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On the One-Dimensional Nonlinear Monodomain Equations with Moving Controls

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Karl Kunisch* Diego A. Souza†

Abstract

In this paper local exact controllability to the trajectories for the one-dimensional monodomain equations with the FitzHugh-Nagumo and Rogers-McCulloch ionic models using distributed controls with a moving support is investigated. In a first step a new Carleman inequality for the linearized monodomain equations, under assumptions on the movement of the control region is presented. It leads to null controllability at any positive time. Subsequently, a local result concerning the exact controllability to the trajectories for the nonlinear monodomain equations is deduced.

Keywords: exact controllability to the trajectories, monodomain equations, moving controls, FitzHugh-Nagumo ionic model, Rogers-McCulloch ionic model.

Mathematics Subject Classification (2010):

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1 Introduction

The main objective of this paper is to study controllability properties of nonlinear reaction-diffusion systems which model the electrical activity in the heart. Our reference model for the heart's electrical activity is the so-called *bidomain model*, formulated mathematically in [20], see also e.g. [19, Chapter 2] and the references therein. Next, we describe such a model in order to motivate the controllability results studied. Since the bidomain model is not frequently discussed in mathematical publications we allow more space for our description. At the end of this section we shall relate our results to results on the exact controllability of the heat equation with memory terms.

We start by saying that heart muscle cells belong to a class of cells known as *excitable cells*, which have the ability to respond actively to an electrical stimulus. In absence of an electrical stimulus cells remain electrically quiescent at a given potential difference across the cell membrane. At rest the potential inside the cells, called the *intracellular potential*, is negative compared to the *extracellular potential*, which is the potential in the interstitial space between the cells and the potential difference is referred to as *transmembrane voltage*. When such cells are stimulated electrically they depolarize the transmembrane voltage towards less negative or positive values. If the delivered stimulus is strong enough to depolarize the cell above an intrinsic firing threshold an active response is elicited, otherwise the cell returns to its resting state. The active response of the cell is of *all or none* type, that is, the elicited active response is always the same independently of the applied stimulus strength. This threshold behavior discriminating between active non-linear and passive linear response is referred to as *excitability*. The depolarization of cells above the firing threshold is a very fast process which is followed by a slower *repolarization* that restores the potential difference to its resting value. The complete cycle of depolarization and repolarization is called an *action potential*. In tissue the intracellular spaces of cells are interconnected and thus an ongoing action potential in one cell can depolarize the resting potential in neighboring cells up to the firing threshold and thus provide a mechanism for the propagation of electrical signals. This ability enables an electric activation occurring in one part of the heart to propagate through the muscle and activate the entire heart.

The bidomain model is a macroscopic model based on the assumption that, at mesoscopic scale, cardiac tissue can be viewed as partitioned into two ohmic conducting media separated by the cell membrane: the *intracellular* medium formed by the interior space of cardiac cells and the *extracellular* medium which represents the space between cells. Both domains are assumed to be continuous, and they both fill the complete volume of the heart muscle. This latter assumption of *interpenetrating domains*, that is intracellular space, extracellular space and membrane co-exist at any point in space, does not reflect biophysical reality at a cellular size scale, but can be justified at a mesoscopic size scales based on homogenization arguments. The justification for viewing the intracellular space as continuous is that the muscle cells are interconnected via conducting pores referred to as *gap junctions*. Because of the gap junctions, substances such as ions or small molecules may pass directly from one cell to another, without entering the space between the two cells (the extracellular domain). Having said this, in each of the two domains a macroscopic electric potential is defined and the membrane acts as an electrical insulator between the two domains, since otherwise we could not have a potential difference between the intracellular and extracellular domains. Although the resistance of the cell membrane itself is very high, it allows electrically charged molecules (ions) to pass through specific channels embedded in the membrane. Then, an electrical current referred to as *ionic current* will cross the membrane, the

magnitude of which will depend on the *driving force* across the membrane, that is the difference between transmembrane voltage and the equilibrium potential for a given ion species, and on the channel's permeability to the this ion species. The *transmembrane voltage* is defined as the potential difference across the membrane for every point in the heart. The bidomain formulation recognizes that cardiac tissue is electrically anisotropic and that current flows in both extracellular and intracellular domains. Bidomain models are necessary to simulate defibrillation and the biophysical mechanisms underlying the initiation of propagation with pacing stimuli where the current is injected in the extracellular domain, and have the further advantage that the most commonly measured cardiac electrical signals – extracellular potential and transmembrane voltage – are direct model outputs.

In the following let Ω denote a sample cardiac tissue in dimension two or three and denote by ν the outward unit normal vector. Let $u_i = u_i(t,x)$ and $u_e = u_e(t,x)$ be the intracellular and extracellular electric potentials, respectively, and denote by $v = u_i - u_e$ the transmembrane electric potential. The anisotropic properties of the media are modeled by intracellular and extracellular conductivity tensors $\sigma_i(x)$ and $\sigma_e(x)$, respectively. We assume that σ_e, $\sigma_i \in C^1(\overline{\Omega}; \mathcal{M}_N(\mathbb{R}))$ ($N = 2$ ou $N = 3$) with $(\sigma_e(x)\xi, \xi) \geq \sigma_{e,0}|\xi|^2$ in $\overline{\Omega}$ ($\sigma_{e,0} > 0$) and $(\sigma_i(x)\xi, \xi) \geq \sigma_{i,0}|\xi|^2$ in $\overline{\Omega}$ ($\sigma_{i,0} > 0$). In this way the propagation of the electrical signal through the cardiac tissue is described by the following parabolic system:

$$\begin{cases} a_m(c_m v_t + \mathcal{I}_{ion}) - \nabla \cdot [\sigma_i(x)\nabla u_i] = \mathcal{I}_{s,i} & \text{in} \quad \Omega \times \mathbb{R}_{>0}, \\ a_m(c_m v_t + \mathcal{I}_{ion}) + \nabla \cdot [\sigma_e(x)\nabla u_e] = \mathcal{I}_{s,e} & \text{in} \quad \Omega \times \mathbb{R}_{>0}, \\ [\sigma_i(x)\nabla u_i] \cdot \nu = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_{>0}, \\ [\sigma_e(x)\nabla u_e] \cdot \nu = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_{>0}, \\ v(\cdot, 0) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $c_m > 0$ is the *capacitance of the cell membrane*, $a_m > 0$ is the homogenized *surface-to-volume ratio of the cell membrane*, $\mathcal{I}_{ion}$ is the *ionic current across the membrane*, $\mathcal{I}_{s,i}$ and $\mathcal{I}_{s,e}$ model the *intracellular* and *extracellular stimulation current* used to trigger the action potential of the cell. The boundary conditions imply that the heart is surrounded by a non-conductive medium, and thus we require that the normal component of both the intracellular and extracellular current to be zero.

The ionic current term $\mathcal{I}_{ion}$ in (1) for a given ion species is a function of the transmembrane voltage, the equilibrium potential of the ion species and additional cellular state variables $\mathbf{w}$ (ionic concentrations and gating variables). Let us explain briefly the appearance of these state variables: although the cell membrane itself is impermeable to ions, it has embedded in it a number of large proteins that form channels through the membrane where the ions can pass. Some transport proteins form pumps and exchangers, which are important for maintaining the correct ionic concentrations in the cells. Both pumps and exchangers have the ability to transport ions in the opposite direction of the flow generated by concentration gradients and electrical fields. In addition to the pumps and exchangers, certain proteins form channels in the membrane, through which ions may flow passively along the direction of the electrochemical gradient which is a function of transmembrane voltage and ion concentrations. These channels are extremely important for the behavior of excitable cells because most of the channels are highly selective regarding which ions are allowed to pass. This property of the channels is essential for generating and maintaining the potential difference across the membrane. The channels, so-called *gating channels*, also have the ability to open and close in response to changes in the transmembrane voltage or the presence of

ligand molecules, and this ability is essential for the signal propagation in excitable tissue. Together with the constitutive equations for the cellular state variables $\mathbf{w}$ system (1) can be rewritten as:

$$\begin{cases} a_m(c_m v_t + \mathcal{I}_{ion}(v,\mathbf{w})) - \nabla\cdot(\sigma_i(x)\nabla u_i) = \mathcal{I}_{s,i} & \text{in} \quad \Omega\times\mathbb{R}_{>0}, \\ a_m(c_m v_t + \mathcal{I}_{ion}(v,\mathbf{w})) + \nabla\cdot(\sigma_e(x)\nabla u_e) = \mathcal{I}_{s,e} & \text{in} \quad \Omega\times\mathbb{R}_{>0}, \\ \mathbf{w}_t + \mathbf{g}(v,\mathbf{w}) = \mathbf{0} & \text{in} \quad \Omega\times\mathbb{R}_{>0}, \\ (\sigma_i(x)\nabla u_i)\cdot\nu = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_{>0}, \\ (\sigma_e(x)\nabla u_e)\cdot\nu = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_{>0}, \\ v(\cdot,0) = v_0, \quad \mathbf{w}(\cdot,0) = \mathbf{w}_0 & \text{in} \quad \Omega, \end{cases} \tag{2}$$

where $\mathbf{g}(v,\mathbf{w})$ is a vector function that depends on the electrophysiological behavior of the heart cells. For simplicity, let us consider that we have only one celullar state variable w and therefore a scalar function g.

Some typical models for the ionic current include the FitzHugh-Nagumo model (see [10])

$$\mathcal{I}_{ion}(v,w) = bv(a-v)(1-v) + c\,w \quad \text{and} \quad g(v,w) = -\gamma(v-\beta w) \tag{3}$$

as well as the Rogers-McCulloch model (see [18])

$$\mathcal{I}_{ion}(v,w) = bv(a-v)(1-v) + c\,vw \quad \text{and} \quad g(v,w) = -\gamma(v-\beta w) \tag{4}$$

where a, b, c, γ, β are positive "membrane" parameters that define the shape of the action potential pulse. For additional discussion of related physiological models leading to systems comparable to (2) we refer to [13] and for one dimensional models [16, 17].

Since the bidomain model for the electrical activity in the heart which is difficult to solve and analyze, by making an assumption on the conductivity tensors σ_i and σ_e, it is possible to simplify the model. Precisely, if we assume *equal anisotropy rates*, i.e. $\sigma_e = \mu\sigma_i$, where μ is a constant scalar, then σ_i can be "eliminated" from (2), resulting in

$$\begin{cases} a_m(c_m v_t + \mathcal{I}_{ion}(v,w)) - \frac{1}{1+\mu}\nabla\cdot(\sigma_e(x)\nabla v) = \frac{1}{1+\mu}(\mathcal{I}_{s,e} + \mu\,\mathcal{I}_{s,i}) & \text{in} \quad \Omega\times\mathbb{R}_{>0}, \\ w_t + g(v,w) = 0 & \text{in} \quad \Omega\times\mathbb{R}_{>0}, \\ (\sigma_e(x)\nabla v)\cdot\nu = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_{>0}, \\ v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0 & \text{in} \quad \Omega, \end{cases} \tag{5}$$

this particular reduction of the bidomain model is the so-called *monodomain model*.

In this paper we shall study controllability properties for a simplified 1D nonlinear monodomain version of (5). From now on, let us consider $\mathcal{I}_{ion}$ and g to be given by (4) (in fact, the results presented in this paper holds for both FitzHugh-Nagumo and Rogers-McCulloch models) Assuming that $L > 0$ be a positive length and $T > 0$ a positive time, system (5) takes the form:

$$\begin{cases} a_m(c_m v_t + \mathcal{I}_{ion}(v,w)) - \frac{1}{1+\mu}(\sigma_e(x)v_x)_x = \frac{\mu}{1+\mu}\,\mathcal{I}_{s,i} + \frac{1}{1+\mu}\mathcal{I}_{s,e} & \text{in} \quad (0,L)\times(0,T), \\ w_t + g(v,w) = 0 & \text{in} \quad (0,L)\times(0,T), \\ \sigma_e(x)v_x\big|_{x=0} = \sigma_e(x)v_x\big|_{x=L} = 0 & \text{in} \quad (0,T), \\ v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0 & \text{in} \quad (0,L). \end{cases} \tag{6}$$

The aim is to prove that we can steer the *transmembrane voltage-state variable* pair from any initial state (v_0, w_0) to a desired state (the final datum of a given trajectory), with the help of

an extracellular stimulation $\mathcal{I}_{s,e}$, also called *the control*, acting only on arbitrary small ω part of $(0,L)$ which moves into $(0,L)$ during an arbitrary small time interval $(0,T)$. This idea of a moving control domain to guarantee controllability has been used for many different problems in the past few years. See [15] for the pioneer work. Here we use the approach introduced in [5], relying on Carleman inequalities, which allows us to treat problems posed on bounded domains. The result will give the exact controllability to the trajectories for the monodomain model (6) as long as the control domain ω moves in an appropriate manner and covers the whole domain $(0,L)$. Without loss of generality we can suppose that $c_m = \mu = 1$ and $a_m = \frac{1}{2}$. We will deal with the controllability for the following system

$$\begin{cases} v_t + \mathcal{I}_{ion}(v,w) - (\sigma_i(x)v_x)_x = \mathcal{I}_{s,i} + \mathcal{I}_{s,e} & \text{in} \quad (0,L)\times(0,T), \\ w_t + g(v,w) = 0 & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)v_x\big|_{x=0} = \sigma_i(x)v_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0 & \text{in} \quad (0,L), \end{cases} \tag{7}$$

where $\mathcal{I}_{s,i} \in H^1(0,T;H^1(0,L))$ and $\mathcal{I}_{s,e} \in L^2((0,L)\times(0,T))$, with $\operatorname{supp}\mathcal{I}_{s,e}(\cdot,t) \subset \omega(t)\ \forall t \in (0,T)$.

