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T. Breiten K. Kunisch

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# BOUNDARY FEEDBACK STABILIZATION OF THE MONODOMAIN EQUATIONS

Tobias Breiten*

Institute for Mathematics and Scientic Computing
Karl-Franzens-Universität Graz
Heinrichstr. 36, 8010 Graz, Austria

Karl Kunisch

Institute for Mathematics and Scientic Computing
Karl-Franzens-Universität Graz
Heinrichstr. 36, 8010 Graz, Austria
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenbergerstraße 69, A-4040 Linz, Austria

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Abstract. Boundary feedback control for a coupled nonlinear PDE-ODE system (in the two and three dimensional cases) is studied. Particular focus is put on the monodomain equations arising in the context of cardiac electrophysiology. Neumann as well as Dirichlet based boundary control laws are obtained by an algebraic operator Riccati equation associated with the linearized system. Local exponential stability of the nonlinear closed loop system is shown by a fixed-point argument. Numerical examples are given for a finite element discretization of the two dimensional monodomain equations.

## 1. Introduction.

This paper is concerned with the problem of boundary feedback stabilization for a coupled nonlinear reaction diffusion system. Both Neumann and Dirichlet boundary control is studied. The dynamics of interest are described by the so-called monodomain equations, a reasonably accurate simplification of the bidomain equations (see, e.g., [19]) which are frequently used to model the electrophysiological activity of the human heart. It is well-known that anomalous behavior such as fibrillation processes can be recognized within the mathematical model in the form of spiral or reentry waves. In this context, the clinical goal consists of a successful termination of these waves by external stimuli, usually called *defibrillation*. In [7], we have shown that the monodomain equations can locally be stabilized around a stationary solution by means of a state feedback law obtained from the linearized system. In contrast to the setup considered therein, a more realistic scenario assumes that the stimulating electrodes are situated at the boundary of the domain. With this in mind, our goal is to extend the concepts from [7] to the case of a boundary control. We further employ slightly different Lipschitz continuity

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* Corresponding author.

estimates than the ones used in [7]. Proceeding this way, we are able to generalize the local stabilization results for the nonlinear system to a broader class of initial perturbations. Mathematically the monodomain equations are an evolution system consisting of a diffusion equation with a polynomial, more specifically, cubic nonlinearity coupled with an ordinary differential equation. Such systems play an important role far beyond their use for the description of the electrical activity of the heart. In fact, many models of this kind arise in cellular biology as described in [14], for example. One such application area is the modeling of wave propagation in excitable systems, as for instance nerve axons. The version as system of ordinary differential system arises in the modeling of excitable cells occurring in ionic flow across cell membranes.

Thus, let us consider the following nonlinear reaction diffusion system

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v,w) + f_v \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial w}{\partial t} &= \iota v - \kappa w + f_w \quad \text{in } \Omega \times (0,\infty), \\ \theta \frac{\partial v}{\partial \nu} &= (\theta - 1)v + m\tilde{u} + g \quad \text{on } \Gamma \times (0,\infty), \\ v(x,0) &= \bar{v} + y_0 \text{ and } w(x,0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$ denotes a bounded open set with smooth boundary $\Gamma = \partial\Omega$ and constants $\iota, \kappa > 0$. The factor $\theta \in \{0,1\}$ is used to model both the Neumann and the Dirichlet boundary control setup, respectively. The nonlinearity $I_{\text{ion}}(v,w)$ is assumed to be of FitzHugh-Nagumo type, i.e.

$$I_{\text{ion}}(v,w) = av^3 - bv^2 + cv + dw, \tag{2}$$

with $a,b,c,d \in \mathbb{R}_+$. Here, $f_v, f_w$ and $g$ are external forcing functions and $(\bar{v},\bar{w}) \in H^{\frac{n}{2}+s}(\Omega) \times L^\infty(\Omega), s > 0$ denotes a stationary solution to

$$\begin{aligned} 0 &= \Delta \bar{v} - I_{\text{ion}}(\bar{v},\bar{w}) + f_v \quad \text{in } \Omega \\ 0 &= \iota \bar{v} - \kappa \bar{w} + f_w \quad \text{in } \Omega \\ \theta \frac{\partial \bar{v}}{\partial \nu} &= (\theta - 1)\bar{v} + g \quad \text{on } \Gamma. \end{aligned} \tag{3}$$

The function $m$ is assumed to localize the control in a part of the boundary $\Gamma$. For the precise assumptions on $m$, we follow the setup from [24]. In particular, if $\Gamma$ is of class $C^3$, we assume that $m \in C^2(\Gamma), m \geq 0$ and $m(x) = 1, x \in \Gamma_c$, where $\Gamma_c$ denotes an open control subset in $\Gamma$.

