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Infinite-Horizon Bilinear Optimal Control Problems: Sensitivity Analysis and Polynomial Feedback Laws

Tobias Breiten* Karl Kunisch† Laurent Pfeiffer‡

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Abstract

An infinite-horizon optimal control problem subject to an infinite-dimensional state equation with state and control variables appearing in a bilinear form is investigated. A sensitivity analysis with respect to the initial condition is carried out. We show in particular that the value function is infinitely differentiable in the neighborhood of the steady state, under a stabilizability assumption. In a second part, we analyse the efficiency of polynomial feedback laws, which are derived from a Taylor expansion of the value function.

Keywords: Infinite-horizon optimal control, bilinear control, regularity of the value function, polynomial feedback laws, sensitivity relations.

AMS Classification: 49J20, 49N35, 49Q12, 93D15.

1 Introduction

In this article, we consider a bilinear optimal control problem of the following form:

$$\begin{aligned} &\inf_{u \in L^2(0,\infty)} \mathcal{J}(u,y_0) := \frac{1}{2}\int_0^\infty \|Cy(t)\|_Z^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2 \,\mathrm{d}t, \\ &\text{where: } \begin{cases} \dot{y}(t) = Ay(t) + (Ny(t)+B)u(t), & \text{for } t > 0 \\ y(0) = y_0 \in Y. \end{cases} \end{aligned} \tag{1}$$

Here $V \subset Y \subset V^*$ is a Gelfand triple of real Hilbert spaces, where the embedding of V into Y is dense and compact, and V^* denotes the topological dual of V. The operator $A\colon \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of an analytic C_0-semigroup e^{At} on Y, $B \in Y$, $C \in \mathcal{L}(Y,Z)$, $N \in \mathcal{L}(V,Y)$, $\alpha > 0$ and $\mathcal{D}(A)$ denotes the domain of A. The precise conditions on A, B, C, and N are given further below. We denote by $\mathcal{V}$ the associated value function, i.e. $\mathcal{V}(y_0)$ is the value of Problem (1) with initial condition y_0.

The optimal control problem is posed over an infinite-time horizon and the state equation is nonlinear, since it contains a bilinear term, Nyu. We have in mind the situation where A is a second-order differential operator and N is a lower-order operator containing zero- and first-order differentiation terms. The operator N, considered as an operator in Y, is unbounded. Some optimal control problems of the Fokker-Planck equation can typically be written in the above form, see [7] and [9, Section 8].

In the first part of the paper, we prove that the solution to the problem, seen as a function of the initial condition y_0, is infinitely differentiable. The result is proved for initial conditions close to the steady state 0. This result implies in particular that the value function is infinitely differentiable in the neighborhood of 0. We also prove a sensitivity relation: for an initial condition y_0, the derivative of $\mathcal{V}$ at y_0 is equal to the associated costate at time 0.

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The second part of the paper is dedicated to the analysis of polynomial feedback laws. Polynomial feedback laws are derived from Taylor approximations of the value function of the form:

$$\mathcal{V}(y) \approx \sum_{j=2}^{k} \frac{1}{j!} \mathcal{T}_j(y, ..., y),$$

where $\mathcal{T}_2$,$\mathcal{T}_3$,...,$\mathcal{T}_k$ are bounded multilinear forms of order 2,3,...,k. The bilinear form $\mathcal{T}_2$ is characterized as the unique solution to an algebraic Riccati equation and the multilinear forms of order 3 and more are characterized as the unique solutions to generalized Lyapunov equations. The specific structure of the Taylor expansion has been known since the 60s (see [21] and the review paper [17]) for a general class of finite-dimensional stabilization problems. We have extended these results to the case of infinite-dimensional bilinear systems in a recent work [9] (see also the numerical tests in [8]). In that work, we have obtained the following estimate:

$$\|u_k - \bar{u}\|_{L^2(0,\infty)} = O(\|y_0\|_Y^{(k+1)/2}),$$

where u_k denotes the open-loop control generated by the feedback law derived from a Taylor expansion of order k and where $\bar{u}$ denotes the solution to Problem (1) (for the initial condition y_0, assumed to be close enough to 0). The main result of the second part of the present article is the following (improved) estimate:

$$\|u_k - \bar{u}\|_{L^2(0,\infty)} = O(\|y_0\|_Y^{k}).$$

Let us point out that this estimate is new, even for the finite-dimensional setting.

Both parts of the article rely on a stability analysis of the optimality conditions associated with Problem (1). This approach is described in abstract frameworks in [6, 16] and has been used for the sensitivity analysis of optimal control problems in many different settings. For the case of infinite-dimensional systems with finite-time horizon, we can mention [14, 15, 24].

Let us briefly comment on the literature on infinite-horizon optimal control problems. Many authors have considered the case of nonlinear ordinary differential equations. In fact, this area of research is still quite active, in part motivated by problems in economics. We refer the reader to the most recent articles [1, 3, 5, 22, 23] and to the references therein. The article [11] gives a very interesting account of the different approaches for investigating infinite-horizon optimal control problems. In this reference, a sensitivity relation is also obtained for problems with control constraints. The case of partial differential equations has received significantly less attention. Much research was dedicated to the linear-quadratic case and the development of proper frameworks for deriving algebraic Riccati equations, see e.g. [13, 18]. The quadratic programming approach for linear-quadratic infinite-horizon optimal control problems was discussed in [19]. For the case of nonlinear partial differential equations, we mention the articles [12] and [26], where optimality conditions are derived for a class of optimal control problems of semilinear parabolic equations. In [12], a sparsity-promoting cost function is considered. In [26], a quadratic cost function (similar to ours) is considered and a sensitivity relation is proved.

We now give a brief account of the contents of the paper. In Section 2, we provide the precise problem formulation and give results on the well-posedness of the state equation. Section 3 is devoted to existence results for optimality systems related to linear-quadratic infinite-horizon optimal control problems. They are used for justifying the applicability of the inverse function theorem used in the sensitivity analysis performed in Section 4. While the results of Section 4 are of local nature, we provide in Section 5 optimality conditions for an arbitrary initial condition. We describe in Section 6 the construction of polynomial feedback laws and summarize the main results obtained in the error analysis of [9]. The improved rate of convergence is established in Section 7. Some proofs of technical results are moved to the Appendix.

2 Formulation of the problem and first properties

2.1 Formulation of the problem

Throughout the article we assume that the following four assumptions hold true.

(A1) The operator $-A$ can be associated with a V-Y coercive bilinear form $a\colon V \times V \to \mathbb{R}$ such that there exist $\lambda \in \mathbb{R}$ and $\delta > 0$ satisfying

$$a(v,v) \geq \delta \|v\|_V^2 - \lambda \|v\|_Y^2 \quad \text{for all } v \in V.$$

(A2) The operator N is such that $N \in \mathcal{L}(V,Y) \cap \mathcal{L}(\mathcal{D}(A),V)$ and $N^* \in \mathcal{L}(V,Y)$.

(A3) [Stabilizability] There exists an operator $F \in \mathcal{L}(Y,\mathbb{R})$ such that the semigroup $e^{(A+BF)t}$ is exponentially stable on Y.

(A4) [Detectability] There exists an operator $K \in \mathcal{L}(Z,Y)$ such that the semigroup $e^{(A-KC)t}$ is exponentially stable on Y.

Conditions (A3) and (A4) are well-known and analysed in infinite-dimensional systems theory, see [13], for example. In particular, there has been ongoing interest on stabilizability of infinite-dimensional parabolic systems by finite-dimensional controllers. We refer to [2, 25] and the references given there.

While the results of this article are obtained for scalar controls, the generalisation to the case of systems of the form

$$\dot{y} = Ay + \sum_{j=1}^{m} (N_j y(t) + B_j) u_j(t),$$

with $B_j \in Y$, can easily be achieved. In this more general case, one has to assume that the operators N_1,...,N_m satisfy Assumption (A2) and Assumption (A3) must be replaced by the following one: there exist operators F_1,...,F_m in $\mathcal{L}(Y,\mathbb{R})$ such that the semigroup $e^{(A+\sum_{j=1}^m B_j F_j)t}$ is exponentially stable.

With (A1) holding the operator A associated to the form a generates an analytic semigroup that we denote by e^{At}, see e.g. [27, Sections 3.6 and 5.4]. Let us set $A_0 = A - \lambda I$, if $\lambda > 0$ and $A_0 = A$ otherwise. Then $-A_0$ has a bounded inverse in Y, see [27, page 75], and in particular it is maximal accretive, see [27, 20]. We have $\mathcal{D}(A_0) = \mathcal{D}(A)$ and the fractional powers of $-A_0$ are well-defined. In particular, $\mathcal{D}((-A_0)^{\frac{1}{2}}) = [\mathcal{D}(-A_0), Y]_{\frac{1}{2}} := (\mathcal{D}(-A_0), Y)_{\frac{1}{2},2}$ the real interpolation space with indices 2 and $\frac{1}{2}$, see [4, Proposition 6.1, Part II, Chapter 1]. Assumption (A5) below will only be used in Sections 6 and 7 for the proof of Lemma 23 and Lemma 30. The assumption is not needed for the sensitivity analysis performed in Section 4 and for the derivation of optimality conditions in Section 5.

(A5) It holds that $[\mathcal{D}(-A_0), Y]_{\frac{1}{2}} = [\mathcal{D}(-A_0^*), Y]_{\frac{1}{2}} = V$.

We are now prepared to state the problem under consideration. For $y_0 \in Y$, consider

$$\inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0) := \frac{1}{2} \int_0^\infty \|CS(u, y_0; t)\|_Z^2 \, \mathrm{d}t + \frac{\alpha}{2} \int_0^\infty u(t)^2 \, \mathrm{d}t, \tag{P}$$

where $S(u, y_0; \cdot)$ is the solution to

$$\begin{cases} \dot{y}(t) = Ay(t) + Ny(t)u(t) + Bu(t), & \text{for } t > 0, \\ y(0) = y_0. \end{cases} \tag{2}$$

Here $y = S(u, y_0)$ is referred to as solution of (2) if for all each $T > 0$, it lies in the space

$$W(0,T) = \{ y \in L^2(0,T;V) \,|\, \dot{y} \in L^2(0,T;V^*) \}.$$

The well-posedness of the state equation is ensured by Lemma 1 below. We recall that $W(0,T)$ is continuously embedded in $C([0,T],Y)$ [20, Theorem 3.1]. We abbreviate

$$W_\infty = W(0,\infty).$$

This space is continuously embedded in $C_b([0,\infty],Y)$, see e.g. the proof of [9, Lemma 1]. We fix $M_0 > 0$ such that for all $y \in W_\infty$,

$$\|y\|_{L^\infty(0,\infty;Y)} \leq M_0 \|y\|_{W_\infty}. \tag{3}$$

Let us mention that for $y \in W_\infty$,

$$\lim_{t\to\infty} \|y(t)\|_Y = 0.$$

A short proof can be found in [9, Lemma 1]. We also set

$$W_\infty^0 = \big\{ y \in W_\infty \,|\, y(0) = 0 \big\}.$$

Finally, we denote by $\mathcal{V}$ the value function associated with Problem (P), defined by

$$\mathcal{V}(y_0) = \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0).$$

Note that origin is a steady state of the uncontrolled system (2) and that $\mathcal{V}(0) = 0$.

2.2 State equation

The first lemma ensures that the state equation is well-posed. The lemma is a simple generalization of [9, Lemma 1]. Unless stated otherwise, y_0 is an initial condition in Y and f lies in $L^2(0, \infty; V^*)$. All along the article, the constant $M > 0$ is a generic constant whose value may change.

Lemma 1. *For all $T > 0$ and $u \in L^2(0,T)$, there exists a unique solution $y \in W(0,T)$ to the following system:*

$$\dot{y} = Ay + Nyu + Bu + f, \quad y(0) = y_0.$$

Moreover, there exists a continuous function c such that

$$\|y\|_{W(0,T)} \leq c(T, \|y_0\|_Y, \|u\|_{L^2(0,T)}, \|f\|_{L^2(0,T;V^*)}). \tag{4}$$

Finally, if $y \in L^2(0,\infty;Y)$, then $y \in W_\infty$.

Using the stabilizability assumption (A3) and the techniques of [4, Theorem 2.2, Part II, Chapter 3] and [28], one can show that for all $f \in L^2(0,\infty;V^*)$ and for all $y_0 \in Y$, the following nonhomogeneous system:

$$\dot{y} = (A + BF)y + f, \quad y(0) = y_0 \tag{5}$$

has a unique solution $y \in W_\infty$. Moreover, there exists a constant $M_s > 0$ independent of f and y_0 such that

$$\|y\|_{W_\infty} \leq M_s(\|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y). \tag{6}$$

Similarly, as a consequence of the detectability assumption (A4), the following nonhomogeneous system:

$$\dot{y} = (A - KC)y + f, \quad y(0) = y_0 \tag{7}$$

has a unique solution $y \in W_\infty$. Moreover, there exists a constant M_d independent of f and y_0 such that

$$\|y\|_{W_\infty} \leq M_d(\|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y). \tag{8}$$

In the following, we address the stability of a class of perturbations of the linear system (7).

Lemma 2. *Let $P \in \mathcal{L}(W_\infty, L^2(0,\infty;V^*))$ be such that $\|P\| < \frac{1}{M_d}$, where $\|P\|$ denotes the operator norm of P. Then there exists a unique solution to the following system:*

$$\dot{y}(t) = (A - KC)y(t) + (Py)(t) + f(t), \quad y(0) = y_0. \tag{9}$$

Moreover,

$$\|y\|_{W_\infty} \leq \frac{M_d}{1 - M_d\|P\|}(\|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y).$$

Proof. We first prove the existence of a solution, by using a classical fixed-point argument. Let $M' = \frac{M_d}{1 - M_d\|P\|}$. Consider the set $\mathcal{M} \subset W_\infty$, defined by

$$\mathcal{M} = \big\{ y \in W_\infty \,|\, \|y\|_{W_\infty} \leq M'(\|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y) \big\}.$$

We consider the mapping $\mathcal{Z} \colon y \in \mathcal{M} \mapsto \mathcal{Z}(y) \in W_\infty$, where $z = \mathcal{Z}(y)$ is the unique solution to

$$\dot{z}(t) = (A - KC)z(t) + (Py)(t) + f(t), \quad z(0) = y_0.$$

We prove that the mapping $\mathcal{Z}$ has a fixed point, which is then a solution to (9). By (8), we have

$$\begin{aligned}\|\mathcal{Z}(y)\|_{W_\infty} &\leq M_d\big(\|Py\|_{L^2(0,\infty;V^*)} + \|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y\big)\\ &\leq \underbrace{M_d(1+\|P\|M')}_{=M'}(\|f\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y).\end{aligned}$$

Therefore $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}$. Now for y_1 and $y_2 \in \mathcal{M}$, we set $z = \mathcal{Z}(y_2) - \mathcal{Z}(y_1)$. Then,

$$\dot{z}(t) = (A - KC)z(t) + (P(y_2 - y_1))(t), \quad z(0) = 0,$$

and by estimate (8), we obtain

$$\|\mathcal{Z}(y_2) - \mathcal{Z}(y_1)\|_{W_\infty} = \|z\|_{W_\infty} \leq M_d\|P\|\|y_2 - y_1\|_{W_\infty}.$$

This proves that $\mathcal{Z}$ is a contraction, since $M_d\|P\| < 1$. Therefore, by the fixed-point theorem, there exists $y \in \mathcal{M}$ such that $\mathcal{Z}(y) = y$, which proves the existence of a solution to (9).

Observe now that the mapping $\mathcal{Z}$, defined on the whole space W_∞, is still a contraction. This proves the uniqueness of the solution to (9) in W_∞. □

Remark 3. The result is also true when the operator $(A - KC)$ is replaced by $(A + BF)$ and the constant M_d by M_s.

In the next lemma, we utilize the previous result and assumption (A4) to establish a detectability property for the bilinear system.