Let us recall the definitions of some usual spaces in the context of parabolic equations with boundary conditions of Neumann type

$$H^2_\nu(0,L) := \{u \in H^2(0,L) : u_x\big|_{x=0} = u_x\big|_{x=L} = 0\} \quad \text{and} \quad H^3_\nu(0,L) := H^3(0,L) \cap H^2_\nu(0,L).$$

Note that, for every $(v_0,w_0) \in H^1(0,L)\times L^2(0,L)$ and every $\mathcal{I}_{s,i}, \mathcal{I}_{s,e} \in L^2((0,L)\times(0,L))$, there exists a unique solution (v,w) to (7) that satisfies (among other things)

$$(v,w) \in C^0([0,T];H^1(0,L)\times L^2(0,L)),$$

see for instance [3, Theorem 1.1].

Let us now fix a *trajectory* $(\bar{v},\bar{w})$, that is, a sufficiently regular solution to the related uncontrolled system

$$\begin{cases} \bar{v}_t + \mathcal{I}_{ion}(\bar{v},\bar{w}) - (\sigma_i(x)\bar{v}_x)_x = \mathcal{I}_{s,i} & \text{in} \quad (0,L)\times(0,T), \\ \bar{w}_t + g(\bar{v},\bar{w}) = 0 & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)\bar{v}_x\big|_{x=0} = \sigma_i(x)\bar{v}_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ \bar{v}(\cdot,0) = \bar{v}_0, \quad \bar{w}(\cdot,0) = \bar{w}_0 & \text{in} \quad (0,L), \end{cases} \tag{8}$$

with $(\bar{v}_0,\bar{w}_0) \in H^1(0,L)\times L^2(0,L)$.

The main result of this paper is the following:

Theorem 1.1 *Assume that $\sigma_i \in C^1(\overline{\Omega};\mathbb{R})$, $T>0$, $\mathcal{I}_{s,i} \in H^1(0,T;H^1(0,L))$, $(\bar{v},\bar{w})$ satisfies (8) with $(\bar{v}_0,\bar{w}_0) \in H^3_\nu(0,L)\times H^2_\nu(0,L)$ and that the control support $\omega : (0,T) \to 2^{(0,L)}$ contains a subset satisfying Assumption 1 below. Then there exists $\delta > 0$ such that whenever $(v_0,w_0) \in H^1(0,L)\times H^2_\nu(0,L)$ and*

$$\|(v_0,w_0) - (\bar{v}_0,\bar{w}_0)\|_{L^2(0,L)\times H^2_\nu(0,L)} \le \delta,$$

we can find a control $\mathcal{I}_{s,e} \in L^2((0,L)\times(0,T))$, with $\operatorname{supp}\mathcal{I}_{s,e}(\cdot,t) \subset \omega(t)\ \forall t \in (0,T)$, such that the associated state (v,w), solution of system (7), satisfies

$$(v,w)(\cdot,T) = (\bar{v},\bar{w})(\cdot,T) \quad \textit{in} \quad (0,L). \tag{9}$$

The property which is established in Theorem 1.1 is called *local exact controllability to trajectories*. It is proof will be given at the end of Section 3. It relies, in part, on arguments and results from [5] and [6] which need to be changed to account for the PDE-ODE coupling and to Neumann boundary conditions. Thus, let us set $v = \bar{v} + y$, $w = \bar{w} + z$ and let us use these identities in (7). Taking into account that $(\bar{v}, \bar{w})$ solves (8), we find:

$$\begin{cases} y_t - (\sigma_i(x)y_x)_x + \mathcal{I}_{ion}(y,z) + 3b\bar{v}y^2 + \ell_y(\bar{v},\bar{w})y + c\bar{v}z = \mathcal{I}_{s,e} & \text{in} \quad (0,L)\times(0,T), \\ z_t + g(y,z) = 0 & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)y_x\big|_{x=0} = \sigma_i(x)y_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0, \quad z(\cdot,0) = z_0 & \text{in} \quad (0,L), \end{cases} \tag{10}$$

where $\ell_y(\bar{v},\bar{w}) := 3b\bar{v}^2 - 2b(1+a)\bar{v} + c\bar{w}$ is the coefficient of the linear term in y and $y_0 := v_0 - \bar{v}_0$ and $z_0 := w_0 - \bar{w}_0$ are the initial data.

In this way the local exact controllability to the trajectories for system (7) is reduced to a local null controllability problem for the solution (y,z) to the nonlinear problem (10).

Notice that if we introduce the variables $p = \gamma e^{\gamma\beta t}y$ and $q = e^{\gamma\beta t}z$, then null controllability for system (10) is equivalent to null controllability for the system

$$\begin{cases} p_t - (\sigma_i(x)p_x)_x + \ell_p(\bar{v},\bar{w})p + \ell_q(\bar{v},\bar{w})q + \mathcal{N}(p,q) = h1_\omega & \text{in} \quad (0,L)\times(0,T), \\ q_t = p & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)p_x\big|_{x=0} = \sigma_i(x)p_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ p(\cdot,0) = p_0, \quad q(\cdot,0) = q_0 & \text{in} \quad (0,L), \end{cases} \tag{11}$$

where $\mathcal{N}(p,q) := b\gamma^{-1}e^{-\gamma\beta t}[3\bar{v} - (1+a)]p^2 + ce^{-\gamma\beta t}pq + b\gamma^{-2}e^{-2\gamma\beta t}\,p^3$ is the nonlinear term, and $\ell_p(\bar{v},\bar{w}) := \ell_y(\bar{v},\bar{w}) - \gamma\beta + ab$ and $\ell_q(\bar{v},\bar{w}) := \gamma c\bar{v}$ are the coefficients of the linear terms in p and q, respectively, $p_0 := \gamma y_0$ and $q_0 := z_0$ are the initial data, and $h := \gamma e^{\gamma\beta t}\mathcal{I}_{s,e}$ is the control and 1_ω is the characteristic function of ω.

In order to solve the latter, following a standard approach, we will first deduce (global) null controllability of a suitable linearized version, namely:

$$\begin{cases} p_t - (\sigma_i(x)p_x)_x + \ell_p(\bar{v},\bar{w})p + \ell_q(\bar{v},\bar{w})q = G + h1_\omega & \text{in} \quad (0,L)\times(0,T), \\ q_t = p & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)p_x\big|_{x=0} = \sigma_i(x)p_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ p(\cdot,0) = p_0, \quad q(\cdot,0) = q_0 & \text{in} \quad (0,L), \end{cases} \tag{12}$$

where G is an appropriate function that decays exponentially as $t \to T^-$.

Then, appropriate and rather classical arguments will be used to deduce the local null controllability of the nonlinear system (11).

Remark 1.2 *In the case of the FitzHugh-Nagumo model, i.e. with $\mathcal{I}_{ion}$ chosen as in* (3)*, the minor difference consist in $\ell_q = \gamma c$, $\ell_y(\bar{v},\bar{w}) := 3b\bar{v}^2 - 2b(1+a)\bar{v} + c$, and $\mathcal{N}(p,q) := b\gamma^{-1}e^{-\gamma\beta t}[3\bar{v} - (1+a)]p^2 + ce^{-\gamma\beta t}q + b\gamma^{-2}e^{-2\gamma\beta t}\,p^3$.*
Let us note in particular that for $q_0 = 0$, $\ell_q = 1$, and $G = 0$, system (12) *becomes*

$$\begin{cases} p_t - (\sigma_i(x)p_x)_x + d\,p + \displaystyle\int_0^t p(x,s)\,ds = h1_\omega & \textit{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)p_x\big|_{x=0} = \sigma_i(x)p_x\big|_{x=L} = 0 & \textit{on} \quad (0,T), \\ p(\cdot,0) = p_0 & \textit{in} \quad (0,L), \end{cases} \tag{13}$$

with $d = \ell_p(\bar{v}, \bar{w})$. Proposition 3 below will imply (global) null controllability for the heat equation with memory.

Exact controllability is a challenging topic which has received a tremendous amount of attention in the literature. We only comment on a few, closely related publications. For this purpose we first note that the linearized system (12) above can be transformed into a parabolic system with a distributed memory term, and an additional time-dependent source term if $q_0 \neq 0$. In [3] optimal control of the monodomain equations is considered and the question of approximate controllability of the heat equation with Dirichlet boundary conditions and with memory term is raised as interesting open problem. In [12] it was verified that the heat equation with Dirichlet boundary conditions and memory terms is not exactly null controllable. A related negative result was obtained in [4] were it was shown that the linearized monodomain system with Neumann boundary conditions is not exactly null controllable for a range of physically relevant parameters. All these negative results refer to the case that the control domain is a fixed domain ω strictly contained in Ω. In the recent publication [7] exact null controllability of the linear heat equation with memory terms and Dirichlet boundary condition is obtained if the control domain sweeps the domain Ω appropriately.

In this paper, we are going to treat different situations which lead to new difficulties compared to the previous works on parabolic equations with memory terms. Let us discuss these differences:

- Neumann boundary condition. The main strategy consist of a Carleman inequality with suitable weights which allows to simultaneously treat the parabolic equation with Neumann boundary conditions and the ODEs. The Carleman inequalities obtained in [7] are inequalities for parabolic equations with Dirichlet boundary condition and cannot be applied for our situation.

- Nonlinearities. Here we are going to consider two kinds of (cubic) nonlinearities arising in the FitzHugh Nagumo and Rogers-McCulloch models which differ due to the way in which the memory term w appears in the PDE equation, compare (3) and (4). The main argument to obtain the exact control result for the nonlinear system relies on the Liusternik inverse mapping theorem.

- Inhomogeneities. Another difficulty to verify exact null controllability arises due to the inhomogeneities G and q_0 in the linear PDE-ODE system.

The paper is organized as follows. In Section 2, we shall present Carleman inequalities with moving observations and regularity results for trajectories. Section 3 deals with the null controllability for linear monodomain equations and the local exact controllability to trajectories for the nonlinear monodomain equations. Finally, in Section A, we present a Carleman inequality for a parabolic equation with Neumann boundary conditions and moving observations.

2 Preliminaries

This section is devoted to introduce some appropriate Carleman inequalities with moving observations for ODE's and PDE's and regularity results for the trajectory solutions for (8).

