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SHAPE OPTIMIZATION FOR MIXED BOUNDARYVALUE PROBLEMS BASED ON AN EMBEDDING DOMAIN METHOD

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Abstract. An embedding domain method for mixed boundary value problems is derived and applied to a class of shape optimization problems. The derivative of the cost with respect to the shape is characterized in terms of the Lagrange multipliers for the primal and the adjoint problem. Numerical results illustrating the efficiency of the method which is related to a mixed finite element method are presented.

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1. Introduction and motivation

The objective of this paper is the study of an embedding domain technique for shape optimization problems associated with mixed boundary value problems. We consider

$$(\mathcal{P}) \quad \begin{cases} \min J(u) \\ \text{subject to } \gamma \in \mathcal{S} \\ \quad -\Delta u = f \quad \text{in } \Omega_\gamma \\ \quad u = 0 \quad \text{on } \Gamma_1(\gamma) \\ \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_2(\gamma), \end{cases}$$

where $\mathcal{S}$ is an index set characterizing the set of admissible domains Ω_γ and J is a cost functional depending on u and hence on γ. For each $\gamma \in \mathcal{S}$, Ω_γ is a bounded connected domain in $\mathbb{R}^2$ with boundary $\partial\Omega_\gamma$. Further $f \in L^2(\mathbb{R}^2)$, and $\Gamma_1(\gamma), \Gamma_2(\gamma)$ are disjoint subsets of $\partial\Omega_\gamma$, each consisting of finitely many connected components and satisfying $\bar{\Gamma}_1(\gamma) \cup \bar{\Gamma}_2(\gamma) = \partial\Omega_\gamma$, with the bar denoting closure.

One of the main contributions of this paper is a new characterization of the gradient of J in terms of the primal and dual variables based on the embedding domain formulation related to $(\mathcal{P})$. This characterization can be interpreted as a splitting or mixed finited element method since the outer normal derivative along $\Gamma_1(\gamma)$ that commonly appears in the representation of the gradient is replaced by an independent variable.

Shape optimization problems such as $(\mathcal{P})$ are treated in several recent monographs. The reader may consult [13], [15], [16] where, however, primarily the case of pure Dirichlet ($\Gamma_2(\gamma) = \emptyset$) and pure Neumann boundary conditions ($\Gamma_1(\gamma) = \emptyset$) are studied. Just as in the general theory of boundary value problems the mixed case also represents additional difficulties in the context of shape optimization problems, due, in part, to the possible lack of regularity of the solution in the neighborhood of the boundary where $\Gamma_1(\gamma)$ and $\Gamma_2(\gamma)$ meet.

The theory of shape optimization concerns itself with existence and regularity of solutions to $(\mathcal{P})$, with differentiability properties of J and with the numerical realization of $(\mathcal{P})$. This paper will cover these topics, but its main focus is an appropriate reformulation of $(\mathcal{P})$ that is amenable to a simple numerical treatment. If $(\mathcal{P})$ was treated in a direct manner based on discretization of Ω_γ and subsequent use of an optimization routine to find the minimizer γ^* in $\mathcal{S}$, then this would require a new grid generation step at each level of the iteration. The aim of this paper is to avoid this step by working in a larger domain $\hat{\Omega}$, the fictitious domain, with the property that $\Omega_\gamma \subset \hat{\Omega}$ for all $\gamma \in \mathcal{S}$. This requires finding an equation (or rather a system of equations) defined on $\hat{\Omega}$ with the property that the restriction to Ω_γ of the solution to the equation on $\hat{\Omega}$ provides a solution to the boundary value problem arising in $(\mathcal{P})$. The authors became aware of this idea from a paper by Haslinger, Hoffmann and Kocvara [11]. There the system on the fictitious domain $\hat{\Omega}$ contains two additional two dimensional variables and one therefore aimed at finding a smaller system with the desired properties. Such a system, which requires only one additional one dimensional variable could be found and is explained in Section 3. This system was recently also used by other authors, see [7], [12]. While Glowinski and collaborators use the em-

bedding or fictitious domain approach as a technique to efficiently solve the boundary value problem in $(\mathcal{P})$ on complicated domains Ω_γ, Haslinger and Klarbing also use the embedding domain approach in the context of shape optimization for Dirichlet problems. Differently from [12] we make consistent use of the dual variables, and as a consequence we obtain a compact form of the derivative of the cost functional with respect to γ. The derivative of the cost involves the outer normal of the state as well as the costate, which can be expressed by certain Lagrange multipliers. The usage of these Lagrange multipliers allows an interpretation of the embedding domain technique as a mixed finite element method. For further related research on the embedding domain technique we refer to [5], [8], [4], [17], [18]. An approximation analysis for the embedding domain technique for Dirichlet problems was carried out in [6].

In the embedding technique of the present paper the inhomogeneity f is extended by zero on the complement $\Omega_\gamma^c = \hat{\Omega} - \Omega_\gamma$ and it thus differs from the previous work that we are aware of. If one was only interested in solving the partial differential equation in $(\mathcal{P})$ then the extension of f outside of Ω_γ would not matter. For optimization problems this is different, however, as can be seen from the following simple example:

$$\begin{cases} \min_{\alpha\in[3,4]} J(\alpha) = \frac{1}{2}\int_1^2 |u(x)-z(x)|^2 dx \\ \text{subject to} \quad \begin{array}{rcll} -u_{xx} & = & f & \text{on } (0,\alpha) \\ u(0) & = & u(\alpha) = 0, & \end{array} \end{cases} \tag{1.1}$$

where $f \in L^2(0,\infty), z \in L^2(1,2)$. For the derivative of J with respect to α we find

$$J'(\alpha) = \int_1^2 (u(x)-z(x))u_\alpha(x)dx,$$

where u_α is the solution to

$$\begin{cases} -u_{\alpha,xx} = 0 & \text{on } (0,\alpha), \\ u_\alpha(0) = 0,\ (u_\alpha)(\alpha) = -u_x(\alpha), \end{cases} \tag{1.2}$$

and u is the solution to the boundary value problem in 1.1. Here the computation was carried out on the varying domains $\Omega_\alpha = (0,\alpha)$, $\alpha \in [3,4]$, and can be made rigorous by the method of mappings, for example [P]. To realize the boundary value problem in (1.1) by an embedding domain technique we choose $\hat{\Omega} = (0,5)$, define

$$\tilde{f} = \begin{cases} f & \text{on } (0,\alpha) \\ 0 & \text{on } \hat{\Omega} \setminus (0,\alpha) \end{cases}$$

and consider

$$\begin{cases} \min \int_0^5 (\frac{1}{2}|\hat{u}_x|^2 - \tilde{f}\hat{u})dx \\ \hat{u} \in H_0^1(\hat{\Omega}) \\ \hat{u}(\alpha) = 0. \end{cases} \tag{1.3}$$

Then the solution to (1.3) satisfies $\hat{u}|\Omega_\gamma = u$ (with u the solution to the boundary value problem in (1.1)) and the Lagrange multiplier g associated to the constraint $\hat{u}(\alpha) = 0$ in (1.3) satisfies $g = -u_x(\alpha)$! The Neumann boundary data of u required in the representation of the gradient of J thus appears naturally as the dual variable if the embedding domain formulation is chosen as in (1.3). In Section 5 we shall see that this allows a simple characterization of the gradient of the cost functional. If, on the other hand, instead of (1.3) one would consider

$$\begin{cases} \min \int_0^5 (\frac{1}{2}|\hat{u}_x|^2 - f\hat{u})dx \\ \hat{u} \in H_0^1(\hat{\Omega}) \\ \hat{u}(\alpha) = 0, \end{cases} \tag{1.4}$$

then $u = \hat{u}|(0, \alpha)$ gives the solution to the boundary value problem in (1.1) and $u^c = \hat{u}|\hat{\Omega} \setminus (0, \alpha]$ is the solution to

$$\begin{aligned} -u_{xx} &= f \qquad \text{on } (\alpha, 5) \\ u(\alpha) &= u(5) = 0. \end{aligned}$$

The Lagrange variable g associated to the constraint $\hat{u}(\alpha) = 0$ in (1.4) satisfies $g = -u_x(\alpha) + u_x^c(\alpha)$, from which one cannot recover $u_x(\alpha)$. Therefore the embedding domain formulation (1.3) is advantageous over (1.4) for the optimization problem (1.1).

We end this section by briefly explaining our motivation to study shape optimization problems. In a collaboration with a branch of the Austrian steel industry we treated the problem of determining the inner surface of a steel furnace from thermal measurements from inside the furnace's wall. In Figure 1 the idealized cross section through part of the wall of a steel furnace is depicted.

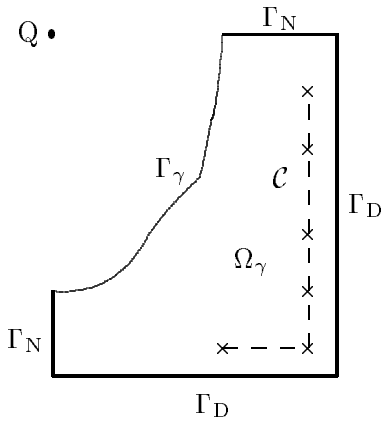

Figure 1

Here Γ_γ denotes the inner wall of the steel furnace the location of which changes due to chemical and physical phenomena during the production process. To obtain information on the location of the inner wall without shutdown of the furnace the engineer typically uses his experience combined with thermal measurements taken at locations ξ_i inside the wall and designated by stars in Figure 1. It is desirable to improve upon this situation. The temperature distribution u inside the wall can be approximated by the solution to

$$-\operatorname{div}(\lambda(u)\operatorname{grad} u) = 0 \text{ in } \Omega_\gamma, \tag{1.5}$$

where λ is the temperature–dependent diffusion coefficient, combined with an inhomogeneous Dirichlet condition on Γ_D and a Neumann condition on Γ_N. Then one poses a least squares problem minimizing a quadratic cost

$$\sum_i |u(\xi_i, \gamma) - \bar{u}_i|^2, \tag{1.6}$$

over a class of admissible boundary curves Γ_γ, where $\bar{u}_i$ denotes the measured temperature at measurement locations ξ_i. The boundary conditions and the diffusion coefficient need to be provided by the practitioner. While λ and the data on Γ_D are readily available, the Neumann condition Γ_N is artificial. It corresponds to the fact that the actual wall of the furnace continues beyond Γ_N with Ω_γ being only a part of it. Minimizing (1.6) with respect to γ can be considered as a special parameter estimation problem with unknown parameter appearing in the boundary of the physical domain. In [2] an approach based on a variable grid technique (isoparametric elements for

the discretization of Ω_γ) with part of the grid described by rays originating from Q, and with Γ_γ described by either Bezier curves or linear splines is described. Alternative approaches are the method of mappings and embedding domain techniques. While the present work was motivated by the practical application described above, the analysis is carried out for a simpler problem, which retains, however, many of the essential features.

In Section 2 a precise problem formulation together with some technical results are given. Section 3 is devoted to the embedding domain technique. Continuous dependence results are obtained in Section 4. In Section 5 we derive a characterization for the derivative of the cost functional with respect to the unknown boundary. Numerical results are summarized in Section 6. To enhance the clarity of the main argument the proofs of some technical results are moved to Appendix A.

2. Problem formulation

We consider the class of admissible Lipschitzian domains

$$\Omega_\gamma = \{(x,y)\colon \gamma(y) < x < 1, y \in (0,1)\} \quad \text{for } \gamma \in \mathcal{S},$$

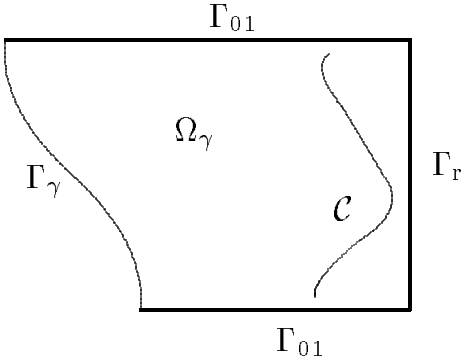

Figure 2

where

$$\mathcal{S} = \{\gamma \in H^2(0,1)\colon \mu \le \gamma(y) \le \nu, y \in [0,1], |\gamma|_{H^2} \le \rho\},$$

with $0 < \mu < \nu < 1$ and $\mu < \rho$. Let Γ_γ denote the curved part of $\partial\Omega_\gamma$, i.e.:

$$\Gamma_\gamma = \{(\gamma(y), y)\colon y \in (0,1)\},$$

and split the rectangular part of $\partial\Omega_\gamma$ as follows

$$\Gamma_r = \{1\} \times (0,1),\ \Gamma_{01} = (\gamma(0),1) \times \{0\} \cup (\gamma(1),1) \times \{1\},$$

so that $\partial\Omega_\gamma = \overline{\Gamma_\gamma} \cup \overline{\Gamma_r} \cup \overline{\Gamma_{01}}$. For $\gamma \in \mathcal{S}$ we define

$$W_\gamma = \{v \in H^1(\Omega_\gamma) \colon \tau_0 v|_{\Gamma_\gamma \cup \Gamma_r} = 0\},$$

where τ_0 is the trace operator on $\partial\Omega_\gamma$. For $f \in L^2(\mathbb{R}^2)$ and $\gamma \in \mathcal{S}$ we consider the state equation in variational form

$$(\nabla u, \nabla v)_{\Omega_\gamma} = (f, v)_{\Omega_\gamma} \quad \text{for all } v \in W_\gamma. \tag{2.1}$$

Clearly there exists a unique solution $u = u_\gamma \in W_\gamma$ of (2.1). Above $(\cdot,\cdot)_{\Omega_\gamma}$ denotes the L^2-inner product on Ω_γ. Note that (2.1) is formally equivalent to

$$\begin{cases} -\Delta u = f & \text{in } \Omega_\gamma \\ \tau_0 u|_{\Gamma_\gamma \cup \Gamma_r} = 0 \\ \frac{\partial u}{\partial n}|_{\Gamma_{01}} = 0, \end{cases} \tag{2.2}$$

where $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u to Ω_γ.

Henceforth we denote by $J_\gamma \colon W_\gamma \to \mathbb{R}$ the cost functional. It could be of the form

$$J_\gamma(u) = \int\limits_{\Omega_\gamma} |u|^2 dx \tag{2.3}$$

or

$$J_\gamma(u) = \int\limits_{\mathcal{C}} |\tau_{\mathcal{C}} u - z|^2 ds, \tag{2.4}$$

where $\mathcal{C}$ is a smooth curve in $R = (\bar\nu, 1) \times (0,1)$, with $\bar\nu \in (\nu, 1)$, $\tau_{\mathcal{C}}$ denotes the trace on $\mathcal{C}$ and $z \in L^2(\mathcal{C})$. In the context of the example of the previous section, $\mathcal{C}$ is a smooth interpolation of the measurement of location points.