Lemma 4. *Let $0 < \delta < \big(\|N\|_{\mathcal{L}(Y,V^*)}M_0M_d\big)^{-1}$ and let $u \in L^2(0,\infty)$ be such that $\|u\|_{L^2(0,\infty)} \leq \delta$. Assume that the unique solution y to the following system:*

$$\dot{y} = Ay + Nyu + Bu + f, \quad y(0) = y_0$$

is such that $Cy \in L^2(0,\infty;Z)$. There exists a constant $M > 0$, independent of y_0, u, f, and y, such that

$$\|y\|_{W_\infty} \leq M\big(\|y_0\|_Y + \|u\|_{L^2(0,\infty)} + \|f\|_{L^2(0,\infty;V^*)} + \|Cy\|_{L^2(0,\infty;Z)}\big).$$

Proof. Consider the following system:

$$\dot{z} = Az + Nzu + Bu + f + KC(y - z), \quad z(0) = y_0. \tag{10}$$

For proving its well-posedness, we introduce the operator $P \in \mathcal{L}(W_\infty, L^2(0,\infty;V^*))$, defined by $(Pv)(t) = Nv(t)u(t)$ for $v \in W_\infty$. We have

$$\begin{aligned}\|Pv\|_{L^2(0,\infty;V^*)} &\leq \|N\|_{\mathcal{L}(Y,V^*)}\|u\|_{L^2(0,\infty)}\|v\|_{L^\infty(0,\infty;Y)}\\ &\leq \underbrace{\|N\|_{\mathcal{L}(Y,V^*)}\|u\|_{L^2(0,\infty)}M_0}_{<M_d^{-1}}\|v\|_{W_\infty}.\end{aligned}$$

Therefore, $\|P\| := \|P\|_{\mathcal{L}(W_\infty,L^2(0,\infty;V^*))} < M_d^{-1}$. Note that (10) can be expressed as

$$\dot{z}(t) = (A - KC)z(t) + (Pz)(t) + (Bu(t) + f(t) + KCy(t)).$$

Thus by Lemma 2, system (10) has a unique solution, which satisfies

$$\begin{aligned}\|z\|_{W_\infty} &\leq M\big(\|Bu + f + KCy\|_{L^2(0,\infty;V^*)} + \|y_0\|_Y\big)\\ &\leq M(\|u\|_{L^2(0,\infty)} + \|f\|_{L^2(0,\infty;V^*)} + \|Cy\|_{L^2(0,\infty;Z)} + \|y_0\|_Y),\end{aligned}$$

where the constant M in the last inequality does not depend on y_0, u, f and y. Finally, we observe that $e := z - y$ satisfies

$$\dot{e}(t) = (A - KC)e(t) + u(t)N(t)e(t) = (A - KC)e(t) + (Pe)(t), \quad e(0) = 0,$$

which proves that $e = 0$, using once again Lemma 2. Therefore, $y = z$ and the lemma is proved. □

Remark 5. The result of Lemma 4 remains true if the bilinear term Nyu is removed. In this case, no restriction on $\|u\|_{L^2(0,\infty)}$ is necessary, since then the well-posedness of z (defined by (10)) follows directly from estimate (6).

Remark 6. In an abstract non-convex setting, a sensitivity analysis can be performed if the linearized constraints are surjective and if the sufficient second-order optimality conditions are satisfied. These two properties are satisfied in the current framework for $(y,u)=(0,0)$, the solution to (P) with initial condition $y_0=0$.

- A consequence of the stabilizability assumption (A3) is that for all $f \in L^2(0,\infty;V^*)$, there exists a pair $(z,v) \in W_\infty \times L^2(0,\infty)$ satisfying: $\dot{z} = Az + Bv + f$, $z(0)=0$ (see the proof of Lemma 9).
- A consequence of the detectability assumption (A4), obtained with Lemma 4 and Remark 5, is the following property: for all $(z,v) \in W_\infty \times L^2(0,\infty)$ satisfying $\dot{z} = Az + Bv$, $z(0)=0$, there exists a constant M independent of (z,v) such that

$$\frac{1}{2}\|Cz\|^2_{L^2(0,\infty;Z)} + \frac{\alpha}{2}\|v\|^2_{L^2(0,\infty)} \geq M(\|z\|^2_{L^2(0,\infty;Y)} + \|v\|^2_{L^2(0,\infty)}).$$

  This property corresponds to the sufficient second-order optimality conditions for (P) with initial condition $y_0=0$.

3 Linear optimality systems

This section is dedicated to the proof of Proposition 7 below, which is a key result for the sensitivity analysis performed in Section 4 and for the error analysis of Section 7. The proof can be found at the end of the section, page 9. For finite-horizon control problems, results like Proposition 7 are quite well-known. The case of infinite-time horizons, however, needs special attention. It should also be pointed out that the proof is not based on PDE techniques, but rather, an associated linear-quadratic optimal control problem is investigated. Before stating the proposition in detail, we recall that W_∞^0 is continuously embedded into $L^2(0,\infty;V)$ and therefore $L^2(0,\infty;V^*)$ is continuously embedded into $(W_\infty^0)^*$. We further introduce the space

$$X := L^2(0,\infty;V^*) \times (W_\infty^0)^* \times L^2(0,\infty).$$

Proposition 7. *For all $(f,g,h) \in X$, there exists a unique triplet $(y,u,p) \in W_\infty^0 \times L^2(0,\infty) \times L^2(0,\infty;V)$ such that*

$$\begin{cases} \dot{y} - (Ay+Bu) = f & \text{in } L^2(0,\infty;V^*) \\ -\dot{p} - A^*p - C^*Cy = g & \text{in } (W_\infty^0)^* \\ \alpha u + \langle B,p\rangle_Y = -h & \text{in } L^2(0,\infty). \end{cases} \tag{11}$$

Moreover there exists a constant $M>0$, independent of (f,g,h), such that

$$\|(y,u,p)\|_{W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)} \leq M\|(f,g,h)\|_X. \tag{12}$$

Assume further that $g \in L^2(0,\infty;V^*)$. Then $p \in W_\infty$ and there exists a constant M, independent of (f,g,h), such that

$$\|p\|_{W_\infty} \leq M\big(\|f\|_{L^2(0,\infty;V^*)} + \|g\|_{L^2(0,\infty;V^*)} + \|h\|_{L^2(0,\infty)}\big). \tag{13}$$

Note that the costate equation in (11) must be understood as follows:

$$\langle p,\dot{\varphi}\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} = \langle A^*p,\varphi\rangle_{L^2(0,\infty;V^*),L^2(0,\infty;V)} + \langle C^*Cy,\varphi\rangle_{L^2(0,\infty;Y)} + \langle g,\varphi\rangle_{(W_\infty^0)^*,W_\infty^0}, \text{ for all } \varphi \in W_\infty^0.$$

The main idea for proving the above result is the following: the linear system (11) constitutes the optimality conditions for the linear-quadratic optimal control problem (LQ) defined below. Given $f \in L^2(0,\infty;V^*)$, $g \in (W_\infty^0)^*$, and $h \in L^2(0,\infty)$, we consider:

$$\min_{(y,u)\in W_\infty^0 \times L^2(0,\infty)} J[g,h](y,u) \quad \text{subject to: } e[f](y,u)=0, \tag{LQ}$$

where

$$J[g,h](y,u) := \frac{1}{2}\int_0^\infty \|Cy(t)\|_Z^2 \,\mathrm{d}t + \langle g, y\rangle_{(W_0^\infty)^*, W_0^\infty} + \frac{\alpha}{2}\int_0^\infty \big(u(t)^2 + h(t)u(t)\big)\,\mathrm{d}t,$$

and where

$$e[f](y,u) = \dot{y} - (Ay + Bu + f) \in L^2(0,\infty;V^*).$$

Note that the initial condition $y(0) = 0$ need not be specified as a constraint since $y \in W_\infty^0$.

Let us prove the existence of a solution to Problem (LQ).

Lemma 8. *There exists a constant $M > 0$ such that for all $(f,g,h) \in X$, the linear-quadratic problem (LQ) has a unique solution (y,u) satisfying the following bounds:*

$$\|y\|_{W_\infty} \leq M\|(f,g,h)\|_X \quad \textit{and} \quad \|u\|_{L^2(0,\infty)} \leq M\|(f,g,h)\|_X. \tag{14}$$

Proof. Let $\tilde{y} \in W_\infty^0$ be defined by $\dot{\tilde{y}} = (A + BF)\tilde{y} + f$. Then by (6) we have

$$\|\tilde{y}\|_{W_\infty} \leq M_s \|f\|_{L^2(0,\infty;V^*)}.$$

Setting $\tilde{u} = F\tilde{y}$, we obtain that

$$\|\tilde{u}\|_{L^2(0,\infty)} \leq M\|f\|_{L^2(0,\infty;V^*)} \quad \text{and} \quad e[f](\tilde{y},\tilde{u}) = 0,$$

and consequently

$$J[g,h](\tilde{y},\tilde{u}) \leq M\|(f,g,h)\|_X^2.$$

Therefore the problem is feasible.

Let us consider now a minimizing sequence $(y_n, u_n)_{n\in\mathbb{N}}$. We can assume that for all $n \in \mathbb{N}$,

$$J[g,h](y_n,u_n) \leq M\|(f,g,h)\|_X^2. \tag{15}$$

We begin by proving that the sequence (y_n, u_n) is bounded in $W_\infty \times L^2(0,\infty)$. To this purpose, we first compute a lower bound of $J[g,h](y_n,u_n)$. Using Young's inequality, we obtain for all $\varepsilon > 0$ that

$$\begin{aligned} J[g,h](y_n,u_n) &\geq \frac{1}{2}\|Cy_n\|_{L^2(0,\infty;Z)}^2 - \|g\|_{(W_\infty^0)^*}\|y_n\|_{W_\infty} \\ &\qquad + \frac{\alpha}{2}\|u_n\|_{L^2(0,\infty)}^2 - \|h\|_{L^2(0,\infty)}\|u_n\|_{L^2(0,\infty)} \\ &\geq \frac{1}{2}\|Cy_n\|_{L^2(0,\infty;Z)}^2 - \frac{1}{2\varepsilon}\|g\|_{(W_\infty^0)^*}^2 - \frac{\varepsilon}{2}\|y_n\|_{W_\infty}^2 \\ &\qquad + \frac{\alpha}{2}\Big(\|u_n\|_{L^2(0,\infty)} - \frac{\|h\|_{L^2(0,\infty)}}{\alpha}\Big)^2 - \frac{\|h\|_{L^2(0,\infty)}^2}{2\alpha}. \end{aligned} \tag{16}$$

Combining (15) and (16), we obtain that there exists a constant M independent of $\varepsilon > 0$ such that

$$\|Cy_n\|_{L^2(0,\infty;Z)}^2 + \alpha\Big(\|u_n\|_{L^2(0,\infty)} - \frac{\|h\|_{L^2(0,\infty)}}{\alpha}\Big)^2 \leq M\Big(\|(f,g,h)\|_X^2 + \varepsilon\|y_n\|_{W_\infty}^2 + \frac{1}{\varepsilon}\|g\|_{(W_\infty^0)^*}^2\Big)$$

and therefore

$$\|Cy_n\|_{L^2(0,\infty;Z)} \leq M\Big(\|(f,g,h)\|_X + \sqrt{\varepsilon}\|y_n\|_{W_\infty} + \frac{1}{\sqrt{\varepsilon}}\|g\|_{(W_\infty^0)^*}\Big), \tag{17}$$

$$\|u_n\|_{L^2(0,\infty)} \leq M\Big(\|(f,g,h)\|_X + \sqrt{\varepsilon}\|y_n\|_{W_\infty} + \frac{1}{\sqrt{\varepsilon}}\|g\|_{(W_\infty^0)^*}\Big). \tag{18}$$

Applying Lemma 4 (taking into account Remark 5) and using (17), we obtain that

$$\begin{aligned} \|y_n\|_{W_\infty} &\leq M\big(\|u_n\|_{L^2(0,\infty)} + \|f\|_{L^2(0,\infty;V^*)} + \|Cy_n\|_{L^2(0,\infty;Z)}\big) \\ &\leq M\big(\|(f,g,h)\|_X + \sqrt{\varepsilon}\|y_n\|_{W_\infty} + \frac{1}{\sqrt{\varepsilon}}\|g\|_{(W_\infty^0)^*}\big) \end{aligned}$$

Choosing $\varepsilon = \frac{1}{(2M)^2}$ (where M is the constant involved in the last inequality), we obtain the existence of another constant M such that

$$\|y_n\|_{W_\infty} \leq M\|(f,g,h)\|_X.$$

Combining this estimate with (18), we finally obtain that

$$\|u_n\|_{L^2(0,\infty)} \leq M\|(f,g,h)\|_X.$$

The sequence $(y_n, u_n)_{n\in\mathbb{N}}$ is therefore bounded in $W_\infty \times L^2(0,\infty)$ and possesses a weak limit point (y,u) satisfying (14). One can prove the optimality of (y,u) with the same techniques as those used for the proof of [9, Proposition 2].

The uniqueness of the solution directly follows from the linearity of the state equation and the strict convexity of the cost functional. □

We give now optimality conditions for Problem (LQ). The existence of a Lagrange multiplier follows directly from the surjectivity of the derivative of the constraints, which itself is a direct consequence of the stabilizability assumption (A3).

Lemma 9. *For all $(f,g,h) \in X$, there exists a unique costate $p \in L^2(0,\infty;V)$ satisfying the following adjoint equation:*

$$-\dot{p} = A^*p + C^*Cy + g \tag{19}$$

and such that

$$\alpha u + h + \langle B, p\rangle_Y = 0. \tag{20}$$

Here (y,u) denotes the unique solution to (LQ). Moreover, there exists a constant $M > 0$ independent of (f,g,h) such that

$$\|p\|_{L^2(0,\infty;V)} \leq M\|(f,g,h)\|_X.$$

Proof. Observe first that $e[f]$ and $J[g,h]$ are both continuously differentiable. We have

$$\begin{aligned} DJ[g,h](y,u)(z,v) =\ & \langle C^*Cy, z\rangle_{L^2(0,\infty;Y)} + \langle g, z\rangle_{(W_\infty^0)^*, W_\infty^0} + \alpha\langle u,v\rangle_{L^2(0,\infty)} + \langle h,v\rangle_{L^2(0,\infty)} \\ =\ & \left\langle \begin{pmatrix} C^*Cy + g \\ \alpha u + h \end{pmatrix}, \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle_{(W_\infty^0)^* \times L^2(0,\infty), W_\infty^0 \times L^2(0,\infty)} \end{aligned}$$

The derivative $De[f](y,u)$, which is independent of f, y and u, is denoted by T. It is given by

$$T\colon (z,v) \in W_\infty^0 \times L^2(0,\infty) \mapsto \dot{z} - (Az + Bv) \in L^2(0,\infty;V^*).$$

The adjoint operator $T^*\colon L^2(0,\infty;V) \to (W_\infty^0)^* \times L^2(0,\infty)$ satisfies

$$\begin{aligned} &\langle T^*p, (z,v)\rangle_{(W_\infty^0)^* \times L^2(0,\infty), W_\infty^0 \times L^2(0,\infty)} \\ &\quad = \langle p, \dot{z}\rangle_{L^2(0,\infty;V), L^2(0,\infty;V^*)} - \langle p, Az\rangle_{L^2(0,\infty;V), L^2(0,\infty;V^*)} - \langle p, Bv\rangle_{L^2(0,\infty;V), L^2(0,\infty;V^*)} \\ &\quad = \langle -\dot{p}, z\rangle_{(W_\infty^0)^*, W_\infty^0} - \langle A^*p, z\rangle_{L^2(0,\infty;V^*), L^2(0,\infty;V)} - \langle \langle B, p\rangle_Y, v\rangle_{L^2(0,\infty)} \\ &\quad = \left\langle \begin{pmatrix} -\dot{p} - A^*p \\ -\langle B,p\rangle_Y \end{pmatrix}, \begin{pmatrix} z \\ v \end{pmatrix} \right\rangle_{(W_\infty^0)^* \times L^2(0,\infty), W_\infty^0 \times L^2(0,\infty)}. \end{aligned}$$

Let us prove that the operator T is surjective. Choose $\varphi \in L^2(0,\infty;V^*)$, let $\tilde{z}$ be the solution to

$$\dot{\tilde{z}} = (A + BF)\tilde{z} + \varphi, \quad \tilde{z}(0) = 0,$$

and set $\tilde{v} = F\tilde{z}$. By estimate (6), $\|\tilde{z}\|_{W_\infty} \leq M_s\|\varphi\|_{L^2(0,\infty;V^*)}$ and therefore

$$\|\tilde{u}\|_{L^2(0,\infty)} \leq M\|\varphi\|_{L^2(0,\infty;V^*)}.$$

Clearly $T(\tilde{z},\tilde{v}) = \varphi$, which proves the surjectivity of T. Consequently, see e.g. [30], there exists a unique $p \in L^2(0,\infty;V)$ such that

$$DJ[g,h](y,u)(z,v) - \langle T^*p, (z,v)\rangle_{(W_\infty^0)^* \times L^2(0,\infty), W_\infty^0 \times L^2(0,\infty)} = 0.$$

Using the expressions of $DJ[g,h](y,u)$ and T^* previously obtained, we deduce the costate equation (19) and relation (20). By the closed range theorem (see [10, Theorem 2.20]) and (14), there exists a constant $M > 0$ such that

$$\begin{aligned}\|p\|_{L^2(0,\infty;V)} &\leq M\|T^*p\|_{(W^0_\infty)^* \times L^2(0,\infty)} \\ &\leq M\|DJ[g,h](y,u)\|_{(W^0_\infty)^* \times L^2(0,\infty)} \\ &\leq M\big(\|C^*Cy\|_{L^2(0,\infty;Y)} + \|g\|_{(W^0_\infty)^*} + \|u\|_{L^2(0,\infty)} + \|h\|_{L^2(0,\infty)}\big).\end{aligned}$$

Finally, using estimate (14) for the solution (y,u) to (LQ), we obtain that

$$\|p\|_{L^2(0,\infty;V)} \leq M\|(f,g,h)\|_X.$$

This concludes the proof. □

We can finally prove Proposition 7.