2.1 Carleman inequalities with moving observations

In this section, we shall present the main assumptions on moving support of the controls and the observations and their consequences in terms of Carleman inequalities. Many of them were introduced or inspired by [1, 5, 9, 15]. These results will play a crucial role in order to obtain the exact controllability to trajectories for the monodomain equations. The control domain ω is required to contain a subset ω_0 which is required to satisfy the following geometric requirements. They are needed for the construction of a suitable weight function for the Carleman estimate. First, let us use the following notation to stand for the control domain:

$$\omega : [0,T] \to 2^{(0,L)},$$

meaning that any $t \in [0,T]$ is associated to a set $\omega(t) \subset (0,L)$ or also, in the space-time domain $[0,L] \times [0,T]$, the control domain ω can be refer to the set $\bigcup_{t\in[0,T]} \omega(t) \times \{t\}$. Throughout this paper, ω is required to contain (in a sense given below) a subset $\omega_0 : [0,T] \to 2^{(0,L)}$ satisfying the following geometric requirements:

Requirements 1 *There exist two times t_1, t_2 with $0 < t_1 < t_2 < T$ such that:*

a) The interval $\omega_0(t) \neq (0,L)$, for all $t \in (0,T)$;

b) $\bigcup\limits_{t\in(0,T)} \omega_0(t) = (0,L)$;

c) $(0,L) \setminus \omega_0(t)$ is nonempty and connected in $(0,L)$ for any $t \in (0,t_1] \cup [t_2,T)$;

d) $(0,L) \setminus \omega_0(t)$ has two nonempty connected components in $(0,L)$ for any $t \in (t_1,t_2)$;

Let $\omega_1 : [0,T] \to 2^{(0,L)}$ be a subset of ω such that

$$\overline{\omega}_0 \subset \mathring{\overline{\omega}}_1 \subset \overline{\omega}_1 \subset \mathring{\overline{\omega}}, \tag{14}$$

where $\mathring{\overline{\omega}}_1$ and $\mathring{\overline{\omega}}$ denote the relative interiors with respect to $[0,L] \times [0,T]$ of $\overline{\omega}_1$ and $\overline{\omega}$, respectively.

We are now prepared for the construction of a suitable weight function.

Lemma 1 *There exist a positive number $\tau \in (0, \min\{1, T/2\})$, a positive constant $C_0 > 0$, and a function $\eta \in C^\infty([0,L] \times [0,T])$ such that*

$$\eta_x(x,t) \neq 0 \qquad \forall x \in \overline{(0,L) \setminus \omega_0(t)},\ \forall t \in [0,T], \tag{15}$$

$$\eta_t(x,t) \neq 0 \qquad \forall x \in \overline{(0,L) \setminus \omega_0(t)},\ \forall t \in [0,T], \tag{16}$$

$$\eta_t(x,t) > 0 \qquad \forall x \in \overline{(0,L) \setminus \omega_0(t)},\ \forall t \in [0,\tau], \tag{17}$$

$$\eta_t(x,t) < 0 \qquad \forall x \in \overline{(0,L) \setminus \omega_0(t)},\ \forall t \in [T-\tau,T], \tag{18}$$

$$\eta_x(0,t) \geq C_0 \qquad \forall t \in [0,T], \tag{19}$$

$$\eta_x(L,t) \leq -C_0 \qquad \forall t \in [0,T], \tag{20}$$

$$\min_{(x,t)\in[0,L]\times[0,T]} \{\eta(x,t)\} = \frac{3}{4}\|\eta\|_{L^\infty([0,L]\times[0,T])}. \tag{21}$$

The proof of Lemma 1 can by obtained by similar arguments as in [5, Appendix A]. It differs with respect to properties (19) and (20).

Next, we introduce a real function $r \in C^\infty(0,T)$, symmetric with respect to $t = \frac{T}{2}$, i.e. $r(t) = r(T-t)$ for any $t \in (0,T)$, and such that for some $\tau > 0$

$$r(t) = \begin{cases} \dfrac{1}{t} & \text{for} \quad 0 < t \leq \frac{\tau}{2}, \\ \text{strictly decreasing} & \text{for} \quad \frac{\tau}{2} < t < \tau, \\ 1 & \text{for} \quad \tau \leq t \leq \frac{T}{2}. \end{cases} \tag{22}$$

For this choices of η and r let us define the weights

$$\alpha(x,t) := r(t)(e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x,t)}) \quad \text{and} \quad \xi(x,t) := r(t)e^{\lambda\eta(x,t)} \quad \forall (x,t) \in (0,L)\times(0,T), \tag{23}$$

where $\lambda > 0$ is a sufficiently large parameter that will be chosen later. We have the following technical result.

Lemma 2 *There exist positive real numbers $\lambda_1 > 0$, $s_1 > 0$ and $C_1 > 0$ (depending on L and ω_1) such that for all $\lambda \geq \lambda_1$, all $s \geq s_1$ and all $\varphi \in H^1(0,T;L^2(0,L))$, the following inequality holds*

$$s\lambda^2 \int_0^T\!\!\int_0^L e^{-2s\alpha}\xi|\varphi|^2\,dxdt \leq C_1\left(\int_0^T\!\!\int_0^L e^{-2s\alpha}|\varphi_t|^2\,dxdt + s^2\lambda^2\int_0^T\!\!\int_{\omega_1(t)} e^{-2s\alpha}\xi^2|\varphi|^2\,dxdt\right). \tag{24}$$

The proof of this result is obtained by similar arguments as in [5, Appendix C] and relies on ideas from [1].

For our purposes, we also need the following Carleman inequality for the heat equation with Neumann boundary conditions:

Lemma 3 *There exist constants $\lambda_2 > 0$, $s_2 > 0$ and $C_2 > 0$ (depending on L and ω_1) such that for any $\lambda \geq \lambda_2$, any $s \geq s_2(\lambda)$, and any terminal datum $\psi_T \in L^2(0,L)$, and any source term $f \in L^2((0,L)\times(0,T))$, the unique weak solution for*

$$\begin{cases} -\psi_t - (\sigma_i(x)\psi_x)_x = f & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)\psi_x\big|_{x=0} = \sigma_i(x)\psi_x\big|_{x=L} = 0 & \text{in} \quad (0,T), \\ \psi(T) = \psi_T & \text{in} \quad (0,L), \end{cases}$$

satisfies

$$\begin{aligned} & s^{-1}\int_0^T\!\!\int_0^L e^{-2s\alpha}\xi^{-1}(|\psi_{xx}|^2 + |\psi_t|^2)\,dxdt \\ & + s\lambda^2\int_0^T\!\!\int_0^L e^{-2s\alpha}\xi|\psi_x|^2\,dxdt + s^3\lambda^4\int_0^T\!\!\int_0^L e^{-2s\alpha}\xi^3|\psi|^2]\,dxdt \\ & + s^3\lambda^3\int_0^T e^{-2s\alpha}\xi^3|\psi|^2\big|_{x=L} + s^3\lambda^3\int_0^T e^{-2s\alpha}\xi^3|\psi|^2\big|_{x=0} \\ & \leq C_2\left(\int_0^T\!\!\int_0^L e^{-2s\alpha}|f|^2\,dxdt + s^3\lambda^4\int_0^T\!\!\int_{\omega_1(t)} e^{-2s\alpha}\xi^3|\psi|^2\,dxdt\right). \end{aligned} \tag{25}$$

We will give a proof for this result in Appendix A.

2.2 Regular trajectories

In this section let us present some regularity results for the uncontrolled solutions to (8). We have the following result:

Proposition 1 *Let $\mathcal{I}_{s,i} \in L^2(0,T;L^2(0,L))$ and $(\bar{v}_0, \bar{w}_0) \in H^1(0,L) \times H^2_\nu(0,L)$. Then, (8) possesses exactly one solution $(\bar{v}, \bar{w})$, with*

$$\begin{cases} \bar{v} \in L^2(0,T;H^2_\nu(0,L)) \cap H^1(0,T;L^2(0,L)) \cap C^0([0,T];H^1(0,L)) \\ \bar{w} \in H^1(0,T;H^2_\nu(0,L)) \cap C^1([0,T];H^1(0,L)). \end{cases} \tag{26}$$

For the proof, we just have to adapt the proof of [3, Theorem 1.1], taking into account that $\bar{w}_0 \not\equiv 0$ and the boundary conditions are Neumann instead of Dirichlet. In this way, we can also obtain the following result.

Proposition 2 *Let $\mathcal{I}_{s,i} \in H^1(0,T;H^1(0,L))$ and $(\bar{v}_0, \bar{w}_0) \in H^3_\nu(0,L) \times H^2_\nu(0,L)$. Then, (8) possesses exactly one solution $(\bar{v}, \bar{w})$, with $\bar{v}, \bar{w} \in W^{1,\infty}(0,T;L^\infty(0,L))$.*

Proof: By Proposition 1 and that the spatial dimension is 1, the nonlinearity $I_{ion}(\bar{v}, \bar{w})$ belongs to the space $H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L)) \cap L^\infty(0,T;L^\infty(0,L))$. Hence $\mathcal{I}_{s,i} - I_{ion}(\bar{v}, \bar{w})$ belongs to $H^1(0,L;L^2(0,L))$. Since $\bar{v}_0 \in H^2_\nu(0,L)$, thanks to [8, Theorem 5, p. 361], we have

$$\begin{cases} \bar{v} \in L^\infty(0,T;H^2_\nu(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H^1(0,L)) \\ \bar{w} \in W^{1,\infty}(0,T;H^2_\nu(0,L)) \cap H^2(0,T;H^1(0,L)) \cap W^{2,\infty}(0,T;L^2(0,L)). \end{cases} \tag{27}$$

Next, taking the spatial derivative in the first equation of (8) we obtain a parabolic equation for $\bar{v}_x$ with right hand side $\partial_x(\mathcal{I}_{s,i} - I_{ion}(\bar{v}, \bar{w}))$, homogeneous Dirichlet boundary conditions, and initial condition $\bar{v}_{0,x}$. By (27) we have that $\partial_x(\mathcal{I}_{s,i} - I_{ion}(\bar{v}, \bar{w}))$ belongs to $H^1(0,L;L^2(0,L))$. Again, since $\bar{v}_{0,x} \in H^2(0,L) \cap H^1_0(0,L)$ and thanks to [8, Theorem 5, p. 361], we have

$$\bar{v}_x \in L^\infty(0,T;H^2(0,L) \cap H^1_0(0,L)) \cap W^{1,\infty}(0,T;L^2(0,L)) \cap H^1(0,T;H^1_0(0,L)). \tag{28}$$

Finally, from (27) and (28), we deduce that

$$\begin{cases} \bar{v} \in L^\infty(0,T;H^3_\nu(0,L)) \cap W^{1,\infty}(0,T;H^1(0,L)) \cap H^1(0,T;H^2_\nu(0,L)) \\ \bar{w} \in W^{1,\infty}(0,T;H^2_\nu(0,L)) \cap H^2(0,T;H^1(0,L)) \cap W^{2,\infty}(0,T;L^2(0,L)). \end{cases} \tag{29}$$

■

Remark 2.1 *As a consequence of the previous result we conclude that*

$$\ell_p(\bar{v}, \bar{w}), \ell_q(\bar{v}, \bar{w}) \in W^{1,\infty}(0,T;H^1(0,L)).$$

3 Exact controllability to trajectories of the monodomain equations

3.1 Controllability for the linear monodomain model

In this Section, we will present a suitable Carleman inequality for a properly chosen adjoint system. This will lead to the null controllability result for (12), see Proposition 3 below.

Notice that, for every $(p_0, q_0) \in L^2(0,L) \times H^1(0,L)$ and every $G, h1_\omega \in L^2((0,L)\times(0,T))$, there exists a unique weak solution (p,q) to (12) that satisfies

$$p \in L^2(0,T;H^1(0,L)) \cap W^{1,2}(0,T;(H^1(0,L))')$$

and

$$q \in W^{2,2}(0,T;(H^1(0,L))') \cap W^{1,2}(0,T;H^1(0,L))$$

and the variational formulation for a.e. $t \in [0,T]$

$$\left|\begin{array}{l} \langle p_t, w\rangle + (\sigma p_x, w_x) + (\ell_p(\bar v,\bar w)p, w) + (\ell_q(\bar v,\bar w)q, w) = (G + h1_\omega, w),\ \forall w \in H^1(0,L), \\ (q_t, v) = (p, v),\ \forall v \in L^2(0,L), \\ p(\cdot,0) = p_0 \text{ and } q(\cdot,0) = q_0, \end{array}\right. \tag{30}$$

where $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{(H^1(0,L))',H^1(0,L)}$.

The following analysis depends in a crucial manner on a transformation of (12) which takes the control from the PDE to the ODE. In view of the regularity assumed for the initial data and their boundary conditions we will verify that null controllability for (12) is equivalent to null controllability for

$$\left\{\begin{array}{lll} \theta_t + \{\ell_q(\bar v,\bar w) - [\ell_p(\bar v,\bar w)]_t\}q = G + h1_\omega & \text{in} & (0,L)\times(0,T), \\ q_t - (\sigma_i(x)q_x)_x + \ell_p(\bar v,\bar w)q = \theta & \text{in} & (0,L)\times(0,T), \\ \sigma_i(x)q_x\big|_{x=0} = \sigma_i(x)q_x\big|_{x=L} = 0 & \text{on} & (0,T), \\ \theta(\cdot,0) = \theta_0, \quad q(\cdot,0) = q_0 & \text{in} & (0,L), \end{array}\right. \tag{31}$$

where $\theta_0 = p_0 - (\sigma_i(x)q_{0,x})_x + \ell_p(\bar v_0,\bar w_0)q_0$.