Similar to [7], we want to find a control input function $u \in L^2(0,\infty;L^2(\Gamma))$ such that the solution $(v,w)$ to (1) locally decays exponentially, i.e., provided the initial perturbation $\mathbf{y}_0 = (y_0,z_0)$ is small in an appropriate sense. For $0 \leq \sigma < \kappa$, let us introduce $\mathbf{y} = (y,z) := (e^{\sigma t}(v - \bar{v}), e^{\sigma t}(w - \bar{w})), u = e^{\sigma t}\tilde{u}$ and focus on

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + \alpha(x) y - dz - ae^{-2\sigma t} y^3 + (b - 3a\bar{v})e^{-\sigma t} y^2 \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial z}{\partial t} &= \iota y - (\kappa - \sigma) z \quad \text{in } \Omega \times (0,\infty), \\ \theta \frac{\partial y}{\partial \nu} &= (\theta - 1)y + mu \quad \text{on } \Gamma \times (0,\infty), \\ y(x,0) &= y_0 \text{ and } z(x,0) = z_0 \quad \text{in } \Omega, \end{aligned} \tag{4}$$

where $\alpha(x) = -3a\bar{v}^2 + 2b\bar{v} - c + \sigma$. Instead of the stabilization of (1) around a solution of (3), we rather consider the stabilization of (4) around zero. The strategy discussed in the following is based on a linear quadratic infinite horizon optimal control problem for the linearized system and has been successfully employed for many different systems, see, e.g., [1, 2, 4, 21, 24]. In particular, for the linearized system, we closely follow the abstract Riccati theory for boundary control systems as presented in [17].

The structure of the paper is as follows. In Section 2 we study the case of a Neumann boundary control together with a partial Dirichlet boundary observation. Based on the general results from [17, Chapter 2], we analyze the effect of a Riccati-based feedback approach obtained for the linearized system when used in the full nonlinear setting. In particular, we slightly extend the Lipschitz continuity estimates from [7]. This will lead to a local stabilization result (in the two and three dimensional cases) for initial perturbations $\mathbf{y}_0 \in H^s(\Omega) \times H^{s+1}(\Omega)$, $s \in \left(\frac{1}{2}, 1\right]$. In Section 3 we investigate the possibility of using a Dirichlet boundary control. Different from the Neumann case, here the abstract setting from [17, Chapter 2] does not directly yield a locally stabilizing feedback law for the nonlinear system. Similar to [21, 24], for $s \in \left(0, \frac{1}{2}\right)$ we first have to show that the involved Riccati operator $\mathbf{\Pi}$ satisfies $\mathbf{\Pi} \in \mathcal{L}(H^s(\Omega) \times L^2(\Omega), H^{s+2}(\Omega) \times H^{s+2}(\Omega))$. As a result, in the two dimensional case, for $s \in \left(0, \frac{1}{2}\right)$, we obtain a local stabilization result for the nonlinear system provided the initial perturbation $\mathbf{y}_0 \in H^s(\Omega) \times H^{s+1}(\Omega)$ is small enough. In Section 4 we illustrate some of the theoretical results in a numerical setup. We conclude with a short summary of our contributions in Section 5.

**Notation.** For $p \geq 1$ and $s \geq 0$, we denote by $L^p(\Omega)$ and $H^s(\Omega)$ the usual Lebesgue and Sobolev spaces. We define $H^s_\nu(\Omega) := \overline{\left\{ y \in \mathcal{D}(\Omega) \,\middle|\, \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}}$, where $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in $\Omega$ and where the closure is taken with respect to the $\|y\|_{H^s(\Omega)}, s \geq 0$, norm. For $s > \frac{3}{2}$, we have $H^s_\nu(\Omega) = \left\{ y \in H^s(\Omega) \,\middle|\, \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}$ and for $s \in [0, \frac{3}{2})$, we have $H^s_\nu(\Omega) = H^s(\Omega)$, see, e.g., [6, Section II-1]. Analogously, for $s > \frac{1}{2}$, we set $H^s_0(\Omega) = \{ y \in H^s(\Omega) \,|y = 0 \text{ on } \Gamma \}$. Given $p \geq 1$, an interval $I \subset \mathbb{R}$ and a Hilbert space $X$, we denote with $L^p(I; X)$ (Bochner) $p$-integrable functions on $I$ with values in $X$. For $Q_{t_0,T} = \Omega \times (t_0, T)$ and $r \geq 0, s \geq 0$ we define the anisotropic Sobolev spaces $H^{r,s}(Q_{t_0,T})$ by