Proof of Proposition 7. The existence of (y,u,p) and estimate (12) directly follow from Lemma 8 and Lemma 9. Let (y_1,u_1,p_1) and (y_2,u_2,p_2) be two solutions to (11). By the linearity of the system, the difference is a solution to (11) with $(f,g,h) = (0,0,0)$. Estimate (12) implies the uniqueness. Let us assume now that $g \in L^2(0,\infty;V^*)$. In order to prove that $p \in W_\infty$, it suffices to prove that $\dot{p} \in L^2(0,\infty;V^*)$. Using the costate equation (19) and estimate (12), we obtain that

$$\begin{aligned}\|\dot{p}\|_{L^2(0,\infty;V^*)} &\leq \big(\|A^*p\|_{L^2(0,\infty;V^*)} + \|C^*Cy\|_{L^2(0,\infty;V^*)} + \|g\|_{L^2(0,\infty;V^*)}\big) \\ &\leq M\big(\|p\|_{L^2(0,\infty;V)} + \|y\|_{L^2(0,\infty;Y)} + \|g\|_{L^2(0,\infty;V^*)}\big) \\ &\leq M\big(\|f\|_{L^2(0,\infty;V^*)} + \|g\|_{L^2(0,\infty;V^*)} + \|h\|_{L^2(0,\infty)}\big).\end{aligned}$$

This implies (13) and concludes the proof of the proposition. □

4 Sensitivity analysis

In this section, after proving the existence and uniqueness of a solution to (P) for all initial conditions y_0 close enough to the origin, we verify that locally, the unique solution, the associated trajectory, and the costate (in W_∞) are infinitely differentiable functions of the initial condition y_0. In particular, this will imply that the value function $\mathcal{V}$ is C^∞ in a neighborhood of the origin.

A first step in the analysis is the derivation of first-order necessary optimality conditions for (P) in a weak form (Proposition 11), i.e. for a costate $p \in L^2(0,\infty;V)$ and an adjoint equation satisfied in $(W^0_\infty)^*$. Then, we prove the existence of a mapping

$$y_0 \mapsto \big(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0)\big),$$

defined for $y_0 \in Y$ close to 0, which is such that $(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0))$ is the unique triplet (y,u,p) in a neighbourhood of $(0,0,0)$ satisfying the weak optimality conditions (Lemma 13). It follows then that $\mathcal{U}(y_0)$ is the unique solution to (P) (Proposition 14), for y_0 close enough to 0.

Optimality conditions in a strong form, involving a costate in W_∞, require an extra step. We first prove the existence of a mapping

$$y_0 \mapsto \big(\tilde{\mathcal{Y}}(y_0), \tilde{\mathcal{U}}(y_0), \tilde{\mathcal{P}}(y_0)\big),$$

defined for $y_0 \in Y$ close to 0, which is such that $(\tilde{\mathcal{Y}}(y_0), \tilde{\mathcal{U}}(y_0), \tilde{\mathcal{P}}(y_0))$ is the unique triplet (y,u,p) in a neighbourhood of $(0,0,0)$ satisfying the strong optimality conditions (Lemma 16). To conclude the sensitivity analysis, it suffices then to check that the mappings $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$ and $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}})$ coincide around 0 (Lemma 17).

We start by proving the existence of a solution to (P), assuming the existence of a feasible control u and the bound (21). This bound enables us to derive estimates on the trajectory for the W_∞-norm, using Lemma 4.

Lemma 10. *Let* $0 \leq \delta_0 \leq \frac{1}{2}\big(\|N\|_{\mathcal{L}(Y,V^*)} M_0 M_d\big)^{-1}$. *Assume that there exists a control* $u \in L^2(0,\infty)$ *such that*

$$\mathcal{J}(u, y_0) \leq \frac{\alpha}{2}\delta_0^2. \tag{21}$$

Then (P) *possesses a solution* $\bar{u}$. *Moreover, there exists a constant* $M > 0$, *independent of* δ_0, *such that*

$$\|\bar{u}\|_{L^2(0,\infty)} \leq \delta_0 \quad \textit{and} \quad \|\bar{y}\|_{W_\infty} \leq M(\|y_0\|_Y + \delta_0), \tag{22}$$

where $\bar{y} = S(\bar{u}, y_0)$.

Proof. We follow the same approach as in Lemma 8. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence, and set $y_n = S(u_n, y_0)$. We can assume that for all $n \in \mathbb{N}$,

$$\mathcal{J}(u_n, y_0) \leq \frac{\alpha}{2}\delta_0^2.$$

This implies that for all $n \in \mathbb{N}$,

$$\|u_n\|_{L^2(0,\infty)} \leq \delta_0 \quad \text{and} \quad \|Cy_n\|_{L^2(0,\infty;Z)} \leq \sqrt{\alpha}\delta_0.$$

By Lemma 4 with $\delta = \frac{1}{2}\big(\|N\|_{\mathcal{L}(Y,V^*)} M_0 M_d\big)^{-1}$, we obtain the existence of M, independent of δ_0, such that for all $n \in \mathbb{N}$,

$$\|y_n\|_{W_\infty} \leq M(\|y_0\|_Y + \|u_n\|_{L^2(0,\infty)} + \|Cy_n\|_{L^2(0,\infty;Z)}) \leq M(\|y_0\|_Y + \delta_0).$$

Therefore the sequence $(y_n, u_n)_{n\in\mathbb{N}}$ possesses a weak limit point $(\bar{y}, \bar{u})$ in $W_\infty \times L^2(0,\infty)$, satisfying estimate (22). One can prove that $\bar{y} = S(\bar{u}, y_0)$ and that $\bar{u}$ is optimal with the same techniques as those used for the proof of [9, Proposition 2]. □

In the following lemma, we state and prove first-order necessary optimality conditions in a weak form. The approach is similar to the one employed for Lemma 9: we formulate the problem as an abstract optimization problem and obtain the existence of a costate in $L^2(0,\infty;V)$ as a Lagrange multiplier. An a-priori estimate must be done on the solution and its associated trajectory in order to prove the surjectivity of the linearized constraints.

Proposition 11. *There exists* $\delta_1 > 0$ *such that, if for* $y_0 \in Y$, *Problem* (P) *possesses a solution* $\bar{u}$ *such that*

$$\|\bar{u}\|_{L^2(0,\infty)} \leq \delta_1 \quad \textit{and} \quad \|\bar{y}\|_{L^2(0,\infty;Y)} \leq \delta_1,$$

where $\bar{y} = S(y_0, \bar{u})$, *then there exists a unique costate* $p \in L^2(0,\infty;V)$ *such that*

$$-\dot{p} = A^*p + \bar{u}N^*p + C^*C\bar{y} \quad \textit{(in } (W_\infty^0)^*\textit{)} \tag{23}$$

and

$$\alpha\bar{u} + \langle N\bar{y} + B, p\rangle_Y = 0. \tag{24}$$

Moreover, there exists a constant $M > 0$, *independent of* $(\bar{y}, \bar{u})$, *such that*

$$\|p\|_{L^2(0,\infty;V)} \leq M(\|\bar{y}\|_{L^2(0,\infty;Y)} + \|\bar{u}\|_{L^2(0,\infty)}). \tag{25}$$

The proof is given in the Appendix, page 24.

Remark 12. At this stage, it is not possible to prove that $p \in W_\infty$. More precisely, it is not possible to prove that $\dot{p} \in L^2(0,\infty;V^*)$ because of the term $\bar{u}N^*p$. Indeed, since $\bar{u} \in L^2(0,\infty)$, one would need to prove that $N^*p \in L^\infty(0,\infty;V^*)$. However, we do not know for the moment whether $p \in L^\infty(0,\infty;Y)$.

Consider now the mapping Φ, defined as follows:

$$\Phi\colon \begin{cases} W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V) & \rightarrow \quad Y \times L^2(0,\infty;V^*) \times (W_\infty^0)^* \times L^2(0,\infty) \\ (y,u,p) & \mapsto \quad \begin{pmatrix} y(0) \\ \dot{y} - (Ay + (Ny+B)u) \\ -\dot{p} - A^*p - uN^*p - C^*Cy \\ \alpha u + \langle Ny + B, p\rangle_Y \end{pmatrix}. \end{cases} \tag{26}$$

This mapping is defined in such a way that for a given y_0, for $(y,u,p) \in W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)$, $\Phi(y,u,p) = (y_0,0,0,0)$ if and only if (y,u,p) satisfies the first-order optimality conditions associated with (P) with initial condition y_0 (in weak form).

From now on, we denote by $B_Y(\delta)$ the closed ball of Y with radius δ and center 0.

Lemma 13. *There exist $\delta_2 > 0$, $\delta_2' > 0$, and three infinitely differentiable mappings*

$$y_0 \in B_Y(\delta_2) \mapsto \big(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0)\big) \in W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)$$

such that for all $y_0 \in B_Y(\delta_2)$, the triplet $\big(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0)\big)$ is the unique solution to

$$\Phi(y,u,p) = (y_0,0,0,0), \quad \max\big(\|y\|_{W_\infty}, \|u\|_{L^2(0,\infty)}, \|p\|_{L^2(0,\infty;V)}\big) \leq \delta_2'$$

in $W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)$. Moreover, there exists a constant $M > 0$ such that for all $y_0 \in B_Y(\delta_2)$,

$$\max\big(\|\mathcal{Y}(y_0)\|_{W_\infty}, \|\mathcal{U}(y_0)\|_{L^2(0,\infty)}, \|\mathcal{P}(y_0)\|_{L^2(0,\infty;V)}\big) \leq M\|y_0\|_Y. \tag{27}$$

Proof. The result is a consequence of the inverse function theorem. The reader can check that Φ is well-defined and infinitely differentiable (note that the derivatives of order 3 and more are null, since Φ contains only linear terms and three bilinear terms: Nyu, $-uN^*p$ and $\langle Ny,p\rangle_Y$). We also have $\Phi(0,0,0) = (0,0,0,0)$. It remains to prove that $D\Phi(0,0,0)$ is an isomorphism. Let us investigate its inverse. Let $(y,u,p) \in W_\infty \times L^2(0,\infty) \times (W_\infty^0)^*$, let $(w_1,w_2,w_3,w_4) \in Y \times L^2(0,\infty;V^*) \times (W_\infty^0)^* \times L^2(0,\infty)$, we have

$$D\Phi(0,0,0)(y,u,p) = (w_1,...,w_4) \Longleftrightarrow \left\{\begin{array}{rcl} y(0) & = & w_1 \\ \dot{y} - Ay - Bu & = & w_2 \\ -\dot{p} - A^*p - C^*Cy & = & w_3 \\ \alpha u + \langle B,p\rangle_Y & = & w_4. \end{array}\right. \tag{28}$$

Denote by $y[w_1]$ the solution y to the following system:

$$\dot{y} = (A+BF)y, \quad y(0) = w_1.$$

By estimate (6), we have $\|y[w_1]\|_{W_\infty} \leq M_s\|w_1\|_Y$. For $u[w_1] = Fy[w_1]$ we obtain

$$\|u[w_1]\|_{L^2(0,\infty)} \leq M\|w_1\|_Y.$$

Let us set $z = y - y[w_1]$. Then the following equivalence holds true:

$$D\Phi(0,0,0)(y,u,p) = (w_1,...,w_4) \Longleftrightarrow \left\{\begin{array}{rcl} z(0) & = & 0 \\ \dot{z} - Az - Bu & = & w_2 + Bu[w_1] \\ -\dot{p} - A^*p - C^*Cz & = & w_3 - C^*Cy[w_1] \\ \alpha u + \langle B,p\rangle_Y & = & w_4. \end{array}\right.$$

We recognize here the optimality conditions associated with a linear-quadratic optimal control problem of the form (LQ). By Proposition 7, the linear system on the right-hand side of the above equivalence has a unique solution (z,u,p), which is the solution to (11) with

$$(f,g,h) = (w_2 + Bu[w_1], w_3 - C^*Cy[w_1], -w_4).$$

Moreover, by Proposition 7, there exists a constant $M > 0$ independent of (f,g,h) such that

$$\begin{aligned} \|(z,u,p)\|_{W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)} &\leq M\|(f,g,h)\|_X \\ &\leq M\|(w_2 + Bu[w_1], w_3 - C^*Cy[w_1], w_4)\|_X \\ &\leq M\|(w_1,w_2,w_3,w_4)\|_{Y\times X}. \end{aligned}$$

Therefore $(y := z + y[w_1], u, p)$ is the unique solution to $D\Phi(0,0,0)(y,u,p) = (w_1,w_2,w_3,w_4)$. In addition

$$\|(y,u,p)\|_{W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)} \leq M\|(w_1,w_2,w_3,w_4)\|_{Y\times X}.$$

This proves that $D\Phi(0,0,0)$ is an isomorphism as well as the existence of $\delta_2 > 0$, $\delta_2' > 0$, and infinitely differentiable mappings $\mathcal{Y}$, $\mathcal{U}$, and $\mathcal{P}$ satisfying the equivalence (28).

It remains to prove (27). Reducing if necessary δ_2, we can assume that the norms of the derivatives of the three mappings are bounded on $B_Y(\delta_2)$ by some constant $M > 0$. The three mappings are therefore Lipschitz continuous with modulus M. Estimate (27) follows, since $\big(\mathcal{Y}(0), \mathcal{U}(0), (\mathcal{P}(0)\big) = (0,0,0)$. □

In the following proposition we prove that for y_0 close enough to 0, $\mathcal{U}(y_0)$ is the unique solution to (P) with initial condition y_0.

Proposition 14. *There exists $\delta_3 \in (0, \delta_2]$ such that for all $y_0 \in B_Y(\delta_3)$, $\mathcal{U}(y_0)$ is the unique solution to (P) with initial condition y_0. Moreover, $\mathcal{Y}(y_0) = S(y_0, \mathcal{U}(y_0))$ and $\mathcal{P}(y_0)$ is the unique associated costate.*

Proof. For the moment, let $\delta_3 = \delta_2$. The value of δ_3 will (possibly) be reduced in the proof. Let $y_0 \in B_Y(\delta_3)$. Our approach consists in proving the existence of a solution $\bar{u}$ to (P), with associated trajectory $\bar{y}$ and costate p. We also show that necessarily,

$$\max\big(\|\bar{y}\|_{W_\infty}, \|\bar{u}\|_{L^2(0,\infty)}, \|p\|_{L^2(0,\infty;V)}\big) \leq \delta_2'.$$

Since then the optimality conditions are satisfied, it holds that $\Phi(\bar{y}, \bar{u}, p) = (y_0, 0, 0, 0)$ and we obtain by Lemma 13 that the solution to (P) is unique and that it is given by $(\bar{y}, \bar{u}, p) = (\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0))$.

