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SFB-Report No. 2016-008 October 2016

A–8010 GRAZ, HEINRICHSTRASSE 36, AUSTRIA

Supported by the
Austrian Science Fund (FWF)

Der Wissenschaftsfonds.

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A reduction method for Riccati-based control of the Fokker-Planck equation

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Abstract: A control strategy which allows to speed up the convergence of a bilinear system governed by the Fokker-Planck equation to its stationary distribution is developed. After linearization of the state equation, a linear feedback control is computed by solving the Riccati equation associated with the linearized problem. A reduction method for approximating this feedback is proposed. From a numerical point of view, this method avoids the resolution of a high-dimensional Riccati equation. Numerical results are provided for a double-well potential and the efficiency of the reduction method is demonstrated.

Keywords: Output feedback control, control of partial differential equations, bilinear systems, Fokker-Planck equation, Riccati equation.

1. INTRODUCTION

A controlled version of the Fokker-Planck equation is considered in this paper. Consider a very large set of dragged particles and assume, following (Risken, 1996, Chapter 1), that the position of each particle is described by the following stochastic differential equation (SDE), called Smoluchowski equation:

$$\mathrm{d}x(t) = -\nabla V(x(t),t)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion. The following partial differential equation, called Fokker-Planck equation, describes the evolution of the probability density function ρ of the particles:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot J(x,t), \quad \text{where: } J(x,t) = \nu\nabla\rho + \rho\nabla V.$$

The variable J is called probability current. See (Risken, 1996, Section 10.4) and (Gardiner, 2004, Section 4.3.4) for details on the connection between SDEs and the Fokker-Planck equation. In this article, the Fokker-Planck equation is considered with reflective boundary conditions:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu\Delta\rho + \nabla\cdot(\rho\nabla V) && \text{in } \Omega\times(0,\infty),\\ 0 &= (\nu\nabla\rho + \rho\nabla V)\cdot \boldsymbol{n} && \text{on } \Gamma\times(0,\infty),\\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ denotes a bounded domain with smooth boundary $\Gamma = \partial\Omega$, and where ρ_0 denotes an initial probability density function with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$. The used boundary conditions model the fact that particles are reflected at Γ. They ensure that the total mass is conserved over time:

$$\int_\Omega \rho(x,t)\mathrm{d}x = 1, \quad \text{for a. e. } t \geq 0.$$

As in Hartmann et al. (2013), let us assume that we can interact with the particles by means of an optical tweezer (see Jones et al. (2015)). From a mathematical point of view, we assume that the potential V is controlled at time t through a real number $u(t)$ in the following way:

$$V(x,t) = W(x) + \alpha(x)u(t), \tag{2}$$

where W and α are fixed functions in $W^{2,\max(2,n+\varepsilon)}(\Omega)$ (with $\varepsilon > 0$). The function α is called control shape function. We assume that

$$\nabla\alpha\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma. \tag{3}$$

The following function is refered to as stationary distribution:

$$\rho_\infty(x) = \frac{\exp(-W(x)/\nu)}{\int_\Omega \exp(-W(z)/\nu)\mathrm{d}z}. \tag{4}$$

The goal of this article is to design a linear feedback controller of the form:

$$u(t) = \mathcal{K}(\rho(t) - \rho_\infty),$$

for (approximately) solving:

$$\begin{aligned} \inf_{u\in L^2(0,\infty)} \mathcal{J}(y,u) &:= \int_0^\infty e^{2\delta t}\big(\langle y(t), \mathcal{M}y(t)\rangle + u(t)^2\big)\mathrm{d}t\\ \text{subject to: } y(t) &= \rho(t) - \rho_\infty, \text{ (1) and (2)}, \end{aligned} \tag{5}$$

where $\mathcal{M}$ is a self-adjoint positive definite operator and where $\delta > 0$ is given. In this way, we speed up the convergence of the system to ρ_∞. We follow and extend the approach developed in a recent article by the same authors, Breiten et al. (2016). In this reference, problem (5) is approximated by a linear-quadratic optimal control problem, obtained by linearizing the state equation for y close to 0 (i. e. ρ close to ρ_∞) and by projecting it on the set of functions with a space-integral equal to 1. A solution in feedback form can be found by solving the corresponding

Riccati equation. The method of approximating an optimal control problem with a linear-quadratic one, in the context of distributed parameter systems has been used in e. g. Barbu et al. (2006); Raymond and Thevenet (2010); Raymond (2006).

The novelty of this article is the use of a reduction method for constructing the feedback law. The method first consists in splitting the state space into an unstable and a stable part. We then project the state variable on the unstable part of the state space. The projected state variable satisfies a reduced state equation, which we stabilize by solving a (reduced) Riccati equation. The obtained feedback eventually enables us to stabilize the whole state variable. This idea goes back to (at least) Triggiani (1975). In the context of the present article, the unstable and the stable subspaces are orthogonal, moreover, the unstable subspace is finite-dimensional, which allows an efficient implementation of this method. In particular, the resolution of a high-dimensional Riccati equation is avoided.

In section 2, we provide results for the well-posedness of the Fokker-Planck equation. We also formulate the state equation as an abstract Cauchy problem and project it on an appropriate hyperspace. Section 3 is dedicated to the design of a Riccati-based feedback law. In section 4, we introduce the reduction method. We finally provide numerical results in section 5, for a two-dimensional problem. The two feedback laws are compared.