$$H^{r,s}(Q_{t_0,T}) = L^2(t_0, T; H^r(\Omega)) \cap H^s(t_0, T; L^2(\Omega)).$$

For $t_0 = 0$ and $T = \infty$ we simply write $Q_\infty := Q_{0,\infty}$. Similar, we use $\Sigma_{t_0,T} := \Gamma \times (t_0, T)$ and $\Sigma_\infty$. We further use $\mathbf{L}^2(\Omega)$ instead of $L^2(\Omega) \times L^2(\Omega)$. Similarly, we define

$$\mathbf{H}^{r,s}(Q_{t_0,T}) := \left(L^2(t_0, T; H^r(\Omega)) \cap H^s(t_0, T; L^2(\Omega))\right) \times \left(L^2(t_0, T; H^r(\Omega)) \cap H^s(t_0, T; L^2(\Omega))\right).$$

In more general, boldface letters are always associated with the PDE-ODE system while italic letters refer to either the PDE or the ODE part. Let $\mathbf{X}$ and $\mathbf{Y}$ denote Hilbert spaces. For a closed densely defined linear operator $\mathbf{A}\colon \mathcal{D}(\mathbf{A}) \subset \mathbf{Y} \to \mathbf{Y}$, the resolvent set of $\mathbf{A}$ is denoted by $\rho(\mathbf{A})$. Given its infinitesimal generator $\mathbf{A}$ with $e^{\mathbf{A}t}$ we denote the associated semigroup. For the definition and calculus involving interpolation spaces $[\mathbf{X}, \mathbf{Y}]_\theta$, we refer to e.g. [6, 18]. By $C, C_1$ and $C_2$ we denote generic constants that may vary throughout consecutive calculations.

2. **Neumann boundary control.** We start with the linearization of (4) for $\theta = 1$. Hence, let us consider

$$\begin{aligned} \frac{\partial y}{\partial t} &= \Delta y + \alpha(x) y - dz \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial z}{\partial t} &= \iota y - (\kappa - \sigma) z \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial y}{\partial \nu} &= mu \quad \text{on } \Gamma \times (0,\infty), \\ y(x,0) &= y_0 \text{ and } z(x,0) = z_0 \quad \text{in } \Omega, \end{aligned} \tag{5}$$

as a boundary control system as described in, e.g., [6, 17, 26]. For this purpose, we introduce the operators $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ and $(\mathbf{A}^*, \mathcal{D}(\mathbf{A}^*))$ with

$$\begin{aligned} \mathbf{A}\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x) y - dz \\ \iota y - (\kappa - \sigma) z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}) = H^2_\nu(\Omega) \times L^2(\Omega), \\ \mathbf{A}^*\mathbf{y} &= \begin{pmatrix} \Delta y + \alpha(x) y + \iota z \\ -dy - (\kappa - \sigma) z \end{pmatrix}, \quad \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\mathbf{A}). \end{aligned} \tag{6}$$

For the rest of this section, let $\lambda_0 \in \rho(\mathbf{A})$ be such that $\widehat{\mathbf{A}} := \lambda_0 I - \mathbf{A}$ with $\mathcal{D}(\widehat{\mathbf{A}}) = \mathcal{D}(\mathbf{A})$ satisfies

$$\begin{aligned} \left(\widehat{\mathbf{A}}\mathbf{y}, \mathbf{y}\right)_{\mathbf{L}^2(\Omega)} &\geq \|\mathbf{y}\|^2_{H^1(\Omega)\times L^2(\Omega)} \quad \text{for all } \mathbf{y} \in \mathcal{D}(\mathbf{A}), \\ \left(\widehat{\mathbf{A}}^*\mathbf{y}, \mathbf{y}\right)_{\mathbf{L}^2(\Omega)} &\geq \|\mathbf{y}\|^2_{H^1(\Omega)\times L^2(\Omega)} \quad \text{for all } \mathbf{y} \in \mathcal{D}(\mathbf{A}^*). \end{aligned}$$

As a consequence, $-\widehat{\mathbf{A}}$ generates an analytic exponentially stable semigroup on $\mathbf{Y} = \mathbf{L}^2(\Omega)$ and, additionally, its fractional powers $\widehat{\mathbf{A}}^\gamma$ (see [20]) are well-defined. Further, we introduce the Neumann map $\mathbf{N}_{\widehat{\mathbf{A}}}$ by $\mathbf{N}_{\widehat{\mathbf{A}}} u = \mathbf{y}$ iff

$$\begin{aligned} \lambda_0 y - \Delta y - \alpha(x) y + dz = 0 \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial \nu} = u \quad \text{on } \Gamma, \\ \lambda_0 z - \iota y + (\kappa - \sigma) z = 0 \quad \text{in } \Omega. \end{aligned} \tag{7}$$