The shape optimization problem is stated as

$$(\mathcal{P}) \qquad \begin{array}{l} \min J_\gamma(u_\gamma) \text{ subject to } \gamma \in \mathcal{S} \\ \text{with } u_\gamma \text{ the solution to (2.1).} \end{array}$$

Let us now turn to the discussion of the relationship between (2.1) and (2.2). We fix some $\gamma \in \mathcal{S}$ and define

$$H(\Delta, \Omega_\gamma) = \{u \in H^1(\Omega_\gamma) \colon \Delta u \in L^2(\Omega_\gamma)\}.$$

Then the trace mapping

$$\tau_1 u = \nabla u \cdot n$$

can be extended from $C^\infty(\Omega_\gamma)$ to a continuous linear operator from $H(\Delta, \Omega_\gamma)$ to $H^{-1/2}(\partial\Omega_\gamma)$. Moreover the following generalized Green's formula holds:

$$\langle \tau_1 u, \tau_0 v\rangle_{\partial\Omega_\gamma} = (\Delta u, v)_{\Omega_\gamma} + (\nabla u, \nabla v)_{\Omega_\gamma} \tag{2.5}$$

for all $u \in H(\Delta, \Omega_\gamma)$ and $v \in H^1(\Omega_\gamma)$. If u_γ is the solution to (2.1), then the first and second equations in (2.2) are satisfied in the sense of $L^2(\Omega_\gamma)$ respectively $L^2(\Gamma_\gamma \cup \Gamma_r)$. It remains to see in which sense, the third equation in (2.2) is satisfied. From (2.1) and (2.5) it follows that

$$\langle \tau_1 u_\gamma, \tau_0 v\rangle_{\partial\Omega_\gamma} = 0 \text{ for all } v \in W_\gamma. \tag{2.6}$$

If $u_\gamma \in H^2(\Omega_\gamma)$, then $\tau_1 u_\gamma \in L^2(\partial\Omega_\gamma)$ and (2.6) implies that $\frac{\partial u_\gamma}{\partial n}|_{\Gamma_{01}} = 0$ in the sense of $L^2(\Gamma_{01})$. To interpret (2.6) in the general case we define

$$H_{00}^{1/2}(\Gamma_{01}) = \{\varphi \in L^2(\Gamma_{01})\colon \exists w \in W_\gamma \text{ satisfying } \tau_0 w\,|_{\Gamma_{01}} = \varphi\}.$$

It is known that the trace operator $\tau_0\,|_{\Gamma_{01}}\colon W_\gamma \to H_{00}^{1/2}(\Gamma_{01})$ is continuous and surjective and hence it follows that (2.6) implies

$$\frac{\partial u_\gamma}{\partial n}\,|_{\Gamma_{01}} = 0 \text{ in } H_{00}^{1/2}(\Gamma_{01})^*, \tag{2.7}$$

where the star denotes the (anti) dual of $H_{00}^{1/2}(\Gamma_{01})$, c.f. ([3] pg.397). Conversely, if $u_\gamma \in H(\Delta, \Omega_\gamma)$ satisfies (2.2) with the third equation in (2.2) interpreted in the sense of (2.7), then u_γ is a solution to (2.1). In the remainder of the paper we shall frequently use the fact that

$$|u_\gamma|_{H^1(\Omega_\gamma)} \le K, \tag{2.8}$$

where the constant K is independent of $\gamma \in \mathcal{S}$.

In Section 5 we need H^2–regularity of the solution u_γ uniformly in γ. For this purpose we consider a restricted class of admissible boundaries and define for $\varepsilon > 0$

$$\bar{\mathcal{S}} = \mathcal{S} \cap \{\gamma \in H^{\frac{5}{2}+\varepsilon}(0,1)\colon \gamma|_{[0,c]} = \alpha_0, \gamma|_{[1-c,c]} = \alpha_1, |\gamma|_{H^{\frac{5}{2}+\varepsilon}} \le \kappa, \mu < \kappa < \nu\}.$$

For $\gamma \in \bar{\mathcal{S}}$ the domain Ω_γ is a curvilinear $C^{1,1}$– polygon with four rectilinear corners. In the definition of $\bar{\mathcal{S}}$ the constants α_0, $\alpha_1 \in [\mu, \nu]$ may depend on γ, furthermore, $c \in (0, \frac{1}{2})$.

Lemma 2.1. *For every $\gamma \in \bar{\mathcal{S}}$ the solution to 2.1 satisfies $u_\gamma \in H^2(\Omega_\gamma)$ and the set $\{|u_\gamma|_{H^2(\Omega_\gamma)}\}_{\gamma\in\bar{\mathcal{S}}}$ is uniformly bounded.*

The proofs of this and the following result will be given in the appendix. In the context of characterizing the derivative of J_γ with respect to γ we shall utilize the adjoint equation to (2.1), which is of the form

$$(\nabla\lambda, \nabla v)_{\Omega_\gamma} = \langle F, v\rangle_{W_\gamma^*, W_\gamma} \quad \text{for all } v \in W_\gamma \tag{2.9}$$

where $(\cdot,\cdot)_{W_\gamma^*,W_\gamma}$ denotes the duality pairing between W_γ and W_γ^*, and $F \in Z^*$, with $Z = \{v \in H^1((\bar\nu, 1)\times(0,1))\colon v|_{\Gamma_r} = 0\}$, the constant $\bar\nu$ was defined after (2.4). In particular

$$\langle F, v\rangle_{W_\gamma^*, W_\gamma} = 0 \quad \text{for all } v \in \mathcal{D}((0,\bar\nu)\times(0,1)).$$

It is straightforward to see that (2.9) has a unique solution $\lambda_\gamma \in W_\gamma$ for every $\gamma \in \mathcal{S}$ and that $\{|\lambda_\gamma|_{W_\gamma}\}_{\gamma\in\mathcal{S}}$ is uniformly bounded. Moreover the following result holds.

Lemma 2.2. *For every $\gamma \in \bar{\mathcal{S}}$ the solution to (2.9) satisfies $\lambda \in H^2(R_\gamma)$ where $R_\gamma = \{(x,y)\colon y \in (0,1), \gamma(y) < x < \frac{\nu+\bar\nu}{2}\}$. Moreover $\{|\lambda|_{H^2(R_\gamma)}\}_{\gamma\in\bar{\mathcal{S}}}$ is uniformly bounded.*

Remark 2.1. *Lemmas 2.1 and 2.2 remain valid if the corners of Ω_γ at $P_1 = (\gamma(0), 0)$ and $P_2 = (\gamma(1), 1)$ are such that γ is linear in neighborhoods of 0 and 1 and the interior angles at P_1 and P_2 are less than or equal to $\frac{\pi}{2}$.*

3. An embedding domain method

The aim of this section is to derive an equation on a simple domain $\hat\Omega$ containing Ω_γ, with the property that the solution of the new system restricted to Ω_γ represents a solution $u \in W_\gamma$ to

$$(\nabla u, \nabla v)_{\Omega_\gamma} = (f, v)_{\Omega_\gamma} \quad \text{for all } v \in W_\gamma. \tag{3.1}$$

Throughout this section $\gamma \in \mathcal{S}$ is fixed, and we note that it would suffice to assume that γ is Lipschitz continuous. We define the fictitious domain $\hat\Omega = (0,1)\times(0,1)$ and set

$$\Omega_\gamma^c = \hat\Omega \setminus \bar\Omega_\gamma.$$

Note that Ω_γ^c is a Lipschitz domain. Furthermore we introduce the spaces

$$\begin{aligned} \hat V &= \{\hat u \in H^1(\hat\Omega) : \tau_0 \hat u|_{\Gamma_r} = 0\}, \\ V_\gamma &= \{u \in H^1(\Omega_\gamma) : \tau_0 u|_{\Gamma_r} = 0\}. \end{aligned} \tag{3.2}$$

We shall use

$$|\hat{u}|_{\hat{V}} = \sqrt{(\nabla\hat{u}, \nabla\hat{u})_{\hat{\Omega}}}$$

as norm on $\hat{V}$. In view of Friedrich's inequality it is equivalent to the norm induced by $H^1(\hat{\Omega})$. The norms on V_γ and W_γ are defined analogously. For any $u \in L^2(\Omega_\gamma)$ we define the extension by zero to $\hat{\Omega}$ as

$$\tilde{u} = \begin{cases} u & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega_\gamma^c. \end{cases} \tag{3.3}$$

We shall utilize the interior trace operator $\tau_\gamma : \hat{V} \to H^{1/2}(\Gamma_\gamma)$ defined by

$$\tau_\gamma \hat{u} = \hat{u}|_{\Gamma_\gamma}.$$

Lemma 3.1. *Let $\varphi \in H^1(\Omega_\gamma)$ and $\psi \in H^1(\Omega_\gamma^c)$ satisfy $\tau_\gamma\varphi = \tau_\gamma\psi$ and define*

$$\hat{\varphi} = \begin{cases} \varphi \text{ in } \Omega_\gamma, \\ \psi \text{ in } \Omega_\gamma^c. \end{cases}$$

Then $\hat{\varphi}$ is in $H^1(\hat{\Omega})$.

The proof of this lemma and of the following one are given in the appendix. From Lemma 3.1 it follows that $\tilde{u} \in \hat{V}$ whenever $u \in W_\gamma$.

Lemma 3.2. *(i) $\tau_\gamma \in \mathcal{L}(\hat{V}, H^{1/2}(\Gamma_\gamma))$ and $\|\tau_\gamma\|_{\mathcal{L}(\hat{V}, H^{1/2}(\Gamma_\gamma))} \le c$ for a constant c independent of $\gamma \in \mathcal{S}$.*

(ii) τ_γ is surjective.

In this section we show that the mixed boundary value problem (3.1) on the curvilinear domain Ω_γ is equivalent to the following problem:

Find a pair $(\hat{u}, \hat{g}) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ satisfying

$$\begin{cases} (\nabla\hat{u}, \nabla\hat{v})_{\hat{\Omega}} - (\tilde{f}, \hat{v})_{\hat{\Omega}} - \langle \hat{g}, \tau_\gamma\hat{v}\rangle_{\Gamma_\gamma} = 0, & \text{for all } \hat{v} \in \hat{V}, \\ \tau_\gamma\hat{u} = 0. \end{cases} \tag{3.4}$$

Proposition 3.1. *There exists a unique solution $(\hat{u}, \hat{g}) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ of (3.4).*

Proof. Define $\mathcal{J} : \hat{V} \to \mathbb{R}$ by

$$\mathcal{J}(\hat{v}) = \frac{1}{2}|\nabla\hat{v}|^2_{\hat{\Omega}} - (\hat{v}, \tilde{f})_{\hat{\Omega}}, \tag{3.5}$$

and set

$$K = \{\hat{v} \in \hat{V} : \tau_\gamma \hat{v} = 0\}.$$

Clearly $\mathcal{I}$ is a strictly convex functional and by Lemma 3.2 K is a convex and closed subset of $\hat{V}$. Hence there exists a unique element $\hat{u} \in K$ satisfying

$$\mathcal{I}(\hat{u}) \leq \mathcal{I}(\hat{v}) \quad \text{for all } \hat{v} \in K.$$

Since by Lemma 3.2 τ_γ is surjective it follows that $\hat{u}$ is a regular point with respect to the constraint $\tau_\gamma \hat{v} = 0$ and hence there exists a Lagrange multiplier $\hat{g} \in H^{1/2}(\Gamma_\gamma)^*$ which renders the Lagrangian

$$\mathcal{L}(\hat{v}, \hat{g}) = \mathcal{I}(\hat{v}) - \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma}$$

stationary at $\hat{u}$, i.e.:

$$D_{\hat{v}} \mathcal{L}(\hat{u}, \hat{g})(\hat{v}) = (\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}, \hat{v})_{\hat{\Omega}} - \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} = 0$$

for all $\hat{v} \in \hat{V}$. Here $D_{\hat{v}}$ denotes the Fréchet derivative of $\mathcal{L}(\hat{v}, \hat{g})$ with respect to $\hat{v}$. We have thus shown that $(\hat{u}, \hat{g})$ solves (3.4). Uniqueness $(\hat{u}, \hat{g})$ follows from the uniqueness of $\hat{u}$ and the first equation in (3.4) together with the surjectivity of τ_γ. □

Theorem 3.1. *(i) Let $(\hat{u}, \hat{g}) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ denote the solution of (3.4) and define $u = \hat{u}|_{\Omega_\gamma} \in W_\gamma$. Then u is the unique solution of (3.1), $\hat{u}|_{\Omega_\gamma^c} = 0$ and*

$$\langle \tau_1 u, \tau_0 v \rangle_{\partial\Omega_\gamma} = \langle \hat{g}, \tau_\gamma v \rangle_{\Gamma_\gamma} \quad \textit{for all } v \in V_\gamma. \tag{3.6}$$

(ii) Conversely, let $u \in W_\gamma$ be the solution of (3.1). Then (3.6) uniquely defines an element $\hat{g} \in H^{1/2}(\Gamma_\gamma)^*$ and $(\tilde{u}, \hat{g})$ is the solution to (3.4). Here $\tilde{u}$ denotes the extension by zero of u.

We note that (3.6) implies that

$$\hat{g} = \frac{\partial u}{\partial n}|_{\Gamma_\gamma} \text{ in } H_{00}^{1/2}(\Gamma_\gamma)^*,$$

and thus loosely speaking, we can say that the Lagrange multiplier associated with the constraint $\tau_\gamma u = 0$ is given by $\frac{\partial u}{\partial n}|_{\Gamma_\gamma}$.

Proof of Theorem 3.1. (i) By (3.4)

$$(\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} = 0 \text{ for all } \hat{v} \in \hat{V} \text{ with support in } \Omega_\gamma^c.$$

Together with $\tau_\gamma \hat{u} = 0$ this implies $\hat{u}|_{\Omega_\gamma^c} = 0$ in $H^1(\Omega_\gamma^c)$. For $v \in W_\gamma$ we obtain

$$(\nabla \hat{u}, \nabla \tilde{v})_{\hat{\Omega}} - (\tilde{f}, \tilde{v})_{\hat{\Omega}} = (\nabla u, \nabla v)_{\Omega_\gamma} - (f, v)_{\Omega_\gamma} = 0,$$

with $\tilde{v}$ defined in 3.3. Since $u = \hat{u}|_{\Omega_\gamma} \in W_\gamma$ this implies that u is the unique solution of (3.1). By Green's formula we obtain from (3.4) for every $\hat{v} \in \hat{V}$

$$(\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} - (f, \hat{v})_{\Omega_\gamma} = \langle \tau_1 u, \tau_0 \hat{v} \rangle_{\partial\Omega_\gamma} = \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma}$$

which gives (3.6).
(ii) Let u be the solution to (3.1). We shall verify that

$$v \to \langle \tau_1 u, \tau_0 v \rangle_{\partial\Omega_\gamma}, \quad v \in V_\gamma, \tag{3.7}$$

uniquely defines an element $\hat{g} \in H^{1/2}(\Gamma_\gamma)^*$ for which (3.6) holds. In particular this requires to verify that for $v \in V_\gamma$ the functional defined in (3.7) only depends on $\tau_\gamma v$. Thus let v and w be elements in V_γ with $\tau_\gamma v = \tau_\gamma w$. It follows that $v - w \in W_\gamma$ and hence by (2.6)

$$\langle \tau_1 u, \tau_0 v \rangle_{\partial\Omega_\gamma} = \langle \tau_1 u, \tau_0 w \rangle_{\partial\Omega_\gamma} + \langle \tau_1 u, \tau_0 (v - w) \rangle_{\partial\Omega_\gamma} = \langle \tau_1 u, \tau_0 w \rangle_{\partial\Omega_\gamma}.$$

Since $\tau_\gamma : V_\gamma \to H^{1/2}(\Gamma_\gamma)$ is surjective, it follows that (3.7) uniquely defines a linear functional $\hat{g}$ on $H^{1/2}(\Gamma_\gamma)$ for which

$$\langle \hat{g}, \tau_\gamma v \rangle_{\Gamma_\gamma} = \langle \tau_1 u, \tau_0 v \rangle_{\partial\Omega_\gamma} \quad \text{for all } v \in V_\gamma$$

holds. To argue continuity of this functional, let $E : H^{1/2}(\Gamma_\gamma) \to V_\gamma$ denote a (bounded) extension operator. Then

$$\begin{aligned} |\langle \hat{g}, \varphi \rangle_{\Gamma_\gamma}| &= |\langle \tau_1 u, \tau_0 E\varphi \rangle_{\partial\Omega_\gamma}| \\ &\leq |\tau_1 u|_{H^{-1/2}(\partial\Omega_\gamma)} \|\tau_0\|_{\mathcal{L}(H^1(\Omega_\gamma), H^{1/2}(\partial\Omega_\gamma))} \|E\| |\varphi|_{H^{1/2}(\Gamma_\gamma)} \end{aligned}$$

and it follows that $\hat{g} \in H^{1/2}(\Gamma_\gamma)^*$. For arbitrary $\hat{v} \in \hat{V}$ we obtain with $v = \hat{v}|\Omega_\gamma$

$$\begin{aligned} (\nabla \tilde{u}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}, \hat{v})_{\hat{\Omega}} - \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} &= (\nabla u, \nabla v)_{\Omega_\gamma} - (f, v)_{\Omega_\gamma} - \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} \\ &= \langle \tau_1 u, \tau_0 v \rangle_{\partial\Omega_\gamma} - \langle \hat{g}, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} = 0, \end{aligned}$$

and thus $(\tilde{u}, \hat{g})$ is the solution to (3.4). □

A common characterization of the norm in $H^{1/2}(\Gamma_\gamma)$ (see [10, pg.20] is given by

$$|\varphi|^2 = \int_{\Gamma_\gamma} |\varphi(x)|^2 d\sigma + \int_{\Gamma_\gamma} \int_{\Gamma_\gamma} \frac{|\varphi(x) - \varphi(y)|^2}{(x - y)^2} d\sigma(x) d\sigma(y).$$

Since $\gamma \in \mathcal{S}$ this norm is equivalent to

$$|\varphi|^2 = \int_0^1 |\varphi(\gamma(y), y)|^2 dy + \int_0^1 \int_0^1 \frac{|\varphi(\gamma(y), y) - \varphi(\gamma(x), x)|^2}{(x-y)^2} dx dy. \tag{3.8}$$

We remark that this equivalence is uniform with respect to $\gamma \in \mathcal{S}$. In the following it will be necessary to compare the trace of $\hat{u} \in \hat{V}$ along the inner boundary Γ_γ for different choices of γ. It will be convenient to identify $H^{1/2}(\Gamma_\gamma)$ with $H^{1/2}(0,1)$ by the following procedure. For $\gamma \in \mathcal{S}$ define the bijection $G_\gamma : (0,1) \to \Gamma_\gamma$ by

$$G_\gamma(y) = (\gamma(y), y)$$

and the embedding $\mathcal{J}_\gamma : H^{1/2}(\Gamma_\gamma) \to H^{1/2}(0,1)$ by

$$\mathcal{J}_\gamma \varphi = \varphi \circ G_\gamma. \tag{3.9}$$

Endowing $H^{1/2}(\Gamma_\gamma)$ with the norm of (3.8) $\mathcal{J}_\gamma$ becomes an isometry.