2. ABSTRACT FORMULATION AND PROJECTION OF THE SYSTEM

2.1 Well-posedness

For arbitrary $T > 0$ and for $u \in L^2(0,T)$, we shall refer to ρ as (variational) solution of (1)-(2) on $(0,T)$ if ρ lies in the space

$$W(0,T) := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*),$$

if $\rho(0) = \rho_0$ and if for a. e. $t \in (0,T)$, for all $v \in H^1(\Omega)$,

$$\langle \rho_t(t), v\rangle + \langle \nu\nabla\rho(t) + \rho(t)\nabla W, \nabla v\rangle + u(t)\langle \rho(t)\nabla\alpha, \nabla v\rangle = 0.$$

Let us recall that $W(0,T) \subset C([0,T], L^2(\Omega))$, so that $\rho(0)$ is well-defined.

Theorem 1. For all $u \in L^2(0,T)$ and $\rho_0 \in L^2(\Omega)$, there exists a unique solution ρ to (1)-(2). If $\rho_0 \in H^1(\Omega)$ and $\Delta\alpha \in L^\infty(\Omega)$, then:

- $\rho_t \in L^2(0,T;L^2(\Omega))$,
- $\rho \in C([0,T];H^1(\Omega))$,
- $\nabla\cdot(\nu\Delta\rho + \rho\nabla W) \in L^2(0,T;L^2(\Omega))$
- $(\nu\nabla\rho + \rho\nabla W)\cdot\boldsymbol{n} = 0$ in $L^2(0,T;H^{-1/2}(\Gamma))$.

Moreover, for all $t \in [0,T]$, $\int_\Omega \rho(x,t)\,\mathrm{d}x = 1$. Finally, if $\rho_0 \geq 0$ a. e. on Ω, then $\rho(x,t) \geq 0$ for all $t > 0$ and a. e. $x \in \Omega$.

This result is proved in (Breiten et al., 2016, Section 2).

2.2 Formulation as a Cauchy problem

We formulate now the controlled system (1)-(2) as an abstract bilinear system, so that the linearized system can be studied by means of semigroup methods:

$$\dot\rho(t) = \mathcal{A}\rho(t) + \mathcal{N}\rho(t)u(t), \quad \rho(0) = \rho_0, \tag{6}$$

where the operators $\mathcal{A}$ and $\mathcal{N}$ are defined by

$$\begin{aligned}\mathcal{A}&\colon \rho \in \mathcal{D}(\mathcal{A}) \mapsto \nu\Delta\rho + \nabla\cdot(\rho\nabla W) \in L^2(\Omega),\\ \mathcal{N}&\colon \rho \in H^1(\Omega) \mapsto \nabla\cdot(\rho\nabla\alpha) \in L^2(\Omega),\end{aligned} \tag{7}$$

where:

$$\mathcal{D}(\mathcal{A}) = \left\{\rho \in H^2(\Omega) \,|\, (\nu\nabla\rho + \rho\nabla W)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

Observe that $\mathrm{im}(\mathcal{A}) \subseteq \mathbb{1}^\perp$ and $\mathrm{im}(\mathcal{N}) \subseteq \mathbb{1}^\perp$, where $\mathbb{1}$ is the constant function (with value 1) and

$$\mathbb{1}^\perp = \left\{\rho \in L^2(\Omega) \,|\, \langle \mathbb{1}, \rho\rangle_{L^2(\Omega)} = 0\right\}.$$

Let us recall (see Adams (1975)) that we have the following embeddings:

$$W^{2,2}(\Omega) \hookrightarrow \begin{cases} C(\Omega) & \text{if } n = 1,2,3,\\ L^q(\Omega),\ q \in [1,\infty) & \text{if } n = 4,\\ L^{\frac{2n}{n-4}}(\Omega) & \text{if } n \geq 5.\end{cases}$$

Since by assumption, α and $W \in W^{2,\max(2,n+\varepsilon)}(\Omega)$, a short computation involving the Hölder inequality shows that $\mathcal{A}$ and $\mathcal{N}$ are well-defined. Their $L^2(\Omega)$-adjoints are given by

$$\begin{aligned}\mathcal{A}^*&\colon \varphi \in \mathcal{D}(\mathcal{A}^*) \mapsto \nu\Delta\varphi - \nabla W\cdot\nabla\varphi \in L^2(\Omega),\\ \mathcal{N}^*&\colon \varphi \in H^1(\Omega) \mapsto -\nabla\varphi\cdot\nabla\alpha \in L^2(\Omega),\end{aligned} \tag{8}$$

where

$$\mathcal{D}(\mathcal{A}^*) = \left\{\varphi \in H^2(\Omega) \,|\, (\nu\nabla\varphi)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

Note that due to (3), a solution $\rho \in \mathcal{D}(\mathcal{A})$ of (6) automatically satisfies the reflective boundary conditions of (1).

Setting $y = \rho - \rho_\infty$, the bilinear system (6) is equivalent to:

$$\dot y(t) = \mathcal{A}y(t) + \mathcal{N}y(t)u(t) + \mathcal{B}u(t), \quad y_0 = \rho_0 - \rho_\infty, \tag{9}$$

where:

$$\begin{aligned}\mathcal{B}&\colon u \in \mathbb{R} \mapsto \mathcal{N}\rho_\infty \in L^2(\Omega)\\ \mathcal{B}^*&\colon \rho \in L^2(\Omega) \mapsto \langle \rho, \mathcal{N}\rho_\infty\rangle_{L^2(\Omega)}.\end{aligned}$$