First we verify the equivalence of systems (12) and (31). Note that, for every $(p_0, q_0) \in L^2(0,L) \times H^1(0,L)$ and every $G, h1_\omega \in L^2((0,L)\times(0,T))$, there exists a unique weak solution (θ, q) to (31) that satisfies

$$\theta \in W^{1,2}(0,T;(H^1(0,L))')$$

and

$$q \in W^{1,2}(0,T;H^1(0,L)) \cap W^{2,2}(0,T;(H^1(0,L))')$$

and the variational formulation for a.e. $t \in [0,T]$

$$\left|\begin{array}{l} \langle\theta_t, v\rangle + (\{\ell_q(\bar v,\bar w) - [\ell_p(\bar v,\bar w)]_t\}q, v) = (G + h1_\omega, v),\ \forall v \in H^1(0,L) \\ \langle q_t, w\rangle + (\sigma q_x, w_x) + (\ell_p(\bar v,\bar w)q, w) = \langle\theta, w\rangle,\ \forall w \in H^1(0,L) \\ \theta(\cdot,0) = \theta_0 \text{ and } q(\cdot,0) = q_0, \end{array}\right. \tag{32}$$

where again $\theta_0 = p_0 - (\sigma_i(x)q_{0,x})_x + \ell_p(\bar v_0,\bar w_0)q_0$.

To check the equivalence one can start by defining $\theta \in W^{1,2}(0,T;(H^1(0,L))')$ such that

$$\left|\begin{array}{lll} \theta_t + \{\ell_q(\bar v,\bar w) - [\ell_p(\bar v,\bar w)]_t\}q = G + h1_\omega & \text{in} & (0,L)\times(0,T), \\ \theta(\cdot,0) = \theta_0 & \text{in} & (0,L), \end{array}\right.$$

Then for each $w \in H^1(0,L)$ and integrating w.r.t t the first equation in (30)

$$\begin{aligned}\langle q_t, w\rangle + (\sigma q_x, w_x) =& \left(\int_0^t [G + h1_\omega - \ell_p(\bar{v},\bar{w})p - \ell_q(\bar{v},\bar{w})q]ds, w\right)\\ &+ (p_0, w) + (\sigma q_{0,x}, w_x)\\ =&\left\langle \int_0^t \theta_t ds, w\right\rangle + \left(\int_0^t (\ell_p(\bar{v},\bar{w})q)_t ds, w\right) + (p_0, w) + (\sigma q_{0,x}, w_x)\\ =&\langle \theta, w\rangle + (\ell_p(\bar{v},\bar{w})q, w) + \langle \theta(0) - \theta_0, w\rangle\\ =&\langle \theta, w\rangle + (\ell_p(\bar{v},\bar{w})q, w).\end{aligned}$$

Thus (θ, q) is the variational solution of (32). Conversely, let $(p_0, q_0) \in L^2(0,L) \times H^1(0,L)$ then $\theta_0 = p_0 - (\sigma_i(x)q_{0,x})_x + \ell_p(\bar{v}_0,\bar{w}_0)q_0 \in (H^1(0,L))'$ and

$$\theta \in W^{1,2}(0,T;(H^1(0,L))')$$

and

$$q \in W^{1,2}(0,T;H^1(0,L)) \cap W^{2,2}(0,T;(H^1(0,L))')$$

Consider

$$\langle q_t, w\rangle + (\sigma q_x, w_x) + (\ell_p(\bar{v},\bar{w})q, w) = \langle \theta, w\rangle \ \forall w \in H^1(\Omega), \text{ a.e. } t \in [0,T].$$

Regularity allows to differentiate w.r.t t, setting $p = q_t$

$$\begin{aligned}\langle p_t, w\rangle + (\sigma p_x, w_x) + (\ell_p(\bar{v},\bar{w})p, w) =& \langle \theta_t, w\rangle - (\{\ell_p(\bar{v},\bar{w})\}_t q, w)\\ =& -(\ell_q(\bar{v},\bar{w})q, w) + (G + h1_\omega, w).\end{aligned}$$

Thus (p, q) is the variational solution of (30).

Let us introduce the function $A(x,t) := \{\ell_q(\bar{v},\bar{w}) - [\ell_p(\bar{v},\bar{w})]_t\}$ and the non-autonomous elliptic operator $Kq := -(\sigma_i(x)q_x)_x + \ell_p(\bar{v},\bar{w})q$. Then, the controllability for system (31) is formulated as follows: for any $(\theta_0, q_0) \in L^2(0,L) \times L^2(0,L)$ and any G satisfying (50) below, there exists a control $h \in L^2((0,L) \times (0,T))$, where its support contains a subset satisfying Assumption 1, such that the associated solution to

$$\begin{cases} \theta_t + A(x,t)\, q = G + h1_\omega & \text{in} \quad (0,L) \times (0,T),\\ q_t + Kq = \theta & \text{in} \quad (0,L) \times (0,T),\\ \sigma_i(x)q_x\big|_{x=0} = \sigma_i(x)q_x\big|_{x=L} = 0 & \text{on} \quad (0,T),\\ \theta(\cdot,0) = \theta_0, \quad q(\cdot,0) = q_0 & \text{in} \quad (0,L), \end{cases} \tag{33}$$

satisfies

$$\theta(\cdot,T) = q(\cdot,T) = 0 \quad \text{in} \quad (0,L).$$

3.1.1 Carleman inequality

In this Section, we will present a suitable Carleman inequality for the so called adjoint of (33), namely:

$$\begin{cases} -\varphi_t - \psi = R & \text{in} \quad Q,\\ -\psi_t + K\psi + A(x,t)\varphi = S & \text{in} \quad Q,\\ \sigma_i(x)\psi_x\big|_{x=0} = \sigma_i(x)\psi_x\big|_{x=L} = 0 & \text{on} \quad \Sigma,\\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & \text{in} \quad \Omega. \end{cases} \tag{34}$$

where $R, S \in L^2((0,L)\times(0,T))$.

Note that, for every $(\varphi_T, \psi_T) \in L^2(0,L)\times L^2(0,L)$ and every $R, S \in L^2((0,L)\times(0,T))$ there exists a unique weak solution (φ,ψ) to (34) that satisfies

$$\varphi \in H^1(0,T;L^2(0,L)) \quad \text{and} \quad \psi \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

and the variational formulation

$$\left|\begin{array}{l} -(\varphi_t, v) - (\psi, v) = (R, v)\ \forall v \in L^2(0,L),\ \text{a.e. } t\in[0,T] \\ -\dfrac{d}{dt}(\psi, w) + (\sigma\psi_x, w_x) + (\ell_p(\bar v,\bar w)\psi, w) + (A\varphi, w) = (S, w)\ \forall w \in H^1(\Omega),\ \text{a.e. } t\in[0,T] \\ \varphi(\cdot,T) = \varphi_T \text{ and } \psi(\cdot,T) = \psi_T. \end{array}\right.$$

In order to prove the linear controllability result, we have to establish an appropriate Carleman estimate. Precisely, we have the following result:

Theorem 3.1 *There exist positive constants s_3, $\lambda_3 \geq 1$ and $C_3 > 0$, only depending on L and ω, such that, for any $\varphi_T, \psi_T \in L^2(0,L)$ the solution (ψ,φ) to the adjoint system* (34) *satisfies:*

$$\begin{aligned} &\int_0^T\!\!\int_0^L [(s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2 + \lambda^2(s\xi)|\varphi|^2]e^{-2s\alpha}dxdt \\ &\leq C_3\left(\int_0^T\!\!\int_0^L [\lambda^4(s\xi)^3|R|^2 + |S|^2]e^{-2s\alpha}dxdt + s^7\lambda^6\int_0^T\!\!\int_{\omega(t)} \xi^7|\varphi|^2e^{-2s\alpha}dxdt\right). \end{aligned}$$

for all $s \geq s_3(T+T^2)$ and for all $\lambda \geq \lambda_3$.

Proof: First, applying Lemma 2 to $(34)_1$, we obtain

$$s\lambda^2\int_0^T\!\!\int_0^L \xi|\varphi|^2e^{-2s\alpha}dxdt \leq C_1\left(\int_0^T\!\!\int_0^L |\psi+R|^2e^{-2s\alpha}dxdt + s^2\lambda^2\int_0^T\!\!\int_{\omega_1(t)} \xi^2|\varphi|^2e^{-2s\alpha}dxdt\right). \tag{35}$$

Next, applying Lemma 3 to $(34)_2$, we obtain

$$\begin{aligned} &\int_0^T\!\!\int_0^L [(s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2]e^{-2s\alpha}\,dx\,dt \\ &\leq C_2\left(\int_0^T\!\!\int_0^L |S - \ell_y(\bar v,\bar w)\psi - A(x,t)\varphi|^2e^{-2s\alpha}\,dx\,dt + \int_0^T\!\!\int_{\omega_1(t)} \lambda^4(s\xi)^3|\psi|^2e^{-2s\alpha}\,dx\,dt\right). \end{aligned} \tag{36}$$

Adding (35) and (36), and absorbing the lower order terms from the right hand side by taking λ large enough, we get

$$\begin{aligned} &\int_0^T\!\!\int_0^L [((s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2 + \lambda^2(s\xi)|\varphi|^2]e^{-2s\alpha}dxdt \\ &\leq C\left(\int_0^T\!\!\int_0^L (|R|^2 + |S|^2)e^{-2s\alpha}dxdt + \int_0^T\!\!\int_{\omega_1(t)} [\lambda^2(s\xi)^2|\varphi|^2 + \lambda^4(s\xi)^3|\psi|^2]e^{-2s\alpha}dxdt\right). \end{aligned} \tag{37}$$

Now, we need to eliminate the local integral of ψ appearing in the right hand side of (37). To do that, let us introduce a function $\zeta \in C^\infty([0,L]\times[0,T])$ satisfying

$$\begin{array}{ll} 0 \leq \zeta \leq 1 & \forall (x,t) \in [0,L]\times[0,T], \\ \zeta(x,t) = 1 & \forall x \in \omega_1(t),\ \forall t \in [0,T], \\ \zeta(x,t) = 0 & \forall x \in [0,L]\setminus\overline{\omega(t)},\ \forall t \in [0,T]. \end{array} \tag{38}$$

This way, we have that

$$s^3\lambda^4\int_0^T\int_{\omega'(t)}\xi^3|\psi|^2e^{-2s\alpha}dxdt\leq s^3\lambda^4\int_0^T\int_0^L\zeta\xi^3|\psi|^2e^{-2s\alpha}dxdt. \tag{39}$$

Then, using $(34)_1$ and integrating by parts with respect to t, we obtain

$$\begin{aligned}
s^3\lambda^4\int_0^T\int_0^L\zeta\xi^3|\psi|^2e^{-2s\alpha}dxdt =& -s^3\lambda^4\int_0^T\int_0^L\zeta\xi^3\psi(\varphi_t+R)e^{-2s\alpha}dxdt\\
=& s^3\lambda^4\int_0^T\int_0^L\zeta\xi^3\psi_t\varphi e^{-2s\alpha}dxdt+s^3\lambda^4\int_0^T\int_0^L\zeta_t\xi^3\varphi\psi e^{-2s\alpha}dxdt\\
&+s^3\lambda^4\int_0^T\int_0^L\zeta\xi^2(3\xi_t-2s\xi\alpha_t)\varphi\psi e^{-2s\alpha}dxdt\\
&-s^3\lambda^4\int_0^T\int_0^L\zeta\xi^3\psi Re^{-2s\alpha}dxdt\\
:=& D_1+D_2+D_3+D_4.
\end{aligned}$$

For D_1, we notice that for every $\epsilon>0$, we obtain

$$\begin{aligned}
|D_1|&\leq\epsilon\int_0^T\int_0^L(s\xi)^{-1}|\psi_t|^2e^{-2s\alpha}dxdt+C_\epsilon s^7\lambda^8\int_0^T\int_0^L\zeta^2\xi^7|\varphi|^2e^{-2s\alpha}dxdt\\
&\leq\epsilon\int_0^T\int_0^L(s\xi)^{-1}|\psi_t|^2e^{-2s\alpha}dxdt+C_\epsilon s^7\lambda^8\int_0^T\int_{\omega(t)}\xi^7|\varphi|^2e^{-2s\alpha}dxdt.
\end{aligned}\tag{40}$$

Next, D_2 is estimated like the term D_1:

$$|D_2|\leq\epsilon s^3\lambda^4\int_0^T\int_0^L\xi^3|\psi|^2e^{-2s\alpha}dxdt+C_\epsilon s^3\lambda^4\int_0^T\int_{\omega(t)}\xi^3|\varphi|^2e^{-2s\alpha}dxdt. \tag{41}$$

for any $\epsilon>0$.