By explicitly solving the second equation in (7) for $z$, we equivalently obtain

$$\lambda_0 y - \Delta y - \alpha(x) y + \frac{d\iota}{\lambda_0 + \kappa - \sigma} y = 0 \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial \nu} = u \quad \text{on } \Gamma.$$

As in [17, Section 3.3] we thus conclude that

$$\mathbf{N}_{\widehat{\mathbf{A}}} \colon \text{continuous } H^s(\Gamma) \to \mathbf{H}^{s+\frac{3}{2}}(\Omega), \quad s \geq 0. \tag{8}$$

In particular, for $s = 0$ let us emphasize that

$$\mathbf{N}_{\widehat{\mathbf{A}}} \colon \text{continuous } L^2(\Gamma) \to \mathbf{H}^{\frac{3}{2}}(\Omega) \subset H^{\frac{3}{2}-2\varepsilon}(\Omega) \times L^2(\Omega) = \mathcal{D}(\widehat{\mathbf{A}}^{\frac{3}{4}-\varepsilon}), \quad \varepsilon > 0. \tag{9}$$

We now may rewrite the linearized system as a boundary control system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B} M u(t), \quad \mathbf{y}_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \tag{10}$$

where $\mathbf{B} = \widehat{\mathbf{A}} \mathbf{N}_{\widehat{\mathbf{A}}} \in \mathcal{L}(L^2(\Gamma), [\mathcal{D}(\mathbf{A}^*)]')$ and $M$ is the multiplication operator associated with $m$.

**Lemma 2.1.** *For all* $\mathbf{p} = (p,q) \in \mathcal{D}(\mathbf{A}^*)$, *it holds that*

$$\mathbf{B}^*\mathbf{p} = \mathbf{N}^*_{\widehat{\mathbf{A}}}\widehat{\mathbf{A}}^*\mathbf{p} = p|_\Gamma. \tag{11}$$

*For* $\mathbf{p} \in H^{\frac{1}{2}+s}(\Omega) \times L^2(\Omega), s > 0,$ *it further holds*

$$\|\mathbf{B}^*\mathbf{p}\|_{H^s(\Gamma)} \le C\|\mathbf{p}\|_{H^{\frac{1}{2}+s}(\Omega)\times L^2(\Omega)}.$$

*Proof.* Assume that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$ and $u \in L^2(\Gamma)$. With $\mathbf{y} = (y,z)$ let us denote $\mathbf{N}_{\widehat{\mathbf{A}}}u = \mathbf{y}$. We then have

$$\begin{aligned}(\mathbf{B}u,\mathbf{p})_{\mathbf{L}^2(\Omega)} &= \left(\mathbf{N}_{\widehat{\mathbf{A}}}u, \widehat{\mathbf{A}}^*\mathbf{p}\right)_{\mathbf{L}^2(\Omega)} = \left(\mathbf{y}, \widehat{\mathbf{A}}^*\mathbf{p}\right)_{\mathbf{L}^2(\Omega)} \\ &= (y, \lambda_0 p - \Delta p - \alpha(x)p - \iota q)_{L^2(\Omega)} + (z, \lambda_0 q + (\kappa-\sigma)q + dp)_{L^2(\Omega)} \\ &= (\lambda_0 y - \Delta y - \alpha(x)y + dz, p)_{L^2(\Omega)} + (\lambda_0 z - \iota y + (\kappa - \sigma)z, q)_{L^2(\Omega)} \\ &\quad - \left(y, \frac{\partial p}{\partial \nu}\right)_{L^2(\Gamma)} + \left(\frac{\partial y}{\partial \nu}, p\right)_{L^2(\Gamma)}.\end{aligned}$$

By (7) and the fact that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$, we thus obtain

$$(\mathbf{B}u,\mathbf{p})_{\mathbf{L}^2(\Omega)} = (u,p)_{L^2(\Gamma)} = (u, \mathbf{B}^*\mathbf{p})_{L^2(\Gamma)}.$$

For the estimate, see, e.g., [6, Part II, Chapter 3, Section 4]. $\square$

Finally, note that (9) implies that

$$\widehat{\mathbf{A}}^{-\gamma}\mathbf{B} = \widehat{\mathbf{A}}^{1-\gamma}\mathbf{N}_{\widehat{\mathbf{A}}} \in \mathcal{L}(U,\mathbf{Y}), \quad \gamma = \frac{1}{4} + \varepsilon, \quad \varepsilon > 0. \tag{12}$$

