beta|$ one obtains the estimate

$$\alpha^2 + (\alpha\xi\dot{\gamma} - \beta\gamma)^2 \ge \frac{1}{2}\alpha^2 + \frac{1}{2}\frac{\mu^2}{4\rho^2}|\beta|^2 \ge \min(\frac{1}{2}, \frac{\mu^2}{8\rho^2})|z|^2.$$

Hence the assertion of Lemma A.3 holds with $\kappa = \min(\frac{\mu^2}{4}, \frac{\mu^2}{8\rho^2})\frac{1}{\nu}$. □

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