Let us start by proving the existence of a solution. By (27), there exists a constant M such that for all $y_0 \in B_Y(\delta_3)$,

$$\|\mathcal{U}(y_0)\|_{L^2(0,\infty)} \leq M\|y_0\|_Y \quad \text{and} \quad \|\mathcal{Y}(y_0)\|_{W_\infty} \leq M\|y_0\|_Y.$$

Therefore,

$$\mathcal{J}(\mathcal{U}(y_0), y_0) \leq M\|y_0\|_Y^2. \tag{29}$$

We reduce now the value of δ_3 so that

$$\sqrt{\frac{2}{\alpha}M}\delta_3 \leq \frac{1}{2}\big(\|N\|_{\mathcal{L}(Y,V^*)}M_0M_d\big)^{-1}.$$

Let us set

$$\delta_0 = \sqrt{\frac{2}{\alpha}M}\|y_0\|_Y.$$

It follows from the two above inequalities that

$$\delta_0 \leq \sqrt{\frac{2}{\alpha}M}\delta_3 \leq \frac{1}{2}\big(\|N\|_{\mathcal{L}(Y,V^*)}M_0M_d\big)^{-1}.$$

Moreover, by (29),

$$\mathcal{J}(\mathcal{U}(y_0), y_0) \leq M\|y_0\|_Y^2 = M\Big(\sqrt{\frac{\alpha}{2M}}\delta_0\Big)^2 = \frac{\alpha}{2}\delta_0^2.$$

The conditions of Lemma 10 are satisfied. Therefore (P) has a solution $\bar{u}$, which satisfies

$$\|\bar{u}\|_{L^2(0,\infty)} \leq \delta_0 \quad \text{and} \quad \|\bar{y}\|_{W_\infty} \leq M(\|y_0\|_Y + \delta_0),$$

where $\bar{y} = S(\bar{u}, y_0)$. Using the definition of δ_0, we obtain the existence of a constant $M > 0$ such that

$$\|\bar{u}\|_{L^2(0,\infty)} \leq M\|y_0\|_Y \quad \text{and} \quad \|\bar{y}\|_{W_\infty} \leq M\|y_0\|_Y. \tag{30}$$

Let us prove now that the optimality conditions are satisfied. Reducing if necessary the value of δ_3, we obtain that $\|\bar{u}\|_{L^2(0,\infty)} \leq \delta_1$ and that $\|\bar{y}\|_{L^2(0,\infty;Y)} \leq \delta_1$ (where $\delta_1 > 0$ is given by Lemma 11). Therefore there exists $p \in L^2(0,\infty;V)$ such that the costate equation (23) and relation (24) hold. Moreover, we obtain

$$\|p\|_{L^2(0,\infty;V)} \leq M(\|\bar{y}\|_{L^2(0,\infty;Y)} + \|\bar{u}\|_{L^2(0,\infty)}) \leq M\|y_0\|_Y. \tag{31}$$

It follows from (30) and (31) that we can reduce for the last time, if necessary, the value of δ_3 so that

$$\max\big(\|\bar{u}\|_{L^2(0,\infty)}, \|\bar{y}\|_{L^2(0,\infty;Y)}, \|p\|_{L^2(0,\infty;V)}\big) \leq \delta_2'.$$

Since $\Phi(\bar{y},\bar{u},p) = (y_0,0,0,0)$, we finally obtain by Lemma 13 that $(\bar{y},\bar{u},p) = (\mathcal{Y}(y_0),\mathcal{U}(y_0),\mathcal{P}(y_0))$. The lemma is proved. □

Corollary 15. *The value function $\mathcal{V}$ is infinitely differentiable on $B_Y(\delta_3)$.*

Proof. The following mapping

$$(y,u) \in W_\infty \times L^2(0,\infty) \mapsto \frac{1}{2}\|Cy\|^2_{L^2(0,\infty;Z)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,\infty)}$$

is clearly infinitely differentiable. By Proposition 14, we find for all $y_0 \in B_Y(\delta_3)$ that

$$\mathcal{V}(y_0) = \frac{1}{2}\|C\mathcal{Y}(y_0)\|^2_{L^2(0,\infty;Z)} + \frac{\alpha}{2}\|\mathcal{U}(y_0)\|^2_{L^2(0,\infty)},$$

with infinitely differentiable mappings $\mathcal{Y}$ and $\mathcal{U}$. The corollary follows, since $\mathcal{V}$ can be expressed as the composition of infinitely differentiable mappings. □

Consider now the mapping $\tilde{\Phi}$ defined as follows:

$$\tilde{\Phi}\colon \begin{cases} W_\infty \times L^2(0,\infty) \times W_\infty & \rightarrow \quad Y \times L^2(0,\infty;V^*) \times L^2(0,\infty;V^*) \times L^2(0,\infty) \\ (y,u,p) & \mapsto \quad \begin{pmatrix} y(0) \\ \dot{y} - (Ay + (Ny+B)u) \\ -\dot{p} - A^*p - uN^*p - C^*Cy \\ \alpha u + \langle Ny+B, p\rangle_Y \end{pmatrix}. \end{cases}$$

The action of $\tilde{\Phi}$ is the same as Φ, but for different choices of spaces for the domain of the adjoint variable p and for the costate equation in the image of $\tilde{\Phi}$. We have already mentioned in Remark 12 the impossibility to prove in a direct way the fact that the adjoint lies in W_∞. Remarkably, the mapping $\tilde{\Phi}$ is well-defined and the well-posedness of the nonlinear equation $\Phi(y,u,p) = (y_0,0,0,0)$ can be easily established.

Lemma 16. *There exist $\delta_4 > 0$, $\delta_4' > 0$, and three infinitely differentiable mappings*

$$y_0 \in B_Y(\delta_4) \mapsto \big(\tilde{\mathcal{Y}}(y_0),\tilde{\mathcal{U}}(y_0),\tilde{\mathcal{P}}(y_0)\big) \in W_\infty \times L^2(0,\infty) \times W_\infty$$

such that for all $y_0 \in B_Y(\delta_4)$, the triplet $\big(\mathcal{Y}(y_0),\mathcal{U}(y_0),\mathcal{P}(y_0)\big)$ is the unique solution to

$$\Phi(y,u,p) = (y_0,0,0,0), \quad \max\big(\|y\|_{W_\infty}, \|u\|_{L^2(0,\infty)}, \|p\|_{W_\infty}\big) \leq \delta_4'$$

in $W_\infty \times L^2(0,\infty) \times W_\infty$.

Proof. The proof is the same as the proof of Lemma 13. The reader can check that $\tilde{\Phi}$ is well-defined and infinitely differentiable. For proving that $D\tilde{\Phi}(0,0,0)$ is an isomorphism, one has to rely on estimate (13) of Proposition 7. □

We can now prove that the mappings $(\mathcal{Y},\mathcal{U},\mathcal{P})$ and $(\tilde{\mathcal{Y}},\tilde{\mathcal{U}},\tilde{\mathcal{P}})$ coincide for y_0 close enough to 0.

Proposition 17. *There exists $\delta_5 \in (0,\min(\delta_2,\delta_4))$ such that for all $y_0 \in B_Y(\delta_5)$,*

$$\big(\mathcal{Y}(y_0),\mathcal{U}(y_0),\mathcal{P}(y_0)\big) = \big(\tilde{\mathcal{Y}}(y_0),\tilde{\mathcal{U}}(y_0),\tilde{\mathcal{P}}(y_0)\big). \tag{32}$$

Proof. By continuity of the mappings $\tilde{\mathcal{Y}}$, $\tilde{\mathcal{U}}$, and $\tilde{\mathcal{P}}$, there exists $\delta_5 \in (0,\min(\delta_2,\delta_4))$ such that for all $y_0 \in B_Y(\delta_5)$,

$$\max\big(\|\tilde{\mathcal{Y}}(y_0)\|_{W_\infty}, \|\tilde{\mathcal{U}}(y_0)\|_{L^2(0,\infty)}, \|\tilde{\mathcal{P}}(y_0)\|_{L^2(0,\infty;V)}\big) \leq \delta_2'. \tag{33}$$

By construction of $\big(\tilde{\mathcal{Y}}(y_0),\tilde{\mathcal{U}}(y_0),\tilde{\mathcal{P}}(y_0)\big)$,

$$\tilde{\Phi}\big(\tilde{\mathcal{Y}}(y_0),\tilde{\mathcal{U}}(y_0),\tilde{\mathcal{P}}(y_0)\big) = (y_0,0,0,0).$$

Therefore

$$\Phi\big(\tilde{\mathcal{Y}}(y_0),\tilde{\mathcal{U}}(y_0),\tilde{\mathcal{P}}(y_0)\big) = (y_0,0,0,0).$$

Combined with (33), we obtain (32) by Lemma 13. □

As a direct consequence of this result, (P) has a unique solution for all $y_0 \in B_Y(\min(\delta_3, \delta_5))$. Moreover, the optimality conditions are satisfied with a costate in W_∞.

5 Optimality conditions for an arbitrary initial condition

In this section, we first prove a sensitivity relation: locally, the costate and the derivative of the value function coincide. This enables us to prove optimality conditions in strong form for (P) for arbitrary initial conditions.

Lemma 18. *There exists $\delta_6 \in (0, \min(\delta_3, \delta_5)]$ such that for all $y_0 \in B_Y(\delta_6)$,*

$$\|\mathcal{Y}(y_0)\|_{L^\infty(0,\infty;Y)} \leq \min(\delta_3, \delta_5) \tag{34}$$

and

$$p(t) = D\mathcal{V}(y(t)), \quad \forall t \geq 0, \tag{35}$$

where $y = \mathcal{Y}(y_0)$ and $p = \mathcal{P}(y_0)$.

Proof. By continuity of the mapping $\mathcal{Y}$, there exists $\delta_6 \in (0, \min(\delta_3, \delta_5)]$ such that for all $y_0 \in B_Y(\delta_6)$, inequality (34) holds.

We now claim the following: for all $y_0 \in B_Y(\delta_6)$, we have $p(0) = D\mathcal{V}(y_0)$, where $p = \mathcal{P}(y_0)$. To verify this this claim, let y_0 and $\tilde{y}_0 \in B_Y(\delta_6)$, and set $(y, u, p) = (\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0))$ and $(\tilde{y}, \tilde{u}) = (\mathcal{Y}(\tilde{y}_0), \mathcal{U}(\tilde{y}_0))$. We have

$$\begin{aligned}\mathcal{V}(\tilde{y}_0) - \mathcal{V}(y_0) = {} & \Big(\frac{1}{2}\|C\tilde{y}\|^2_{L^2(0,\infty;Z)} + \frac{\alpha}{2}\|\tilde{u}\|^2_{L^2(0,\infty)}\Big) - \Big(\frac{1}{2}\|Cy\|^2_{L^2(0,\infty;Z)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,\infty)}\Big) \\ & - \big\langle p, \dot{\tilde{y}} - (A\tilde{y} + (N\tilde{y} + B)\tilde{u})\big\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} \\ & + \big\langle p, \dot{y} - (Ay + (Ny + B)u)\big\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)}. \end{aligned} \tag{36}$$

Indeed, u and $\tilde{u}$ are optimal and the last two terms (in brackets) are null. The four following relations can be easily verified:

$$\begin{aligned}\frac{1}{2}\|C\tilde{y}\|^2_{L^2(0,\infty;Z)} - \frac{1}{2}\|Cy\|^2_{L^2(0,\infty;Z)} &= \langle C^*Cy, \tilde{y} - y\rangle_{L^2(0,\infty;Y)} + \frac{1}{2}\|C(\tilde{y} - y)\|^2_{L^2(0,\infty;Z)}, \\ \frac{\alpha}{2}\|\tilde{u}\|^2_{L^2(0,\infty)} - \frac{\alpha}{2}\|u\|^2_{L^2(0,\infty)} &= \alpha\langle u, \tilde{u} - u\rangle_{L^2(0,\infty)} + \frac{\alpha}{2}\|\tilde{u} - u\|^2_{L^2(0,\infty)}, \\ N\tilde{y}\tilde{u} - Nyu &= Ny(\tilde{u} - u) + N(\tilde{y} - y)u + N(\tilde{y} - y)(\tilde{u} - u), \\ -\langle p, \dot{\tilde{y}} - \dot{y}\rangle_{L^2(V),L^2(V^*)} &= \langle p(0), \tilde{y}_0 - y(0)\rangle_Y + \langle \dot{p}, \tilde{y} - y\rangle_{L^2(V^*),L^2(V)}. \end{aligned} \tag{37}$$

Combining (36) and (37) yields

$$\begin{aligned}\mathcal{V}(\tilde{y}_0) - \mathcal{V}(y_0) = {} & \langle p(0), \tilde{y}(0) - y(0)\rangle_Y + \frac{1}{2}\|C(\tilde{y} - y)\|^2_{L^2(0,\infty;Y)} + \frac{\alpha}{2}\|\tilde{u} - u\|^2_{L^2(0,\infty)} \\ & + \big\langle p, N(\tilde{y} - y)(\tilde{u} - u)\big\rangle_{L^2(0,\infty;V);L^2(0,\infty;V^*)} \\ & + \big\langle \underbrace{\dot{p} + A^*p + uN^*p + C^*Cy}_{=0}, \tilde{y} - y\big\rangle_{L^2(0,\infty;V^*);L^2(0,\infty;V)} \\ & + \big\langle \underbrace{\alpha u + \langle Ny + B, p\rangle_Y}_{=0}, \tilde{u} - u\big\rangle_{L^2(0,\infty)}. \end{aligned}$$

For $\tilde{y}_0 = y_0 + h$, we have $\|\tilde{y} - y\|_{W_\infty} \leq M\|h\|_Y$ and $\|\tilde{u} - u\|_{L^2(0,\infty)} \leq M\|h\|_Y$, by the Lipschitz-continuity of the mappings $\mathcal{Y}$ and $\mathcal{U}$. It follows that the three quadratic terms in the above relation are of order $\|h\|_Y^2$ and thus that

$$\begin{aligned} & |\mathcal{V}(\tilde{y}_0) - \mathcal{V}(y_0) - \langle p(0), \tilde{y}_0 - y_0\rangle_Y| \\ & \quad = \Big|\frac{1}{2}\|C(\tilde{y} - y)\|^2_{L^2(0,\infty;Y)} + \frac{\alpha}{2}\|\tilde{u} - u\|^2_{L^2(0,\infty)} + \big\langle p, N(\tilde{y} - y)(\tilde{u} - u)\big\rangle_{L^2(0,\infty;V);L^2(0,\infty;V^*)}\Big| \\ & \quad \leq M\|h\|_Y^2. \end{aligned}$$

This proves that $D\mathcal{V}(y_0) = p(0)$, as announced.

To verify (35) for arbitrary $t \geq 0$, let $y_0 \in B_Y(\delta_6)$, set $(y, u, p) = (\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0))$ and choose $t \geq 0$. We define

$$\tilde{y}\colon s \geq 0 \mapsto y(t+s), \quad \tilde{u}\colon s \geq 0 \mapsto u(t+s), \quad \tilde{p}\colon s \geq 0 \mapsto p(t+s).$$

By the dynamic programming principle, $\tilde{u}$ is the solution to (P) with initial condition $\tilde{y}(0) = y(t)$. The associated trajectory and costate are $\tilde{y}$ and $\tilde{p}$. Since $\|y(t)\|_Y \leq \min(\delta_3, \delta_5)$, we can apply the previous claim. We obtain that $D\mathcal{V}(\tilde{y}(0)) = \tilde{p}(0)$ and finally that $D\mathcal{V}(y(t)) = p(t)$. □

Using the optimality condition (24), we directly obtain the following corollary, which states that the mapping $y \in Y \mapsto -\frac{1}{\alpha} D\mathcal{V}(y)(Ny + B)$ is an optimal feedback law.

Corollary 19. *For all $y_0 \in B_Y(\delta_6)$,*

$$u(t) = -\frac{1}{\alpha} D\mathcal{V}(y(t))(Ny(t) + B), \quad \text{for a.e. } t > 0,$$

where $y = \mathcal{Y}(y_0)$ and $u = \mathcal{U}(y_0)$.

We can now prove the optimality conditions for any initial condition (assuming the existence of a solution). Roughly speaking, the proof consists in showing that the optimality conditions are satisfied for (T_1, ∞), with T_1 sufficiently large, using a dynamic programming principle argument. Optimality conditions for the whole interval $(0, \infty)$ can then be obtained easily, using once again a dynamic programming principle argument and Lemma 18.