2.1. **Existing Riccati theory for the linearized system.** With system (4), we associate the following cost functional

$$\begin{aligned}\mathcal{J}(u,\mathbf{y}) &= \int_0^\infty \left(\left\|\widetilde{M}\mathbf{C}\mathbf{y}(t)\right\|^2_{L^2(\Gamma)} + \|u(t)\|^2_{L^2(\Gamma)}\right)\mathrm{d}t \\ &= \int_0^\infty \left(\left\|\widetilde{M}Cy(t)\right\|^2_{L^2(\Gamma)} + \|u(t)\|^2_{L^2(\Gamma)}\right)\mathrm{d}t,\end{aligned} \tag{13}$$

where the multiplication operator $\widetilde{M}$ denotes the observable part of $y$ on the boundary $\Gamma$ and $\mathbf{C}\mathbf{y} = Cy$ where $C\colon H^{\frac{1}{2}+s}(\Omega) \to L^2(\Gamma), s > 0$, is the Dirichlet trace operator $Cy = y_{|\Gamma}$. Based on the results in [17], our goal now is to find $u$ in feedback form such that the cost functional (13) is minimized. We are going to apply the results from [17, Chapter 2] and therefore consider the assumptions:

(H1) $\mathbf{A}$ is the infinitesimal generator of a strongly continuous analytic semigroup on $\mathbf{Y} := \mathbf{L}^2(\Omega)$.

(H2) $\widehat{\mathbf{A}}^{-\gamma}\mathbf{B}M \in \mathcal{L}(U,\mathbf{Y}),\ U := L^2(\Gamma)$, for some fixed constant $\gamma$, $0 \le \gamma < 1$.

(H3) $\widetilde{M}\mathbf{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), Z),\ Z := L^2(\Gamma)$, for some fixed $0 < \delta < \min\{1-\gamma, \frac{1}{2}\}$.

(H4) For each $\mathbf{y}_0 \in \mathbf{Y}$, there exists $u \in L^2(0,\infty;U)$ such that the corresponding solution $\mathbf{y}$ satisfies $\widetilde{M}\mathbf{C}\mathbf{y} \in L^2(0,\infty;Z)$ so that $\mathcal{J}(u,\mathbf{y}) < \infty$.

(H5) There exists an operator $\mathbf{K} \in \mathcal{L}(Z, \mathcal{D}(\widehat{\mathbf{A}}^\delta))$, such that the strongly continuous analytic semigroup $e^{(\mathbf{A}+\mathbf{K}\widetilde{M}\mathbf{C})t}$ on $\mathcal{D}(\widehat{\mathbf{A}}^\delta)$, generated by $\mathbf{A} + \mathbf{K}\widetilde{M}\mathbf{C}$ is exponentially stable.

Let us comment on these assumptions. As we have shown in [7, Lemma 3.1], $\mathbf{A}$ indeed is the infinitesimal generator of a strongly continuous analytic semigroup on $\mathbf{Y}$. Hence, (H1) is satisfied. From (12), it follows that (H2) holds for $U = L^2(\Gamma)$ and

$$\frac{1}{4} < \gamma < 1. \tag{14}$$

Lemma 2.1 implies that $\mathbf{C} = \mathbf{B}^*$ and we thus have that $\widetilde{M}\mathbf{C} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), Z)$ for

$$\frac{1}{4} < \delta < \min\left\{1-\gamma, \frac{1}{2}\right\} \tag{15}$$

such that (H3) is satisfied. It remains to show the validity of (H4) and (H5). For this, we may follow the strategy used in [7] and first consider the decoupled problem

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial t} &= \Delta \tilde{y} + \alpha(x)\tilde{y} + \omega \tilde{y} \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial \tilde{y}}{\partial \nu} &= mu \quad \text{on } \Gamma \times (0,\infty), \\ \tilde{y}(x,0) &= y_0, \end{aligned} \tag{16}$$

where $\omega > 0$, together with the cost functional

$$J(u,\tilde{y}) = \int_0^\infty \left( \|\tilde{y}(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Gamma)} \right) \mathrm{d}t, \tag{17}$$