Since $|\xi_t|+|\alpha_t|\leq C\lambda\xi^2$, for every $\epsilon>0$, we infer that

$$\begin{aligned}
|D_3|&\leq Cs^4\lambda^5\int_0^T\int_0^L\zeta\xi^5|\varphi\psi|e^{-2s\alpha}dxdt\\
&\leq\epsilon s^3\lambda^4\int_0^T\int_0^L\xi^3|\psi|^2e^{-2s\alpha}dxdt+C_\epsilon s^5\lambda^6\int_0^T\int_{\omega(t)}\xi^7|\varphi|^2e^{-2s\alpha}dxdt.
\end{aligned}$$

Finally, D_4 is estimated as follows:

$$|D_4|\leq\epsilon s^3\lambda^4\int_0^T\int_0^L\xi^3|\psi|^2e^{-2s\alpha}dxdt+C_\epsilon s^3\lambda^4\int_0^T\int_0^L\xi^3|R|^2e^{-2s\alpha}dxdt. \tag{42}$$

for any $\epsilon>0$.

Combining (37) and (39)-(42), and absorbing the lower order terms, we get

$$\begin{aligned}
&\int_0^T\int_0^L[((s\xi)^{-1}(|\psi_{xx}|^2+|\psi_t|^2)+\lambda^2(s\xi)|\psi_x|^2+\lambda^4(s\xi)^3|\psi|^2+\lambda^2(s\xi)|\varphi|^2]e^{-2s\alpha}dxdt\\
&\leq C_3\left(\int_0^T\int_0^L[\lambda^4(s\xi)^3|R|^2+|S|^2]e^{-2s\alpha}dxdt+s^7\lambda^6\int_0^T\int_{\omega(t)}\xi^7|\varphi|^2e^{-2s\alpha}dxdt\right).
\end{aligned}\tag{43}$$

■

3.1.2 Null controllability

We will now deduce a second Carleman inequality with weights that do not vanish at $t = 0$. More precisely, let us consider the function

$$l(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le T/2, \\ r(t) & \text{for} \quad T/2 \le t \le T, \end{cases}$$

with r defined in (22), and the following associated weight functions:

$$\begin{aligned}
&\bar{\alpha}(x,t) := l(t)(e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x,t)}) && \forall (x,t) \in (0,L)\times(0,T),\\
&\bar{\xi}(x,t) := l(t)e^{\lambda\eta(x,t)} && \forall (x,t) \in (0,L)\times(0,T),\\
&\bar{\alpha}^*(t) := \min_{x\in[0,L]} \bar{\alpha}(x,t) = l(t)(e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta(\cdot,t)\|_\infty}), && \forall t \in (0,T),\\
&\bar{\xi}^*(t) := \max_{x\in[0,L]} \bar{\xi}(x,t) = l(t)e^{\lambda\|\eta(\cdot,t)\|_\infty} && \forall t \in (0,T),\\
&\hat{\bar{\alpha}}(t) := \max_{x\in[0,L]} \bar{\alpha}(x,t), && \forall t \in (0,T),\\
&\hat{\bar{\xi}}(t) := \min_{x\in[0,L]} \bar{\xi}(x,t) && \forall t \in (0,T),
\end{aligned}$$

and

$$\rho_1(t) = (\bar{\xi}^*)^{-3/2}e^{s\bar{\alpha}^*}, \;\; \rho_2(t) = e^{s\bar{\alpha}^*}, \;\; \rho_3(t) = (\bar{\xi}^*)^{-7/2}e^{s\bar{\alpha}^*}, \;\; \rho_4(t) = (\hat{\bar{\xi}})^{-1/2}e^{s\hat{\bar{\alpha}}}.$$

Remark 3.2 *Notice that $\bar{\alpha}^*$ and $\hat{\bar{\alpha}}$ blow up exponentially as $t \to T^-$ and $\bar{\xi}^*$ and $\hat{\bar{\xi}}$ blow up polynomially as $t \to T^-$.*

By combining Theorem 3.1 and classical energy estimates satisfied by φ and ψ, we easily deduce the following result:

Lemma 4 *Under the assumptions of Theorem 1.1, there exist positive constants s_4, $\lambda_4 \ge 1$ and $C_4 > 0$, only depending on L and ω such that, for any $\varphi_T, \psi_T \in L^2(0,L)$ and any $R, S \in L^2((0,L)\times(0,T))$, the solution to the adjoint system (34) satisfies:*

$$\begin{aligned}
&\int_0^T\!\!\int_0^L [\bar{\xi}|\psi_x|^2 + \bar{\xi}^3|\psi|^2 + \bar{\xi}|\varphi|^2]e^{-2s\bar{\alpha}}dxdt + \|\varphi(\cdot,0)\|^2_{L^2(0,L)} + \|\psi(\cdot,0)\|^2_{L^2(0,L)}\\
&\le C_4\left(\int_0^T\!\!\int_0^L [\rho_1^{-2}|R|^2 + \rho_2^{-2}|S|^2]dxdt + \int_0^T\!\!\int_{\omega(t)} \rho_3^{-2}|\varphi|^2 dxdt\right). \qquad (44)
\end{aligned}$$

for all $s \ge s_4(T + T^2)$ and for all $\lambda \ge \lambda_4$.

Proof: Let us start by proving the following estimate for system (34):

$$\|\varphi\|^2_{H^1(0,T/2;L^2(0,L))} + \|\psi\|^2_{L^2(0,T/2;H^1_\nu(0,L))\cap C^0([0,T/2];L^2(0,L))} \qquad (45)$$

$$\le C\left(\|(R,S)\|^2_{L^2(0,3T/4;[L^2(0,L)]^2)} + \frac{1}{T^2}\|(\varphi,\psi)\|^2_{L^2(T/2,3T/4;[L^2(0,L)]^2)}\right). \qquad (46)$$

To do that, let us introduce a function $\kappa \in C^1([0,T])$ with

$$\kappa \equiv 1 \quad \text{in} \quad [0,T/2], \qquad \kappa \equiv 0 \quad \text{in} \quad [3T/4,T], \qquad |\kappa'| \le C/T,$$

for some $C > 0$. Using classical energy estimates for the system satisfied by $(\kappa\varphi, \kappa\psi)$, which is analogous to (34), we obtain

$$\begin{aligned}&\|\kappa\varphi\|^2_{H^1(0,T;L^2(0,L))} + \|\kappa\psi\|^2_{L^2(0,T;H^1_\nu(0,L))\cap C^0([0,T];L^2(0,L))}\\ \leq& C\left(\|(\kappa R, \kappa S)\|^2_{L^2(0,T;[L^2(0,L)]^2)} + \|(\kappa'\varphi, \kappa'\psi)\|^2_{L^2(0,T;[L^2(0,L)]^2)}\right),\end{aligned}$$

which leads to (45).

Since the weights are bounded from above and below, using (45), we obtain a first estimate in $(0, L) \times (0, T)$:

$$\begin{aligned}&\int_0^{T/2}\int_0^L [\bar\xi|\psi_x|^2 + \bar\xi^3|\psi|^2 + \bar\xi|\varphi|^2]e^{-2s\bar\alpha}dxdt + \|\varphi(\cdot,0)\|^2_{L^2(0,L)} + \|\psi(\cdot,0)\|^2_{L^2(0,L)}\\ \leq& C(T,s,\lambda)\left(\int_0^{3T/4}\int_0^L [\rho_1^{-2}|R|^2 + \rho_2^{-2}|S|^2]dxdt + \int_{T/2}^{3T/4}\int_0^L [\bar\xi^3|\psi|^2 + \bar\xi|\varphi|^2]e^{-2s\bar\alpha}dxdt\right).\end{aligned} \tag{47}$$

On the other hand, since $\alpha = \bar\alpha$ and $\xi = \bar\xi$ in $(0, L) \times (T/2, T)$, we have:

$$\begin{aligned}&\int_{T/2}^T\int_0^L [\lambda^2(s\bar\xi)|\psi_x|^2 + \lambda^4(s\bar\xi)^3|\psi|^2 + \lambda^2(s\bar\xi)|\varphi|^2]e^{-2s\bar\alpha}dxdt\\ =&\int_{T/2}^T\int_0^L [\lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2 + \lambda^2(s\xi)|\varphi|^2]e^{-2s\alpha}dxdt\\ \leq&\int_0^T\int_0^L [((s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2 + \lambda^2(s\xi)|\varphi|^2]e^{-2s\alpha}dxdt.\end{aligned} \tag{48}$$

In this way we obtain by Theorem 3.1,

$$\begin{aligned}&\int_{T/2}^T\int_0^L [\bar\xi|\psi_x|^2 + \bar\xi^3|\psi|^2 + \bar\xi|\varphi|^2]e^{-2s\bar\alpha}dxdt\\ \leq& C(T,s,\lambda)\left(\int_0^T\int_0^L [\xi^3|R|^2 + |S|^2]e^{-2s\alpha}dxdt + \int_0^T\int_{\omega(t)} \xi^7|\varphi|^2e^{-2s\alpha}dxdt\right).\end{aligned}$$

Finally, from the definition of $\bar\alpha$, $\bar\xi$, $\bar\alpha^*$ and $\bar\xi^*$, we deduce:

$$\begin{aligned}&\int_{T/2}^T\int_0^L [\bar\xi|\psi_x|^2 + \bar\xi^3|\psi|^2 + \bar\xi|\varphi|^2]e^{-2s\bar\alpha}dxdt\\ \leq& C(T,s,\lambda)\left(\int_0^T\int_0^L [\rho_1^{-2}|R|^2 + \rho_2^{-2}|S|^2]dxdt + \int_0^T\int_{\omega(t)} \rho_3^{-2}|\varphi|^2dxdt\right).\end{aligned}$$

which, combined with (47), provides (44). ■

The next step is to prove null controllability of the linear system (12). Of course, we will need some specific conditions on the source G. Thus, let us introduce the linear operators $\mathcal{M}_1$ and $\mathcal{M}_2$ by

$$\mathcal{M}_1(\theta) = \theta_t \quad \text{and} \quad \mathcal{M}_2(q) = q_t + Kq, \tag{49}$$

and the spaces

$$\begin{aligned} E_0 = \{ (q,h) : & \rho_1 \mathcal{M}_2(q), \rho_2 q, \rho_3 h 1_\omega \in L^2((0,L)\times(0,T)), \\ & \rho_4^{1/2} q \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2_\nu(0,L)) \cap L^\infty(0,T;H^1(0,L)), \\ & \rho_4^{1/2}\mathcal{M}_2(q) \in H^1(0,T;L^2(0,L)) \}\end{aligned}$$

and

$$\begin{aligned} E = \{ (q,h) : & (q,h) \in E_0,\ \rho_4[\mathcal{M}_1(\mathcal{M}_2(q)) + Aq - h1_\omega] \in L^2((0,L)\times(0,T)) \\ & \rho_4^{1/2}\mathcal{M}_1(q) \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2_\nu(0,L)) \cap L^\infty(0,T;H^1(0,L)), \\ & \rho_4^{1/2}\mathcal{M}_1(q) \in L^4((0,L)\times(0,T)), \rho_4^{1/3}\mathcal{M}_1(q) \in L^6((0,L)\times(0,T)),\ q(\cdot,0) \in H^2_\nu(0,L) \}.\end{aligned}$$

It is clear that E_0 and E are Banach spaces for the norms $\|\cdot\|_{E_0}$ and $\|\cdot\|_E$, where

$$\begin{aligned} \|(q,h)\|_{E_0} = \big(& \|\rho_1\mathcal{M}_2(q)\|^2_{L^2((0,L)\times(0,T))} + \|\rho_2 q\|^2_{L^2((0,L)\times(0,T))} + \|\rho_3 h 1_\omega\|^2_{L^2((0,L)\times(0,T))} \\ & + \|\rho_4^{1/2}(\mathcal{M}_2(q),q)\|^2_{H^1(0,T;L^2(0,L))} + \|\rho_4^{1/2} q\|^2_{L^2(0,T;H^2_\nu(0,L))\cap L^\infty(0,T;H^1(0,L))}\big)^{1/2}\end{aligned}$$

and

$$\begin{aligned} \|(q,h)\|_E = \big(& \|(q,h)\|^2_{E_0} + \|\rho_4[\mathcal{M}_1(\mathcal{M}_2(q)) + Aq - h1_\omega]\|^2_{L^2((0,L)\times(0,T))} \\ & \|\rho_4^{1/2}\mathcal{M}_1(q)\|^2_{L^4((0,L)\times(0,T))} + \|\rho_4^{1/3}\mathcal{M}_1(q)\|^2_{L^6((0,L)\times(0,T))} + \|q(\cdot,0)\|^2_{H^2_\nu(0,L)}\big)^{1/2}.\end{aligned}$$