2.3 Change of variables

Let us set

$$\Phi(x) = \log\nu + \frac{W(x)}{\nu}.$$

Following (Risken, 1996, Section 6.3), we define

$$\mathcal{A}_s\colon \rho \in \mathcal{D}(\mathcal{A}_s) \mapsto e^{\Phi/2}\mathcal{A}e^{-\Phi/2} \in L^2(\Omega), \tag{10}$$

where:

$$\mathcal{D}(\mathcal{A}_s) = \left\{\rho \in H^2(\Omega) \,\middle|\, (\nu\nabla\rho + \frac{1}{2}\rho\nabla W)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

A short computation shows that

$$\mathcal{A}(e^{-\Phi/2}\rho) = \nu e^{-\Phi/2}\Big(\Delta\rho + \frac{1}{2}\rho\Delta\Phi - \frac{1}{4}\rho\nabla\Phi\cdot\nabla\Phi\Big).$$

Using the previously mentioned embeddings and Hölder inequality, it can be shown that the image of $\mathcal{A}_s$ is indeed included into $L^2(\Omega)$ for $\rho \in H^2(\Omega)$.

Lemma 2. The operator $\mathcal{A}_s$ is self-adjoint. The spectrum $\sigma(\mathcal{A}_s)$ of $\mathcal{A}_s$ is a pure point spectrum, contained in $\overline{\mathbb{R}}_-$, with only accumulation point $-\infty$. The eigenfunctions $\{\Psi_i\}_{i=0}^\infty$ form a complete orthogonal set. Moreover, $\mathcal{A}_s$ generates an analytic semigroup of contractions on $L^2(\Omega)$ and consequently, $\mathcal{A}$ generates a semigroup of class $G(M,0)$ on $L^2(\Omega)$.

This lemma is proved in (Breiten et al., 2016, Section 3.1). The following lemma follows from the symmetry of $\mathcal{A}_s$.

Lemma 3. The operators $\mathcal{A}_s$ and $\mathcal{A}$ have the same spectrum. Moreover, for all $(\psi, \lambda) \in (\mathcal{D}(\mathcal{A}) \times \mathbb{R})$,

$$\mathcal{A}\psi = \lambda\psi \Longleftrightarrow \mathcal{A}_s\varphi = \lambda\varphi,\ \varphi = e^{\Phi/2}\psi$$
$$\mathcal{A}\psi = \lambda\psi \Longleftrightarrow \mathcal{A}^*\varphi = \lambda\varphi,\ \varphi = e^{\Phi}\psi.$$

Note that in particular, 0 is an eigenvalue of $\mathcal{A}$, $\mathcal{A}^*$, and $\mathcal{A}_s$ with the eigenvectors ρ_∞, $\mathbb{1}$, and $e^{-\Phi/2}$, respectively.

We now define:

$$\mathcal{X} = \{e^{-\Phi/2}\}^\perp = \big\{\rho \in L^2(\Omega) \,|\, \langle e^{-\Phi/2}, \rho\rangle_{L^2(\Omega)} = 0\big\}.$$

Since $\mathcal{A}_s$ is a self-adjoint operator and since $e^{-\Phi/2} \in \mathrm{Ker}(\mathcal{A}_s)$, the range of $\mathcal{A}_s$ is included in $\mathcal{X}$. For $u \in L^2(0,\infty)$, let us introduce the shifted variable z, defined by

$$z(t) = e^{\Phi/2}(\rho(t) - \rho_\infty) \in \mathcal{X}, \tag{11}$$

where ρ is a solution to (6). Note that $z \in \mathcal{X}$, since by Theorem 1, $\langle \mathbb{1}, \rho(t)\rangle = \langle \mathbb{1}, \rho_0\rangle$ for all $t \geq 0$. The shifted variable is the solution to the following Cauchy problem:

$$\dot{z}(t) = \widetilde{\mathcal{A}}_s z(t) + \widetilde{\mathcal{N}} z(t) u(t) + \widetilde{\mathcal{B}} u, \tag{12}$$

where $\widetilde{\mathcal{A}}_s$ is the restriction of $\mathcal{A}_s$ to $\mathcal{D}(\mathcal{A}_s) \cap \mathcal{X}$ and where the operators $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{B}}$ are defined by:

$$\widetilde{\mathcal{N}}\colon \rho \in H^1(\Omega) \cap \mathcal{X} \mapsto e^{\Phi/2}\mathcal{N}(e^{-\Phi/2}\rho) \in \mathcal{X}$$
$$\widetilde{\mathcal{B}}\colon u \in \mathbb{R} \mapsto \big(e^{\Phi/2}\mathcal{N}\rho_\infty\big)u \in \mathcal{X}.$$

Note that $\widetilde{A}_s$ is a self-adjoint operator and that the images of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{B}}$ are included in $\mathcal{X}$ since the image of $\mathcal{N}$, as already mentioned, is itself included into $\mathbb{1}^\perp$. Note also that the adjoint of $\widetilde{\mathcal{B}}$ is given by:

$$\widetilde{\mathcal{B}}^*\colon \rho \in \mathcal{X} \mapsto \big\langle \rho, e^{\Phi/2}\mathcal{N}\rho_\infty\big\rangle_{L^2(\Omega)} \in \mathbb{R}.$$

The linearization of (12) around 0 reads:

$$\dot{z}(t) = \widetilde{\mathcal{A}}_s z(t) + \widetilde{\mathcal{B}} u(t). \tag{13}$$

The change of variables (11) has a twofold interest: first, the reduction of the state space to $\mathcal{X}$ enables us to investigate the stabilizability of (13) with the Hautus criterion. Moreover, the fact that $\mathcal{A}$ is replaced by a self-adjoint operator, $\widetilde{\mathcal{A}}_s$, enables us to implement the reduction method described in section 4.