Theorem 20. *Let $y_0 \in Y$ and assume that there exists a solution $\bar{u}$ to (P) with initial condition y_0. Then the associated trajectory $\bar{y} = S(\bar{u}, y_0)$ lies in W_∞. Moreover, there exists a costate $p \in W_\infty$ such that for a.e. $t \geq 0$,*

$$-\dot{p} = A^* p + \bar{u} N^* p + C^* C \bar{y} \tag{38}$$

and

$$\alpha \bar{u} + \langle N\bar{y} + B, p \rangle_Y = 0. \tag{39}$$

Proof. Let $\delta_0 = \frac{1}{2} \big(\|N\|_{\mathcal{L}(Y,V^*)} M_0 M_d \big)^{-1}$ and let $T_0 > 0$ be sufficiently large, so that

$$\frac{1}{2} \int_{T_0}^\infty \|C\bar{y}(t)\|_Z^2 \, \mathrm{d}t + \frac{\alpha}{2} \int_{T_0}^\infty \bar{u}(t)^2 \, \mathrm{d}t \leq \frac{\alpha}{2} \delta_0^2.$$

We define $\tilde{u}\colon t \geq 0 \mapsto \bar{u}(T_0 + t)$ and $\tilde{y}\colon t \geq 0 \mapsto \bar{y}(T_0 + t)$. By the dynamic programming principle, $\tilde{u}$ is a solution to (P) with initial condition $\tilde{y}(0) = y(T_0)$, and associated trajectory $\tilde{y}$. Since $\mathcal{J}(\tilde{y}_0, \tilde{u}) \leq \frac{\alpha}{2} \delta_0^2$, we obtain by Lemma 10 that $\tilde{y} \in W_\infty$, thus $\bar{y} \in W_\infty$. As a consequence, $\lim_{t \to \infty} \|\bar{y}(t)\|_Y = 0$ and there exists $T_1 \geq 0$ such that $\|\bar{y}(T_1)\|_Y \leq \delta_6$.

Let $\hat{u}\colon t \geq 0 \mapsto \bar{u}(T_1 + t)$ and $\hat{y}\colon t \geq 0 \mapsto \bar{y}(T_1 + t)$. Again by the dynamic programming principle, $\hat{u}$ is a solution to (P) with initial condition $\hat{y}(0) = \bar{y}(T_1)$ and associated trajectory $\hat{y}$. Since $\|\hat{y}(0)\| \leq \delta_3$, Proposition 14 implies that

$$\hat{y} = \mathcal{Y}\big(\bar{y}(T_1)\big) \quad \text{and} \quad \hat{u} = \mathcal{U}\big(\bar{y}(T_1)\big)$$

Moreover, by Proposition 17, the associated costate $\hat{p} = \mathcal{P}\big(\bar{y}(T_1)\big)$ lies in W_∞.

Let us now define $p \in W_\infty(T_1, \infty)$ by $p(t) = \hat{p}(t - T_1)$, for all $t \in [T_1, \infty)$. Clearly the costate equation (38) and relation (39) hold true for $t \geq T_1$. Uniqueness of p on $[T_1, \infty)$ directly follows from the uniqueness of the costate associated with the optimal control $\hat{u}$.

Let us construct p on $[0, T_1]$. Observe first that by Lemma 18, we have $\hat{p}(0) = D\mathcal{V}(\hat{y}(0))$, and thus

$$p(T_1) = D\mathcal{V}(\hat{y}(0)) = D\mathcal{V}(\bar{y}(T_1)). \tag{40}$$

Let the extension of p on $[0, T_1]$ be the unique solution to the following system:

$$-\dot{p} = A^* p + \bar{u} N^* p + C^* C \bar{y}, \quad p(T_1) = D\mathcal{V}(\bar{y}(T_1)). \tag{41}$$

Existence and the uniqueness of the solution to this system in $W(0, T_1)$ can be obtained with the same methods as those used for Lemma 1. The terminal condition in the above system is compatible with (40). Therefore p satisfies the costate equation (38) on the whole interval $(0,\infty)$ and $p \in W_\infty$.

It remains to prove that (39) is satisfied on $(0,T_1)$. We only sketch the proof, which is classical. Observe first that by the dynamic programming principle, the control $\bar{u}_{|(0,T_1)}$ is a solution to the following problem:

$$\min_{u\in L^2(0,T_1)} J_{T_1}(u) := \frac{1}{2}\int_0^{T_1} \|CS(y_0,u;t)\|_Z^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^{T_1} u(t)^2 \,\mathrm{d}t + \mathcal{V}(y(T_1)).$$

Note also that (41) is the associated costate equation. It can be easily established that the control-to-state mapping $u \in L^2(0,T_1) \mapsto S(y_0,u)_{|(0,T_1)}$ is continuously differentiable and that its derivative can be described as the linearization of the state equation. It follows that $J_{T_1}(\cdot)$ is differentiable. A well-known computation (relying on an integration by parts) yields that

$$DJ_{T_1}(\bar{u})v = \int_0^{T_1} \big(\alpha\bar{u} + \langle N\bar{y}(t) + B, p(t)\rangle_Y\big)v(t)\,\mathrm{d}t, \quad \forall v \in L^2(0,T_1).$$

Since $\bar{u}$ is optimal, $DJ_{T_1}(\bar{u}) = 0$ and (39) follows. The theorem is proved. □

6 Construction and properties of polynomial feedback laws

We recall in this section the relevant definitions and main results obtained in [9] for polynomial feedback laws. These are described by bounded multilinear forms.

For $k \geq 1$ we make use of the following norm:

$$\|(y_1,...,y_k)\|_{Y^k} = \max_{i=1,\dots,k} \|y_i\|_Y. \tag{42}$$

We denote by $B_{Y^k}(\delta)$ the closed ball in Y^k with radius δ and center 0. For $k \geq 1$ we say that $\mathcal{T}\colon Y^k \to \mathbb{R}$ is a bounded multilinear form if $\mathcal{T}$ is linear in each variable separately and

$$\|\mathcal{T}\| := \sup_{y\in B_{Y^k}(1)} |\mathcal{T}(y)| < \infty. \tag{43}$$

We denote by $\mathcal{M}(Y^k,\mathbb{R})$ the set of bounded multilinear forms. For all $\mathcal{T} \in \mathcal{M}(Y^k,\mathbb{R})$ and for all $(z_1,...,z_k) \in Y^k$,

$$|\mathcal{T}(z_1,...,z_k)| \leq \|\mathcal{T}\| \prod_{i=1}^k \|z_i\|_Y. \tag{44}$$

Bounded multilinear forms $\mathcal{T} \in \mathcal{M}(Y^k,\mathbb{R})$ are called symmetric if for all z_1,...,$z_k \in Y^k$ and for all permutations σ of $\{1,...,k\}$,

$$\mathcal{T}(z_{\sigma(1)},...,z_{\sigma(k)}) = \mathcal{T}(z_1,...,z_k).$$

Given two multilinear forms $\mathcal{T}_1 \in \mathcal{M}(Y^k,\mathbb{R})$ and $\mathcal{T}_2 \in \mathcal{M}(Y^\ell,R)$, we denote by $\mathcal{T}_1\otimes\mathcal{T}_2$ the bounded multilinear mapping which is defined for all $(y_1,...,y_{k+\ell}) \in Y^{k+\ell}$ by

$$(\mathcal{T}_1 \otimes \mathcal{T}_2)(y_1,\dots,y_{k+\ell}) = \mathcal{T}_1(y_1,...,y_k)\mathcal{T}_2(y_{k+1},...,y_{k+\ell}).$$

For $y \in Y$, we denote

$$y^{\otimes k} = (y,...,y) \in Y^k.$$

6.1 Taylor approximation

For all $k \geq 2$, we construct a polynomial approximation $\mathcal{V}_k$ of $\mathcal{V}$ of the following form:

$$\mathcal{V}_k\colon Y \to \mathbb{R}, \quad \mathcal{V}_k(y) = \sum_{j=2}^k \frac{1}{j!}\mathcal{T}_j(y,\dots,y), \tag{45}$$

where $\mathcal{T}_2$,...,$\mathcal{T}_j$,...,$\mathcal{T}_k$ are bounded multilinear forms of order 2,...,j,..., k. The first multilinear form, the bilinear form $\mathcal{T}_2$, is obtained as the solution to an algebraic operator Riccati equation and the other multilinear forms are obtained as the solutions to linear operator equations which we call generalized Lyapunov equations.

Let us denote by $\Pi \in \mathcal{L}(Y)$ the unique nonnegative self-adjoint operator satisfying the following algebraic operator Riccati equation:

$$\langle A^*\Pi z_1, z_2\rangle + \langle \Pi A z_1, z_2\rangle + \langle Cz_1, Cz_2\rangle_Z - \frac{1}{\alpha}\langle B, \Pi z_1\rangle_Y \langle B, \Pi z_2\rangle_Y = 0, \quad \text{for all } z_1, z_2 \in \mathcal{D}(A). \tag{46}$$

It is well-known, see [13, Theorem 6.2.7] that, as a consequence of assumptions (A3) and (A4), the linearized closed-loop operator

$$A_\Pi := A - \frac{1}{\alpha} BB^*\Pi \tag{47}$$

generates an exponentially stable semigroup on Y.

The precise structure of the generalized Lyapunov equations is given in Theorem 21 below. In the definition of the right-hand sides of these equations, we make use of a specific symmetrization technique that we define now. For i and $j \in \mathbb{N}$, consider the following set of permutations:

$$S_{i,j} = \big\{\sigma_{i+j} \in S_{i+j} \,|\, \sigma(1) < ... < \sigma(i) \text{ and } \sigma(i+1) < ... < \sigma(i+j)\big\},$$

where S_{i+j} is the set of permutations of $\{1, ..., i+j\}$. Let $\mathcal{T}$ be a multilinear form of order $i+j$. We denote by $\text{Sym}_{i,j}(\mathcal{T})$ the multilinear form defined by

$$\text{Sym}_{i,j}(\mathcal{T})(z_1, ..., z_{i+j}) = \binom{i+j}{i}^{-1} \Big[\sum_{\sigma \in S_{i,j}} \mathcal{T}(z_{\sigma(1)}, ..., z_{\sigma(i+j)})\Big], \quad \forall (z_1, ..., z_{i+1}) \in Y^{i+j}.$$

Theorem 21 (Theorem 16, [9])**.** *There exists a unique sequence of bounded symmetric multilinear forms $(\mathcal{T}_j)_{j\geq 2,}$ with $\mathcal{T}_j : Y^j \to \mathbb{R}$, and a unique sequence of bounded multilinear forms $(\mathcal{R}_j)_{j\geq 3}$ with $\mathcal{R}_j : \mathcal{D}(A)^j \to \mathbb{R}$ such that for all $(z_1, z_2) \in Y^2$,*

$$\mathcal{T}_2(z_1, z_2) := (z_1, \Pi z_2) \tag{48}$$

and such that for all $j \geq 3$, for all $(z_1, ..., z_j) \in \mathcal{D}(A)^j$,

$$\sum_{i=1}^{j} \mathcal{T}_k(z_1, ..., z_{i-1}, A_\Pi z_i, z_{i+1}, ..., z_j) = \frac{1}{2\alpha}\mathcal{R}_j(z_1, ..., z_j), \tag{49a}$$

where

$$\begin{aligned} \mathcal{R}_j = {} & 2j(j-1)\text{Sym}_{1,j-1}\big(\mathcal{C}_1 \otimes \mathcal{G}_{j-1}\big) \\ & + \sum_{i=2}^{j-2} \binom{k}{i} \text{Sym}_{i,j-i}\big((\mathcal{C}_i + i\mathcal{G}_i) \otimes (\mathcal{C}_{j-i} + (j-i)\mathcal{G}_{j-i})\big), \end{aligned} \tag{49b}$$

and where

$$\begin{cases} \mathcal{C}_i(z_1, ..., z_i) = & \mathcal{T}_{i+1}(B, z_1, ..., z_i), \quad \text{for } i = 1, ..., j-2, \\ \mathcal{G}_i(z_1, ..., z_i) = & \frac{1}{i}\Big[\sum_{\ell=1}^{i} \mathcal{T}_i(z_1, ..., z_{\ell-1}, Nz_\ell, z_{\ell+1}, ..., z_i)\Big], \quad \text{for } i = 1, ..., j-1. \end{cases} \tag{49c}$$

6.2 Feedback laws and associated closed-loop systems

A polynomial feedback law $\mathbf{u}_k : y \in V \to \mathbb{R}$ can now be obtained by replacing the value function $\mathcal{V}$ by its approximation $\mathcal{V}_k$ in the optimal feedback law given by Corollary 19:

$$\begin{aligned} \mathbf{u}_k(y) = & -\frac{1}{\alpha} D\mathcal{V}_k(y)(Ny + B) \\ = & -\frac{1}{\alpha}\Big(\sum_{i=2}^{k} \frac{1}{(i-1)!}\mathcal{T}_i(Ny + B, y, \dots, y)\Big). \end{aligned} \tag{50}$$

A justification of the differentiability of $\mathcal{V}_k$ and a formula for its derivative, used in the above expression, can be found in [9, Lemma 7]. We consider now the closed-loop system associated with the feedback law $\mathbf{u}_k$:

$$\dot{y}(t) = Ay(t) + (Ny(t) + B)\mathbf{u}_k(y(t)), \quad y(0) = y_0. \tag{51}$$

For a given initial condition y_0, its solution is denoted by $S(\mathbf{u}_k, y_0)$. We also denote by $\mathbf{U}_k(y_0)$ the open-loop control defined by

$$\mathbf{U}_k(y_0; t) = \mathbf{u}_k(S(\mathbf{u}_k, y_0; t)), \quad \text{for a.e. } t > 0. \tag{52}$$

The following theorem states that for $\|y_0\|_Y$ small enough, the closed-loop system (51) has a unique solution and generates an open-loop control in $L^2(0, \infty)$.

Theorem 22 (Theorem 22 and Corollary 23, [9])**.** *For all $k \geq 2$, there exist two constants $\delta_7 > 0$ and $M > 0$ such that for all $y_0 \in B_Y(\delta_7)$, the closed-loop system* (51) *admits a unique solution $S(\mathbf{u}_k, y_0) \in W_\infty$ satisfying*

$$\|S(\mathbf{u}_k, y_0)\|_{W_\infty} \leq M\|y_0\|_Y. \tag{53}$$

Moreover, $\mathbf{U}_k(y_0) \in L^2(0, \infty)$ and the two mappings $y_0 \in B_Y(\delta_7) \mapsto S(\mathbf{u}_k, y_0)$ and $y_0 \in B_Y(\delta_7) \mapsto \mathbf{U}_k(y_0)$ are Lipschitz-continuous.

6.3 Error analysis

We finally recall some of the key lemmas used in the error analysis of [9], since they will be useful for the extension provided in the next section.

The main idea consists in defining a perturbed cost function $\mathcal{J}_k$ which has the property that $\mathcal{V}_k$ is its value function. This is achieved by constructing a remainder term r_k, defined for $k \geq 2$ and $y \in V$ by

$$r_k(y) = \frac{1}{2\alpha} \sum_{i=k+1}^{2k} \sum_{j=i-k}^{k} q_{k,j}(y) q_{k,i-j}(y), \tag{54}$$

where the mappings $q_{k,1}$, $q_{k,2}$,..., and $q_{k,k}$ are given by

$$\begin{cases} q_{k,1}(y) = \mathcal{C}_1(y), \\ q_{k,i}(y) = \frac{1}{i!}\big(\mathcal{C}_i(y^{\otimes i}) + i\mathcal{G}_i(y^{\otimes i})\big), & \forall i = 2, ..., k-1, \\ q_{k,k}(y) = \frac{1}{(k-1)!}\mathcal{G}_k(y^{\otimes k}). \end{cases}$$

We recall that the definitions of $\mathcal{C}_i$ and $\mathcal{G}_i$ are given by (49c). Note also that $r_k \colon V \to \mathbb{R}$ is infinitely differentiable. The perturbed cost function $\mathcal{J}_k$ is defined by

$$\mathcal{J}_k(u, y_0) := \frac{1}{2} \int_0^\infty \|CS(u, y_0; t)\|_Y^2 \,\mathrm{d}t + \frac{\alpha}{2} \int_0^\infty u^2(t) \,\mathrm{d}t + \int_0^\infty r_k\big(S(u, y_0; t)\big) \,\mathrm{d}t.$$

The well-posedness of $\mathcal{J}_k$ is guaranteed if $S(y_0, u) \in W_\infty$, see [9, Proposition 26]. We point out that r_k is not necessarily non-negative.