4. Continuous dependence and existence to $(\mathcal{P})$

In this section we establish continuous dependence of the solution $(\hat{u}, \hat{g}) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ on the interior boundary γ as well as existence to problem $(\mathcal{P})$. For this purpose we identify $H^{1/2}(\Gamma_\gamma)$ and $H^{1/2}(0,1)$ by means of the isometry $\mathcal{J}_\gamma$ defined in (3.9). Since $(\mathcal{J}_\gamma^{-1})^* : H^{1/2}(\Gamma_\gamma)^* \to H^{1/2}(0,1)^*$ is an isometry we may also interpret $\hat{g}$ as a functional on $H^{1/2}(0,1)$. More precisely we define for any $\hat{g} \in H^{1/2}(\Gamma_\gamma)^*$ the element $G \in H^{1/2}(0,1)^*$ by

$$\langle G, \psi \rangle_{(0,1)} = \langle \hat{g}, \mathcal{J}_\gamma^{-1} \psi \rangle_{\Gamma_\gamma}, \quad \text{for all } \psi \in H^{1/2}(0,1), \tag{4.1}$$

satisfying

$$|G|_{H^{1/2}(0,1)^*} = |\hat{g}|_{H^{1/2}(\Gamma_\gamma)^*}.$$

In addition we introduce $\mathcal{T}_\gamma \in \mathcal{L}(\hat{V}, H^{1/2}(0,1))$ by

$$\mathcal{T}_\gamma = \mathcal{J}_\gamma \circ \tau_\gamma. \tag{4.2}$$

By Lemma 3.2 $\mathcal{T}_\gamma$ is a surjection. Moreover there exists a constant c such that

$$\|\mathcal{T}_\gamma\| = \|\tau_\gamma\| \leq c \quad \text{for all } \gamma \in \mathcal{S}.$$

Subsequently $(\hat{u}_\gamma, \hat{g}_\gamma)$ will denote the solution to (3.4) for $\gamma \in \mathcal{S}$ and with f replaced by f_γ, i.e., the inhomogeneity can depend on the location of γ. This

situation can arise, for example, if a problem with inhomogenous boundary conditions, such as the model problem of Section 2, is transformed to an equation with homogeneous boundary conditions.

Theorem 4.1. *Let $\gamma_n, \bar{\gamma} \in \mathcal{S}$ and $f_{\gamma_n} \in L^2(\Omega_{\gamma_n})$, $f_{\bar{\gamma}} \in L^2(\Omega_{\bar{\gamma}})$ satisfy $\lim_{n\to\infty} \gamma_n = \bar{\gamma}$ in $W^{1,\infty}(0,1)$ and $\lim_{n\to\infty} \tilde{f}_{\gamma_n} = \tilde{f}_{\bar{\gamma}}$ in $L^2(\hat{\Omega})$. Then the solutions $(\hat{u}_{\gamma_n}, \hat{g}_{\gamma_n}) \in \hat{V} \times H^{1/2}(\Gamma_{\gamma_n})^*$ satisfy*

$$\lim_{n\to\infty} \hat{u}_{\gamma_n} = \hat{u}_{\bar{\gamma}} \text{ in } \hat{V} \text{ and } w^* - \lim_{n\to\infty} (\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n} = (\mathcal{J}_{\bar{\gamma}}^{-1})^* \hat{g}_{\bar{\gamma}},$$

where $(\hat{u}_{\bar{\gamma}}, \hat{g}_{\bar{\gamma}})$ is the unique solution of (3.4) with (γ, f) replaced by $(\bar{\gamma}, f_{\bar{\gamma}})$.

The proof of this result is given at the end of the section.

We turn to existence of a solution to $(\mathcal{P})$ and assume that

$$\text{(H)} \quad \begin{cases} \gamma_n \to \bar{\gamma} \text{ in } W^{1,\infty}(0,1), \text{ with } \gamma_n \in \mathcal{S}, \text{ and } \tilde{u}_n \to \tilde{u} \text{ in } \hat{V} \text{ implies} \\ J_{\gamma_n}(u_{\gamma_n}) \to J_{\bar{\gamma}}(u). \end{cases}$$

In view of Lemma 4.2 below (H) is satisfied for the functionals defined in (2.3) and (2.4).

Theorem 4.2. *If (H) holds, then $(\mathcal{P})$ has at least one solution $\gamma^* \in \mathcal{S}$.*

Proof. Let $\{\gamma_n\}$ be a minimizing sequence in $\mathcal{S}$. Then there exists a subsequence, again denoted by $\{\gamma_n\}$, that converges strongly in $W^{1,\infty}(0,1)$ to some $\gamma^* \in \mathcal{S}$. Defining $f_{\gamma_n} = f|\Omega_{\gamma_n}$ it follows that $\tilde{f}_{\gamma_n} \to \tilde{f}_{\gamma^*}$ in $L^2(\hat{\Omega})$. The claim now follows from Theorem 4.1 and (H). □

For the proof of Theorem 4.1 we shall utilize the following three lemmas.

Lemma 4.1. *Let $\{\tilde{f}_\gamma : \gamma \in \mathcal{S}\}$, be uniformly bounded in $L^2(\hat{\Omega})$. Then the solutions of (3.4) $(\hat{u}_\gamma, \hat{g}_\gamma) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ satisfy*

$$|(\hat{u}_\gamma, \hat{g}_\gamma)|_{\hat{V}\times H^{1/2}(\Gamma_\gamma^*)} \le k$$

for a constant k independent of $\gamma \in \mathcal{S}$.

Proof. Recall that $\hat{u}_\gamma$ is the unique minimizer of $\mathcal{I}$ defined in (3.5) in K. In particular this implies that $\mathcal{I}(\hat{u}_\gamma) \le \mathcal{I}(0) = 0$, and hence

$$|\hat{u}_\gamma|_{\hat{V}}^2 \le 2|f_\gamma|_{L^2(\Omega_\gamma)} |\hat{u}_\gamma|_{L^2(\Omega_\gamma)},$$

which implies that $\{\hat{u}_\gamma\}_{\gamma\in\mathcal{S}}$ is uniformly bounded in $\hat{V}$. To obtain the bound on $\hat{g}_\gamma$ choose for arbitrary $\varphi \in H^{1/2}(\Gamma_\gamma)$ any $\hat{v} \in \hat{V}$ satisfying $\tau_\gamma \hat{v} = \varphi$. Then (3.4) gives

$$\langle \hat{g}_\gamma, \varphi \rangle_{\Gamma_\gamma} = \langle \hat{g}_\gamma, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} = (\nabla \hat{u}_\gamma, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_\gamma, \hat{v})_{\hat{\Omega}},$$

which implies

$$|\langle \hat{g}_\gamma, \varphi \rangle_{\Gamma_\gamma}| \leq \tilde{k} |\hat{v}|_{\hat{V}}$$

for a constant $\tilde{k}$ independent of $\gamma \in \mathcal{S}$. The last inequality implies

$$|\langle \hat{g}, \varphi \rangle_{\Gamma_\gamma}| \leq \tilde{k} |||\varphi|||_{H^{1/2}(\Gamma_\gamma)},$$

where we have set

$$|||\varphi|||_{H^{1/2}(\Gamma_\gamma)} = \inf\{|\hat{v}|_{\hat{V}} : \hat{v} \in \hat{V}, \tau_\gamma \hat{v} = \varphi\},$$

which is an equivalent norm on $H^{1/2}(\Gamma_\gamma)$, and this equivalence is uniform in $\gamma \in \mathcal{S}$. This ends the proof. □

The next result establishes the strong continuous dependence of the interior trace operator on γ.

Lemma 4.2. *Let $\gamma_n, \bar{\gamma} \in \mathcal{S}$ satisfy $\lim_{n\to\infty} \gamma_n = \bar{\gamma}$ in $W^{1,\infty}(0,1)$. Then $\mathcal{T}_{\gamma_n}$ converges strongly to $\mathcal{T}_{\bar{\gamma}}$.*

The proof of Lemma 4.2 is given in the appendix.

Lemma 4.3. *Assume that $\gamma_n \to \gamma$ in $C(0,1)$ with $\gamma_n \in \mathcal{S}$, and let $\hat{u}_n$ be any sequence in $\hat{V}$ satisfying $\tau_{\gamma_n} \hat{u}_n = 0$, for all n. If $\hat{u}_n$ converges weakly to $\hat{u}$ in $\hat{V}$ then $\tau_\gamma \hat{u} = 0$.*

Proof. For every n we have

$$|\mathcal{T}_\gamma \hat{u}|_{L^2(0,1)} \leq |\mathcal{T}_\gamma(\hat{u} - \hat{u}_n)|_{L^2(0,1)} + |\mathcal{T}_\gamma \hat{u}_n - \mathcal{T}_{\gamma_n} \hat{u}_n|_{L^2(0,1)}.$$

Due to the compact embedding of $H^{1/2}(0,1)$ into $L^2(0,1)$, the first term on the right hand side of the above inequality converges to zero. The second is estimated as follows

$$|\mathcal{T}_\gamma \hat{u}_n - \mathcal{T}_{\gamma_n} \hat{u}_n|^2_{L^2(0,1)} = \int_0^1 |\hat{u}_n(\gamma(s), s) - \hat{u}_n(\gamma_n(s), s)|^2 ds$$

$$\leq \int_0^1 \Big| \int_{\gamma_n(s)}^{\gamma(s)} \frac{\partial}{\partial x} \hat{u}_n(x, s) dx \Big|^2 ds \leq |\gamma - \gamma_n|_\infty |\hat{u}_n|_{H^1(\hat{\Omega})}.$$

Since $\{\hat{u}_n\}$ is uniformly bounded in $\hat{V}$, the result follows. □

Proof of Theorem 4.1. Lemma 4.1 implies the existence of a subsequence of $\hat{u}_{\gamma_n}$, denoted by the same symbol, with the property that $\hat{u}_{\gamma_n}$ converges weakly in $\hat{V}$ to an element $\hat{u} \in \hat{V}$. Lemma 4.3 implies that $\tau_{\bar{\gamma}} \hat{u} = 0$.

Next observe that for every $\hat{v} \in \hat{V}$

$$\lim_{n\to\infty} \langle \hat{g}_{\gamma_n}, \tau_{\gamma_n} \hat{v} \rangle_{\Gamma_{\gamma_n}} = \lim_{n\to\infty} \left[(\nabla \hat{u}_{\gamma_n}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\gamma_n}, \hat{v})_{\hat{\Omega}} \right] \tag{4.3}$$

$$= (\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} - (f_{\bar{\gamma}}, \hat{v})_{\hat{\Omega}} \tag{4.4}$$

For $\varphi \in H^{1/2}(0,1)$ choose $\hat{v} \in \hat{V}$ such that $\varphi = \mathcal{T}_{\bar{\gamma}} \hat{v}$ and calculate

$$\langle (\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n}, \varphi \rangle_{(0,1)} = \langle (\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n}, \mathcal{T}_{\bar{\gamma}} \hat{v} - \mathcal{T}_{\gamma_n} \hat{v} \rangle_{(0,1)} + \langle (\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n}, \mathcal{T}_{\gamma_n} \hat{v} \rangle_{(0,1)}.$$

The first term on the right hand side of the above equality converges to 0 by Lemma 4.1 and 4.2, and the second one equals $\langle \hat{g}_n, \tau_{\gamma_n} \hat{v} \rangle_{\Gamma_{\gamma_n}}$. Hence, in view of (4.3), $(\mathcal{J}_{\gamma_n}^{-1})^* \hat{g}_{\gamma_n}$ converges weak–$*$ to a uniquely determined element $\hat{G} \in H^{1/2}(0,1)^*$ which satisfies

$$\langle \hat{G}, \varphi \rangle_{(0,1)} = (\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\bar{\gamma}}, \hat{v})_{\hat{\Omega}}.$$

Then $\hat{g} = \mathcal{J}_{\bar{\gamma}}^* \hat{G}$ satisfies for arbitrary $\hat{v} \in \hat{V}$ and $\varphi = \mathcal{T}_{\bar{\gamma}} \hat{v}$

$$\langle \hat{g}, \tau_{\bar{\gamma}} \hat{v} \rangle_{\Gamma_{\bar{\gamma}}} = (\nabla \hat{u}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\bar{\gamma}}, \hat{v})_{\hat{\Omega}}.$$

It follows that $(\hat{u}, \hat{g})$ is the unique solution $(\hat{u}_{\bar{\gamma}}, \hat{g}_{\bar{\gamma}})$ of (3.4) with (γ, f) replaced by $(\bar{\gamma}, \tilde{f}_{\bar{\gamma}})$ and that for the original sequence w–$\lim \hat{u}_{\gamma_n} = \hat{u}_{\bar{\gamma}}$ and w^*–$\lim \mathcal{J}_{\gamma_n}^{-1} \hat{g}_{\gamma_n} = \mathcal{J}_{\bar{\gamma}}^{-1} \hat{g}_{\bar{\gamma}}$. It remains to show that $\hat{u}_{\gamma_n} \to \hat{u}_{\bar{\gamma}}$ strongly in $\hat{V}$. In this respect we argue as follows:

$$(\nabla \hat{u}_{\gamma_n}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\gamma_n}, \hat{v})_{\hat{\Omega}} - \langle \hat{g}_{\gamma_n}, \tau_{\gamma_n} \hat{v} \rangle_{\Gamma_{\gamma_n}} = 0, \text{ and}$$
$$(\nabla \hat{u}_{\bar{\gamma}}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\bar{\gamma}}, \hat{v})_{\hat{\Omega}} - \langle \hat{g}_{\bar{\gamma}}, \tau_{\bar{\gamma}} \hat{v} \rangle_{\Gamma_{\bar{\gamma}}} = 0$$

for all $\hat{v} \in \hat{V}$. Inserting $\hat{v} = \hat{u}_{\gamma_n}$ in the first and $\hat{v} = \hat{u}_{\bar{\gamma}}$ in the second equation we obtain the relationships

$$|\hat{u}_{\gamma_n}|_{\hat{V}}^2 = (\tilde{f}_{\gamma_n}, \hat{u}_{\gamma_n})_{\hat{\Omega}} \text{ and } |\hat{u}_{\bar{\gamma}}|_{\hat{V}}^2 = (\tilde{f}_{\bar{\gamma}}, \hat{u}_{\bar{\gamma}})_{\Omega} \text{ for all } n.$$