Proposition 3 *Let the assumptions of Theorem* 1.1 *hold, and assume that* $p_0 \in L^2(0,L)$, $q_0 \in H^2_\nu(0,L)$, *and*

$$\rho_4 G \in L^2((0,L)\times(0,T)). \tag{50}$$

Then, there exists a control h *such that for the solution* (q,p) *to* (12) *we have that* $(q,h) \in E$. *In particular, we have that*

$$p(\cdot,T) = q(\cdot,T) = 0. \tag{51}$$

Proof: We will follow the general method introduced and used in [11] for linear parabolic problems. The existence proof will be based in a Lax-Milgram argument. To motivate the introduction of the appropriate bilinear form it is useful to introduce the following auxiliary extremal problem

$$\begin{cases} \text{Minimize} \quad J(\theta,q,h) = \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_0^L \left[\rho_1^2|\theta|^2 + \rho_2^2|q|^2 + \rho_3^2|h|^2 1_\omega\right] dx\,dt \\ \text{subject to } h \in L^2((0,L)\times(0,T)),\ \operatorname{supp}(h(\cdot,t)) \subset \omega(t)\ \ \forall t \in (0,T) \text{ and} \\ \begin{cases} \mathcal{M}_1(\theta) + A(x,t)\,q - h1_\omega = G & \text{in} \quad (0,L)\times(0,T), \\ \mathcal{M}_2(q) = \theta & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x) q_x\big|_{x=0} = \sigma_i(x) q_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ \theta(\cdot,0) = \theta_0, \quad q(\cdot,0) = q_0 & \text{in} \quad (0,L). \end{cases} \end{cases} \tag{52}$$

Observe that due to the behavior of the weights ρ_i at $t = T$ a solution $(\widehat{\theta},\widehat{q},\widehat{h})$ to (52) is a good candidate to satisfy $(\widehat{q},\widehat{h}) \in E$.

Let us suppose for the moment that $(\widehat{\theta}, \widehat{q}, \widehat{h})$ solves (52). Then, by the *Lagrange's multipliers formalism* the dual variables $\widehat{\varphi}$ and $\widehat{\psi}$ satisfy the following system

$$\begin{cases} \widehat{\theta} = \rho_1^{-2}[\mathcal{M}_1^*(\widehat{\varphi}) - \widehat{\psi}] & \text{in} \quad (0,L)\times(0,T), \\ \widehat{q} = \rho_2^{-2}[M_2^*(\widehat{\psi}) + A\widehat{\varphi}] & \text{in} \quad (0,L)\times(0,T), \\ \widehat{h} = -\rho_3^{-2}\widehat{\varphi}1_\omega & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)\widehat{\psi}_x\big|_{x=0} = \sigma_i(x)\widehat{\psi}_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \end{cases} \tag{53}$$

where $\mathcal{M}_i^*$ is the adjoint operator of $\mathcal{M}_i$ $(i = 1, 2)$, i.e.,

$$\mathcal{M}_1^*(\varphi) = -\varphi_t, \quad \text{and} \quad \mathcal{M}_2^*(\psi) = -\psi_t + K\psi.$$

Now let us set:

$$P_0 = \left\{ (\varphi,\psi) \in C^2([0,L]\times[0,T];\mathbb{R}^2) : \sigma_i\psi_x\big|_{x=0} = \sigma_i\psi_x\big|_{x=L} = 0 \text{ in } (0,T) \right\},$$

the bilinear form on $P_0 \times P_0$

$$\begin{aligned} \mathcal{B}((\widetilde{\varphi},\widetilde{\psi}),(\varphi,\psi)) \;&=\; \int_0^T\!\!\int_0^L \rho_1^{-2}[\mathcal{M}_1^*(\widetilde{\varphi}) - \widetilde{\psi}][\mathcal{M}_1^*(\varphi) - \psi]\,dx\,dt \\ &\quad + \int_0^T\!\!\int_0^L \rho_2^{-2}[\mathcal{M}_2^*(\widetilde{\psi}) + A\widetilde{\varphi}][\mathcal{M}_2^*(\psi) + A\varphi]\,dx\,dt + \int_0^T\!\!\int_{c_l(t)}^{c_r(t)} \rho_3^{-2}\widehat{\varphi}\,\varphi\,dx\,dt \end{aligned}$$

and the linear form on P_0

$$\langle \mathcal{L}, (\varphi,\psi)\rangle = \int_0^T\!\!\int_0^L G\,\varphi\,dx\,dt + \int_0^L \theta_0\,\varphi(0)\,dx + \int_0^L q_0\,\psi(0)\,dx.$$

For these definitions we note that $(\widehat{\varphi},\widehat{\psi})$ should satisfies

$$\mathcal{B}((\widehat{\varphi},\widehat{\psi}),(\varphi,\psi)) = \langle \mathcal{L}, (\varphi,\psi)\rangle \quad \forall (\varphi,\psi) \in P_0, \tag{54}$$

i.e., the solution to (52) satisfies (54). Conversely, if we are able to solve (54) in suitable sense and then use (53) to define $(\widehat{\theta}, \widehat{q}, \widehat{h})$, then we will be able to prove that we have found a solution to 52.

Next we will focus on the Lax-Milgram problem (54). It is clear that $\mathcal{B} : P_0 \times P_0 \to \mathbb{R}$ is a symmetric, definite positive and bilinear form on P_0, i.e. a scalar product in this linear space (thanks to the Carleman estimate (44)). We will denote by P the completion of P_0 for the norm induced by $\mathcal{B}$. Then P is a Hilbert space for the scalar product $\mathcal{B}$. On the other hand, in view of the Carleman estimate (44), (50) and the fact that $(\bar{\xi})^{-1/2}e^{s\bar{\alpha}} \le \rho_4$, the linear form $(\varphi,\psi) \mapsto \langle \mathcal{L}, (\varphi,\psi)\rangle$ is well-defined and continuous on P. Hence, from *Lax-Milgram's lemma,* we deduce that the variational problem

$$\mathcal{B}((\widehat{\varphi},\widehat{\psi}),(\varphi,\psi)) = \langle \mathcal{L}, (\varphi,\psi)\rangle \quad \forall (\varphi,\psi) \in P, \tag{55}$$

possesses exactly one solution $(\widehat{\varphi},\widehat{\psi}) \in P$.

With $(\widehat{\varphi},\widehat{\psi})$ given let $\widehat{\theta}$, $\widehat{q}$ and $\widehat{h}$ be given by (53). It is readily seen that

$$J(\widehat{\theta},\widehat{q},\widehat{h}) = \frac{1}{2}\mathcal{B}((\widehat{\varphi},\widehat{\psi}),(\widehat{\varphi},\widehat{\psi})) < +\infty \tag{56}$$

and, also, that $(\widehat{\theta},\widehat{q})$ is the unique weak solution to the system in (52) for $h = \widehat{h}$.

Finally, it remains to prove that $(\widehat{q},\widehat{h}) \in E$. Using (56) and the linear system in (52), we can easily check that $\rho_1\mathcal{M}_2(\widehat{q})$, $\rho_2\widehat{q}$, $\rho_3\widehat{h}1_\omega \in L^2((0,L)\times(0,T))$, $\rho_4[\mathcal{M}_1(\mathcal{M}_2(\widehat{q}))+A\widehat{q}-\widehat{h}1_\omega] \in L^2((0,L)\times(0,T))$ and $\widehat{q}(\cdot,0) \in H^2_\nu(0,L)$. We set $(\tilde{\theta},\tilde{q},\tilde{h}) := \rho_4^{1/2}(\widehat{\theta},\widehat{q},\widehat{h})$ and we can see that $(\widetilde{\theta},\widetilde{q},\widetilde{h})$ solves the following problem

$$\begin{cases} \mathcal{M}_1(\widetilde{\theta}) = -A(x,t)\,\widetilde{q} + \rho_4^{1/2}G + \widetilde{h}1_\omega + (\rho_4^{1/2})_t\widehat{\theta} & \text{in} \quad (0,L)\times(0,T), \\ \mathcal{M}_2(\widetilde{q}) = \widetilde{\theta} + (\rho_4^{1/2})_t\widehat{q} & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)\widetilde{q}_x\big|_{x=0} = \sigma_i(x)\widetilde{q}_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ \widetilde{\theta}(\cdot,0) = \rho_4^{1/2}(0)\theta_0, \quad \widetilde{q}(\cdot,0) = \rho_4^{1/2}(0)q_0 & \text{in} \quad (0,L). \end{cases} \tag{57}$$

Notice that, thanks to the fact that $\rho_1\mathcal{M}_2(\widehat{q})$ and $\rho_2\widehat{q}$ belong to $L^2((0,L)\times(0,T))$, we deduce that $(\rho_4^{1/2})_t\widehat{q}, (\rho_4^{1/2})_t\widehat{\theta} \in L^2((0,L)\times(0,T))$. Thus, since $\theta_0 \in L^2(0,L)$, $q_0 \in H^2_\nu(0,L)$, we have that

$$\left| \begin{array}{l} \widetilde{\theta} \in H^1(0,T;L^2(0,L)) \\ \widetilde{q} \in H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2_\nu(0,L)) \cap C^0([0,T];H^1(0,L)). \end{array} \right. \tag{58}$$

Notice that the linear system in (52) is equivalent to (12) for $(\widehat{p},\widehat{q})$, with $\widehat{p} = \widehat{q}_t$, and control $h = \widehat{h}$. In fact, by a similar argument as used to obtain (58), we get that $\rho_4^{3/4}p \in L^2((0,L)\times(0,T))$. We set $(\widetilde{p},\widetilde{q},\widetilde{h}) := \rho_4^r(\widehat{p},\widehat{q},\widehat{h})$ (with $r = 1/3$ and $r = 1/2$) and we observe that $(\widetilde{p},\widetilde{q},\widetilde{h})$ solves the following problem

$$\begin{cases} \widetilde{p}_t - (\sigma_i(x)\widetilde{p}_x)_x + \ell_p(\bar{v},\bar{w})\widetilde{p} + \ell_q(\bar{v},\bar{w})\widetilde{q} = \rho_4^r G + \widetilde{h}1_\omega + (\rho_4^r)_t\widehat{p} & \text{in} \quad (0,L)\times(0,T), \\ \widetilde{q}_t = \widetilde{p} + (\rho_4^r)_t\widehat{q} & \text{in} \quad (0,L)\times(0,T), \\ \sigma_i(x)\widetilde{p}_x\big|_{x=0} = \sigma_i(x)\widetilde{p}_x\big|_{x=L} = 0 & \text{on} \quad (0,T), \\ \widetilde{p}(\cdot,0) = \rho_4^r(0)p_0, \quad \widetilde{q}(\cdot,0) = \rho_4^r(0)q_0 & \text{in} \quad (0,L), \end{cases} \tag{59}$$

Since $p_0 \in L^2(0,L)$, $q_0 \in H^2_\nu(0,L)$ and $(\rho_4^r)_t\widehat{q}, (\rho_4^r)_t\widehat{p} \in L^2((0,L)\times(0,T))$, we deduce that

$$\left| \begin{array}{l} \widetilde{q} \in H^1(0,T;L^2(0,L)) \\ \widetilde{p} \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L)). \end{array} \right. \tag{60}$$

We conclude using [14, Chapter II. §3] to get $\widetilde{p} \in L^{\tilde{\kappa}}((0,L)\times(0,T))$, for $1 \le \tilde{\kappa} \le 6$.