which is finite for $\tilde{y} \in L^2(Q_\infty)$ and $u \in L^2(\Sigma_\infty)$. Using available null controllability results from [11, Theorem 2] and a well-known extension argument (see, e.g., [12, Theorem 2.3]) we particularly conclude that for each $y_0 \in L^2(\Omega)$ there exists $u \in L^2(\Sigma_\infty)$ such that the corresponding solution $\tilde{y}$ satisfies $\tilde{y} \in L^2(Q_\infty)$ so that $J(u,\tilde{y}) < \infty$. In other words, the decoupled system (16) is stabilizable. According to [7, Lemma 3.2], if we choose $\omega = \varepsilon + \frac{\gamma d}{\delta - \varepsilon}, 0 < \varepsilon < \delta$, we obtain exponential stability of the semigroup on $\mathbf{Y}$ (with decay rate $\varepsilon$) even for the coupled problem (5). Hence, assumption (H4) is fulfilled. Finally, the detectability condition (H5) of the pair $(\mathbf{A}, \widetilde{M}\mathbf{C})$ is equivalent to the stabilizability of the pair $(\mathbf{A}^*, \mathbf{C}^*\widetilde{M})$. Since we already know that $\mathbf{B} = \mathbf{C}^*$, the arguments provided for (H4) apply here as well and imply (H5). In summary, all of the assumptions (H1)-(H5) are indeed fulfilled for the monodomain equations and we can use the results from [17, Section 2.5] that guarantee the existence of a unique nonnegative self-adjoint solution $\mathbf{\Pi} = \mathbf{\Pi}^* \in \mathcal{L}(\mathbf{Y})$ to the algebraic Riccati equation for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}(\widehat{\mathbf{A}}^{\delta+s}), s > 0:$

$$(\mathbf{A}^*\mathbf{\Pi}\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + (\mathbf{\Pi}\mathbf{A}\mathbf{z}_1, \mathbf{z}_2)_{\mathbf{Y}} + \left(\widetilde{M}\mathbf{C}\mathbf{z}_1, \widetilde{M}\mathbf{C}\mathbf{z}_2\right)_Z = (M\mathbf{B}^*\mathbf{\Pi}\mathbf{z}_1, M\mathbf{B}^*\mathbf{\Pi}\mathbf{z}_2)_U .$$

Additionally, by [17, Theorem 2.5.2], it holds that

$$(\widehat{\mathbf{A}}^*)^\gamma \mathbf{\Pi} \widehat{\mathbf{A}}^{-\delta} \in \mathcal{L}(\mathbf{Y}), \quad \mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), U) \tag{18}$$

and $(\mathbf{A}_{\mathbf{\Pi}}, \mathcal{D}(\mathbf{A}_{\mathbf{\Pi}}))$, where $\mathbf{A}_{\mathbf{\Pi}} = \mathbf{A} - \mathbf{B}M^2\mathbf{B}^*\mathbf{\Pi}$, is the infinitesimal generator of a strongly continuous semigroup exponentially stable on $\mathcal{D}(\widehat{\mathbf{A}}^\delta)$. In [17, Section 2.5], these results are actually derived by a change of variables

$$\begin{aligned} \bar{\mathbf{B}}M &\equiv \widehat{\mathbf{A}}^\delta \mathbf{B}M \colon \text{continuous } U \to (\mathcal{D}(\widehat{\mathbf{A}}^*)^{1+\delta})', \\ \widehat{\mathbf{A}}^{-\bar{\gamma}}\bar{\mathbf{B}}M &= \widehat{\mathbf{A}}^{-(\bar{\gamma}-\delta)}\mathbf{B}M = \widehat{\mathbf{A}}^{-\gamma}\mathbf{B}M \in \mathcal{L}(U,\mathbf{Y}), \quad \bar{\gamma} = \gamma + \delta < 1, \\ \widetilde{M}\bar{\mathbf{C}} &\equiv \widetilde{M}\mathbf{C}\widehat{\mathbf{A}}^{-\delta} \in \mathcal{L}(\mathbf{Y}, Z) \end{aligned} \tag{19}$$

for the transformed system

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{y}} = \mathbf{A}\bar{\mathbf{y}} + \bar{\mathbf{B}}Mu, \quad \bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0,$$

where $\bar{\mathbf{y}} = \widehat{\mathbf{A}}^{\delta}\mathbf{y}$ and the modified cost functional

$$\bar{J}(u, \bar{\mathbf{y}}_0) = \int_0^{\infty} \left( \left\| \widetilde{M}\bar{\mathbf{C}}\bar{\mathbf{y}}(t) \right\|_Z^2 + \|u(t)\|_U^2 \right) \mathrm{d}t.$$

In particular, for the corresponding Riccati operator $\bar{\mathbf{\Pi}}$ it is shown that

$$M\bar{\mathbf{B}}^*\bar{\mathbf{\Pi}} \in \mathcal{L}(\mathbf{Y}, U) \tag{20}$$

and that the semigroup generated by $\mathbf{A}_{\bar{\mathbf{\Pi}}} = \mathbf{A} - \bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\bar{\mathbf{\Pi}}$ is exponentially stable on $\mathbf{Y}$, see [17, Theorem 2.5.1/2.5.2]. Moreover, we have the relation

$$\mathbf{A}_{\mathbf{\Pi}} = \mathbf{A} - \mathbf{B}M^2\mathbf{B}^*\mathbf{\Pi} = \widehat{\mathbf{A}}^{-\delta}\mathbf{A}_{\bar{\mathbf{\Pi}}}\widehat{\mathbf{A}}^{\delta}. \tag{21}$$

2.2. **The nonhomogeneous linear system.** Before we turn to the nonlinear setting, we first consider the nonhomogeneous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{A}_{\mathbf{\Pi}}\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{22}$$

with $\mathbf{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. For what follows, we recall a regularity result from [6].