Subtracting yields

$$\left| |\hat{u}_{\gamma_n}|_{\hat{V}}^2 - |\hat{u}_{\bar{\gamma}}|_{\hat{V}}^2 \right| \le |\tilde{f}_{\gamma_n} - \tilde{f}_{\bar{\gamma}}|_{L^2(\hat{\Omega})} |\hat{u}_{\gamma_n}|_{\hat{V}} + |\tilde{f}_{\bar{\gamma}}|_{L^2(\hat{\Omega})} |\hat{u}_{\gamma_n} - u_{\bar{\gamma}}|_{L^2(\hat{\Omega})}$$

and hence $\lim_{n\to\infty} |\hat{u}_{\gamma_n}|_{\hat{V}} = |\hat{u}_{\bar{\gamma}}|_{\bar{V}}$. Together with weak convergence this implies strong convergence of $\hat{u}_{\gamma_n}$ to $\hat{u}_{\bar{\gamma}}$. □

5. Differentiability

In this section we discuss the differentiability of the cost functional

$$J(u_\gamma) = \int_{\mathcal{C}} |\tau_{\mathcal{C}} u_\gamma - z|^2 ds, \tag{5.1}$$

where u_γ is the solution to (2.1) with respect to γ. We shall characterize the Fréchet derivative at points $\gamma \in \bar{\mathcal{S}}$ in directions $\bar{\gamma} \in D$, where

$$D = \{\bar{\gamma} \in H^{5/2+\varepsilon} : \bar{\gamma}|[0,c] = \beta_0, \bar{\gamma}|[1-c,1] = \beta_1\},$$

with $\varepsilon > 0$ as in the definition of $\bar{\mathcal{S}}$. The constants β_0, β_1 may depend on $\bar{\gamma}$. Note that D is a linear subspace of $H^{5/2+\varepsilon}$. The regularity requirement in the definition of $\bar{\mathcal{S}}$ is stronger than needed for the main result of this section. We require that $\gamma \in H^2$ and we need conclusions of Lemmas 2.1 and 2.2 to hold, for which it would suffice if γ was in $C^{1,1}$ in addition to being linear in the neighborhoods of 0 and 1, with interior angles of Ω_γ less than or equal to $\frac{\pi}{2}$ at $(\gamma(0), 0)$ and $(\gamma(1), 1)$. Assuming $\gamma \in H^{5/2+\varepsilon}$ implies $\gamma \in C^{1,1} \cap H^2$ and in addition it guarantees the existence of a solution to

$$\begin{array}{l} \min J_\gamma(u_\gamma) \\ \text{subject to } \gamma \in \bar{\mathcal{S}}, \end{array} \tag{$\tilde{\mathcal{P}}$}$$

compare Theorem 4.2. Moreover, by Lemma 2.1 $u_\gamma \in H^2(\Omega_\gamma)$ for every $\gamma \in \bar{\mathcal{S}}$ and $\{|u_\gamma|_{H^2(\Omega_\gamma)}\}_{\gamma \in \bar{\mathcal{S}}}$ is uniformly bounded. As a consequence, the functionals $\hat{g}_\gamma$ (and analogously $\hat{\mu}_\gamma$) admit the following representation (using the same symbol also for their Riesz representatives)

$$\langle \hat{g}_\gamma, \phi \rangle_{\Gamma_\gamma} = \int_0^1 \hat{g}_\gamma(y)\phi(y)\sqrt{1 + \dot{\gamma}^2(y)}\, dy, \qquad \phi \in H^{1/2}(\Gamma_\gamma).$$

The derivative of $\gamma \to J(u_\gamma)$ will be characterized in terms of the primal and the adjoint state variable. The Lagrange formalism suggests to consider the variational problem

$$(\nabla \lambda_\gamma, \nabla v)_{\Omega_\gamma} = -2 \int_{\mathcal{C}} (\tau_{\mathcal{C}} u_\gamma - z)\tau_{\mathcal{C}} v ds \quad \text{for all } v \in W_\gamma. \tag{5.2}$$

For every $\gamma \in \mathcal{S}$ there exists a unique solution $\lambda_\gamma \in W_\gamma$ to (5.2). Moreover $\lambda_\gamma \in H^2(R_\gamma)$ for every $\gamma \in \bar{\mathcal{S}}$ and $\{|\lambda_\gamma|_{H^2(R_\gamma)}\}_{\gamma \in \bar{\mathcal{S}}}$ is uniformly bounded by

Lemma 2.2 (see also Lemma 2.2 for the definition of R_γ). Together with (5.2) we also consider the system in $\hat{V} \times H^{1/2}(\Gamma_\gamma)^*$:

$$\begin{cases} (\nabla \hat{\lambda}_\gamma, \nabla \hat{v})_{\hat{\Omega}} + 2 \int\limits_{\mathcal{C}} (\tau_{\mathcal{C}} u_\gamma - z) \tau_{\mathcal{C}} \hat{v} ds - \langle \hat{\mu}_\gamma, \tau_\gamma \hat{v} \rangle_{\Gamma_\gamma} = 0 \\ \tau_\gamma \hat{\lambda}_\gamma = 0, \quad \text{for all } \hat{v} \in \hat{V}. \end{cases} \tag{5.3}$$

A minor modification of Proposition 3.1 implies that there exists a unique solution $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma) \in \hat{V} \times H^{1/2}(\Gamma_\gamma)^*$ of (5.3) for every $\gamma \in \mathcal{S}$. The relationship between λ_γ and $\hat{\lambda}_\gamma$ is clarified in the following corollary which follows from Theorem 3.1 and the fact that $\lambda_\gamma \in H^1(\Omega_\gamma)$.

Corollary 5.1. *Let $\gamma \in \bar{\mathcal{S}}$.*

1. *Let $\lambda_\gamma \in W_\gamma$ denote the solution to (5.2) and let $\hat{\lambda}_\gamma$ stand for the extension of λ_γ by zero to $\hat{\Omega}$. Then $\frac{\partial \lambda_\gamma}{\partial n}|_{\Gamma_\gamma} =: \hat{\mu}_\gamma \in H^{\frac{1}{2}}(\Gamma_\gamma)$ and $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma)$ is the solution to (5.3).*
2. *Conversely, if $(\hat{\lambda}_\gamma, \hat{\mu}_\gamma) \in \hat{V} \times H^{\frac{1}{2}}(\Gamma_\gamma)^*$ is the solution to (5.3) then $\lambda_\gamma := \hat{\lambda}_\gamma|_{\Omega_\gamma}$ is the solution to (5.2) and $\hat{\mu}_\gamma = \frac{\partial \hat{\lambda}_\gamma}{\partial n}|_{\Gamma_\gamma}$.*

We are now prepared to characterize the derivative of the cost functional (5.1) with respect to $\gamma \in \bar{\mathcal{S}}$.

Theorem 5.1. *Let $f \in L^\infty(\hat{\Omega})$. Then $\gamma \mapsto J(u_\gamma)$ has a Fréchet derivative at every $\gamma \in \bar{\mathcal{S}}$ characterized by*

$$D_\gamma J(u_\gamma)\bar{\gamma} = \int_0^1 \hat{\mu}_\gamma(y) \hat{g}_\gamma(y) \bar{\gamma}(y)\, dy \qquad \text{for every } \bar{\gamma} \in D. \tag{5.4}$$

As a direct consequence we obtain the following optimality condition for $(\tilde{\mathcal{P}})$.

Corollary 5.2. *Let γ^* be a solution to $(\tilde{\mathcal{P}})$. Then the following optimality condition holds:*

$$D_\gamma J(u_\gamma)(\gamma - \gamma^*) = \int_0^1 \hat{\mu}_{\gamma^*}(y) \hat{g}_{\gamma^*}(y)(\gamma(y) - \gamma^*(y)) dy \geq 0 \quad \text{for every } \gamma \in \bar{\mathcal{S}},$$

where

$$\begin{cases} (\nabla \hat{u}_{\gamma^*}, \nabla \hat{v})_{\hat{\Omega}} - (\tilde{f}_{\gamma^*}, \hat{v})_{\hat{\Omega}} - \langle g_{\hat{\gamma}^*}, \tau_{\gamma^*} \hat{v} \rangle_{\Gamma_{\gamma^*}} = 0, \\ \tau_\gamma \hat{u}_{\gamma^*} = 0. \qquad \text{for all } \hat{v} \in \hat{V} \end{cases}$$

$$\begin{cases} (\nabla \hat{\lambda}_{\gamma^*}, \nabla \hat{v})_{\hat{\Omega}} + 2 \int\limits_{\mathcal{C}} (\tau_{\mathcal{C}} u_{\gamma^*} - z) \tau_{\mathcal{C}} \hat{v} ds - \langle \hat{\mu}_{\gamma^*}, \tau_{\gamma^*} \hat{v} \rangle_{\Gamma_{\gamma^*}} = 0 \\ \tau_\gamma \hat{\lambda}_{\gamma^*} = 0, \qquad \text{for all } \hat{v} \in \hat{V}. \end{cases}$$

The proof of Theorem 5.1 is based on several technical results which are stated next. Their proofs are given in the appendix. To verify Theorem 5.1 we shall need to consider $\gamma + t\bar{\gamma}$ with $\gamma \in \bar{\mathcal{S}}$, $\bar{\gamma} \in D$, $t \in \mathbb{R}$ for small t. Consequently we need to consider a neighborhood $\bar{\mathcal{S}}_N$ of $\bar{\mathcal{S}}$. A convenient choice is given by

$$\bar{\mathcal{S}}_N = \{\gamma \in \bar{\mathcal{S}} \colon \frac{\mu}{2} \le \gamma \le \frac{\nu + \bar{\nu}}{2}, |\gamma|_{H^{\frac{5}{2}+\varepsilon}} \le 2\rho\}.$$

We note that $\Gamma_\gamma \cap \mathcal{C}$ is empty for every $\gamma \in \bar{\mathcal{S}}_N$. It is obvious that for $\gamma \in \bar{\mathcal{S}}_N$ there exist solutions to (2.2) and to (5.3), and, moreover $\{|u_\gamma|_{H^2(\Omega_\gamma)}\}_{\gamma \in \bar{\mathcal{S}}_N}$ and $\{|\lambda_\gamma|_{H^2(R_\gamma)}\}_{\gamma \in \bar{\mathcal{S}}_N}$ are uniformly bounded. This uniform $H^2(\Omega_\gamma)$ regularity implies Lipschitz continuity of $\gamma \mapsto u_\gamma$. For a precise statement of this fact we introduce for $\gamma, \tilde{\gamma} \in \bar{\mathcal{S}}_N$ the bijection $\psi \colon \Omega_\gamma \to \Omega_{\tilde{\gamma}}$

$$\psi(x, y) = (\frac{1 - \tilde{\gamma}}{1 - \gamma} x + \frac{\tilde{\gamma} - \gamma}{1 - \gamma}, y), \tag{5.5}$$

and we define

$$U_{\tilde{\gamma}} = u_{\tilde{\gamma}} \circ \psi. \tag{5.6}$$

Proposition 5.1. *Let $f \in L^\infty(\hat{\Omega})$. Then there exists a constant K such that*

$$|U_{\tilde{\gamma}} - u_\gamma|_{H^1(\Omega_\gamma)} \le K |\gamma - \tilde{\gamma}|_{H^2(0,1)}$$

for all $\tilde{\gamma}, \gamma \in \bar{\mathcal{S}}_N$. Moreover K can be chosen such that

$$|u_{\tilde{\gamma}} - u_\gamma|_{H^1((\frac{\nu+\bar{\nu}}{2}, 1) \times (0,1))} \le K |\gamma - \tilde{\gamma}|_{H^2(0,1)}.$$

Corollary 5.3. *For every $\gamma \in \bar{\mathcal{S}}$ and $\bar{\gamma} \in D$ we have*

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathcal{C}} |\tau_{\mathcal{C}}(u_{\gamma + t\bar{\gamma}} - u_\gamma)|^2 \, ds = 0.$$

We shall also use the following result:

Lemma 5.1. *For every $v \in W_\gamma \cap H^2(R_\gamma)$ with $\gamma \in \bar{\mathcal{S}}_N$ we have*

$$\frac{\partial v}{\partial n}|_{\Gamma_\gamma} = -\tau_\gamma(v_x \sqrt{1 + \dot{\gamma}^2}) \qquad \text{a.e. on } (0,1).$$

Proof of Theorem 5.1. We first calculate the directional derivative. Let $\gamma \in \bar{\mathcal{S}}$, $\bar{\gamma} \in D$ and let $t > 0$ be sufficiently small so that $\gamma + t\bar{\gamma} \in \bar{\mathcal{S}}_N$. We have

$$\begin{aligned} \frac{1}{t}(J(u_{\gamma+t\bar{\gamma}}) - J(u_\gamma)) &= \frac{1}{t}[\int_{\mathcal{C}} |\tau_{\mathcal{C}} u_{\gamma+t\bar{\gamma}} - z|^2 \, ds - \int_{\mathcal{C}} |\tau_{\mathcal{C}} u_\gamma - z|^2 \, ds] \\ &= \frac{1}{t}[\int_{\mathcal{C}} |\tau_{\mathcal{C}}(u_{\gamma+t\bar{\gamma}} - u_\gamma)|^2 \, ds + 2 \int_{\mathcal{C}} \tau_{\mathcal{C}}(u_{\gamma+t\bar{\gamma}} - u_\gamma)(\tau_{\mathcal{C}} u_\gamma - z) \, ds]. \end{aligned}$$

In view of Corollary 5.3 the first term on the right hand side of the above equality converges to zero for $t \to 0$ and we therefore concentrate on the last term. Using the adjoint equation (5.3) this term equals

$$\begin{aligned}
&\frac{2}{t}(\tau_{\mathcal{C}}(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma), \tau_{\mathcal{C}}\hat{u}_\gamma - z)_{L^2(\mathcal{C})} \\
&\quad = -\frac{1}{t}(\nabla\hat{\lambda}_\gamma, \nabla(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma))_{\hat{\Omega}} + \frac{1}{t}\langle\hat{\mu}_\gamma, \tau_\gamma(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\rangle_{\Gamma_\gamma} \\
&\quad = \frac{1}{t}\langle\hat{\mu}_\gamma, \tau_\gamma(\hat{u}_{\gamma+t\bar{\gamma}} - \hat{u}_\gamma)\rangle_{\Gamma_\gamma} - \frac{1}{t}(\tilde{f}_{\gamma+t\bar{\gamma}} - \tilde{f}_\gamma, \hat{\lambda}_\gamma)_{\hat{\Omega}} \\
&\quad - \frac{1}{t}[\langle\hat{g}_{\gamma+t\bar{\gamma}}, \tau_{\gamma+t\bar{\gamma}}\hat{\lambda}_\gamma\rangle_{\Gamma_{\gamma+t\bar{\gamma}}} - \langle\hat{g}_\gamma, \tau_\gamma\hat{\lambda}_\gamma\rangle_{\Gamma_\gamma}] =: A + B + C,
\end{aligned} \tag{5.7}$$

where $\tilde{f}_\gamma$ is the trivial extension of $f|_{\Omega_\gamma}$ to $\hat{\Omega}$. To estimate these terms separately we define the disjoint sets

$$E^+ = \{y \in [0,1] \colon \bar{\gamma}(y) \geq 0\} \quad \text{and} \quad E^- = \{y \in [0,1] \colon \bar{\gamma}(y) < 0\}.$$

Since $\hat{\lambda}_\gamma$ vanishes on $\hat{\Omega} \setminus \Omega_\gamma$ we obtain

$$B = \frac{1}{t}\int_{E^+}\int_\gamma^{\gamma+t\bar{\gamma}} f\hat{\lambda}_\gamma \, dxdy = \frac{1}{t}\int_{E^+}\int_\gamma^{\gamma+t\bar{\gamma}}\int_\gamma^x \lambda_{\gamma,\xi}(\xi, y)\, d\xi f(x,y)\, dxdy,$$

which implies the estimate

$$\begin{aligned}
|B| &\leq \frac{1}{t}|f|_{L^\infty(\hat{\Omega})}\int_{E^+}\int_\gamma^{\gamma+t\bar{\gamma}} |\gamma + t\bar{\gamma} - \xi||\lambda_{\gamma,\xi}(\xi, y)|\, d\xi dy \\
&\leq t^{\frac{1}{2}}|f|_{L^\infty(\hat{\Omega})}|\bar{\gamma}|_\infty^{\frac{3}{2}}|\lambda_\gamma|_{H^1(\Omega_\gamma)},
\end{aligned}$$

and hence B tends to zero for $t \to 0^+$.