The next lemma states that $\mathcal{V}_k$ is the value function associated with the problem of minimization of $\mathcal{J}_k$ over controls which guarantee trajectories in W_∞. Moreover, the control $\mathbf{U}_k(y_0)$ given by (50) and (52) is a solution to the problem. Let us emphasize the fact that the result is stated for an initial condition in $B_Y(\delta_7) \cap V$.

Lemma 23 (Lemma 29, [9])**.** *Let $k \geq 2$ and $y_0 \in B_Y(\delta_7) \cap V$. Then $\mathcal{J}_k(u, y_0)$ and $\mathcal{J}_k(\mathbf{U}_k(y_0), y_0)$ are finite and*

$$\mathcal{V}_k(y_0) = \mathcal{J}_k(\mathbf{U}_k(y_0), y_0) \leq \mathcal{J}_k(u, y_0),$$

for all $u \in L^2(0, \infty)$ with $S(u, y_0) \in W_\infty$.

The loss of optimality when using $\mathbf{U}_k(y_0)$ is estimated in Theorem 25 below. The proof relies on Lemma 23 and on the two estimates given in the next lemma.

Lemma 24 (Lemma 28, [9])**.** *Let $k \geq 2$. There exists a constant $M > 0$ such that for all $y_0 \in B_Y(\delta_8)$,*

$$\int_0^\infty r_k\big(\bar{y}(t)\big)\, dt \leq M\|y_0\|_Y^{k+1} \quad \text{and} \quad \int_0^\infty r_k\big(S(\mathbf{u}_k, y_0; t)\big)\, dt \leq M\|y_0\|_Y^{k+1},$$

where $\bar{y}$ is an optimal trajectory for problem (P) with initial value y_0.

Finally, the following theorem asserts that $\mathcal{V}_k$ is an approximation of $\mathcal{V}$ of order $k+1$ in the neighbourhood of 0 and gives an error estimate on the efficiency of the open-loop control generated by $\mathbf{u}_k$.

Theorem 25 (Proposition 2, Theorem 30, and Theorem 32, [9])**.** *Let $k \geq 2$. There exist $\delta_8 \in (0, \delta_7]$ and a constant $M > 0$ such that for all $y_0 \in B_Y(\delta_8)$, the following estimates hold:*

$$\begin{aligned} \mathcal{J}(\mathbf{U}_k(y_0), y_0) &\leq \mathcal{V}(y_0) + M\|y_0\|_Y^{k+1}, \\ |\mathcal{V}(y_0) - \mathcal{V}_k(y_0)| &\leq M\|y_0\|_Y^{k+1}. \end{aligned}$$

In addition, for all $y_0 \in B_Y(\delta_8)$, Problem (P) with initial condition y_0 possesses a solution $\bar{u}$ satisfying

$$\begin{aligned} \|\bar{u} - \mathbf{U}_k(y_0)\|_{L^2(0,\infty)} &\leq M\|y_0\|_Y^{(k+1)/2} \\ \|S(\bar{u}, y_0) - S(\mathbf{u}_k, y_0)\|_{W_\infty} &\leq M\|y_0\|_Y^{(k+1)/2}. \end{aligned}$$

We finish this section with an observation of the multilinear forms $\mathcal{T}_k$. The analysis of [9] performed for obtaining the results presented in this section does not rely on the C^∞-regularity of the value function. It was therefore not clear that the multilinear forms $\mathcal{T}_2$, $\mathcal{T}_3$,... are the derivatives of $\mathcal{V}$ of order 2, 3,... evaluated at 0. This relation can now be established.

Theorem 26. *For all $k \geq 2$, $\mathcal{T}_k = D^k\mathcal{V}(0)$.*

Proof. The proof is based on the following result (referred to as polarization identity), proved in [29, Theorem 1]: for all symmetric multilinear forms $\mathcal{T} \in \mathcal{M}(Y^k, \mathbb{R})$, for all $y = (y_1, ..., y_k) \in Y^k$,

$$\mathcal{T}(y_1, ..., y_k) = \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 ... \partial\lambda_k} f[y](0),$$

where the function $f[y]$ is a polynomial function defined by

$$f[y] \colon \lambda \in \mathbb{R}^k \mapsto \mathcal{T}\Big(\Big(\sum_{i=1}^k \lambda_i y_i\Big)^{\otimes k}\Big).$$

As a direct corollary, we obtain that if two symmetric multilinear forms coincide on the set of diagonal terms $\{y^{\otimes k} \,|\, y \in Y^k\}$, they are equal.

Let us come back to the proof of the lemma. Let $k \geq 2$ and let $y \in Y$. By Theorem 25, we have the following Taylor expansion (with respect to $\theta \in \mathbb{R}$):

$$\mathcal{V}(\theta y) = \sum_{j=2}^k \frac{\theta^j}{j!} \mathcal{T}_j(y^{\otimes j}) + o(|\theta|^{k+1}).$$

We have proved in Corollary 15 that $\mathcal{V}$ is C^∞, therefore,

$$\mathcal{V}(\theta y) = \sum_{j=2}^k \frac{\theta^j}{j!} D^j\mathcal{V}(0)(y^{\otimes j}) + o(|\theta|^{k+1}).$$

By the uniqueness of the Taylor expansion of functions of real variables, we deduce that for all $y \in Y$, $\mathcal{T}_k(y^{\otimes k}) = D^k\mathcal{V}(0)(y^{\otimes k})$. Since $\mathcal{T}_k$ and $D^k\mathcal{V}(0)$ are both symmetric and coincide on the set of diagonal terms, they are equal, which concludes the proof. □

7 Error analysis: new estimates

In this section we improve the estimates obtained in Theorem 25. The approach consists of two main steps. First we use the fact that the control $\mathbf{U}_k(y_0)$ is the solution to an optimal control problem with a specific perturbation. The corresponding optimality conditions lead to a perturbed adjoint equation, see Lemma 29. In a second step, we analyze the influence of the perturbation of the optimality conditions.

We consider the perturbation term in the definition of $\mathcal{J}_k$ and define

$$R_k \colon y \in W_\infty \mapsto \int_0^\infty r_k(y(t))\, \mathrm{d}t \in \mathbb{R}.$$

In the following lemma, we prove the differentiability of R_k and give an estimate of the norm of its derivative. The derivative of R_k will appear as an additional term in the perturbed costate equation.

Lemma 27. *The mapping R_k is continuously differentiable. Its derivative is given by*

$$DR_k(y)z = \int_0^\infty Dr_k(y(t))z(t)\, dt,$$

for all y and $z \in W_\infty$. Moreover, for all $\delta > 0$, there exists a constant M such that

$$|DR_k(y)z| \leq M\|y\|_{W_\infty}^k \|z\|_{W_\infty}, \tag{55}$$

for all $y \in W_\infty$ with $\|y\|_{W_\infty} \leq \delta$ and for all $z \in W_\infty$. Finally, if $y \in W_\infty \cap L^\infty(0,\infty;V)$, then $DR_k(y) \in L^2(0,\infty;V^*)$.

This lemma is proved in the Appendix, page 25. As was already pointed out in Section 6, the optimality of $\mathbf{U}_k(y_0)$ for the minimization problem of $\mathcal{J}_k(y_0,\cdot)$ has only been proved for an initial condition in $B_Y(\delta_7) \cap V$. The next technical lemma will enable us to prove the optimality of $\mathbf{U}_k(y_0)$ for initial conditions close to 0 but not necessarily in V.

Lemma 28. *There exist constants $\delta_9 > 0$ and $M > 0$ such that for all $y_0 \in B_Y(\delta_9)$ and u with $\|u\|_{L^2(0,\infty)} \leq \delta_9$, we have: If $\|y\|_{W_\infty} \leq \delta_9$ where $y = S(y_0,u)$, then for all $\tilde{y}_0 \in B_Y(\delta_9)$, there exists $\tilde{u} \in L^2(0,\infty)$ such that*

$$\|\tilde{u} - u\|_{L^2(0,\infty)} \leq M\|\tilde{y}_0 - y_0\|_Y \quad \text{and} \quad \|\tilde{y} - y\|_{W_\infty} \leq M\|\tilde{y}_0 - y_0\|_Y,$$

where $\tilde{y} = S(y_0,\tilde{u})$.

This lemma is proved in the Appendix, page 26.

Lemma 29. *Let $k \geq 2$. There exists $\delta_{10} > 0$ with the following property: If $y_0 \in B_Y(\delta_{10})$, then there exists a unique costate $p_k \in L^2(0,\infty;V)$ such that*

$$-\dot{p}_k = A^* p_k + u_k N^* p_k + C^*Cy_k + DR_k(y_k) \qquad \text{in } (W_\infty^0)^*, \tag{56}$$

and

$$\alpha u_k + \langle Ny_k + B, p_k \rangle_Y = 0, \tag{57}$$

where $y_k = S(\mathbf{u}_k, y_0)$ and $u_k = \mathbf{U}_k(y_0)$. Moreover, there exists a constant M, independent of y_0, such that

$$\|p_k\|_{L^2(0,\infty;V)} \leq M\|y_0\|_Y. \tag{58}$$

Proof. Since $S(\mathbf{u}_k,\cdot)$ is continuous, there exists $\delta_{10} \in (0,\delta_7)$ such that for all $y_0 \in B_Y(\delta_{10})$, $\|S(\mathbf{u}_k,y_0)\|_{L^\infty(0,\infty;Y)} < \delta_9$.

For a given $y_0 \in B_Y(\delta_{10})$, consider the following problem:

$$\inf_{\substack{y \in W_\infty \\ u \in L^2(0,\infty)}} J_k(y,u) := \frac{1}{2}\int_0^\infty \|Cy(t)\|_Z^2\, \mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2\, \mathrm{d}t + R_k(y),$$

$$\text{subject to: } e_k(y,u) := \big(\dot{y} - (Ay + (Ny+B)u), y(0) - y_0\big) = (0,0). \tag{59}$$

From Lemma 23 we know that for $y_0 \in B_Y(\delta_{10}) \cap V$, the control $\mathbf{U}_k(y_0)$ is a global solution to this problem. We claim now that if $y_0 \in B_Y(\delta_{10})$, then $(S(\mathbf{u}_k, y_0), \mathbf{U}_k(y_0))$ is a local solution. Let us fix $y_0 \in B_Y(\delta_{10})$ and denote $(y_k, u_k) = (S(\mathbf{u}_k, y_0), \mathbf{U}_k(y_0))$. Let us set $\varepsilon = \frac{1}{M_0}(\delta_9 - \|y_k\|_{L^\infty(0,\infty;Y)})$, and let $(y, u) \in W_\infty \times L^2(0, \infty)$ be such that $e(y, u) = 0$ and $\|y - y_k\|_{W_\infty} \leq \varepsilon$. Then

$$\|y - y_k\|_{L^\infty(0,\infty;Y)} \leq M_0 \varepsilon$$

and thus $\|y\|_{L^\infty(0,\infty;Y)} \leq \delta_9$. Let $(y_0^n)_{n \in \mathbb{N}}$ be a sequence in $B_Y(\delta_9) \cap V$ converging to y_0 in Y. By Lemma 28, there exists for all $n \in \mathbb{N}$ a control u_n such that

$$\|u_n - u\|_{L^2(0,\infty)} \leq M\|y_0^n - y_0\|_Y \quad \text{and} \quad \|y_n - y\| \leq M\|y_0^n - y_0\|_Y,$$

where $y_n = S(u_n, y_0^n)$. Since J_k is continuous, $J_k(y_n, u_n) \underset{n\to\infty}{\longrightarrow} J_k(y, u)$. Using the continuity of the mappings $y_0 \mapsto S(\mathbf{u}_k, y_0)$ and $y_0 \mapsto \mathbf{U}_k(y_0)$, we also obtain that

$$J_k(S(\mathbf{u}_k, y_0^n), \mathbf{U}_k(y_0^n)) \underset{n\to\infty}{\longrightarrow} J_k(S(\mathbf{u}_k, y_0), \mathbf{U}_k(y_0)) = J_k(y_k, u_k).$$

From the optimality of $(S(\mathbf{u}_k, y_0^n), \mathbf{U}_k(y_0^n))$, we deduce that for all $n \in \mathbb{N}$,

$$J_k(S(\mathbf{u}_k, y_0^n), \mathbf{U}_k(y_0^n)) \leq J_k(y_n, u_n)$$

and finally, passing to the limit in n

$$J_k(y_k, u_k) \leq J_k(y, u).$$

This proves the local optimality of (y_k, u_k).

The derivation of the optimality conditions, the proof of uniqueness of p_k as well as the proof of estimate (58), can be done exactly in the same way as in Lemma 11. The only difference is the presence of the term $DR_k(y_k)$ in the costate equation, which can be estimated with Lemma 27. □

Lemma 30. *For $y_0 \in B_Y(\delta_{10}) \cap V$, $y_k = S(\mathbf{u}_k, y_0)$, and $u_k = \mathbf{U}_k(y_0)$, let p_k be the unique costate given by the previous lemma. Then, $p_k \in W_\infty$. Moreover,*

$$p_k(t) = D\mathcal{V}_k(y_k(t)), \quad \text{for all } t \geq 0. \tag{60}$$

Proof. We first prove that $y_k \in L^\infty(0, \infty; V)$. We set $z = (-A_0)^{\frac{1}{2}} y_k$. Then, z is the solution to

$$\dot{z} = Az + \tilde{N} z u_k + \tilde{B} u_k, \quad z(0) = (-A_0)^{\frac{1}{2}} y_0,$$

where $\tilde{N} = (-A_0)^{\frac{1}{2}} N (-A_0)^{-\frac{1}{2}}$ and where $\tilde{B} = (-A_0)^{\frac{1}{2}} B$. Since $(-A_0)^{-\frac{1}{2}} \in \mathcal{L}(V, \mathcal{D}(-A_0))$ and $(-A_0)^{\frac{1}{2}} \in \mathcal{L}(V, Y)$, we have $\tilde{N} \in \mathcal{L}(V, Y)$ and $\tilde{B} \in V^*$, as a consequence of (A5). We have $z(0) \in Y$, since $y_0 \in V$, moreover, $z \in L^2(0, \infty; Y)$, since $y_k \in L^2(0, \infty; V)$. Therefore, by Lemma 1, $z \in L^\infty(0, \infty; Y)$ and finally, $y_k = (-A_0)^{-\frac{1}{2}} z \in L^\infty(0, \infty; V)$.

Let us prove now that $p_k \in W_\infty$. To this purpose, it is sufficient to prove that the right-hand side of (56) lies in $L^2(0, \infty; V^*)$. Since $y_k \in L^\infty(0, \infty; V)$, we have by Lemma 27 that $DR_k(y_k) \in L^2(0, \infty; V^*)$. It remains to check that $u_k N^* p_k \in L^2(0, \infty; V^*)$. We have $N y_k \in L^\infty(0, \infty; Y)$, moreover, $u_k(t) = \mathbf{u}_k(y_k(t))$, for a.e. $t \geq 0$. Using the definition of the feedback law $\mathbf{u}_k$ (given by (50)), we deduce that $u_k \in L^\infty(0, \infty)$. Moreover, $N^* p_k \in L^2(0, \infty; V^*)$, thus, $u_k N^* p_k \in L^2(0, \infty; V^*)$, and finally $p_k \in W_\infty$.

The proof of equality (60) is the same as the one for Lemma 18, observing that $\mathcal{V}_k$ is the value function of Problem (59). Note that it was necessary to prove that $p_k \in W_\infty$ (and thus to assume that $y_0 \in V$), so that the integration by part in (37) is well-defined. □

We finally obtain the desired improvement of Theorem 25.