**Theorem 2.2.** *(*[6, Chapter 3, Theorem 2.2]*) Let $\mathbf{Y}$ be a Hilbert space and suppose that $\mathbf{A}$ is the infinitesimal generator of an analytic semigroup of negative type on $\mathbf{Y}$. Then, for all $0 \le \alpha \le 1$, the mapping*

$$\begin{aligned} \mathbf{y} \ &\mapsto \ \left( \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} - \mathbf{A}\mathbf{y}, \mathbf{y}(0) \right) \\ &L^2(0,\infty; [\mathcal{D}(\mathbf{A}), \mathbf{Y}]_{\alpha}) \cap H^1(0,\infty; [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}) \\ &\quad \to L^2(0,\infty; [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}) \times [[\mathcal{D}(\mathbf{A}), \mathbf{Y}]_{\alpha}, [\mathcal{D}(\mathbf{A}^*), \mathbf{Y}]'_{1-\alpha}]_{\frac{1}{2}} \end{aligned}$$

*is an isomorphism.*

With Theorem 2.2, we have the following regularity result (see also [24, Lemma 4.1]).

**Lemma 2.3.** *If $u \in H^{s,\frac{s}{2}}(\Sigma_\infty)$ with $0 \le s < 1$, then the mild solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\widehat{\mathbf{A}}\mathbf{y} + \mathbf{B}Mu, \quad \mathbf{y}(0) = 0 \tag{23}$$

*satisfies*

$$\|\mathbf{y}\|_{\mathbf{H}^{s+\frac{3}{2}-\mu,\frac{s}{2}+\frac{3}{4}-\frac{\mu}{2}}(Q_\infty)} \le C\|u\|_{H^{s,\frac{s}{2}}(\Sigma_\infty)}, \quad \mu > 0.$$

*Furthermore, if $u \in H^{s,\frac{s}{2}}(\Sigma_\infty)$ with $1 < s \le 2$ and $u(0) = 0$ then the same estimate holds true.*

*Proof.* The idea of the proof is well-known and can be found for the heat equation and the Navier-Stokes equation in, e.g., [16, 21]. For the sake of completeness, we provide the details here as well. For this, let us consider the mild solution

$$\mathbf{y}(t) = \int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{B}Mu(\tau)\,\mathrm{d}\tau = \widehat{\mathbf{A}}\int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\,\mathrm{d}\tau. \tag{24}$$

We start with the case $s = 0$, i.e., let $u \in L^2(\Sigma_\infty)$. From (12) we find that $\widehat{\mathbf{A}}^{\frac{3-\mu}{4}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu \in \mathbf{L}^2(Q_\infty)$. We thus obtain

$$\left\|\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{y}(t)\right\|_{\mathbf{L}^2(\Omega)} \le \int_0^t \left\|\widehat{\mathbf{A}}^{1-\frac{\mu}{4}}e^{-\widehat{\mathbf{A}}(t-\tau)}\right\| \left\|\widehat{\mathbf{A}}^{\frac{3-\mu}{4}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\right\|_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau.$$

Since the semigroup $e^{-\widehat{\mathbf{A}}\cdot}$ is exponentially stable, with [20, Theorem 6.13(c)] this implies for some $\xi > 0$ that

$$\left\|\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{y}(t)\right\|_{\mathbf{L}^2(\Omega)} \le C\int_0^t (t-\tau)^{-1+\frac{\mu}{4}}e^{-\xi(t-\tau)} \left\|\widehat{\mathbf{A}}^{\frac{3-\mu}{4}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\right\|_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau.$$

An application of Young's inequality for convolutions now shows that

$$\|\mathbf{y}\|_{L^2(0,\infty;H^{\frac{3}{2}-\mu}(\Omega))\times L^2(Q_\infty)} \le C\|u\|_{L^2(\Sigma_\infty)}.$$

Explicit integration of the second equation in (23) yields

$$z(t) = \int_0^t \iota e^{-(\kappa+\lambda_0-\sigma)(t-\tau)}y(\tau)\,\mathrm{d}\tau. \tag{25}$$