In the next step we show that

$$\lim_{t\to 0^+} A = -\int_{E^-} \hat{\mu}_\gamma(y)(u_\gamma)_x(\gamma(y), y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\, dy. \tag{5.8}$$

Note that $\tau_\gamma(u_{\gamma,x})$ is well–defined since $u_\gamma \in H^2(\Omega_\gamma)$. Corollary 5.1 ensures that $\hat{\mu}_\gamma \in H^{\frac{1}{2}}(\Gamma_\gamma)$ and since $\tau_\gamma\hat{u}_\gamma = 0$ one obtains

$$\begin{aligned}
A &= \frac{1}{t}\int_0^1 \hat{\mu}_\gamma(y)\hat{u}_{\gamma+t\bar{\gamma}}(\gamma(y), y)\sqrt{1+\dot{\gamma}^2}\, dy \\
&= \frac{1}{t}\int_{E^-} \hat{\mu}_\gamma(y)u_{\gamma+t\bar{\gamma}}(\gamma(y), y)\sqrt{1+\dot{\gamma}^2}\, dy.
\end{aligned}$$

To verify the equality in (5.8) we consider

$$\begin{aligned}
&\frac{1}{t}\int_{E^-}\hat{\mu}_\gamma(y)u_{\gamma+t\bar{\gamma}}(\gamma(y),y)\sqrt{1+\dot{\gamma}^2}\,dy\\
&\quad+\int_{E^-}\hat{\mu}_\gamma(y)u_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\,dy\\
&\quad=\int_{E^-}\hat{\mu}_\gamma(y)\bar{\gamma}(y)\frac{1}{t\bar{\gamma}}\int_{\gamma+t\bar{\gamma}}^{\gamma}\big(u_{\gamma+t\bar{\gamma},\xi}(\xi,y)-u_{\gamma+t\bar{\gamma},\xi}(\gamma(y),y)\big)\,d\xi\sqrt{1+\dot{\gamma}^2}\,dy\\
&\quad-\int_{E^-}\hat{\mu}_\gamma(y)\bar{\gamma}(y)(u_{\gamma+t\bar{\gamma},x}(\gamma(y),y)-u_{\gamma,x}(\gamma(y),y))\sqrt{1+\dot{\gamma}^2}\,dy=:I-II.
\end{aligned}$$

We shall show that both I and II tend to zero, which will imply (5.8). To estimate I we exploit the fact that $u_{\gamma+t\bar{\gamma}}$ is uniformly bounded in $H^2(\Omega_{\gamma+t\bar{\gamma}})$ for all t that are sufficiently small. We find

$$\begin{aligned}
|I|&\le\int_{E^-}|\hat{\mu}_\gamma(y)||\bar{\gamma}(y)||\frac{1}{t\bar{\gamma}(y)}\int_{\gamma+t\bar{\gamma}}^{\gamma}\int_{\gamma}^{\xi}u_{\gamma+t\bar{\gamma},xx}(\tau,y)\,d\tau d\xi|\sqrt{1+\dot{\gamma}^2}\,dy\\
&\le K\int_{E^-}|\hat{\mu}_\gamma(y)||\bar{\gamma}(y)|\int_{\gamma+t\bar{\gamma}}^{\gamma}|u_{\gamma+t\bar{\gamma},xx}(\tau,y)|d\tau dy\\
&\le K|\bar{\gamma}|_{L^\infty}^{\frac{3}{2}}\sqrt{t}|\hat{\mu}_\gamma|_{L^2}|u_{\gamma+t\bar{\gamma}}|_{H^2(\Omega_{\gamma+t\bar{\gamma}})}.
\end{aligned}$$

Here and below K denotes a generic constant that is independent of all t sufficiently small. From the above estimate it follows that $\lim_{t\to 0^+}I=0$. Concerning the second term we argue

$$\begin{aligned}
II&=\int_{E^-}\hat{\mu}_\gamma(y)\bar{\gamma}(y)[u_{\gamma+t\bar{\gamma},x}(\gamma(y),y)-u_{\gamma+t\bar{\gamma},x}((\gamma+t\bar{\gamma})(y),y)]\sqrt{1+\dot{\gamma}^2}\,dy\\
&\quad+\int_{E^-}\hat{\mu}_\gamma(y)\bar{\gamma}(y)[u_{\gamma+t\bar{\gamma},x}((\gamma+t\bar{\gamma})(y),y)-u_{\gamma,x}(\gamma(y),y)]\sqrt{1+\dot{\gamma}^2}\,dy\\
&=:II1+II2.
\end{aligned}$$

An argument similar to the one used in the estimate of I shows that $II1$ tends to zero for $t\to 0^+$. Concerning $II2$ we observe that by Lemma 5.1

this term equals (with $\frac{\partial}{\partial n_\gamma}$ denoting the unit outer normal derivative to Γ_γ)

$$\begin{aligned} II2 = &\int_{E^-} \hat{\mu}_\gamma(y)\bar{\gamma}(y)\frac{\partial u_{\gamma+t\bar{\gamma}}}{\partial n_{\gamma+t\bar{\gamma}}}((\gamma+t\bar{\gamma})(y),y) \\ &[\frac{\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}}{\sqrt{1+\dot{\gamma}^2}} - \frac{\sqrt{1+\dot{\gamma}^2}}{\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}}]dy \\ &- \int_0^1 \frac{\partial u_{\gamma+t\bar{\gamma}}}{\partial n_{\gamma+t\bar{\gamma}}}((\gamma+t\bar{\gamma})(y),y)\varphi(y)\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}dy \\ &+ \int_0^1 \frac{\partial u_\gamma}{\partial n_\gamma}(\gamma(y),y)\varphi(y)\sqrt{1+\dot{\gamma}^2}\,dy \end{aligned}$$

where

$$\varphi(y) = \begin{cases} 0 & \text{for } y \in E^+ \\ \frac{\hat{\mu}_\gamma(y)\,\bar{\gamma}(y)}{\sqrt{1+\dot{\gamma}^2}} & \text{for } y \in E^-. \end{cases}$$

The first term in the above equality tends to zero for $t \to 0^+$. Employing Theorem 3.1 and 4.1 the last two terms can be expressed as

$$-\langle (\mathcal{J}^{-1}_{\gamma+t\bar{\gamma}})^* \hat{g}_{\gamma+t\bar{\gamma}}, \varphi\rangle_{(0,1)} + \langle (\mathcal{J}^{-1}_{\gamma})^* \hat{g}_{\gamma}, \varphi\rangle_{(0,1)}. \tag{5.9}$$

This expression tends to zero by Theorem 4.1 and the uniform boundedness of $|\hat{g}_{\gamma+t\bar{\gamma}}|_{H^{\frac{1}{2}}(0,1)}$ for all t sufficiently small which allows to lift the convergence of the functionals $(\mathcal{J}^{-1}_\gamma)^* g_\gamma$ on $H^{\frac{1}{2}}(0,1)$ to $L^2(0,1)$. The verification of (5.8) is now complete.

Finally we shall prove that

$$\lim_{t\to 0^+} C = -\int_{E^+} \hat{g}_\gamma(y)(\lambda_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\,dy. \tag{5.10}$$

Since $\tau_\gamma \hat{\lambda}_\gamma = 0$ and $\hat{\lambda}_\gamma$ vanishes on $\hat{\Omega} \setminus \Omega_\gamma$ we obtain by Corollary 5.1

$$\begin{aligned} C &= -\frac{1}{t}\int_{E^+} \hat{g}_{\gamma+t\bar{\gamma}}(y)\lambda_\gamma(\gamma+t\bar{\gamma}(y),y)\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}\,dy \\ &= -\frac{1}{t}\int_{E^+} \hat{g}_{\gamma+t\bar{\gamma}}(y)\int_\gamma^{\gamma+t\bar{\gamma}} \lambda_{\gamma,x}(x,y)\,dx\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}\,dy. \end{aligned}$$

We consider the difference

$$
\begin{aligned}
-\frac{1}{t}\int_{E^+}\hat{g}_{\gamma+t\bar{\gamma}}(y)\int_{\gamma}^{\gamma+t\bar{\gamma}}\lambda_{\gamma,x}(x,y)\,dx\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}\,dy \\
+\int_{E^+}\hat{g}_{\gamma}(y)\lambda_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\,dy \\
=-\int_{E^+}\hat{g}_{\gamma+t\bar{\gamma}}(y)\bar{\gamma}\frac{1}{t\bar{\gamma}}\int_{\gamma}^{\gamma+t\bar{\gamma}}[\lambda_{\gamma,x}(x,y)-\lambda_{\gamma,x}(\gamma(y),y)]dx \\
\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}\,dy \\
-[\int_{E^+}\hat{g}_{\gamma+t\bar{\gamma}}(y)\bar{\gamma}\lambda_{\gamma,x}(\gamma(y),y)\sqrt{1+(\dot{\gamma}+t\dot{\bar{\gamma}})^2}\,dy \\
-\int_{E^+}\hat{g}_{\gamma}(y)\bar{\gamma}\lambda_{\gamma,x}(\gamma(y),y)\sqrt{1+\dot{\gamma}^2}\,dy]=:I+II.
\end{aligned}
$$

An argument analogous to the one carried out in the discussion of A shows that I converges to 0 for $t\to 0^+$. Defining $\chi\in L^2(0,1)$ by

$$
\chi(y)=\begin{cases}\lambda_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y) & \text{for } y\in E^+\\ 0 & \text{for } y\in E^-\end{cases}
$$

we have

$$
II=-\langle(\mathcal{J}^{-1}_{\gamma+t\bar{\gamma}})^*\hat{g}_{\gamma+t\bar{\gamma}},\chi\rangle_{(0,1)}+\langle(\mathcal{J}^{-1}_{\gamma})^*\hat{g}_{\gamma},\chi\rangle_{(0,1)},
$$

which tends to zero by the same reasoning as in (5.9) above. Summarizing the discussion so far we have shown that $J(u_\gamma)$ has a right directional derivative $D^+J(u_\gamma)$ determined by

$$
\begin{aligned}
D^+J(u_\gamma)\bar{\gamma}=&-\int_{E^-}\hat{\mu}_\gamma(y)u_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\,dy\\
&-\int_{E^+}\hat{g}_\gamma(y)\lambda_{\gamma,x}(\gamma(y),y)\bar{\gamma}(y)\sqrt{1+\dot{\gamma}^2}\,dy.
\end{aligned}
$$

In view of Lemma 5.1, Theorem 3.1 and Corollary 5.1 we find

$$
\begin{aligned}
D^+J(u_\gamma)\bar{\gamma}&=\int_{E^-}\hat{\mu}_\gamma(y)\frac{\partial u_\gamma}{\partial n_\gamma}\bar{\gamma}(y)\,dy+\int_{E^+}\hat{g}_\gamma(y)\frac{\partial\lambda_\gamma}{\partial n_\gamma}\bar{\gamma}(y)\,dy\\
&=\int_{E^-}\hat{\mu}_\gamma(y)\hat{g}_\gamma(y)\bar{\gamma}(y)\,dy+\int_{E^+}\hat{g}_\gamma(y)\hat{\mu}_\gamma(y)\bar{\gamma}(y)\,dy\\
&=\int_0^1\hat{g}_\gamma(y)\hat{\mu}_\gamma(y)\bar{\gamma}(y)\,dy.
\end{aligned}
$$

Thus the directional derivative of $\gamma \mapsto J(u_\gamma)$ exists and (5.4) holds. It is clearly a bounded linear operator on D and hence $D_\gamma J(u_\gamma)$ is the Gateaux derivative of $\gamma \mapsto J(u_\gamma)$ at $\gamma \in \bar{\mathcal{S}}$.

To argue that $D_\gamma J(u_\gamma)$ is in fact the Fréchet derivative of $\gamma \mapsto J(u_\gamma)$ note that $\gamma \mapsto \hat{g}_\gamma$ is continuous from $\bar{\mathcal{S}}_N$ to $L^2(0,1)$. This follows from Theorem 4.1 and boundedness of $\{|\hat{g}_\gamma|_{H^{\frac{1}{2}}(0,1)}\}_{\gamma\in\bar{\mathcal{S}}}$. Similarly $\gamma \mapsto \hat{\mu}_\gamma$ is continuous from $\bar{\mathcal{S}}_N$ to $L^2(0,1)$, which follows from the boundedness of $\{|\hat{\mu}_\gamma|_{H^{\frac{1}{2}}(0,1)}\}_{\gamma\in\bar{\mathcal{S}}}$ and $w^*\text{–}\lim_{\tilde{\gamma}\to\gamma}(\mathcal{J}_{\tilde{\gamma}}^{-1})^*\hat{\mu}_{\tilde{\gamma}} = (\mathcal{J}_\gamma^{-1})^*\hat{\mu}_\gamma$, which is proven in the same way as Theorem 4.1. □

Proof of Lemma 5.1. For $v \in W_\gamma \cap H^2(R_\gamma)$ the tangential derivative along Γ_γ vanishes. This follows from

$$0 = \lim_{h\to 0}(v(\gamma(y+h), y+h) - v(\gamma(y), y)) = v_x(\gamma(y), y)\dot{\gamma}(y) + v_y(\gamma(y), y)$$

which holds a.e. on $(0,1)$. This relationship implies

$$\begin{aligned}\frac{\partial v}{\partial n}(\gamma(y), y) &= \nabla v(\gamma(y), y)\begin{pmatrix}-1\\ \dot{\gamma}\end{pmatrix}\frac{1}{\sqrt{1+\dot{\gamma}^2}}\\ &= -v_x(\gamma(y), y)\sqrt{1+\dot{\gamma}^2},\ \text{a.e. on } (0,1)\end{aligned}$$

where n is the outer normal to Ω_γ along Γ_γ. This establishes the result. □

6. Numerical Examples

In this section we briefly report on our numerical experience with the embedding domain approach applied to a shape optimization problem. While there are many numerical results based on the embedding domain technique for the forward problem, see e.g. [7] and the references therein, we are only aware of a few publications related to inverse problems. A one–dimensional example can be found in [11] and some two–dimensional examples are presented in [4]. In contrast to our approach distributed controls, defined on $\hat{\Omega} \setminus \Omega_\gamma$ are used in [11] and [4].

We consider the cost functional (2.4)

$$J_\gamma(u) = \int_{\mathcal{C}} |\tau_{\mathcal{C}} u - z|^2 ds,$$

where for simplicity the curve $\mathcal{C}$ is given by

$$\mathcal{C} = \{(\sigma, y) \colon y \in (0,1)\},$$

with a fixed $\sigma \in (0,1)$. In the discretization of the primal and dual problems (3.4) and (5.3) two independent meshes are utilized: on the fictitious domain $\hat{\Omega}$ an equidistant mesh determined by N^2 interior meshpoints $(x_i, y_j) = (\frac{i}{N+1}, \frac{j}{N+1})$, $i,j = 0,\dots,N+1$, is used. Elements $\hat{v}$ of $\hat{V}$ are approximated by bilinear tensor splines

$$\hat{v}^N(x,y) = \sum_{i=1}^{N} \sum_{j=0}^{N+1} v_{ij}\varphi_i(x)\varphi_j(y),$$

where

$$\varphi_i(x) = \begin{cases} 1-(N+1)|x-x_i|, & \text{for } x \in [x_{i-1}, x_i], \\ 0, & \text{else.} \end{cases}$$

The interior boundary γ is discretized with M equally spaced interior meshpoints: $\eta_i = \frac{i}{M+1}$, $i = 0,\dots,M+1$. Let $\{\psi_i\}_{i=0}^{M+1}$ denote the appropriate piecewise linear basis functions and let $\{\chi_i\}_{i=1}^{M+1}$ stand for the characteristic functions of the intervals $[\eta_{i-1}, \eta_i]$, $i = 1,\dots,M+1$. The functions γ and $\hat{g} \in H^{\frac{1}{2}}(0,1)^*$ are approximated by

$$\gamma^M = \sum_{i=0}^{M+1} \gamma_i \psi_i, \quad \text{and} \quad \hat{g}^M = \sum_{i=1}^{M+1} g_i \chi_i$$

respectively.