Finally, we need to argue that (51) holds. For this propose, we use that $(\widehat{q},\widehat{h}) \in E$ and consequently $(\rho_2\widehat{q},\rho_1\widehat{\theta}) \in L^2((0,L)\times(0,T)) \times L^2((0,L)\times(0,T))$. Since $\widehat{q},\widehat{\theta} \in C^0([0,T];L^2(0,L))$ the singularities of ρ_1 and ρ_2 imply (51). ■

3.2 Controllability for the nonlinear monodomain model

We can now end the proof of Theorem 1.1. We will use the following inverse mapping theorem (see [2]):

Theorem 3.3 *Let B_1 and B_2 be two Banach spaces and let $\mathcal{A} : B_1 \mapsto B_2$ satisfy $\mathcal{A} \in C^1(B_1;B_2)$. Assume that $e_0 \in B_1$, $\mathcal{A}(e_0) = i_0$ and $\mathcal{A}'(e_0) : B_1 \mapsto B_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $i \in B_2$ satisfying $\|i - i_0\|_{B_2} < \delta$, there exists one solution to the equation*

$$\mathcal{A}(e) = i, \quad e \in B_1.$$

We shall apply this result with $B_1 = E$, $B_2 = F_1 \times F_2$ and for any $e = (q, h) \in B_1$ we set

$$\mathcal{A}(e) = (\mathcal{M}_1(\mathcal{M}_2(q)) + Aq - h1_\omega + \mathcal{N}(q_t, q), q(0), \mathcal{M}_2(q)(0)).$$

Here, $F_1 = \rho_4^{-1} L^2((0, L) \times (0, T))$ and $F_2 = H^2_\nu(0, L) \times L^2(0, L)$.

Thanks to the definition of the space E, it is not difficult to check that $\mathcal{A}$, which contains linear, bilinear and trilinear terms, is continuous and therefore $\mathcal{A} \in C^1(B_1; B_2)$. Let e_0 be the origin of B. Notice that $\mathcal{A}'(e_0) : B_1 \mapsto B_2$ is the mapping that, to each $(q, h) \in B_1$, associates the function $(\mathcal{M}_1(\mathcal{M}_2(q)) + Aq - h1_\omega, q(0), \mathcal{M}_2(q)(0))$ in B_2. In view of the null controllability result for (12) given in Proposition 4, $\mathcal{A}'(e_0)$ is surjective.

Consequently, we can indeed apply Theorem 3.3 with these data and, in particular, there exists $\delta > 0$ such that, if

$$\|(0, q_0, \theta_0)\|_{B_2} = \|(q_0, \theta_0)\|_{H^2_\nu(0,L)\times L^2(0,L)} \leq \delta,$$

we can find a control h such that the associated solution to (11) satisfies $p(\cdot, T) = 0$ and $q(\cdot, T) = 0$ in $(0, L)$.

A Neumann Carleman inequality for a parabolic equation

To verify Lemma 3 we start with the following remark.

Remark A.1 *Notice that the function r blows up at $t = 0$ and $t = T$. Also, observe that $\partial_t^k r(t) = \frac{(-1)^k k!}{t^{k+1}} = (-1)^k k!(r(t))^{k+1}$ close to $t = 0$ and $\partial_t^k r(t) = \frac{k!}{(T-t)^{k+1}} = k!(r(t))^{k+1}$ close $t = T$.*

Further we have

$$\begin{aligned}
\alpha_x &= \lambda\xi(-\eta_x),\\
\alpha_{xx} &= \lambda^2\xi(-\eta_x^2 - \lambda^{-1}\eta_{xx}),\\
\alpha_{xxx} &= \lambda^3\xi(-\eta_x^3 - 3\lambda^{-1}\eta_{xx}\eta_x - \lambda^{-2}\eta_{xxx}),\\
\alpha_{xxxx} &= \lambda^4\xi(-\eta_x^4 - 6\lambda^{-1}\eta_x^2\eta_{xx} - 3\lambda^{-2}\eta_{xx}^2 - \lambda^{-3}\eta_{xxxx}),\\
\alpha_t &= \lambda\xi^2\left[-\xi^{-1}\eta_t + \lambda^{-1}\left(e^{2\lambda(\|\eta\|_\infty - \eta)} - e^{-\lambda\eta}\right)\left(\frac{1}{r}\right)_t\right],\\
\alpha_{tt} &= \lambda^2\xi^3\Big[-\xi^{-2}\eta_t^2 - \lambda^{-1}\xi^{-2}\eta_{tt} - 2\lambda^{-1}e^{-2\lambda\eta}\eta_t\frac{r_t}{r^3} + \lambda^{-2}\left(e^{\lambda(2\|\eta\|_\infty - 3\eta)} - e^{-2\lambda\eta}\right)\frac{r_{tt}}{r^3}\Big],\\
\alpha_{xt} &= \lambda^2\xi^2\left[-\xi^{-1}\eta_t\eta_x - \lambda^{-1}\xi^{-1}\eta_{xt} + \lambda^{-1}e^{-\lambda\eta}(r^{-1})_t\eta_x\right].
\end{aligned} \tag{61}$$

It follows that there exists a positive constant $C > 0$, such that for $(x, t) \in [0, L] \times (0, T)$we have the following pointwise estimates:

$$\begin{aligned}
|\alpha_{xx}| &\leq C\lambda^2\xi,\\
|\alpha_{xxx}| &\leq C\lambda^3\xi,\\
|\alpha_{xxxx}| &\leq C\lambda^4\xi,\\
|\alpha_t| &\leq C(T + e^{2\lambda\|\eta\|_\infty})\lambda\xi^2,\\
|\alpha_{tt}| &\leq C(T^2 + T + e^{2\lambda\|\eta\|_\infty})\lambda^2\xi^3,\\
|\alpha_{xt}| &\leq C(T + 1)\lambda^2\xi^2.
\end{aligned} \tag{62}$$

We set $w = e^{-s\alpha}\psi$ and observe that

$$w_x = -s\alpha_x w + e^{-s\alpha}\psi_x.$$

Using the boundary conditions of ψ, we deduce $w_x = -s\alpha_x w$ on $\{0, L\} \times (0, T)$.

Remark A.2 *From the definition of α, given in* (23)*, notice that $w(\cdot, T) = w(\cdot, 0) = 0$ and $w_x(\cdot, T) = w_x(\cdot, 0) = 0$.*

Now, let us introduce the partial differential operator $P := \partial_t + \partial_{xx}$. Then, we have the following decomposition

$$e^{-s\alpha} P(e^{s\alpha} w) = P_e\, w + P_k\, w,$$

where

$$P_e w := w_{xx} + (s\alpha_t + s^2\alpha_x^2)w$$

is the self-adjoint part of the operator P and

$$P_k w := w_t + 2s\alpha_x w_x + s\alpha_{xx} w$$

is the skew-adjoint part of P. It follows that

$$\begin{aligned} \|e^{-s\alpha} P(e^{s\alpha} w)\|^2_{L^2((0,L)\times(0,T))} &= \|P_e w\|^2_{L^2((0,L)\times(0,T))} + \|P_k w\|^2_{L^2((0,L)\times(0,T))} \\ &\quad + 2(P_e w, P_k w)_{L^2((0,L)\times(0,T))}. \end{aligned} \tag{63}$$

The rest of the proof is devoted to analyzing the term $(P_s w, P_a w)_{L^2((0,L)\times(0,T))}$. Indeed, from the above definition of the operators P_e and P_k it follows

$$\begin{aligned} 2(P_e w, P_k w)_{L^2((0,L)\times(0,T))} &= 2\,(w_{xx}, w_t) + 2\,(w_{xx}, 2s\alpha_x w_x) + 2\,(w_{xx}, s\alpha_{xx} w) \\ &\quad + 2\left(s\alpha_t w + s^2\alpha_x^2 w, w_t\right) + 2\left(s\alpha_t w + s^2\alpha_x^2 w, 2s\alpha_x w_x\right) \\ &\quad + 2\left(s\alpha_t w + s^2\alpha_x^2 w, s\alpha_{xx} w\right) \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{64}$$

Now, in order to get estimates for the term $2(P_e w, P_k w)_{L^2((0,L)\times(0,T))}$, first let us work with each integral term I_i, $i = 1, \ldots, 6$.

For the first integral term, we integrate by parts in time and we obtain

$$I_1 = -2\int_0^T\!\!\int_0^L w_x w_{xt} + 2\int_0^T w_t w_x \Big|_{x=0}^{x=L}.$$

Then, thanks to Remark A.2 and the fact that $w_x = -s\alpha_x w$ on $\{0, L\} \times (0, T)$, after an integration by parts in the last term we have

$$I_1 = s\int_0^T \alpha_{xt} w^2 \Big|_{x=0}^{x=L}. \tag{65}$$

For the second term, we integrate by parts in space and we deduce

$$I_2 = -2s\int_0^T\!\!\int_0^L \alpha_{xx} w_x^2 + 2s\int_0^T \alpha_x w_x^2 \Big|_{x=0}^{x=L}. \tag{66}$$

For the third term, after two integration by parts, we obtain

$$I_3 = -2s\int_0^T\!\!\int_0^L \alpha_{xxx} w_x w - 2s\int_0^T\!\!\int_0^L \alpha_{xx} w_x^2 + 2s\int_0^T \alpha_{xx} w w_x \Big|_{x=0}^{x=L}.$$

And again integrating by parts in space the first term, we deduce

$$I_3 = s\int_0^T\int_0^L \alpha_{xxxx}w^2 - 2s\int_0^T\int_0^L \alpha_{xx}w_x^2 + 2s\int_0^T \alpha_{xx}ww_x\Big|_{x=0}^{x=L} - s\int_0^T \alpha_{xxx}w^2\Big|_{x=0}^{x=L}. \tag{67}$$

Fourthly, we integrate by parts in time and using Remark A.2, we get

$$I_4 = -s\int_0^T\int_0^L \alpha_{tt}w^2 - 2s^2\int_0^T\int_0^L \alpha_x\alpha_{xt}w^2. \tag{68}$$

And for the fifth term, we conclude

$$\begin{aligned} I_5 = &-\int_0^T\int_0^L \left(2s^2\alpha_x\alpha_{xt} + 4s^3\alpha_x^2\alpha_{xx}\right)w^2 - 2\int_0^T\int_0^L (s^2\alpha_t + s^3\alpha_x^2)\alpha_{xx}w^2 \\ &+ 2\int_0^T (s^2\alpha_t\alpha_x + s^3\alpha_x^3)w^2\Big|_{x=0}^{x=L}. \end{aligned} \tag{69}$$

For the last term, we obtain

$$I_6 = 2\int_0^T\int_0^L (s^2\alpha_t + s^3\alpha_x^2)\alpha_{xx}w^2. \tag{70}$$

From (64)-(70), we get

$$\begin{aligned} 2(P_sw, P_aw) = &-4s\int_0^T\int_0^L \alpha_{xx}w_x^2 + \int_0^T\int_0^L \left(s\alpha_{xxxx} - s\alpha_{tt} - 4s^2\alpha_x\alpha_{xt} - 4s^3\alpha_x^2\alpha_{xx}\right)w^2 \\ &+ \int_0^T \left(2s\alpha_x w_x^2 + 2s\alpha_{xx}ww_x + 2s^2\alpha_t\alpha_x w^2 + 2s^3\alpha_x^3w^2 + s\alpha_{xt}w^2 - s\alpha_{xxx}w^2\right)\Big|_{x=0}^{x=L}. \end{aligned}$$

Recalling that $w_x = -s\alpha_x w$ on $\{0, L\}\times(0,T)$, from the previous identity we deduce

$$\begin{aligned} 2(P_sw, P_aw) = &-4s\int_0^T\int_0^L \alpha_{xx}w_x^2 + \int_0^T\int_0^L \left(s\alpha_{xxxx} - s\alpha_{tt} - 4s^2\alpha_x\alpha_{xt} - 4s^3\alpha_x^2\alpha_{xx}\right)w^2 \\ &+ \int_0^T \left[4s^3\alpha_x^3 + 2s^2\alpha_x\left(\alpha_t - \alpha_{xx}\right) + s(\alpha_t - \alpha_{xx})_x\right]w^2\Big|_{x=0}^{x=L} \\ &= DT_1 + DT_2 + BT, \end{aligned} \tag{71}$$

where DT_1 and DT_2 correspond to the distributed terms and BT corresponds to the boundary terms.