Theorem 31. *Let $k \geq 2$. Then there exist $\delta_{11} > 0$ and $M > 0$ such that for all $y_0 \in B_Y(\delta_{11})$,*

$$\max\left(\|y_k - \bar{y}\|_{W_\infty}, \|u_k - \bar{u}\|_{L^2(0,\infty)}, \|p_k - \bar{p}\|_{L^2(0,\infty;V)}\right) \leq M\|y_0\|_Y^k, \tag{61}$$

where

$$(\bar{y}, \bar{u}, \bar{p}) = \big(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0)\big) \quad \text{and} \quad (y_k, u_k) = \big(S(\mathbf{u}_k, y_0), \mathbf{U}_k(y_0)\big)$$

and where p_k is the costate given by Lemma 29. Moreover,

$$\mathcal{J}(y_0, u_k) \leq \mathcal{V}(y_0) + M\|y_0\|_Y^{2k}. \tag{62}$$

Proof. Step 1: application of the inverse function theorem. We consider again the mapping Φ defined by (26). As was proved in Lemma 13, Φ is infinitely differentiable and $D\Phi(0,0,0)$ is an isomorphism. For a given $\delta > 0$, we denote

$$B(\delta) = \big\{(y,w) \in Y \times (W_\infty^0)^* \,|\, \|y\|_Y \leq \delta,\ \|w\|_{(W_\infty^0)^*} \leq \delta\big\}.$$

Applying the inverse function theorem, we obtain that there exist $\delta > 0$, $\delta' > 0$, and three infinitely differentiable mappings

$$(y_0,w) \in B(\delta) \mapsto \big(\hat{\mathcal{Y}}(y_0,w), \hat{\mathcal{U}}(y_0,w), \hat{\mathcal{P}}(y_0,w)\big) \in W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)$$

such that for all $(y,u,p) \in W_\infty \times L^2(0,\infty) \times L^2(0,\infty;V)$ and for all pairs $(y_0,w) \in B(\delta)$, if $\max(\|y\|_{W_\infty}, \|u\|_{L^2(0,\infty)}, \|p\|_{L^2(0,\infty;V)}) \leq \delta'$, then

$$\Phi(y,u,p) = (y_0,0,w,0) \Longleftrightarrow \begin{cases} y = \hat{\mathcal{Y}}(y_0,w) \\ u = \hat{\mathcal{U}}(y_0,w) \\ p = \hat{\mathcal{P}}(y_0,w). \end{cases}$$

We shall use this fact with $w = DR_k(y_k)$. By the continuity of the mappings $S(\mathbf{u}_k,\cdot)$ and $\mathbf{U}_k(\cdot)$, by Lemma 27 and by Lemma 29, there exists $\delta_{11} \in (0,\delta_{10})$ so that for all $y_0 \in B_Y(\delta_{11})$,

$$\begin{cases} \max\big(\|y_k\|_{W_\infty}, \|u_k\|_{L^2(0,\infty)}, \|p_k\|_{L^2(0,\infty;V)}\big) \leq \delta', \\ \max\big(\|y_0\|_Y, \|DR_k(y_k)\|_{(W_\infty^0)^*}\big) \leq \delta. \end{cases} \tag{63}$$

Step 2: a characterization of (y_k,u_k,p_k). We now claim that for $y_0 \in B_Y(\delta_{11})$,

$$\begin{cases} y_k = \hat{\mathcal{Y}}(y_0, DR_k(y_k)), \\ u_k = \hat{\mathcal{U}}(y_0, DR_k(y_k)), \\ p_k = \hat{\mathcal{P}}(y_0, DR_k(y_k)). \end{cases} \tag{64}$$

Let us first consider the case where $y_0 \in B_Y(\delta_{11}) \cap V$. The key observation is that $\Phi(y_k,u_k,p_k) = (y_0, 0, DR_k(y_k), 0)$. This equality is clearly satisfied for the first three coordinates of Φ, since $y_k(0) = y_0$, and since y_k and p_k satisfy the state and costate equations, respectively. By Lemma 30, $p_k(t) = D\mathcal{V}_k(y_k(t))$ for a.e. $t \geq 0$, therefore,

$$u_k(t) = \mathbf{u}_k(y_k(t)) = -\frac{1}{\alpha} D\mathcal{V}_k(y_k(t))(Ny_k(t)+B) = -\frac{1}{\alpha}\langle p_k(t), Ny_k(t)+B\rangle_Y.$$

This proves that $\alpha u_k + \langle Ny_k + B, p_k\rangle = 0$ and justifies that $\Phi(y_k,u_k,p_k) = (y_0,0,DR_k(y_k),0)$. Combined with (63), we obtain (64) as announced.

Let us consider now the case of an arbitrary initial condition y_0 in $B_Y(\delta_{11})$. Let $(y_0^n)_{n\in\mathbb{N}}$ be a sequence in $B_Y(\delta_{11}) \cap V$ converging to y_0. We set $y_k^n = S(\mathbf{u}_k, y_0^n)$, $u_k^n = \mathbf{U}_k(y_0^n)$ and denote by p_k^n the associated costate. We have already proved that

$$\begin{cases} y_k^n = \hat{\mathcal{Y}}(y_0^n, DR_k(y_k^n)), \\ u_k^n = \hat{\mathcal{U}}(y_0^n, DR_k(y_k^n)). \end{cases}$$

By continuity of the mappings $S(\mathbf{u}_k,\cdot)$, $\hat{\mathcal{Y}}$, $\hat{\mathcal{U}}$, and $DR_k(\cdot)$ (see Lemma 27), we can pass to the limit in the above relations. It follows that

$$\begin{cases} y_k = \hat{\mathcal{Y}}(y_0, DR_k(y_k)), \\ u_k = \hat{\mathcal{U}}(y_0, DR_k(y_k)). \end{cases}$$

Let us set $\tilde{p}_k = \hat{\mathcal{P}}(y_0, DR_k(y_k))$. By construction of $\hat{\mathcal{P}}$, $\tilde{p}_k$ is such that

$$-\dot{\tilde{p}}_k - A^*\tilde{p}_k - u_k N^*\tilde{p}_k - C^*Cy_k = DR_k(y_k), \quad \alpha u_k + \langle Ny_k + B, \tilde{p}_k\rangle_Y = 0,$$

meaning that $\tilde{p}_k$ satisfies the costate equation (56) and relation (57). It must be equal to p_k, by Lemma 29. Therefore, (64) holds and the claim is proved.

Step 3: a characterization of $(\bar{y}, \bar{u}, \bar{p})$. Now, let us reduce δ_{11}, if necessary, so that for all $y_0 \in B_Y(\delta_{11})$,

$$\max\big(\|\hat{\mathcal{Y}}(y_0,0)\|_{W_\infty}, \|\hat{\mathcal{U}}(y_0,0)\|_{L^2(0,\infty)}, \|\hat{\mathcal{P}}(y_0,0)\|_{L^2(0,\infty;V)}\big) \leq \delta_2',$$

with δ_3 defined in Proposition 14. Then we have

$$\big(\hat{\mathcal{Y}}(y_0,0), \hat{\mathcal{U}}(y_0,0), \hat{\mathcal{P}}(y_0,0)\big) = \big(\mathcal{Y}(y_0), \mathcal{U}(y_0), \mathcal{P}(y_0)\big).$$

Step 4: conclusion. The value of δ_{11} can be reduced once again, so that the mappings $\hat{\mathcal{Y}}$, $\hat{\mathcal{U}}$, and $\hat{\mathcal{P}}$ are Lipschitz-continuous. Using the Lipschitz continuity of $S(\mathbf{u}_k, \cdot)$ and Lemma 27, we obtain that

$$\begin{aligned}\|y_k - \bar{y}\|_{W_\infty} &= \big\|\hat{\mathcal{Y}}\big(y_0, DR_k(w_k)\big) - \hat{\mathcal{Y}}(y_0,0)\big\|_{W_\infty} \\ &\leq M\|DR_k(y_k)\|_{(W_\infty^0)^*} \\ &\leq M\|y_k\|_{W_\infty}^k \\ &\leq M\|y_0\|_Y^k.\end{aligned}$$

The remaining estimates on $\|u_k - \bar{u}\|_{L^2(0,\infty)}$ and $\|p_k - \bar{p}\|_{L^2(0,\infty;V)}$ can be proved similarly. Estimate (61) follows.

For proving (62), we use the same technique as in Lemma 18. We have

$$\begin{aligned}\mathcal{J}(y_0,u_k) - \mathcal{J}(y_0,\bar{u}) = &\Big(\frac{1}{2}\|Cy_k\|_{L^2(0,\infty;Z)}^2 + \frac{\alpha}{2}\|u_k\|_{L^2(0,\infty)}^2\Big) - \Big(\frac{1}{2}\|C\bar{y}\|_{L^2(0,\infty;Z)}^2 + \frac{\alpha}{2}\|\bar{u}\|_{L^2(0,\infty)}^2\Big) \\ &- \langle \bar{p}, \dot{y}_k - (Ay_k + (Ny_k + B)u_k)\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} \\ &+ \langle \bar{p}, \dot{\bar{y}} - (A\bar{y} + (N\bar{y} + B)\bar{u})\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} \\ = &\frac{1}{2}\|C(y_k - \bar{y})\|_{L^2(0,\infty;Y)}^2 + \frac{\alpha}{2}\|u_k - \bar{u}\|_{L^2(0,\infty)}^2 \\ &+ \big\langle \bar{p}, N(y_k - \bar{y})(u_k - \bar{u})\big\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} \\ \leq &M\big(\|y_k - \bar{y}\|_{W_\infty}^2 + \|u_k - \bar{u}\|_{L^2(0,\infty)}^2\big) \\ \leq &M\|y_0\|_Y^{2k}.\end{aligned}$$

Estimate (62) follows. The theorem is proved. □

8 Conclusion

We have performed a sensitivity analysis for an infinite-horizon optimal control problem involving an infinite-dimensional state equation. Error estimates for the efficiency of polynomial feedback laws have been derived. The approach that we have used, based on a stability analysis of the optimality conditions, is quite general and can certainly be used for other classes of partial differential equations. Future work will focus on stabilization problems of semilinear parabolic equations, for which the derivation and analysis of polynomial feedback laws are completely open. Non-smooth variants of the implicit function theorem should also enable us to perform a sensitivity analysis for infinite-time horizon control problems with a sparsity-promoting term in the cost function. Finally, our approach could also be used to derive error estimates on the efficiency of other kinds of feedback laws, like State Dependent Riccati Equations based feedback laws.

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A Appendix: technical proofs

Proof of Proposition 11. We fix a value of $\delta_1 > 0$ which is such that $\delta_1 < \big(\|N\|_{\mathcal{L}(Y,V^*)} M_0 M_d\big)^{-1}$. Then, by Lemma 4, $\bar{y} \in W_\infty$. As a consequence, $(\bar{y}, \bar{u})$ is a solution to the following optimization problem:

$$\inf_{(y,u) \in W_\infty \times L^2(0,\infty)} J(y,u), \quad \text{subject to: } e(y,u) = 0,$$

where

$$J(y,u) = \frac{1}{2} \int_0^\infty \|Cy(t)\|_Z^2 \,\mathrm{d}t + \frac{\alpha}{2} \int_0^\infty u(t)^2 \,\mathrm{d}t$$

and where

$$e(y,u) = \big(\dot{y} - (Ay + Nyu + Bu), y(0) - y_0\big) \in L^2(0,\infty;V^*) \times Y.$$

Our approach for deriving optimality conditions is similar to the one of Lemma 9. In order to have a state variable in W_∞^0, we first need to perform a shift of the state equation. Let $u \in L^2(0,\infty)$ and set $y = S(y_0,u)$. Then, $z = y - \bar{y}$ is the solution to the following system:

$$\dot{z} = Az + Nzu + (N\bar{y} + B)u - (N\bar{y}\bar{u} + B\bar{u}), \quad z(0) = 0.$$

We can now consider the following optimization problem:

$$\inf_{(z,u) \in W_\infty^0 \times L^2(0,\infty)} \tilde{J}(z,u), \quad \text{subject to: } \tilde{e}(z,u) = 0, \tag{65}$$

where

$$\tilde{J}(z,u) = J(z + \bar{y}, u) = \frac{1}{2} \int_0^\infty \|C(z(t) + \bar{y}(t))\|_Z^2 \,\mathrm{d}t + \frac{\alpha}{2} \int_0^\infty u(t)^2 \,\mathrm{d}t$$

and where

$$\tilde{e}(z,u) = \dot{z} - \big(Az + Nzu + Bu - (N\bar{y}\bar{u} + B\bar{u})\big) \in L^2(0,\infty;V^*).$$

For all $(y,u) \in W_\infty \times L^2(0,\infty)$, we have:

$$e(y,u) = 0 \Longleftrightarrow \big[\tilde{e}(z,u) = 0 \quad \text{and} \quad z \in W_\infty^0\big], \quad \text{where } z = y - \bar{y}.$$

Since $\tilde{J}(z,u) = J(z + \bar{y}, u)$, we deduce that $(\bar{y} - \bar{y} = 0, \bar{u})$ is a solution to problem (65).

The mappings $\tilde{J}$ and $\tilde{e}$ are continuously differentiable. We have

$$\begin{aligned} D\tilde{J}(0,\bar{u})(\xi,v) &= \langle C^*C\bar{y}, \xi \rangle_{L^2(0,\infty;Y)} + \alpha \langle \bar{u}, v \rangle_{L^2(0,\infty)} \\ D\tilde{e}(0,\bar{u})(\xi,v) &= \dot{\xi} - (A + \bar{u}N)\xi - (N\bar{y} + B)v. \end{aligned}$$

Let us prove now that $D\tilde{e}(0,\bar{u})$ is surjective, if $\delta_1 > 0$ is sufficiently small. For $\varphi \in L^2(0,\infty;V^*)$, let z be the solution to

$$\dot{z} = (A + \bar{u}N)z + (N\bar{y} + B)Fz + \varphi, \quad z(0) = 0.$$

Then we find

$$\dot{z}(t) = (A + BF)z(t) + (Pz)(t) + \varphi(t),$$

where

$$(Pz)(t) = \bar{u}(t)Nz(t) + N\bar{y}(t)Fz(t),$$

and

$$\begin{aligned} \|Pz\|_{L^2(0,\infty;V^*)} &\leq \big(\|N\|_{\mathcal{L}(Y,V^*)} \|\bar{u}\|_{L^2(0,\infty)} + \|N\|_{\mathcal{L}(Y,V^*)} \|\bar{y}\|_{L^2(0,\infty;Y)} \|F\|_{\mathcal{L}(Y,\mathbb{R})}\big) \|z\|_{L^\infty(0,\infty;Y)} \\ &\leq M_0 \big(\|N\|_{\mathcal{L}(Y,V^*)} + \|N\|_{\mathcal{L}(Y,V^*)} \|F\|_{\mathcal{L}(Y,\mathbb{R})}\big) \delta_1 \|z\|_{W_\infty}. \end{aligned}$$

It follows that for $\delta_1 > 0$ chosen sufficiently small

$$\|P\|_{\mathcal{L}(W_\infty, L^2(0,\infty;V^*))} < \frac{1}{M_s}.$$

Therefore, by Lemma 2 and Remark 3, there exists a constant $M > 0$ such that

$$\|z\|_{W_\infty} \leq M\|\varphi\|_{L^2(0,\infty;V^*)}. \tag{66}$$

Setting $v = Fz$, we obtain that

$$\|v\|_{L^2(0,\infty)} \leq M\|\varphi\|_{L^2(0,\infty;V^*)}. \tag{67}$$

Finally we have

$$D\tilde{e}(0,\bar{u})(z,v) = \varphi, \tag{68}$$

which proves that $D\tilde{e}(0,\bar{u})$ is surjective. Let us emphasize the fact that the constant M involved in (66) and (67) does not depend on $(\bar{u},\bar{y})$ (but it depends on δ_1). It follows from the surjectivity of $D\tilde{e}(0,\bar{u})$ that there exists a unique $p \in L^2(0,\infty;V)$ such that for all $(z,v) \in W_\infty^0 \times L^2(0,\infty)$,

$$D\tilde{J}(0,\bar{u})(z,v) - \big\langle p, D\tilde{e}(0,\bar{u})(z,v)\big\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} = 0. \tag{69}$$

The costate equation (23) and relation (24) follow, similarly to the proof of Lemma 9. It remains to prove estimate (25) on the costate. Let $\varphi \in L^2(0,\infty;V^*)$ and (z,v) be taken as in the proof of the surjectivity of $D\tilde{e}(0,u)$. From (68) and (69), we deduce that

$$\begin{aligned}
\langle p,\varphi\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} &= \big\langle p, D\tilde{e}(0,\bar{u})(z,v)\big\rangle_{L^2(0,\infty;V),L^2(0,\infty;V^*)} \\
&= D\tilde{J}(0,\bar{u})(z,v) \\
&\leq M\big(\|\bar{y}\|_{L^2(0,\infty;Y)}\|z\|_{L^2(0,\infty;Y)} + \|\bar{u}\|_{L^2(0,\infty)}\|v\|_{L^2(0,\infty)}\big) \\
&\leq M(\|\bar{y}\|_{L^2(0,\infty;Y)} + \|\bar{u}\|_{L^2(0,\infty)})\|\varphi\|_{L^2(0,\infty;V^*)}.
\end{aligned}$$