The approximation of the constraint $\tau_\gamma \hat{u} = 0$ in (3.4) and of τ_γ^* is obtained by evaluating the line integrals

$$\begin{aligned} \langle \hat{g}^M, \tau_{\gamma^M} \hat{v}^N \rangle_{\Gamma^M} &= \sum_{i=1}^{M+1} g_i \int_{\eta_{i-1}}^{\eta_i} \hat{v}^N(\gamma^M(y), y) \sqrt{1 + (\dot{\gamma}^M)^2(y)}\, dy \\ &= \sum_{i=1}^{M+1} \sum_{r=1}^{N} \sum_{s=0}^{N+1} g_i v_{rs} \int_{\eta_{i-1}}^{\eta_i} \varphi_r(\gamma^M(y)) \varphi_s(y) \sqrt{1 + (\dot{\gamma}^M)^2(y)}\, dy \\ &= G^M T_{\gamma^M}^{N,M} V^N, \end{aligned}$$

where G^M and V^N stand for the coordinate vectors of $\hat{g}^M$ and $\hat{v}^N$ respectively. This procedure results in systems approximating (3.4) of saddle point type:

$$\begin{pmatrix} D & (T_{\gamma^M}^{N,M})^T \\ T_{\gamma^M}^{N,M} & 0 \end{pmatrix} \begin{pmatrix} U^N \\ G^M \end{pmatrix} = \begin{pmatrix} F_{\gamma^M}^{N,M} \\ 0, \end{pmatrix} \tag{6.1}$$

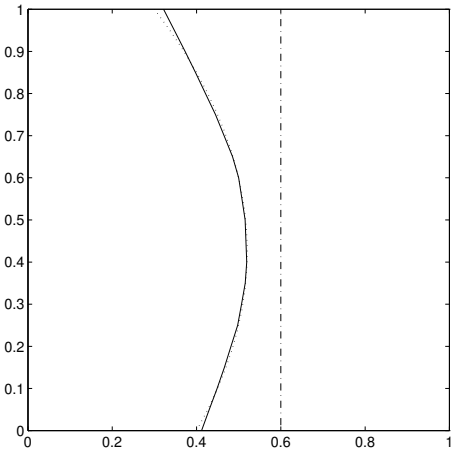

Figure 1

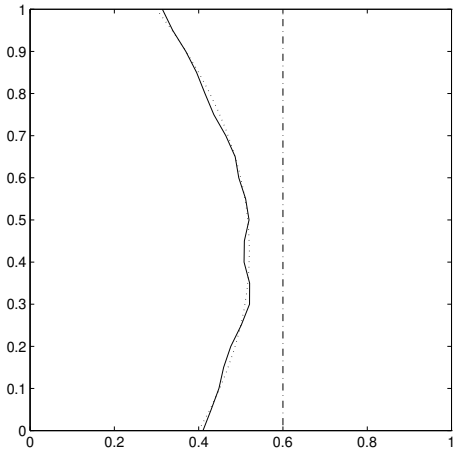

Figure 2

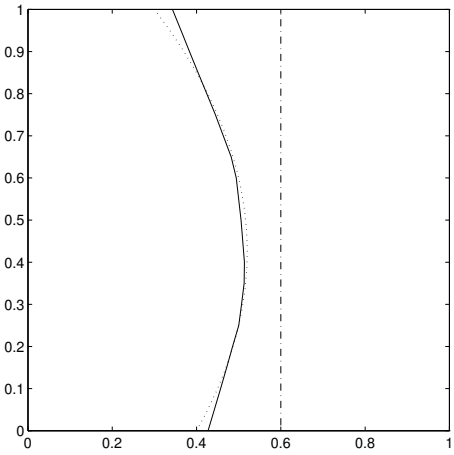

Figure 3

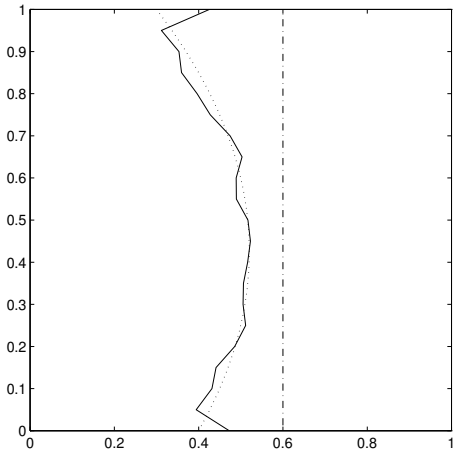

Figure 4

where $F_{\gamma^M}^{N,M}$ is the coordinate vector of the projection of $\tilde{f}_{\gamma^M}$ onto the subspace spanned by the bilinear tensor splines. Approximations of mixed finite element type like the one we propose are known to converge if the discretization of the primal and the dual variables (here $\hat{u}_\gamma$ and $\hat{g}_\gamma$) satisfy the Babuška–Brezzi condition. We took this condition into account by choosing N approximately equal to $2M$. In passing we remark that G^M exhibits strong oscillations if M is taken of the same order as N.

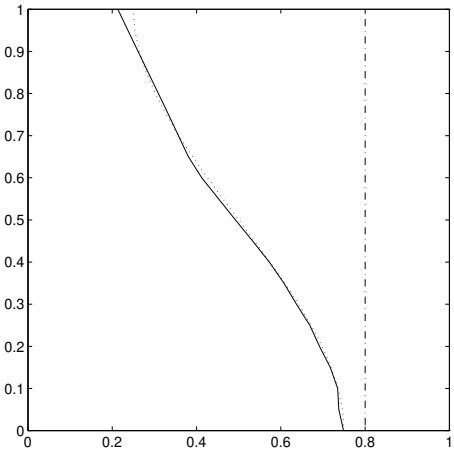

Figure 5

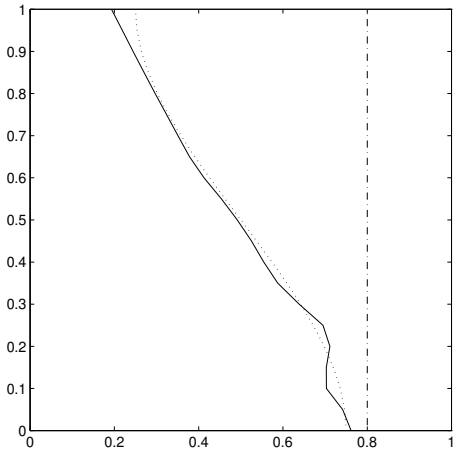

Figure 6

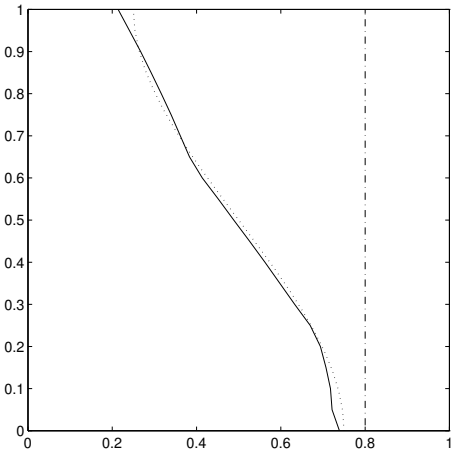

Figure 7

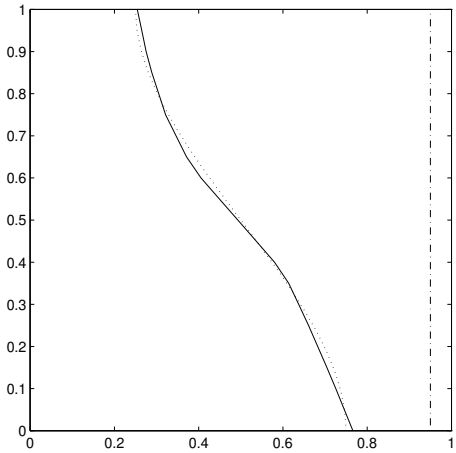

Figure 8

Example 6.1. Here we choose

$$f = 10 \qquad \text{on } \Omega_\gamma,$$
$$\gamma(y) = -\frac{2}{3}y^2 + \frac{1.7}{3}y + 0.4, \quad \text{for } y \in [0,1]$$

in (3.4). The observation z was generated by solving the forward problem with $N = 80$, $M = 40$ and interpolating u^{80} over a 40×40 grid at $\sigma = 0.6$.

Example 6.2. This is an example where the analytical solution is chosen

and f is calculated from the equation:

$$\begin{array}{rcll} u(x,y) & = & 10(1-x)(x-\gamma(y)), & \text{for } x \in (\gamma(y),1), y \in (0,1) \\ f & = & -\Delta u & \\ \gamma(y) & = & y^3 - \frac{3}{2}y^2 + \frac{3}{4}, & \text{for } y \in (0,1). \end{array}$$

The observation z is given by

$$z = u(\sigma, \cdot)$$

with $\sigma = 0.8$ and $\sigma = 0.95$.

The minimization of the approximation to the cost functional (2.4) was performed with the MATLAB–implementation of the BFGS quasi–Newton method using the gradient information obtained from (5.6). The maximum number of iterations was limited by 30. In a future implementation we plan to provide our own minimization routine. Each iteration requires one solve of the adjoint equation and in general several solves of the state equation. For each solve only $F^{N,M}_{\gamma^M}$ and $T^{N,M}_{\gamma^M}$ in (6.1) need to be recomputed. For the practical success of the numerical implementation the realization of a *coarse–to–fine cycle* in the optimization process was of great importance: for fixed N we performed a series of optimization problems with increasing values for M, the initial choice at the n–th step being an interpolation of the minimizer of the $(n-1)$–th step. The figures below show the final result of such cycles with $M = 1, 2, 4, 7, 19$ and $N = 40$.

For each example we performed 3 experiments: optimal (noise free) data, noisy data and noisy — but presmoothed — data. Noisy data were generated by adding 10% relative noise provided by MATLAB random numbers uniformly distributed in $(-0.1, 0.1)$ to the noise free data at the nodal points of the mesh. Presmoothed data z_p were obtained by solving

$$\min_{z_p} \{|z - z_p|^2_{L^2(0,1)} + 0.005|z_p'|^2_{L^2(0,1)}\}.$$

In each figure the original domain is depicted by the dotted line, the initial guess for γ (which always coincides with σ) is indicated by dashdots and the solid line represents the final estimate of γ ($M = 19$). In each group of plots the first one refers to optimal data, the second to noisy data and the third to noisy but smoothened data. Figure 4 should be compared with Figure 1: it shows the estimate of γ if the coarse–to–fine cycle is replaced by one optimization on the finest grid: $N = 40$, $M = 19$ (we limited the maximum number of iterations by 100 in this experiment). Figure 8 demonstrates the behavior of the algorithm for observations which are far from γ: $\sigma = 0.95$.

In this case as well the "unknown" shape of the domain is reconstructed very well from the data on $\mathcal{C}$.

7. Appendix

Proof of Lemma 2.1. The proof is essentially contained in Chapters 1 and 2 of [9] for the case of a polygon with piecewise straight boundary. Here one edge of Ω_γ is curvilinear and we therefore briefly outline the proof referring frequently to results in [9].

Step1: We define

$$W_\gamma^2 = \{\varphi \in H^2(\Omega_\gamma)\colon \tau_0\varphi|_{\Gamma_\gamma \cup \Gamma_r} = 0, \frac{\partial \varphi}{\partial n}|_{\Gamma_{01}} = 0\}.$$

For every $\varphi \in W_\gamma^2$ the following identity holds,

$$\int_{\Omega_\gamma} D_x^2\varphi D_y^2\varphi \, d\omega = \int_{\Omega_\gamma} (D_x D_y \varphi)^2 d\omega, \tag{A.1}$$

where D_x and D_y denote partial derivatives with respect to x and y respectively [9], Lemma 2.2.2.

Step 2: For all $\varphi \in W_\gamma^2$

$$|\varphi|_{H^2(\Omega_\gamma)} \le C|\Delta\varphi|_{L^2(\Omega_\gamma)}, \tag{A.2}$$

for a constant C independent of $\gamma \in \bar{\mathcal{S}}$. In fact (A.1) implies that

$$|\Delta\varphi|^2_{L^2(\Omega_\gamma)} = |D_x^2\varphi|^2_{L^2(\Omega_\gamma)} + |D_y^2\varphi|^2_{L^2(\Omega_\gamma)} + 2|D_x D_y\varphi|^2_{L^2(\Omega_\gamma)}.$$

Furthermore,

$$\int_{\Omega_\gamma} |\nabla\varphi|^2 d\omega = -\int_{\Omega_\gamma} \varphi\Delta\varphi d\omega \le |\varphi|_{L^2(\Omega_\gamma)}|\Delta\varphi|_{L^2(\Omega_\gamma)}$$

and a direct calculation shows that $|\varphi|_{L^2(\Omega_\gamma)} \le |\nabla\varphi|_{L^2(\Omega_\gamma)}$. These estimates imply (A.2).

Step 3: From step 2 it follows that $\Delta\colon W_\gamma^2 \to L^2(\Omega_\gamma)$ is injective and that it has closed range. Denote by $\mathcal{N} \subset L^2(\Omega_\gamma)$ the orthogonal complement of the range of Δ in $L^2(\Omega_\gamma)$. Every element $v \in \mathcal{N}$ necessarily satisfies v and $\Delta v \in L^2(\Omega_\gamma)$ and

$$\Delta v = 0 \quad \text{in } \Omega_\gamma,\ \tau_0 v|_{\Gamma_\gamma \cup \Gamma_r} = 0, \frac{\partial v}{\partial n}|_{\Gamma_{01}} = 0. \tag{A.3}$$

This is proven in the same manner as Lemma 2.3.1 in [9]. Here we use the trace theorem which requires the $C^{1,1}$ property of elements in $\bar{\mathcal{S}}$.

Step 4: Next the regularity of $v \in \mathcal{N}$ is determined. Clearly v is harmonic inside of Ω_γ and one can show as in Lemma 2.3.4 in [9] that v can be extended as a harmonic function along the flat portions of $\partial\Omega_\gamma$ so that in particular $v \in C^\infty(\bar{\Omega}_\gamma \setminus U)$, where U is a neighborhood of the four corners of Ω_γ and of the curvilinear part of Γ_γ. This defines a neighborhood $\tilde{U}$ of the curvilinear part of Γ_γ relative to Ω_γ. Classical elliptic regularity theory implies that $u_\gamma|_{\bar{U}} \in H^2(\tilde{U})$. It follows that there exists a neighborhood $\hat{U}$ of the corners of Ω_γ such that $u \in H^2(\Omega_\gamma \setminus \hat{U})$. One can now proceed as in the proof of Theorem 2.3.7 in [9] to show that the cardinality of $\mathcal{N}$ is zero. Here we use the fact that the interior angles at $(\gamma(0), 0)$ and $(\gamma(1), 1)$ are no larger than $\frac{\pi}{2}$. Consequently $u_\gamma \in H^2(\Omega_\gamma)$ for every $\gamma \in \bar{\mathcal{S}}$.