Then, thanks to the identities (61), we obtain

$$BT = -4s^3\lambda^3\int_0^T \eta_x^3\xi^3w^2\Big|_{x=0}^{x=L} - 2s^2\lambda\int_0^T \left(\alpha_t - \alpha_{xx}\right)\eta_x\xi w^2\Big|_{x=0}^{x=L} + s\int_0^T (\alpha_t - \alpha_{xx})_x w^2\Big|_{x=0}^{x=L}.$$

Therefore, using the estimates (62) and the property (19)-(20) of η, we have the following bound

$$\begin{aligned} BT \geq\ & 4C^3s^3\lambda^3\int_0^T \xi^3w^2\big|_{x=L} - 2s^2\lambda^2C^2\left(T + e^{2\lambda\|\eta\|_\infty}\right)\int_0^T \xi^3w^2\big|_{x=L} - 2s^2\lambda^3C^2\int_0^T \xi^2w^2\big|_{x=L} \\ &- C(T+1)s\lambda^2\int_0^T \xi^2w^2\big|_{x=L} - Cs\lambda^3\int_0^T \xi w^2\Big|_{x=L} \\ &+ 4C^3s^3\lambda^3\int_0^T \xi^3w^2\big|_{x=0} - 2s^2\lambda^2C^2\left(T + e^{2\lambda\|\eta\|_\infty}\right)\int_0^T \xi^3w^2\big|_{x=0} - 2s^2\lambda^3C^2\int_0^T \xi^2w^2\big|_{x=0} \\ &- C(T+1)s\lambda^2\int_0^T \xi^2w^2\big|_{x=0} - Cs\lambda^3\int_0^T \xi w^2\Big|_{x=0}. \end{aligned}$$

Hence, we have for any $\lambda \geq C$ and any $s \geq C\left(1+T+e^{2\lambda\|\eta\|_\infty}\right)$:

$$BT \geq 4C^3 s^3 \lambda^3 \int_0^T \xi^3 w^2\big|_{x=L} + 4C^3 s^3 \lambda^3 \int_0^T \xi^3 w^2\big|_{x=0}. \tag{72}$$

Now, let us estimate the distributed terms in DT. Thanks to (15) and (61), for DT_1 we have:

$$\begin{aligned} DT_1 &= 4s\lambda^2 \int_0^T\!\!\int_0^L \eta_x^2 \xi w_x^2 + 4s\lambda \int_0^T\!\!\int_0^L \eta_{xx}\xi w_x^2 \\ &\geq Cs\lambda^2 \int_0^T\!\!\int_0^L \xi w_x^2 - Cs\lambda^2 \int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2 - Cs\lambda \int_0^T\!\!\int_0^L \xi w_x^2. \end{aligned}$$

Hence, taking $\lambda \geq C$, we obtain

$$DT_1 \geq Cs\lambda^2 \int_0^T\!\!\int_0^L \xi w_x^2 - Cs\lambda^2 \int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2. \tag{73}$$

Also, in order to get an estimate for DT_2, we use (15), (16), (61) and (62) to obtain:

$$\begin{aligned} DT_2 \geq\ & Cs^3\lambda^4 \int_0^T\!\!\int_0^L \xi^3 w^2 - Cs^3\lambda^4 \int_0^T\!\!\int_{\omega_0(t)} \xi^3 w^2 - Cs^3\lambda^3 \int_0^T\!\!\int_0^L \xi^3 w^2 \\ & - Cs\lambda^4 \int_0^T\!\!\int_0^L \xi w^2 - Cs\lambda^2 \int_0^T\!\!\int_0^L \xi^3 w^2 - Cs^2\lambda^3 \int_0^T\!\!\int_0^L \xi^3 w^2. \end{aligned}$$

Therefore, for $\lambda \geq C$, we deduce

$$DT_2 \geq Cs^3\lambda^4 \int_0^T\!\!\int_0^L \xi^3 w^2 - Cs^3\lambda^4 \int_0^T\!\!\int_{\omega_0(t)} \xi^3 w^2. \tag{74}$$

From (63), (71), (72), (73) and (74), we conclude that

$$\begin{aligned} & \|P_e w\|^2_{L^2((0,L)\times(0,T))} + \|P_k w\|^2_{L^2((0,L)\times(0,T))} + s^3\lambda^4 \int_0^T\!\!\int_0^L \xi^3 w^2 + s\lambda^2 \int_0^T\!\!\int_0^L \xi w_x^2 \\ & + s^3\lambda^3 \int_0^T \xi^3 w^2\big|_{x=L} + s^3\lambda^3 \int_0^T \xi^3 w^2\big|_{x=0} \\ \leq\ & C\left(\|e^{-s\alpha} P(e^{s\alpha} w)\|^2_{L^2((0,L)\times(0,T))} + s\lambda^2 \int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2 + s^3\lambda^4 \int_0^T\!\!\int_{\omega_0(t)} \xi^3 w^2 \right). \end{aligned} \tag{75}$$

Now, using that $P_e w := w_{xx} + (s\alpha_t + s^2\alpha_x^2)w$, we can deduce:

$$\begin{aligned} s^{-1} \int_0^T\!\!\int_0^L \xi^{-1} w_{xx}^2 &= s^{-1} \int_0^T\!\!\int_0^L \xi^{-1} |P_e w - (s\alpha_t + s^2\alpha_x^2)w|^2 \\ &\leq Cs^{-1} \int_0^T\!\!\int_0^L \xi^{-1} \left(|P_e w|^2 + s^2\alpha_t^2 w^2 + s^4\alpha_x^4 w^2 \right) \\ &\leq Cs^{-1} \int_0^T\!\!\int_0^L \xi^{-1} \left(|P_e w|^2 + s^2\lambda^2\xi^4 w^2 + s^4\lambda^4\xi^4 w^2 \right) \\ &\leq C\left(s^{-1} \int_0^T\!\!\int_0^L \xi^{-1} |P_e w|^2 + \int_0^T\!\!\int_0^L s^3\lambda^4\xi^3 w^2 \right). \end{aligned} \tag{76}$$

We can do the same for $P_k w := w_t + 2s\alpha_x w_x + s\alpha_{xx} w$ and then

$$\begin{aligned} s^{-1}\int_0^T\!\!\int_0^L \xi^{-1} w_t^2 &= s^{-1}\int_0^T\!\!\int_0^L \xi^{-1}|P_k w - 2s\alpha_x w_x - s\alpha_{xx} w|^2 \\ &\le Cs^{-1}\int_0^T\!\!\int_0^L \xi^{-1}\left(|P_k w|^2 + s^2\alpha_x^2 w_x^2 + s^2\alpha_{xx}^2 w^2\right) \\ &\le Cs^{-1}\int_0^T\!\!\int_0^L \xi^{-1}\left(|P_k w|^2 + s^2\lambda^2\xi^2 w_x^2 + s^4\lambda^4\xi^2 w^2\right) \\ &\le C\left(s^{-1}\int_0^T\!\!\int_0^L \xi^{-1}|P_k w|^2 + \int_0^T\!\!\int_0^L s\lambda^2\xi^2 w_x^2 + \int_0^T\!\!\int_0^L s^3\lambda^4\xi^3 w^2\right). \end{aligned} \tag{77}$$

From (75), (76) and (77), we obtain

$$\begin{aligned} & s^{-1}\int_0^T\!\!\int_0^L \xi^{-1} w_t^2 + s^{-1}\int_0^T\!\!\int_0^L \xi^{-1} w_{xx}^2 + s^3\lambda^4\int_0^T\!\!\int_0^L \xi^3 w^2 + s\lambda^2\int_0^T\!\!\int_0^L \xi w_x^2 \\ & + s^3\lambda^3\int_0^T \xi^3 w^2\big|_{x=L} + s^3\lambda^3\int_0^T \xi^3 w^2\big|_{x=0} \\ \le\ & C\left(\|e^{-s\alpha}P(e^{s\alpha}w)\|^2_{L^2((0,L)\times(0,T))} + s\lambda^2\int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2 + s^3\lambda^4\int_0^T\!\!\int_{\omega_0(t)} \xi^3 w^2\right). \end{aligned} \tag{78}$$

To conclude the proof, we need to eliminate the local term w_x containing on the right hand side of the previous inequality. To do that, let us introduce a function $\bar\zeta \in C^\infty([0,L]\times[0,T])$ satisfying

$$\begin{aligned} &0 \le \bar\zeta \le 1 && \forall (x,t)\in[0,L]\times[0,T], \\ &\bar\zeta(x,t) = 1 && \forall x\in\omega_0(t),\ \forall t\in[0,T], \\ &\bar\zeta(x,t) = 0 && \forall x\in[0,L]\setminus\overline{\omega_1(t)},\ \forall t\in[0,T], \end{aligned}$$

where we use (14).

We have

$$\begin{aligned} s\lambda^2\int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2 &\le s\lambda^2\int_0^T\!\!\int_0^L \bar\zeta\xi w_x^2 \\ &= -s\lambda^2\int_0^T\!\!\int_0^L \bar\zeta\xi w_{xx} w - s\lambda^2\int_0^T\!\!\int_0^L \bar\zeta_x\xi w_x w - s\lambda^2\int_0^T\!\!\int_0^L \bar\zeta\xi_x w_x w \\ &\quad + s\lambda^2\int_0^T \xi w_x w\Big|_{x=0}^{x=L}. \end{aligned}$$

Using the fact that $w_x = -s\alpha_x w$ on $\{0,1\}\times(0,T)$ and (19)-(20) and (61), we deduce

$$\begin{aligned} Cs\lambda^2\int_0^T\!\!\int_{\omega_0(t)} \xi w_x^2 \le\ & -Cs\lambda^2\int_0^T\!\!\int_0^L \bar\zeta\xi w_{xx} w - Cs\lambda^2\int_0^T\!\!\int_0^L \bar\zeta_x\xi w_x w - Cs\lambda^3\int_0^T\!\!\int_0^L \bar\zeta\eta_x w_x w \\ & + Cs^2\lambda^3\int_0^T \xi^2\eta_x w^2\Big|_{x=0}^{x=L} \\ \le\ & \frac{1}{2}s^{-1}\int_0^T\!\!\int_0^L \xi^{-1} w_{xx}^2 + \frac{1}{2}s\lambda^2\int_0^T\!\!\int_0^L \xi w_x^2 + \frac{C}{2}s^3\lambda^4\int_0^T\!\!\int_0^L (\bar\zeta^2 + \bar\zeta_x^2)\xi^3 w^2. \end{aligned}$$

Then, we obtain from the above inequality and (75)

$$\begin{aligned} & s^{-1}\int_0^T\int_0^L \xi^{-1}w_t^2 + s^{-1}\int_0^T\int_0^L \xi^{-1}w_{xx}^2 + s^3\lambda^4\int_0^T\int_0^L \xi^3 w^2 + s\lambda^2\int_0^T\int_0^L \xi w_x^2 \\ & + s^3\lambda^3\int_0^T \xi^3 w^2\big|_{x=L} + s^3\lambda^3\int_0^T \xi^3 w^2\big|_{x=0} \\ \leq\ & C\left(\|e^{-s\alpha}P(e^{s\alpha}w)\|^2_{L^2((0,L)\times(0,T))} + s^3\lambda^4\int_0^T\int_{\omega_1(t)} \xi^3 w^2\right). \end{aligned} \tag{79}$$

We finally can turn back to ψ:

$$\begin{aligned} & s^{-1}\int_0^T\int_0^L \xi^{-1}\psi_t^2 e^{-2s\alpha} + s^{-1}\int_0^T\int_0^L \xi^{-1}\psi_{xx}^2 e^{-2s\alpha} + s\lambda^2\int_0^T\int_0^L \xi\psi_x^2 e^{-2s\alpha} + s^3\lambda^4\int_0^T\int_0^L \xi^3\psi^2 e^{-2s\alpha} \\ & + s^3\lambda^3\int_0^T \xi^3\psi^2 e^{-2s\alpha}\big|_{x=L} + s^3\lambda^3\int_0^T \xi^3\psi^2 e^{-2s\alpha}\big|_{x=0} \\ & \leq C\left(\int_0^T\int_0^L e^{-2s\alpha}f^2 + s^3\lambda^4\int_0^T\int_{\omega_1(t)} \xi^3\psi^2 e^{-2s\alpha}\right) \end{aligned} \tag{80}$$

and hence (25) follows.

$$\begin{aligned} & (M_l H(1/\alpha M_l^{-1}(-\nabla p\cdot\nabla y - \beta D^*\psi)\nabla p, \nabla v) + (y,v) - (z,v) = 0, \\ & (M_l H(1/\alpha M_l^{-1}(-\nabla p\cdot\nabla y - \beta D^*\psi)\nabla y, \nabla v) - (f,v) = 0, \\ & (DM_l H(1/\alpha M_l^{-1}(-\nabla p\cdot\nabla y - \beta D^*\psi)\nabla y\cdot\nabla p, \phi) - (M_l(\partial B_\infty)_\gamma(M_l^{-1}\psi), \phi) = 0, \end{aligned}$$

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