3. A FIRST LINEAR FEEDBACK CONTROLLER

We design a linear feedback controller by linearizing the controlled system (9) for ρ close to ρ_∞. We therefore aim at solving:

$$\begin{aligned} &\inf_{u \in L^2(0,\infty)} \mathcal{J}(y,u), \quad \text{subject to:} \\ &\qquad \dot{y}(t) = \mathcal{A}y(t) + \mathcal{B}v(t), \quad y_0 = \rho_0 - \rho_\infty. \end{aligned} \tag{14}$$

Using the change of variables $z(t) = e^{\Phi/2}y(t)$ (introduced in section 2) and $\hat{z}(t) = e^{\delta t}z(t)$, we obtain the following equivalent formulation of problem (14):

$$\inf_{v \in L^2(0,\infty)} \hat{\mathcal{J}}(\hat{z}, v) := \int_0^\infty \big(\langle \hat{z}, \widetilde{\mathcal{M}}\hat{z}\rangle_{L^2(\Omega)} + v(t)^2\big)\mathrm{d}t, \tag{15}$$

subject to:

$$\dot{\hat{z}}(t) = (\widetilde{\mathcal{A}}_s + \delta I)\hat{z}(t) + \widetilde{\mathcal{B}} v(t), \tag{16}$$

where $\widetilde{\mathcal{M}}\colon \mathcal{X} \to \mathcal{X}$ is the unique self-adjoint operator satisfying: for all ρ_1 and $\rho_2 \in \mathcal{X}$,

$$\langle \rho_1, \widetilde{\mathcal{M}}\rho_2\rangle_{L^2(\Omega)} = \langle e^{-\Phi/2}\rho_1, \mathcal{M}e^{-\Phi/2}\rho_2\rangle_{L^2(\Omega)}.$$

Problem (14) and problem (15)-(16) are equivalent, in so far as for all $v \in L^2(0,\infty)$, v is solution to (15)-(16) if and only if $e^{-\delta t}v$ is a solution to (14).

The Hautus criterion for the stabilizability of infinite-dimensional linear systems provides a sufficient condition for the δ-stabilizability of the pair $(\widetilde{\mathcal{A}}_s, \widetilde{\mathcal{B}})$, see (Curtain and Zwart, 2005, Definition 5.2.1). The criterion writes:

$$\forall \lambda \geq -\delta,\ \mathrm{Ker}(\lambda I - \widetilde{\mathcal{A}}_s) \cap \mathrm{Ker}(\widetilde{\mathcal{B}}^*) = \{0\}. \tag{17}$$

Note that this formulation of the Hautus criterion takes into account the fact that $\widetilde{\mathcal{A}}_s$ is self-adjoint and that therefore, its spectrum is included in $\mathbb{R}$. By Lemma 3, the Hautus criterion (17) is equivalent to the following condition:

$$\forall \lambda \geq -\delta,\ \mathrm{Ker}(\lambda I - \mathcal{A}^*) \cap \mathrm{Ker}(\mathcal{B}^*) \subseteq \mathrm{Span}(\rho_\infty).$$

Consider the following Riccati equation: $\forall \rho_1, \rho_2 \in \mathcal{D}(\widetilde{A}_s)$,

$$\big\langle \rho_1, \big[(\widetilde{\mathcal{A}}_s + \delta I)\Pi + \Pi(\widetilde{\mathcal{A}}_s + \delta I) - \Pi\widetilde{\mathcal{B}}\widetilde{\mathcal{B}}^*\Pi + \widetilde{\mathcal{M}}\big]\rho_2\big\rangle = 0, \tag{18}$$

where the unknown variable $\Pi \in \mathcal{L}(\mathcal{X})$ is a self-adjoint operator.

Lemma 4. Assume that the Hautus criterion (17) is satisfied. Then, the Riccati equation (18) has a unique solution Π and the pair $(\widetilde{\mathcal{A}}_s, \widetilde{\mathcal{B}})$ is δ-stabilizable, with the following feedback control:

$$v = -\widetilde{\mathcal{B}}^*\Pi\hat{z}. \tag{19}$$

Since we assumed $\mathcal{M}$ to be positive definite, the pair $(\mathcal{A}, \mathcal{M})$ is trivially detectable and this lemma follows from (Curtain and Zwart, 2005, Theorem 5.2.11). Note that the corresponding feedback for problem (14), denoted by $\mathcal{K}_0$, is given by:

$$u = -\mathcal{B}^* e^{\Phi/2}\Pi e^{\Phi/2} y =: \mathcal{K}_0 y. \tag{20}$$

As shown in (Breiten et al., 2016, Theorem 4.7), we have the following local stabilization result if the above feedback is applied to the bilinear system (6).

Theorem 5. There exist two constants $C_1 > 0$ and $C_2 > 0$ such that for $\|\rho_0 - \rho_\infty\|_{L^2(\Omega)} \leq C_1$, the system

$$\dot{\rho}(t) = \mathcal{A}\rho(t) - \mathcal{N}\rho(t)\big(\mathcal{K}_0(\rho(t) - \rho_\infty)\big), \quad \rho(0) = \rho_0,$$

admits a unique solution $\rho \in W(0,\infty)$ which satisfies

$$\big\|e^{\delta\cdot}(\rho(\cdot) - \rho_\infty)\big\|_{W(0,\infty)} \leq C_2.$$

4. STABILIZATION BY REDUCTION OF THE STATE VARIABLE

We introduce in this section the reduction method for computing a linear feedback controller, $\mathcal{K}_1$, which is an approximation the feedback $\mathcal{K}_0$, given by (19). In this manner, the resolution of a Riccati equation of large dimension is avoided, at almost no loss of performance. This method could be also used for future developments aiming at controlling the Fokker-Planck equation in higher dimensions. The technique is detailed in (Triggiani, 1975, Sections 4 and 6). It is also discussed in Raymond and Thevenet (2010) and the references therein.