Step 5: The uniform bound for $\{|u_\gamma|_{H^2(\Omega_\gamma)}\}_{\gamma\in\bar{\mathcal{S}}}$ follows from (A.2). This ends the proof. □

Proof of Lemma 2.2. It is simple to argue that λ_γ is analytic in a neighborhood U relative to Ω_γ of the line $L = \{[\frac{\nu+\bar{\nu}}{2}, y] \colon y \in (0,1)\}$. Using a finite differencing argument in the x–direction one shows that

$$|D_x^2\lambda_\gamma|_{L^2(U)} + |D_{xy}^2\lambda_\gamma|_{L^2(U)} \le k|\lambda_\gamma|_{H^1(\Omega_\gamma)}$$

for a constant k independent of $\gamma \in \mathcal{S}$. Using the governing equation for λ_γ implies that $\{|\lambda_\gamma|_{H^2(U)}\}_{\gamma\in\mathcal{S}}$ is uniformly bounded, and hence $\{|\lambda_\gamma|_{H^{\frac{3}{2}}(L)}\}_{\gamma\in\mathcal{S}}$ is uniformly bounded as well. By the trace theorem there exist functions $f_\gamma \in H^2(\hat{R})$, with $\hat{R} = (0, \frac{\nu+\bar{\nu}}{2}) \times (0,1)$, such that $\{|f_\gamma|_{\gamma\in\mathcal{S}}\}$ is uniformly bounded, $\tau_0 f_\gamma|_{\{0\}\times(0,1)} = 0$, $\tau_0 f_\gamma|_L = \lambda_\gamma|_L$, and $\frac{\partial}{\partial n} f_\gamma|_{(0,\frac{\nu+\bar{\nu}}{2})\times\{0,1\}} = 0$. The mapping

$$\psi(x,y) \colon (x,y) \mapsto \Big(\frac{\nu+\bar{\nu}}{\nu+\bar{\nu}-2\gamma}(x-\gamma(y)), y\Big)$$

is an isomorphism from $\hat{R}$ onto B_γ, where $B_\gamma = \{(x,y) \colon y \in (0,1), \gamma(y) < x < \frac{\nu+\bar{\nu}}{2}\}$. Let us define

$$F_\gamma = f_\gamma \circ \psi.$$

It is simple to argue that $\{|F_\gamma|_{H^2(B_\gamma)}\}$ is uniformly bounded and that

$$F_\gamma|_{\Gamma_\gamma} = 0, \qquad F_\gamma|_L = \lambda_\gamma|_L,$$
$$\frac{\partial F_\gamma}{\partial n}|_{(\gamma(0),\frac{\nu+\bar{\nu}}{2})\times\{0\}} = 0, \qquad \frac{\partial F_\gamma}{\partial n}|_{(\gamma(1),\frac{\nu+\bar{\nu}}{2})\times\{1\}} = 0.$$

Let $W(B_\gamma) = \{\varphi \in H^1(B_\gamma)\colon \tau_0\varphi|_{\Gamma_\gamma} = \tau_0\varphi|_L = 0\}$. Then $\tilde{\lambda}_\gamma := \lambda_\gamma|_{B_\gamma} - F_\gamma \in W(B_\gamma)$ and $\tilde{\lambda}_\gamma$ satisfies

$$(\nabla\tilde{\lambda}_\gamma, \nabla v)_{B_\gamma} = (\nabla F_\gamma, \nabla v)_{B_\gamma} = -(\Delta F_\gamma, v)_{B_\gamma} \qquad \text{for all } v \in W(B_\gamma).$$

Now we appeal to Lemma (2.1) with Ω_γ replaced by B_γ and conclude that $\tilde{\lambda}_\gamma \in H^2(B_\gamma)$ and $\{|\tilde{\lambda}_\gamma|_{H^2(B_\gamma)}\}_{\gamma\in\mathcal{S}}$ is uniformly bounded. It follows that $\lambda_\gamma|_{B_\gamma} \in H^2(B_\gamma)$ and that $\{|\lambda_\gamma|_{H^2(B_\gamma)}\}_{\gamma\in\mathcal{S}}$ is uniformly bounded as well. This ends the proof. □

Proof. Proof of Lemma 3.1 Using the transformations

$$\begin{aligned} T_0 &: \Omega_\gamma^c \to \Omega_0, \qquad T_0(x,y) = \left(\frac{x}{2\gamma(y)}, y\right) \\ T_1 &: \Omega_\gamma \to \Omega_1, \qquad T_1(x,y) = \left(\frac{1+x-2\gamma(y)}{2(1-\gamma(y))}, y\right) \end{aligned}$$

the domains Ω_γ^c and Ω_γ are transformed to $\Omega_0 = (0, \frac{1}{2}) \times (0,1)$ and $\Omega_1 = (\frac{1}{2}, 1) \times (0,1)$, respectively. The mappings T_0 and T_1 are bijections which, together with their inverses, are Lipschitz continuous. Hence, defining

$$\begin{aligned} v_0(\xi,\eta) &= (\psi \circ T_0^{-1})(\xi,\eta), \qquad \text{for } (\xi,\eta) \in \Omega_0, \\ v_1(\xi,\eta) &= (\varphi \circ T_1^{-1})(\xi,\eta), \qquad \text{for } (\xi,\eta) \in \Omega_1, \end{aligned}$$

we conclude from Lemma II-3.2 in [14] that $v_0 \in H^1(\Omega_0)$ and $v_1 \in H^1(\Omega_1)$. By construction $v_0(\frac{1}{2}, \cdot) = v_1(\frac{1}{2}, \cdot)$ in $H^{1/2}(0,1)$. Consequently

$$\hat{v} = \begin{cases} v_0 & \text{in } \Omega_0 \\ v_1 & \text{in } \Omega_1 \end{cases}$$

satisfies $\hat{v} \in H^1(\hat{\Omega})$. Next we define the bijection $T : \hat{\Omega} \to \hat{\Omega}$ by

$$T(x,y) = \begin{cases} T_0(x,y) & \text{for } (x,y) \in \Omega_\gamma^c \\ T_1(x,y) & \text{for } (x,y) \in \Omega_\gamma \\ (\frac{1}{2}, y) & \text{for } x = \gamma(y), y \in (0,1), \end{cases}$$

which, together with its inverse, is Lipschitz continuous. Since $\hat{\varphi} = \hat{v} \circ T$ the result follows by invoking Lemma II-3.2 in [14] once again. □

Proof of Lemma 3.2. Choose $\hat{u} \in C^1(\hat{\Omega})$ with $\hat{u}(1,\cdot) \equiv 0$. Since

$$\int_0^1 |\hat{u}(\gamma(y), y)|^2 dy \le |\hat{u}|^2_{H^1(\hat{\Omega})}$$

it suffices to discuss the second term in the norm (3.8). Reflecting $\hat{u}$ across the edges $x = 0$ and $x = 1$ as in [1, pg.158] $\hat{u}$ can be extended to a function $\check{u}$ defined on $\check{\Omega} = (-1, 2) \times (0, 1)$. By Lemma IX-2 in [1] we have $\check{u} \in H^1(\check{\Omega})$ and

$$|\check{u}|_{H^1(\check{\Omega})} \leq 3|\hat{u}|_{H^1(\hat{\Omega})}.$$

Furthermore, consider the change of coordinates

$$T : (s, t) \to (s + \gamma(t) - t, t)$$

and define

$$\hat{v} = \check{u} \circ T.$$

By Lemma II-3.2 in [14] we conclude $\hat{v} \in H^1(\hat{\Omega})$ and moreover

$$|\hat{v}|_{H^1(\hat{\Omega})} \leq c|\check{u}|_{H^1(T\hat{\Omega})} \leq c|\check{u}|_{H^1(\check{\Omega})} \leq 3c|\hat{u}|_{H^1(\hat{\Omega})}$$

with a constant $c > 0$ which can be chosen independently of $\gamma \in \mathcal{S}$. Note that

$$\hat{v}(s, s) = \check{u}(\gamma(s), s) = \hat{u}(\gamma(s), s), s \in (0, 1).$$

Consequently we find

$$\int_0^1 \int_0^1 \frac{|\hat{v}(s, s) - \hat{v}(t, t)|^2}{(s - t)^2} ds dt \leq 8|\hat{v}|^2_{H^1(\hat{\Omega})} \leq 24c|\hat{u}|_{H^1(\hat{\Omega})},$$

where we used Hardy's inequality (Lemma II-5.2 in [14]). A density argument completes the proof of the first statement. As to the surjectivity of τ_γ choose $\varphi \in H^{1/2}(\Gamma_\gamma)$ and extend φ to functions φ_i, $i = 1, 2$, on $\partial\Omega_\gamma$, $\partial\Omega^c_\gamma$ such that $\varphi_1 \in H^{1/2}(\partial\Omega^c_\gamma)$, $\varphi_2 \in H^{1/2}(\partial\Omega_\gamma)$ and $\varphi_i|_{\Gamma_\gamma} = \varphi$. Since $\Omega^c_\gamma, \Omega_\gamma$ are Lipschitz continuous domains there exist functions $\Phi_1 \in H^1(\Omega^c_\gamma)$, $\Phi_2 \in H^1(\Omega_\gamma)$ such that $\Phi_1|_{\partial\Omega^c_\gamma} = \varphi_1$, $\Phi_2|_{\partial\Omega_\gamma} = \varphi_2$. Define Φ by $\Phi|_{\Omega^c_\gamma} = \Phi_1$ and $\Phi|_{\Omega_\gamma} = \Phi_2$. By Lemma 2.1 we have $\Phi \in H^1(\hat{\Omega})$ and $\tau_\gamma \Phi = \varphi$ is satisfied by the construction of Φ. □

Proof of Lemma 4.2. Since the family of trace operators $\{\mathcal{T}_\gamma\}_{\gamma \in \mathcal{S}}$ is uniformly bounded in $\mathcal{L}(\hat{V}, H^{1/2}(0, 1))$, hence it suffices to show that

$$\lim_{\gamma \to \bar{\gamma}} \mathcal{T}_\gamma \hat{v} = \mathcal{T}_{\bar{\gamma}} \hat{v}$$

for all $\hat{v} \in C^2(\hat{\Omega}) \cap \hat{V}$. In the following calculations we denote

$$\Delta\gamma(y) = \gamma(y) - \bar{\gamma}(y)$$

and without loss of generality we assume that $|\Delta\gamma|_{L^\infty} < 1$. Let $\hat{\Omega}$ and $\check{v} \in H^1(\hat{\Omega})$ be defined as in the proof of Lemma 3.2 and let

$$\hat{w}(x,y) = \check{v}(x + \Delta\gamma(y), y), \quad \text{for } (x,y) \in \hat{\Omega}.$$

Note that

$$(\mathcal{T}_{\bar{\gamma}}\hat{w})(y) = (\mathcal{T}_{\gamma}\hat{v})(y) \text{ and } \hat{w} = \check{v} \circ T,$$

where $T(x,y) = (x + \Delta\gamma(y), y)$ is a bijection from $\hat{\Omega}$ onto $T\hat{\Omega}$ which, together with its inverse, is Lipschitz continuous. This implies that $\hat{w} \in H^1(\hat{\Omega})$. Since $\mathcal{T}_{\gamma}\hat{v} - \mathcal{T}_{\bar{\gamma}}\hat{v} = \mathcal{T}_{\bar{\gamma}}(\hat{w} - \hat{v})$, where we extended $\mathcal{T}_{\bar{\gamma}}$ to $H^1(\hat{\Omega})$, since $\hat{w} \notin \hat{V}$ in general, we obtain

$$|\mathcal{T}_{\gamma}\hat{v} - \mathcal{T}_{\bar{\gamma}}\hat{v}|_{H^{1/2}(0,1)} \leq \|\mathcal{T}_{\bar{\gamma}}\||\hat{w} - \hat{v}|_{H^1(\hat{\Omega})}.$$

It remains to show that $|\hat{w} - \hat{v}|_{H^1(\hat{\Omega})} \to 0$ as $|\Delta\gamma|_{W^{1,\infty}} \to 0$. We have

$$\begin{aligned}
|\hat{w} - \hat{v}|^2_{H^1(\hat{\Omega})} \leq & \int_{\hat{\Omega}} |\check{v}(x + \Delta\gamma(y), y) - \hat{v}(x,y)|^2 dxdy \\
& + \int_{\hat{\Omega}} |\check{v}_x(x + \Delta\gamma(y), y) - \hat{v}_x(x,y)|^2 dxdy \\
& + 2\int_{\hat{\Omega}} |\check{v}_y(x + \Delta\gamma(y), y) - \hat{v}_y(x,y)|^2 dxdy \\
& + 2\int_{\hat{\Omega}} |\check{v}_x(x + \Delta\gamma(y), y)|^2 |(\Delta\gamma)_y|^2 dxdy =: I + \cdots + IV.
\end{aligned}$$

Estimating the first term we split the integral according to

$$E_+ = \{y \in (0,1) : \Delta\gamma(y) \geq 0\} \text{ and } E_- = \{y \in (0,1) : \Delta\gamma(y) < 0\},$$

and consider

$$I = (\int_{E^+} + \int_{E^-})(\int_0^1 |\check{v}(x + \Delta\gamma(y), y) - \hat{v}(x,y)|^2 dx)dy.$$

For $y \in E_+$ we find

$$\begin{aligned}\int_0^1 |\check{v}(x+\Delta\gamma(y),y) - \hat{v}(x,y)|^2 dx &= \int_0^{1-\Delta\gamma(y)} |\hat{v}(x+\Delta\gamma(y),y) - \hat{v}(x,y)|^2 dx \\ &+ \int_{1-\Delta\gamma(y)}^1 |\hat{v}(2-x-\Delta\gamma(y),y) - \hat{v}(x,y)|^2 dx \\ &\le |\Delta\gamma|_\infty^2 |\hat{v}_x|_{\infty,\hat{\Omega}}^2 + \int_{1-\Delta\gamma(y)}^1 (2-2x-\Delta\gamma(y))^2 dx |\hat{v}_x|_{\infty,\hat{\Omega}}^2 \\ &\le |\Delta\gamma|_\infty^2 |\hat{v}_x|_{\infty,\hat{\Omega}}^2 + |\Delta\gamma|_\infty^3 \cdot |\hat{v}_x|_{\infty,\hat{\Omega}}^2 \le 2|\Delta\gamma|_\infty^2 |\hat{v}_x|_{\infty,\hat{\Omega}}^2.\end{aligned}$$

This bound also applies for $y \in E_-$ and consequently we find

$$I \le 2|\Delta\gamma|_\infty^2 |\hat{v}_x|_{\infty,\Omega}^2.$$

An analogous argument shows that

$$II + III \le 4|\Delta\gamma|_\infty^2 (|\hat{v}_{xx}|_{\infty,\hat{\Omega}}^2 + |\hat{v}_{yy}|_{\infty,\hat{\Omega}}^2),$$

and

$$IV \le 2|\Delta\gamma|_{W^{1,\infty}}^2 |\hat{v}_x|_{\infty,\hat{\Omega}}^2.$$

These estimates end the proof of the lemma. □

Proof of Proposition 5.1. We recall that ψ maps Ω_γ onto $\Omega_{\tilde{\gamma}}$, and define

$$F = f \circ \psi \text{ and } \xi(x,y) = \frac{1-\tilde{\gamma}}{1-\gamma} x + \frac{\tilde{\gamma}-\gamma}{1-\gamma}, \tag{A.4}$$

where $\tilde{\gamma}$, γ are functions of $y \in [0,1]$. It can be verified that $\{|U_{\tilde{\gamma}}|_{H^2(\Omega_\gamma)}\}_{\tilde{\gamma}\in\bar{\delta}}$ is uniformly bounded. A convenient way to obtain this estimate is to express and estimate the derivatives of $U_{\tilde{\gamma}}$ in the order $U_{\tilde{\gamma},x}$, $U_{\tilde{\gamma},y}$, $U_{\tilde{\gamma},xx}$, $U_{\tilde{\gamma},xy}$, $U_{\tilde{\gamma},yy}$ in terms of derivatives of $u_{\tilde{\gamma}}$. A calculation shows that

$$\begin{cases} -\Delta U_{\tilde{\gamma}} + (1 - \frac{1}{\xi_x^2}) U_{\tilde{\gamma},xx} - (\frac{\xi_y}{\xi_x})^2 U_{\tilde{\gamma},xx} \\ \qquad +2\frac{\xi_y}{\xi_x} U_{\tilde{\gamma},yx} - (2\frac{\xi_{yx}\xi_y}{\xi_x^2} - \frac{\xi_{yy}}{\xi_x}) U_{\tilde{\gamma},x} = F(x,y) \quad \text{in } \Omega_\gamma \\ \tau_0 U_{\tilde{\gamma}}|_{\Gamma_\gamma \cup \Gamma_r} = 0 \\ U_{\tilde{\gamma},y}(\cdot,1) = \frac{\xi_y(\cdot,1)}{\xi_x(\cdot,1)} U_{\tilde{\gamma},x}(\cdot,1) \qquad \text{on } \Gamma_1 \\ U_{\tilde{\gamma},y}(\cdot,0) = \frac{\xi_y(\cdot,0)}{\xi_x(\cdot,0)} U_{\tilde{\gamma},x}(\cdot,0) \qquad \text{on } \Gamma_0. \end{cases} \tag{A.5}$$