Once again, the constant M obtained above does not depend on $(\bar{y},\bar{u})$ and φ, therefore, (25) holds true. □

Proof of Lemma 27. The mapping r_k can be written in the following form:

$$\begin{aligned}
r_k(y) = &\sum_{i=k+1}^{2k}\sum_{j=1}^{j_1(i)} \mathcal{Q}_{1,j}^i(y,...,y) + \sum_{i=k+1}^{2k}\sum_{j=1}^{j_2(i)} \mathcal{Q}_{2,j}^i(y,...,y,Ny,y,...,y) \\
&+ \sum_{i=k+1}^{2k}\sum_{j=1}^{j_3(i)} \mathcal{Q}_{3,j}^i(y,...,y,Ny,y,...,y,Ny,y,...,y),
\end{aligned}$$

where all the mappings $\mathcal{Q}_{\ell,j}^i$ are bounded multilinear forms of order i. To simplify, we prove the result for the following mapping:

$$R\colon y \in W_\infty \mapsto \int_0^\infty r(y(t))\,\mathrm{d}t, \quad \text{where: } r(y) = \mathcal{Q}(Ny,Ny,y,...,y)$$

and $\mathcal{Q}$ is a bounded multilinear form of order $i \geq k+1$. The general case easily follows. For y and $z \in V$, we have

$$\begin{aligned}
Dr(y)z = &\,\mathcal{Q}(Nz,Ny,y,...,y) + \mathcal{Q}(Ny,Nz,y,...,y) \\
&+ \mathcal{Q}(Ny,Ny,z,y,...,y) + ... + \mathcal{Q}(Ny,Ny,y,...,y,z) \in \mathbb{R}.
\end{aligned}$$

We prove that R is continuously differentiable and that

$$DR(y)z = \int_0^\infty Dr(y(t))z(t)\,\mathrm{d}t. \tag{70}$$

Let us define

$$\begin{aligned}
&R_1\colon (y_1,...,y_k) \in (W_\infty)^k \mapsto \int_0^\infty \mathcal{Q}(Ny_1,Ny_2,y_3,...,y_k)\,\mathrm{d}t, \\
&R_2\colon y \in W_\infty \mapsto y^{\otimes k} \in (W_\infty)^k,
\end{aligned}$$

so that $R = R_1 \circ R_2$. The operator R_2 is linear and bounded, thus it is infinitely differentiable. The mapping R_1 is a bounded multilinear form, since

$$\begin{aligned} |R_1(y_1,...,y_k)| &\leq \|\mathcal{Q}\| \|Ny_1\|_{L^2(0,\infty;Y)} \|Ny_2\|_{L^2(0,\infty;Y)} \|y_3\|_{L^\infty(0,\infty;Y)} \cdots \|y_k\|_{L^\infty(0,\infty;Y)} \\ &\leq M\|y_1\|_{L^2(0,\infty;V)} \|y_2\|_{L^2(0,\infty;V)} \|y_3\|_{L^\infty(0,\infty;Y)} \cdots \|y_k\|_{L^\infty(0,\infty;Y)} \\ &\leq M\|y_1\|_{W_\infty} \cdots \|y_k\|_{W_\infty}. \end{aligned}$$

Therefore, R_1 is continuously differentiable (see [9, Lemma 7]), moreover,

$$\begin{aligned} DR_1(y_1,...,y_k)(z_1,...,z_k) = R_1(z_1,y_2,...,y_k) \\ + R_1(y_1,z_2,y_3,...,y_k) + ... + R_1(y_1,...,y_{k-1},z_k). \end{aligned} \tag{71}$$

This proves that R is continuously differentiable. Moreover, by the chain rule,

$$DR(y)z = DR_1(R_2(y))DR_2(y)z.$$

Combined with (71), we obtain (70).

Let us prove estimate (55). For y and $z \in V$, the following estimate holds:

$$|Dr(y)z| \leq M\big(\|y\|_V \|y\|_Y^{i-2} \|z\|_V + \|y\|_V^2 \|y\|_Y^{i-3} \|z\|_Y\big). \tag{72}$$

Therefore, for all y and $z \in W_\infty$,

$$\begin{aligned} \int_0^\infty |Dr(y(t))(z(t))| \,\mathrm{d}t \,&\leq M\big(\|y\|_{L^2(0,\infty;V)} \|y\|_{L^\infty(0,\infty;Y)}^{i-2} \|z\|_{L^2(0,\infty;V)} \\ &\qquad + \|y\|_{L^2(0,\infty;V)}^2 \|y\|_{L^\infty(0,\infty;Y)}^{i-3} \|z\|_{L^\infty(0,\infty;Y)}\big) \\ &\leq M\|y\|_{W_\infty}^{i-1} \|z\|_{W_\infty}. \end{aligned}$$

The constant M involved in the above inequality is independent of y and z, therefore, for a given $\delta > 0$,

$$\Big| \int_0^\infty Dr(y(t))(z(t)) \,\mathrm{d}t \Big| \leq M\|y\|_{W_\infty}^{i-1-k} \|y\|_{W_\infty}^k \|z\|_{W_\infty} \leq M\delta^{i-1-k} \|y\|_{W_\infty}^k \|z\|_{W_\infty},$$

if $\|y\|_{W_\infty} \leq \delta$, since $i \geq k+1$. This proves estimate (55).

Assume now that $y \in W_\infty \cap L^\infty(0,\infty;V)$. As a consequence of (72), there exists a constant $M > 0$, independent of y and z, such that

$$\begin{aligned} |DR(y)z| \leq M\big(&\|y\|_{L^2(0,\infty;V)} \|z\|_{L^2(0,\infty;V)} \|y\|_{L^\infty(0,\infty;Y)}^{i-2} \\ &+ \|y\|_{L^\infty(0,\infty;V)} \|y\|_{L^2(0,\infty;V)} \|z\|_{L^2(0,\infty;V)} \|y\|_{L^\infty(0,\infty;Y)}^{i-3}\big), \end{aligned}$$

which proves that in this case $DR(z) \in L^2(0,\infty;V^*)$. □

Proof of Lemma 28. The main idea is the following: we construct $\tilde{y}$ as the solution to the following system:

$$\dot{\tilde{y}} = A\tilde{y} + (N\tilde{y} + B)(u + F(\tilde{y} - y)), \quad \tilde{y}(0) = \tilde{y}_0. \tag{73}$$

The corresponding control $\tilde{u}$ is then defined by

$$\tilde{u} = u + F(\tilde{y} - y). \tag{74}$$

The key observation is that $z := \tilde{y} - y$ is the solution to the following nonlinear system:

$$\dot{z} = (A + BF + uN + NyF)z + NzFz, \quad z(0) = \tilde{y}_0 - y_0.$$

Steps 1 and 2 below are intermediate results. They will be used in step 3, where we prove the existence of a solution to (73). We finally prove the claim of the lemma in step 4.

Step 1. We first prove that there exist $\delta_a > 0$ and $M_1 > 0$ such that for all

$$(u, y, z_0, f) \in L^2(0,\infty) \times W_\infty \times Y \times L^2(0,\infty;V^*),$$

with $\max(\|u\|_{L^2(0,\infty)}, \|y\|_{W_\infty}) \leq \delta_a$, the following linear system has a unique solution:

$$\dot{z} = (A + BF + uN + NyF)z + f, \quad z(0) = z_0, \tag{75}$$

satisfying

$$\|z\|_{W_\infty} \leq M_1(\|z_0\|_Y + \|f\|_{L^2(0,\infty;V^*)}). \tag{76}$$

This is a direct consequence of Lemma 2. Indeed, let us define $P \in \mathcal{L}(W_\infty, L^2(0,\infty;V^*))$ by

$$(Pz)(t) = u(t)Nz(t) + Ny(t)Fz(t). \tag{77}$$

We have

$$\begin{aligned}
\|Pz\|_{L^2(0,\infty;V^*)} &\leq \|u\|_{L^2(0,\infty)} \|N\|_{\mathcal{L}(Y,V^*)} \|z\|_{L^\infty(0,\infty;Y)} \\
&\quad + \|N\|_{\mathcal{L}(Y,V^*)} \|y\|_{L^2(0,\infty;Y)} \|F\|_{\mathcal{L}(Y,\mathbb{R})} \|z\|_{L^\infty(0,\infty;Y)} \\
&\leq M\delta_a \|z\|_{W_\infty},
\end{aligned}$$

where the constant M is independent of u, y, and z. One can therefore choose $\delta_a > 0$ such that $\|P\|_{\mathcal{L}(W_\infty,L^2(0,\infty;V^*))} \leq 1/(2M_s)$. The well-posedeness of (75) and estimate (76) follow.

Step 2. We prove that there exists $M_2 > 0$ such that for all z_1 and $z_2 \in W_\infty$, the following estimate holds:

$$\|Nz_2Fz_2 - Nz_1Fz_1\|_{L^2(0,\infty;V^*)} \leq M_2 \max\big(\|z_1\|_{L^\infty(0,\infty;Y)}, \|z_2\|_{L^\infty(0,\infty;Y)}\big) \|z_2 - z_1\|_{W_\infty}. \tag{78}$$

For all z_1 and $z_2 \in W_\infty$, we have

$$\begin{aligned}
&\|Nz_2Fz_2 - Nz_1Fz_1\|_{L^2(0,\infty;V^*)} \\
&\quad \leq \|N(z_2 - z_1)Fz_2\|_{L^2(0,\infty;V^*)} + \|Nz_1F(z_2 - z_1)\|_{L^2(0,\infty;V^*)} \\
&\quad \leq \|N\|_{\mathcal{L}(Y,V^*)} \|z_2 - z_1\|_{L^2(0,\infty;Y)} \|F\|_{\mathcal{L}(Y,\mathbb{R})} \|z_2\|_{L^\infty(0,\infty;Y)} \\
&\qquad + \|N\|_{\mathcal{L}(Y,V^*)} \|z_1\|_{L^\infty(0,\infty;Y)} \|F\|_{\mathcal{L}(Y,\mathbb{R})} \|z_2 - z_1\|_{L^2(0,\infty;Y)}.
\end{aligned}$$

Estimate (78) follows.

Step 3. We prove now that there exist $\delta_b > 0$ and $M_3 > 0$ such that for all $(u, y, z_0) \in L^2(0,\infty) \times W_\infty \times B_Y(\delta_b)$ with $\|u\|_{L^2(0,\infty)} \leq \delta_a$ and $\|y\|_{W_\infty} \leq \delta_a$, the following nonlinear system:

$$\dot{z} = (A + BF + uN + NyF)z + NzFz, \quad z(0) = z_0 \tag{79}$$

has a unique solution in W_∞, satisfying

$$\|z\|_{W_\infty} \leq M_3 \|z_0\|_Y.$$

We prove this claim with $M_3 = 2M_1$ and

$$\delta_b = \frac{1}{M_0 M_2 M_3^2}.$$

By definition of M_3, we have

$$\delta_b = \frac{1}{2M_0 M_1 M_2 M_3}.$$

The existence of a solution to (79) is obtained with a fixed-point argument. For a fixed $\|z_0\|_Y \in B_Y(\delta_b)$, let us consider the set $\mathcal{M} \subset W_\infty$ defined by

$$\mathcal{M} = \big\{ z \in W_\infty \,|\, \|z\|_{W_\infty} \leq M_3 \|z_0\|_Y \big\}.$$

We also consider the mapping $\mathcal{Z}\colon \mathcal{M} \to W_\infty$, where $\tilde{z} = \mathcal{Z}(z)$ is defined as the unique solution to the following system:

$$\dot{\tilde{z}} = (A + BF + uN + NyF)\tilde{z} + NzFz, \quad \tilde{z}(0) = z_0.$$

For $z \in \mathcal{M}$ we have

$$\|z\|_{L^\infty(0,\infty;Y)} \leq M_0 \|z\|_{W_\infty} \leq M_0 M_3 \|z_0\|_Y \leq M_0 M_3 \delta_b,$$

and thus using (76) and (78)

$$\begin{aligned}\|\tilde z\|_{W_\infty} &\le M_1(\|z_0\|_Y + \|NzFz\|_{L^2(0,\infty;V^*)})\\ &\le M_1(\|z_0\|_Y + M_2\|z\|_{L^\infty(0,\infty;Y)}\|z\|_{W_\infty})\\ &\le M_1(1+M_0M_2M_3^2\delta_b)\|z_0\|_Y\\ &\le 2M_1\|z_0\|_Y,\end{aligned}$$

which proves that $\mathcal{Z}(z)\in\mathcal{M}$. Next let z_1 and $z_2\in\mathcal{M}$, and set $z=\mathcal{Z}(z_2)-\mathcal{Z}(z_1)$. We find

$$\dot z = (A+BF+uN+NyF)z + (Nz_2Fz_2 - Nz_1Fz_1), \quad z(0)=0.$$

Therefore, by (76), we obtain

$$\begin{aligned}\|z\|_{W_\infty} &\le M_1M_2 \underbrace{\max\big(\|z_1\|_{L^\infty(0,\infty;Y)}, \|z_2\|_{L^\infty(0,\infty;Y)}\big)}_{\le M_0M_3\delta_b}\|z_2-z_1\|_{W_\infty}\\ &\le \frac{1}{2}\|z_2-z_1\|_{W_\infty}.\end{aligned}$$

This proves that $\mathcal{Z}$ is a contraction and that it possesses a unique fixed point. The existence and the uniqueness of a solution to (79) in $\mathcal{M}$ follow.

Let us prove now the uniqueness in W_∞. Let z be the unique solution to (79) in $\mathcal{M}$. As was already proved, $\|z\|_{L^\infty(0,\infty;Y)}\le M_0M_3\delta_b$. Assume that there exists another solution $\tilde z\in W_\infty$. Since $\tilde z(0)=z(0)$, there exists a time $T>0$ such that

$$\|\tilde z\|_{L^\infty(0,T;Y)} \le \frac{3}{2}M_0M_3\delta_b, \quad \tilde z_{|[0,T]}\neq z_{|[0,T]}.$$

One can check that

$$\begin{aligned}\|\tilde z - z\|_{W(0,T)} &\le M_1M_2\max\big(\|\tilde z\|_{L^\infty(0,T;Y)}, \|z\|_{L^\infty(0,T;Y)}\big)\|\tilde z - z\|_{W(0,T)}\\ &\le \big(M_1M_2\big)\Big(\frac{3}{2}M_0M_3\delta_b\Big)\|\tilde z - z\|_{W(0,T)}\\ &\le \frac{3}{4}\|\tilde z - z\|_{W(0,T)}.\end{aligned}$$

Therefore, $\|\tilde z - z\|_{W(0,T)}=0$, which is a contradiction.

Step 4. To conclude, let us set $\delta_9=\min(\delta_a,\delta_b/2)$. Let y_0 and $\tilde y_0\in B_Y(\delta_9)$, let $u\in L^2(0,\infty)$ satisfy $\|u\|_{L^2(0,\infty)}\le\delta_9$, and assume that $\|y\|_{W_\infty}\le\delta_9$, where $y=S(y_0,u)$. Then,

$$\|\tilde y_0 - y_0\|_Y \le 2\delta_9 \le \delta_b$$

and therefore, for $z_0=\tilde y_0 - y_0$, the nonlinear system (73) has a unique solution $z\in W_\infty$, satisfying $z\le M_3\|\tilde y_0-y_0\|_Y$. We set $\tilde y = y+z$ and $\tilde u = u + F(\tilde y - y)$, so that (73) and (74) hold. Moreover, we have

$$\|\tilde y - y\|_{W_\infty}\le M_3\|\tilde y_0 - y_0\|_Y \quad \text{and} \quad \|\tilde u - u\|_{L^2(0,\infty)} \le \|F\|_{\mathcal{L}(Y,\mathbb{R})}\|z\|_{W_\infty} \le M\|\tilde y_0 - y_0\|_Y.$$

The lemma is proved. □

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