4.1 Reduction of the state space

Consider again the shifted linearized system

$$\dot{z} = \widetilde{\mathcal{A}}_s z + \widetilde{\mathcal{B}} u.$$

Let us denote by $\{\Psi_i\}_{i=0}^{+\infty}$ the eigenfunctions of $\mathcal{A}_s$ and by $\{\lambda_i\}_{i=0}^{+\infty}$ the corresponding eigenvalues. They form a complete orthogonal set. We assume that: $\lambda_0 \geq \lambda_1 \geq \lambda_2 ....$ We have: $\lambda_0 = 0$ and $\Psi_0 = e^{-\Phi/2}$. The eigenfunctions $\{\Psi_i\}_{i=1}^{+\infty}$ are all eigenfunctions of $\widetilde{\mathcal{A}}_s$ and form a complete orthogonal set of $\mathcal{X}$.

Let ℓ be a sufficiently large index so that $\lambda_{\ell+1} < -\delta$. Consider the finite-dimensional space $\mathcal{X}_1 = \mathrm{Span}(\Psi_1, ..., \Psi_\ell)$ and its orthogonal complement in $\mathcal{X}$, denoted $\mathcal{X}_2$. Observe that $\mathcal{X}_2$ is the closure of $\mathrm{Span}(\Psi_{\ell+1}, \Psi_{\ell+2}, ...)$. Denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the orthogonal projections on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. Set $z_1(t) = \mathcal{P}_1 z(t)$, and $z_2(t) = \mathcal{P}_2 z(t)$. We have $z(t) = z_1(t) + z_2(t)$, moreover,

$$\dot{z}_1(t) = \widetilde{\mathcal{A}}_{s,1} z_1(t) + \mathcal{P}_1 \widetilde{\mathcal{B}} u(t), \tag{21}$$

$$\dot{z}_2(t) = \widetilde{\mathcal{A}}_{s,2} z_2(t) + \mathcal{P}_2 \widetilde{\mathcal{B}} u(t), \tag{22}$$

where $\widetilde{\mathcal{A}}_{s,1}$ and $\widetilde{\mathcal{A}}_{s,2}$ are obtained by splitting the self-adjoint operator $\widetilde{\mathcal{A}}_s$ according to the orthogonal decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$. The operators $\widetilde{\mathcal{A}}_{s,1}$ and $\widetilde{\mathcal{A}}_{s,2}$ are self-adjoint operators, with eigenfunctions $\{\Psi_i\}_{i=1}^{\ell}$ and $\{\Psi_i\}_{i=\ell+1}^{+\infty}$, respectively.

The method consists now in stabilizing only the component z_1 (described by (21)). The Hautus criterion for system (21) reads:

$$\forall \lambda \geq -\delta,\ \mathrm{Ker}(\lambda I - \widetilde{\mathcal{A}}_{s,1}) \cap \mathrm{Ker}\big((\mathcal{P}_1 \widetilde{\mathcal{B}})^*\big) = \{0\}. \tag{23}$$

It is easy to check that $(\mathcal{P}_1 \widetilde{\mathcal{B}})^*$ is the restriction of $\widetilde{\mathcal{B}}^*$ to $\mathcal{X}_1$. As a direct consequence, if the Hautus criterion (17) (for the original system) is satisfied, then (23) holds. The pair $(\widetilde{\mathcal{A}}_{s,1}, \mathcal{P}_1 \widetilde{\mathcal{B}})$ is therefore δ-stabilizable with a feedback law $u = \mathcal{K} z_1$.

Lemma 6. The system:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} \widetilde{\mathcal{A}}_{s,1} + \mathcal{P}_1 \widetilde{\mathcal{B}} \mathcal{K} & 0 \\ \mathcal{P}_2 \widetilde{\mathcal{B}} \mathcal{K} & \widetilde{\mathcal{A}}_{s,2} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

is δ-exponentially stable.

Proof. This is a direct application of (Triggiani, 1975, Theorem 6.1). The assumptions of the theorem can be easily checked. The spectrum decomposition assumption is satisfied, since $\widetilde{\mathcal{A}}_s$ has a pure point spectrum. Moreover, for all $\rho \in \mathcal{D}(\widetilde{\mathcal{A}}_{s,2}) = \mathcal{D}(\mathcal{A}_s) \cap \mathcal{X}_2$,

$$\langle \rho, \widetilde{\mathcal{A}}_{s,2} \rho \rangle \leq -\delta \|\rho\|_{L^2(\Omega)},$$

thus, by (Bensoussan et al., 2007, Proposition 2.11), $\widetilde{\mathcal{A}}_{s,2}$ generates an analytic semigroup and then by (Triggiani, 1975, Page 387), $\widetilde{\mathcal{A}}_{s,2}$ satisfies the spectrum determined growth assumption.

It follows from the above lemma that the following system:

$$\dot{z}(t) = (\widetilde{\mathcal{A}}_s + \widetilde{\mathcal{B}} \mathcal{K} \mathcal{P}_1) z(t)$$

is δ-exponentially stable.