One can show the existence of constants K and k independent of $\gamma \in \bar{\mathcal{S}}$ and $(x,y) \in \Omega_\gamma$ such that

$$\begin{cases} |1 - \frac{1}{\xi_x^2(x,y)}| \le K|\tilde{\gamma}(y) - \gamma(y)| \\ |\xi_y(x,y)| \le K(|\tilde{\gamma}(y) - \gamma(y)| + |\dot{\tilde{\gamma}}(y) - \dot{\gamma}(y)|) \\ |\xi_{yy}(x,y)| \le K(|\tilde{\gamma}(y) - \gamma(y)| + |\dot{\tilde{\gamma}}(y) - \dot{\gamma}(y)| + |\ddot{\tilde{\gamma}}(y) - \ddot{\gamma}(y)|) \\ 0 < k \le \xi_x(x,y) \le k^{-1}. \end{cases} \tag{A.6}$$

Substracting (2.2) from (A.5) we find for $w = U_{\tilde{\gamma}} - u_\gamma \in W_\gamma \cap H^2(\Omega_\gamma)$

$$\begin{cases} -\Delta w + (1 - \frac{1}{\xi_x^2})U_{\tilde{\gamma},xx} - (\frac{\xi_y}{\xi_x})^2 U_{\tilde{\gamma},xx} \\ \qquad +2\frac{\xi_y}{\xi_x}U_{\tilde{\gamma},xy} - (2\frac{\xi_{yx}\xi_y}{\xi_x^2} - \frac{\xi_{yy}}{\xi_x})U_{\tilde{\gamma},x} = F - f \\ \tau_0 w|_{\Gamma_\gamma \cup \Gamma_r} = 0 \\ \frac{\partial}{\partial n} w|_{\Gamma_1} = \frac{\xi_y(\cdot,1)}{\xi_x(\cdot,1)} U_{\tilde{\gamma},x}(\cdot,1), \quad \frac{\partial}{\partial n} w|_{\Gamma_0} = -\frac{\xi_y(\cdot,0)}{\xi_x(\cdot,0)} U_{\tilde{\gamma},x}(\cdot,0), \end{cases} \tag{A.7}$$

and by Green's formula

$$\begin{aligned} |\nabla w|^2_{L^2(\Omega_\gamma)} &= -(\Delta w, w)_{\Omega_\gamma} + \int_{\gamma(1)}^{1} \frac{\xi_y(x,1)}{\xi_x(x,1)} U_{\tilde{\gamma},x}(x,1)dx \\ &\quad + \int_{\gamma(0)}^{1} \frac{\xi_y(x,0)}{\xi_x(x,0)} U_{\tilde{\gamma},x}(x,0)dx =: A + B + C. \end{aligned} \tag{A.8}$$

Henceforth we use K as a generic constant independent of γ and $\tilde{\gamma} \in \bar{\mathcal{S}}$. By the trace theorem, uniform boundedness of $\{|U_{\tilde{\gamma}}|_{H^2(\Omega_\gamma)}\}_{\gamma \in \bar{\mathcal{S}}}$ and (A.6)

$$|B| + |C| \le K|\tilde{\gamma} - \gamma|_{H^2(0,1)}. \tag{A.9}$$

Inserting (A.7) into A we find by (A.6)

$$\begin{aligned} |A| &\le [|(1 - \xi_x^{-2}U_{\tilde{\gamma},xx}|_{L^2} + |(\frac{\xi_y}{\xi_x})^2 U_{\tilde{\gamma},xx}|_{L^2} + 2|\frac{\xi_y}{\xi_x}U_{\tilde{\gamma},yx}|_{L^2} \\ &\quad + 2|\frac{\xi_{yx}\xi_y}{\xi_x^2}U_{\tilde{\gamma},x}|_{L^2}]|w|_{L^2} + |(\frac{\xi_{yy}}{\xi_x}U_{\tilde{\gamma},x}, w)_{\Omega_\gamma}| + |(F - f, w)_{\Omega_\gamma}| \\ &\le K\big[(|\tilde{\gamma} - \gamma|_C + |\tilde{\gamma} - \gamma|^2_{C^1})|U_{\tilde{\gamma},xx}|_{L^2} + |\tilde{\gamma} - \gamma|_{C^1}|U_{\tilde{\gamma},xy}|_{L^2} \\ &\quad + |\tilde{\gamma} - \gamma|_{C^1}|U_{\tilde{\gamma},x}|_{L^2}\big]|w|_{L^2} + |\xi_{yy}|_{L^2}|U_{\tilde{\gamma},x}|_{L^4}|w|_{L^4} + |(F - f, w)_{\Omega_\gamma}|. \end{aligned}$$

Combining this estimate with (A.8), (A.9) we find

$$|w|^2_{H^1(\Omega_\gamma)} \le K|\tilde{\gamma} - \gamma|_{H^2(0,1)}|w|_{H^1(\Omega_\gamma)} + |(F - f, w)_{\Omega_\gamma}|. \tag{A.10}$$

To estimate the last term we introduce the sets

$$E^+ = \{y \in (0,1) : \tilde{\gamma}(y) \geq \gamma(y)\} \text{ and } E^- = \{y \in (0,1) : \tilde{\gamma}(y) < \gamma(y)\},$$

and write

$$(F - f, w)_{\Omega_\gamma} = (\int\limits_{E^-} + \int\limits_{E^+}) \int\limits_{\gamma(y)}^{1} (F - f) w \, dxdy. \tag{A.11}$$

To estimate the first summand on the right hand side of (A.11) we use Fubini's theorem

$$\begin{aligned} I &:= \int\limits_{E^-} \int\limits_{\gamma(y)}^{1} (F - f) w dxdy = - \int\limits_{E^-} \int\limits_{\gamma(y)}^{1} (F - f) \int\limits_{x}^{1} w(\tau, y) d\tau \, dxdy \\ &= - \int\limits_{E^-} \int\limits_{\gamma(y)}^{1} (\int\limits_{\gamma(y)}^{\tau} (F - f) dx) w_\tau(\tau, y) d\tau dy. \end{aligned} \tag{A.12}$$

In view of $F = f \circ \psi$ we find for almost all $y \in E^+$

$$\begin{aligned} \int\limits_{\gamma(y)}^{\tau} (F - f) dx &= \int\limits_{\gamma(y)}^{\tau} f(\frac{1 - \tilde{\gamma}}{1 - \gamma} x + \frac{\tilde{\gamma} - \gamma}{1 - \gamma}, y) dx - \int\limits_{\gamma(y)}^{\tau} f(x, y) dx \\ &= \frac{1 - \gamma(y)}{1 - \tilde{\gamma}(y)} \int\limits_{\tilde{\gamma}(y)}^{\bar{\gamma}^*(\tau)} f(x, y) dx - \int\limits_{\gamma(y)}^{\tau} f(x, y) dx, \end{aligned}$$

where we have set

$$\gamma^*(\tau) = \frac{1 - \tilde{\gamma}}{1 - \gamma} \tau + \frac{\tilde{\gamma} - \gamma}{1 - \gamma}.$$

Inserting into (A.12) this yields

$$I = - \int\limits_{E^-} \int\limits_{\gamma(y)}^{1} (\frac{1 - \gamma}{1 - \tilde{\gamma}} \int\limits_{\tilde{\gamma}(y)}^{\gamma^*(\tau)} f dx - \int\limits_{\gamma(y)}^{\tau} f dx) w_\tau(\tau, y) d\tau dy.$$

Note that $\text{sign}(\gamma^*(\tau) - \tau) = \text{sign}(\tilde{\gamma}(y) - \gamma(y))$ for $\tau \in [0, 1]$, and hence $y \in E^-$ implies $\gamma^*(\tau) \leq \tau$, for $\tau \in [0, 1]$. We consider two cases.

(i) For $\tau \in (\gamma(y), \gamma(y) + \frac{1-\gamma(y)}{1-\tilde{\gamma}(y)}(\gamma(y) - \tilde{\gamma}(y))$ we obtain

$$\max(|\gamma^*(\tau) - \tilde{\gamma}(y)|, |\tau - \gamma(y)|) \leq K|\gamma(y) - \tilde{\gamma}(y)|$$

and as a consequence we conclude

$$|\frac{1-\gamma(y)}{1-\tilde{\gamma}(y)} \int_{\bar{\gamma}(y)}^{\gamma^*(\tau)} f dx - \int_{\gamma(y)}^{\tau} f dx| \leq K|f|_\infty |\tilde{\gamma}(y) - \gamma(y)|.$$

(ii) For $\tau \in (\gamma(y) + \frac{1-\gamma(y)}{1-\tilde{\gamma}(y)}(\gamma(y) - \tilde{\gamma}(y)), 1)$ we find

$$|\tau - \gamma^*(\tau)| \leq K|\tilde{\gamma}(y) - \gamma(y)|$$

and consequently

$$\begin{aligned}
&|\frac{1-\gamma(y)}{1-\tilde{\gamma}(y)} \int_{\bar{\gamma}(y)}^{\gamma^*(y)} f dx - \int_{\gamma(y)}^{\tau} f dx| \\
&\leq |\frac{1-\gamma(y)}{1-\tilde{\gamma}(y)} \int_{\bar{\gamma}(y)}^{\gamma(\tau)} f dx| + |\frac{1-\gamma(y)}{1-\tilde{\gamma}(y)} - 1||\int_{\bar{\gamma}(y)}^{\gamma^*(\tau)} f dx| + |\int_{\gamma^*(\tau)}^{\tau} f dx| \\
&\leq K|f|_{L^\infty}|\gamma(y) - \tilde{\gamma}(y)|.
\end{aligned}$$

A similar argument can be carried out for $y \in E^+$ and hence we obtain for all $y \in (0,1)$

$$|\int_{\gamma(y)}^{\tau} (F - f) dx| \leq K|f|_{L^\infty}|\gamma(y) - \tilde{\gamma}(y)|.$$

Inserting this estimate into (A.11) and (A.12) we obtain

$$|(F - f, w)_{\Omega_\gamma}| \leq K|f|_{L^\infty}|\gamma - \tilde{\gamma}|_{H^1}|w|_{H^1(\Omega_\gamma)}. \tag{A.13}$$

We note that under additional regularity on f an estimate of the type (A.13) could be obtained more directly than in the case $f \in L^\infty(\hat{\Omega})$ only. Combining (A.10) and (A.13) gives the first estimate of Proposition 5.1.

To verify the second claim it suffices to argue that

$$|u_{\tilde{\gamma}} - U_{\tilde{\gamma}}|_{H^1((\alpha,1)\times(0,1))} \leq K|\gamma - \tilde{\gamma}|_{H^2(0,1)}, \tag{A.14}$$

for a constant K independent of $\gamma, \tilde{\gamma} \in \bar{\mathcal{S}}_N$. Here we put $\alpha = \frac{\nu+\bar{\nu}}{2}$. For the L^2-part of the norm in (A.14) we find

$$\int_{y=0}^{1}\int_{x=\alpha}^{1} |u_{\bar{\gamma}}(\xi(x,y),y) - u_{\bar{\gamma}}(x,y)|^2 dxdy = \int_{y=0}^{1}\int_{x=\alpha}^{1} \Big| \int_{s=x}^{\xi} u_{\bar{\gamma},x}(s,y)ds\Big|^2 dxdy$$
$$\leq \frac{|\gamma-\tilde{\gamma}|_C}{1-\alpha} \int_{y=0}^{1} \Big[\int_{x=\alpha}^{1} \Big| \int_{s=x}^{\xi} |u_{\bar{\gamma},x}|^2 ds\Big| dx\Big] dy, \tag{A.15}$$

where ξ is defined in (A.4) and we used

$$\xi(x,y) - x = \frac{1}{1-\gamma}(\tilde{\gamma}-\gamma)(1-x). \tag{A.16}$$

We next estimate, for *fixed* $y \in [0,1]$ the expression in brackets in (A.15) and consider separately the cases $\tilde{\gamma}(y) < \gamma(y)$ and $\tilde{\gamma}(y) \geq \gamma(y)$. For $\tilde{\gamma}(y) < \gamma(y)$ it follows from (A.16) that $\xi(x,y) - x < 0$ and, suppressing the dependence of u on $\tilde{\gamma}$, we have

$$\int_{x=\alpha}^{1} \Big| \int_{s=x}^{\xi(x,y)} |u_x(s,y)|^2 ds\Big| dx = \int_{x=\alpha}^{1}\int_{s=\xi}^{x} |u_x|^2 dsdx$$
$$= \int_{s=\alpha}^{1} \int_{x=s}^{\frac{s(1-\gamma)+\gamma-\tilde{\gamma}}{1-\tilde{\gamma}}} |u_x(s,y)|^2 dxds + \int_{\frac{\tilde{\gamma}(1-\alpha)+\alpha-\gamma}{1-\gamma}}^{\alpha} \int_{x=\alpha}^{\frac{s(1-\gamma)+\gamma-\tilde{\gamma}}{1-\tilde{\gamma}}} |u_x(s,y)|^2 dxds$$
$$= I + II.$$

We estimate I and II separately:

$$I \leq \int_{s=\alpha}^{1} |u_x(s,y)|^2 \frac{(\gamma-\tilde{\gamma})(1-s)}{1-\tilde{\gamma}} ds \leq \frac{|\gamma-\tilde{\gamma}|_C}{1-\alpha} \int_{s=\alpha}^{1} |u_x(s,y)|^2 ds,$$

$$II = \int_{s=\frac{\tilde{\gamma}(1-\alpha)+\alpha-\gamma}{1-\gamma}}^{\alpha} |u_x(s,y)|^2 \Big(\frac{s(1-\gamma)+\gamma-\tilde{\gamma}-\alpha(1-\tilde{\gamma})}{1-\tilde{\gamma}}\Big) ds$$
$$\leq \int_{s=\frac{\tilde{\gamma}(1-\alpha)+\alpha-\gamma}{1-\gamma}}^{\alpha} |u_x(s,y)|^2 \frac{(\gamma-\tilde{\gamma})(1-\alpha)}{1-\tilde{\gamma}} ds \leq \frac{|\gamma-\tilde{\gamma}|_C}{1-\alpha} \int_{\bar{\gamma}}^{\alpha} |u_x(s,y)|^2 ds.$$

Combining these estimates we have for $y \in [0,1]$ satisfying $\tilde{\gamma}(y) < \gamma(y)$

$$\int_{x=\alpha}^{1} | \int_{s=x}^{\xi(x,y)} |u_x|^2 ds| dx \leq \frac{|\gamma - \tilde{\gamma}|_C}{1-\alpha} \int_{\tilde{\gamma}}^{1} |u_x(s,y)|^2 ds. \tag{A.17}$$

An analogous estimate, with a factor 2 on the right hand side of (A.17) can be obtained for $y \in [0,1]$ with $\tilde{\gamma}(y) \geq \gamma(y)$. Inserting these estimates into (A.15) we find

$$|U_{\tilde{\gamma}} - u_{\tilde{\gamma}}|^2_{L^2(\alpha,1)\times(0,1))} \leq \frac{2}{(1-\alpha)^2} |\gamma - \tilde{\gamma}|^2_C |u_{\tilde{\gamma}}|^2_{H^1(\Omega_{\tilde{\gamma}})}.$$

The terms involving the first derivatives of $U_{\tilde{\gamma}} - u_{\tilde{\gamma}}$ with respect to x and y can be treated analogously. This ends the proof of Proposition 5.1. □

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