4.2 Reduced Riccati equation

It remains to design a feedback law on the finite dimensional space $\mathcal{X}_1$. This can be done with the following cost functional:

$$\mathcal{J}_1(z_1, u) = \int_0^\infty e^{2\delta t} \big(\langle z_1(t), \widetilde{M}_1 y_1(t) \rangle_{L^2(\Omega)} + u(t)^2 \big) \mathrm{d}t,$$

where $\widetilde{\mathcal{M}}_1 \in \mathcal{L}(\mathcal{X}_1)$ is the unique self-adjoint operator satisfying: for all ρ_1 and $\rho_2 \in \mathcal{X}_1$,

$$\langle \rho_1, \widetilde{\mathcal{M}}_1 \rho_2 \rangle_{L^2(\Omega)} = \langle \rho_1, \mathcal{M} \rho_2 \rangle_{L^2(\Omega)}.$$

The cost function $\mathcal{J}_1$ can be seen as a "reduced" version of the original cost function. The corresponding (finite-dimensional) Riccati equation is the following:

$$(\widetilde{\mathcal{A}}_{s,1} + \delta I)^* \Pi_1 + \Pi_1 (\widetilde{\mathcal{A}}_{s,1} + \delta I) - \Pi_1 \mathcal{P}_1 \widetilde{\mathcal{B}} \widetilde{\mathcal{B}}^* P \mathcal{P}_1^* \Pi_1 + \widetilde{\mathcal{M}}_1 = 0,$$

where the unknown variable $\Pi_1 \in \mathcal{L}(\mathcal{X}_1)$ is a finite-dimensional self-adjoint operator. The feedback law reads as:

$$u = -\mathcal{B}^* (\mathcal{P}_1^* \Pi_1 \mathcal{P}_1) z.$$

and transforming back to the state variable y,

$$u = -\mathcal{B}^* e^{\Phi/2} (\mathcal{P}_1^* \Pi_1 \mathcal{P}_1) e^{\Phi/2} y =: \mathcal{K}_1 y. \tag{24}$$

Remark 7. From a theoretical point of view, the reduction method also works in the absence of orthogonality property for the eigenvectors of $\mathcal{A}_s$. However, the computation of the projection $\mathcal{P}_1$ is then much more involved, since it is not anymore orthogonal.

Remark 8. Theorem 5 still holds when the feedback law $\mathcal{K}_1$ is used (instead of $\mathcal{K}_0$). The proof can be easily adapted, observing that $\widetilde{\mathcal{A}}_s + \widetilde{\mathcal{B}} \mathcal{K}_1 \mathcal{P}_1 + \delta I$ is exponentially stable.

As an alternative way for reducing the complexity we also mention specific model reduction approaches, such as the ones investigated in Hartmann (2011); Hartmann et al. (2013).

5. NUMERICAL RESULTS

We present numerical results taken from Breiten et al. (2016) and compare the linear feedback $\mathcal{K}_0$, given in (20) with its approximation, $\mathcal{K}_1$, given in (24).

5.1 Setting

We consider a two-dimension system, with

$$\Omega = (-1.5, 1.5) \times (-1, 1) \subset \mathbb{R}^2,$$

with $\nu = 1$, and with a two-dimensional double-well potential of the form

$$W(x) = 3(x_1^2 - 1)^2 + 6x_2^2.$$

We assume that the particles are initially all located in the center of the right potential well, that is to say, the initial distribution is a Dirac centered at $x_1 = 1, x_2 = 0$.

Figure 1 shows the double well potential W as well as the corresponding stationary distribution.

Two control shape functions, α_1 and α_2, are considered. Their graphs are shown in figure 2. The control shape function α_1 is constructed in such a way that the Hautus criterion is satisfied, see (Breiten et al., 2016, Subsection 4.2) for details. The second control shape function is obtained after rotation of the first control shape function of an angle equal to 90°.

The operator $\mathcal{M}$ is equal to the identity operator and the value of δ has been set to:

$$\delta = 12,26.$$

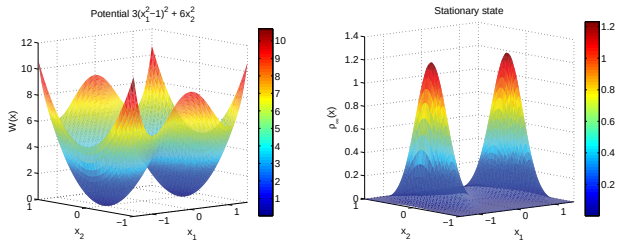


Fig. 1. Confining double well potential (left) and associated stationary state (right).

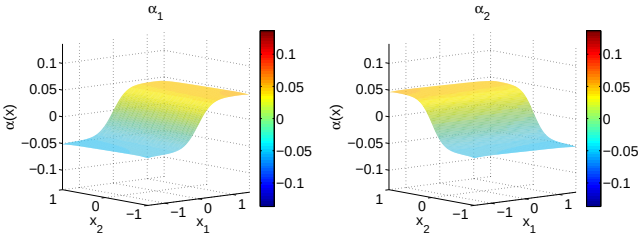


Fig. 2. Control shape functions

5.2 Discretization

For the spatial semidiscretization, a finite difference scheme with $k = n_{x_1} \cdot n_{x_2} = 96 \cdot 64 = 6144$ degrees of freedom was implemented. The discretization $A \in \mathbb{R}^{k\times k}$ of the operator $\mathcal{A}$ defined as in (7) was obtained by first discretizing the operator $\mathcal{A}^*$ as given by (8) and by then taking the transpose of the resulting matrix. The reason for this indirect approach was that the discretization of $\mathcal{A}^*$ only required the incorporation of "standard" Neumann boundary conditions rather than the mixed boundary conditions arising for $\mathcal{A}$. Due to the convective terms included in $\mathcal{A}$ and $\mathcal{A}^*$, a first order upwind scheme was utilized. Let us emphasize that even for the value $\nu = 1$, this turned out to be essential for the accuracy of the discretization. We also mention the possibility of using more advanced discretization schemes that have been proposed in the context of the Fokker-Planck equation, see, e. g. Annunziato and Borzì (2013); Chang and Cooper (1970).

For our numerical tests, the (discrete) stationary distribution ρ_∞^k is computed as the unique vector of norm 1 of the kernel of A (we do not use the formula (4)). Note that numerically, ρ_∞^k is positive and that the kernel of A^T is the set of vector with identical coordinates. Observe that up to a multiplicative constant, ρ_∞ and $e^{-\Phi}$ are equal. The operator $\rho \mapsto e^{-\Phi/2}\rho$ is discretized with the diagonal matrix D whose diagonal terms are equal to square roots of the coordinates of ρ_∞^k. The operator $\rho \mapsto e^{\Phi/2}$ is discretized with the matrix D^{-1}. Finally, the operator $\mathcal{A}_s$ is obtained by computing $A_s = D^{-1}AD$. Numerically, it is almost symmetric and is diagonalizable with non-positive eigenvalues.

All simulations were generated on an Intel®Xeon(R) CPU E31270 @ 3.40 GHz x 8,16 GB RAM, Ubuntu Linux 14.04, MATLAB® Version 8.0.0.783 (R2012b) 64-bit (glnxa64). For solving the Riccati equations, we used the MATLAB® routine `care`. We used the MATLAB® routine `eigs` for computing the greatest eigenvalues. The solutions of the ODE systems were always obtained by the MATLAB® routine `ode23`.

5.3 Comparison of the two linear feedbacks

The reduction method has been tested with a selection of 10 eigenvectors. The highest eigenvalue of A_s on the stable subspace is equal to $-35,0$. Let us denote by K_0 and K_1 the discretized feedback laws associated with $\mathcal{K}_0$ and $\mathcal{K}_1$, respectively. The discretized feedbacks are row vectors of dimension $96{\cdot}64$.

For the first control shape function, we have

$$2\|K_1 - K_0\|/(\|K_0\| + \|K_1\|) = 3,72.10^{-8},$$

for the second control shape function,

$$2\|K_1 - K_0\|/(\|K_0\| + \|K_1\|) = 3,77.10^{-5}.$$

The discretized feedback laws K_0 and K_1 are therefore very close, for the two different control shape functions. As a result, the trajectories obtained for the two different feedback laws are almost identical, for the two different control shape functions.

The resolution of the full Riccati equation takes approximately one hour (the unknown variable is a matrix with 6144 rows and columns), whereas the computation of the first 10 eigenvectors and the resolution of the corresponding "reduced" Riccati equation takes less than 1 second. The computation time is therefore significantly reduced, and the method could allow the design of Riccati-based feedback laws in higher dimension.

5.4 Stabilization results

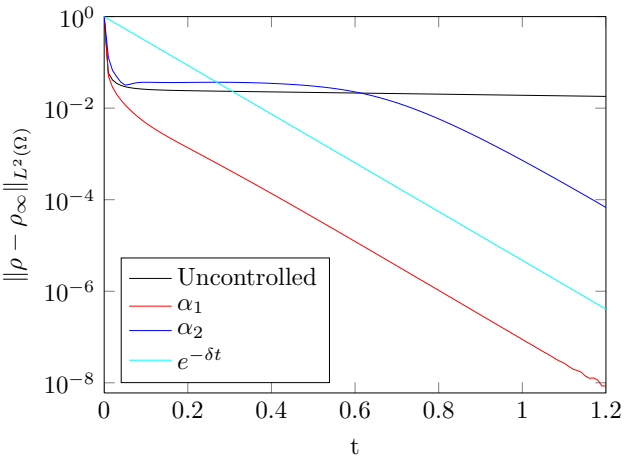
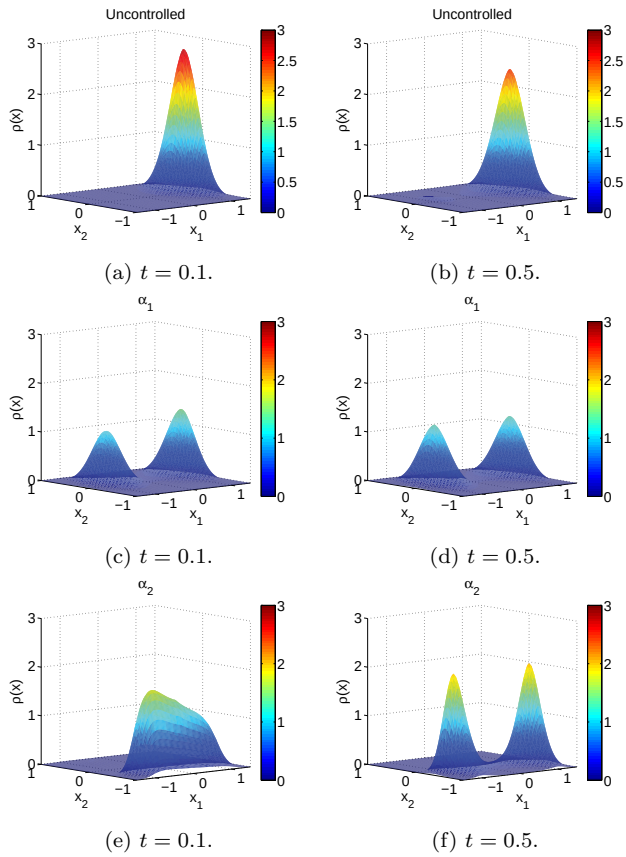


Fig. 3. Comparison of $L^2(\Omega)$-norm evolution

As is shown in Figure 3, the convergence rate of the uncontrolled system is undesirably slow. This mainly reflects the fact that the particle has to overcome the "energy barrier" between the potential wells. A rate of convergence equal to δ is observed when a linear feedback is used (results are the same for K_0 and K_1), for the two control shape functions. Note however that in the case of the control shape function α_2, the feedback is almost ineffective at the beginning.

Observe that when the control shape function α_1 is used, a positive value of the control has the following effect: The left-hand well is lowered and the right-hand well is heightened, which speeds up the transfer from the right-hand well to the left-hand well. The second control shape function is almost constant with respect to the variable x_1,

thus, the linear feedback cannot raise the right-hand well above the left-hand one. The linear feedback acts therefore in a rather different way, in this situation: It first attracts the particles at the lower boundary from where it is slowly moved to the center of the wells. This explains why the convergence to ρ_∞ is very slow, at the beginning. These phenomena are illustrated by Figure 4 and Figure 5, which show the temporal evolution of the state of the systems and the temporal evolution of the potential V, respectively.

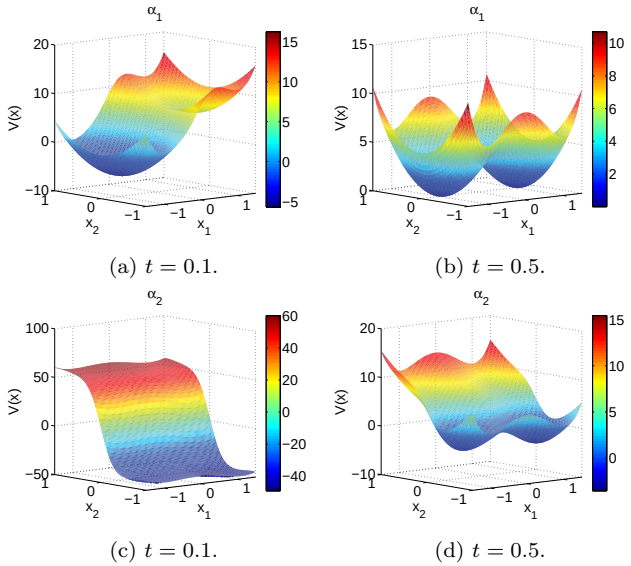


(a) $t = 0.1$. (b) $t = 0.5$.

(c) $t = 0.1$. (d) $t = 0.5$.

(e) $t = 0.1$. (f) $t = 0.5$.

Fig. 4. Temporal evolution of the state ρ.



(a) $t = 0.1$. (b) $t = 0.5$.

(c) $t = 0.1$. (d) $t = 0.5$.

Fig. 5. Temporal evolution of the potential $V(x)$.

ACKNOWLEDGEMENTS

This work was supported in part by the ERC advanced grant 668998 (OCLOC) under the EU's H2020 research program.

REFERENCES

Adams, R. (1975). *Sobolev Spaces*. New York Academic Press.

Annunziato, M. and Borzì, A. (2013). A Fokker-Planck control framework for multidimensional stochastic processes. *Journal of Computational and Applied Mathematics*, (237), 487–507.

Barbu, V., Lasiecka, I., and Triggiani, R. (2006). Tangential boundary stabilization of Navier–Stokes equations. *Memoirs of the American Mathematical Society*, 181(852), 1–128.

Bensoussan, A., Da Prato, G., Delfour, M., and Mitter, S. (2007). *Representation and Control of Infinite Dimensional Systems*. Birkhäuser Boston Basel Berlin. doi: 10.1007/978-0-8176-4581-6.

Breiten, T., Kunisch, K., and Pfeiffer, L. (2016). Control strategies for the Fokker-Planck equation. *SFB-Report*, 2016-003.

Chang, J. and Cooper, G. (1970). A practical scheme for Fokker–Planck equations. *Journal of Computational Physics*, 6, 1–16.

Curtain, R. and Zwart, H. (2005). *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag.

Gardiner, C.W. (2004). *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, volume 13 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, third edition.

Hartmann, C. (2011). Balanced model reduction of partially-observed Langevin equations: an averaging principle. *Mathematical and Computer Modelling of Dynamical Systems*, (17), 463–490.

Hartmann, C., Schäfer-Bund, B., and Thöns-Zueva, A. (2013). Balanced averaging of bilinear systems with applications to stochastic control. *SIAM Journal on Control and Optimization*, 51(3), 2356–2378.

Jones, P.H., Onofrio, M.M., and Volpe, G. (2015). *Optical Tweezers: Principles and Applications*. Cambridge University Press.

Raymond, J.P. (2006). Feedback boundary stabilization of the two-dimensional Navier–Stokes equations. *SIAM Journal on Control and Optimization*, 45(3), 790–828.

Raymond, J. and Thevenet, L. (2010). Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. *Discrete Contin. Dyn. Syst.*, 27(3), 1159–1187.

Risken, R. (1996). *The Fokker-Planck Equation: Methods of Solutions and Applications*. Springer Verlag Berlin.

Triggiani, R. (1975). On the stabilizability problem in Banach space. *Journal of Mathematical Analysis and Applications*, 52(3), 383–403.