In particular, we thus have $z \in L^2(0,\infty;H^{\frac{3}{2}-\mu}(\Omega))$. Next, using (24) let us consider

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) &= \widehat{\mathbf{A}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(t) - \widehat{\mathbf{A}}\int_0^t \widehat{\mathbf{A}}e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\,\mathrm{d}\tau \\
&= \widehat{\mathbf{A}}^{\frac{1}{4}+\frac{\mu}{2}}\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(t) - \widehat{\mathbf{A}}^{\frac{1}{4}+\frac{\mu}{2}}\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\int_0^t \widehat{\mathbf{A}}e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\,\mathrm{d}\tau \\
&= \widehat{\mathbf{A}}^{\frac{1}{4}+\frac{\mu}{2}}\left(\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(t) - \widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{y}(t)\right).
\end{aligned}$$

Consequently, using the regularity of $\mathbf{y}$, we conclude that

$$\begin{aligned}
\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t)\right\|_{L^2(0,\infty;[\mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{4}+\frac{\mu}{2}})]')} &\le C\left(\left\|\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{2}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu\right\|_{\mathbf{L}^2(Q_\infty)} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{H}^{\frac{3}{2}-\mu}(\Omega))}\right) \\
&\le C\|u\|_{L^2(\Sigma_\infty)}.
\end{aligned}$$

By the *intermediate derivatives theorem*, see [18, Theorem 4.1], we also find that

$$\mathbf{y} \in H^{\frac{3}{4}-\frac{\mu}{2}}\left(0,\infty;\left[\mathbf{H}^{\frac{3}{2}-\mu}(\Omega),[\mathcal{D}(\widehat{\mathbf{A}}^{\frac{1}{4}+\frac{\mu}{2}})]'\right]_{\frac{3}{4}-\frac{\mu}{2}}\right) \subset H^{\frac{3}{4}-\frac{\mu}{2}}(0,\infty;\mathbf{L}^2(\Omega)). \tag{26}$$

This shows the assertion for $s = 0$. Assume now that $u \in H^{2,1}(\Sigma_\infty)$ and that $u(0) = 0$. We then have

$$\mathbf{y}(t) = \widehat{\mathbf{A}}\int_0^t e^{-(t-\tau)\widehat{\mathbf{A}}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu(\tau)\,\mathrm{d}\tau = \mathbf{N}_{\widehat{\mathbf{A}}}Mu(t) - \int_0^t e^{-(t-\tau)\widehat{\mathbf{A}}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu'(\tau)\,\mathrm{d}\tau.$$

From (8), we conclude that $\mathbf{N}_{\widehat{\mathbf{A}}}Mu \in L^2(0,\infty;\mathbf{H}^{\frac{7}{2}}(\Omega))$. Also, it holds that

$$\begin{aligned}
&\left\|\widehat{\mathbf{A}}^{\frac{7}{4}-\frac{\mu}{2}}\int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}}Mu'(\tau)\,\mathrm{d}\tau\right\|_{\mathbf{L}^2(\Omega)} \\
&\quad= \left\|\int_0^t \widehat{\mathbf{A}}^{1-\frac{\mu}{4}}e^{-\widehat{\mathbf{A}}(t-\tau)}\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{4}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu'(\tau)\mathrm{d}\tau\right\|_{\mathbf{L}^2(\Omega)} \\
&\quad\le C\int_0^t (t-\tau)^{-1+\frac{\mu}{4}}e^{-\xi(t-\tau)}\left\|\widehat{\mathbf{A}}^{\frac{3}{4}-\frac{\mu}{4}}\mathbf{N}_{\widehat{\mathbf{A}}}Mu'(\tau)\right\|_{\mathbf{L}^2(\Omega)}\,\mathrm{d}\tau.
\end{aligned}$$

Young's inequality for convolutions yields

$$\left\| e^{-\widehat{\mathbf{A}}\cdot} * \left(\mathbf{N}_{\widehat{\mathbf{A}}} M u'(\cdot)\right) \right\|_{L^2(0,\infty;H^{\frac{7}{2}-\mu}(\Omega)\times L^2(\Omega))} \le C\|u\|_{H^{2,1}(\Sigma_\infty)}$$

and, as a consequence, it follows that

$$\|\mathbf{y}\|_{L^2(0,\infty;H^{\frac{7}{2}-\mu}(\Omega)\times L^2(\Omega))} \le C\|u\|_{H^{2,1}(\Sigma_\infty)}.$$

Once more we use integration as in (25) to conclude that

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{H}^{\frac{7}{2}-\mu}(\Omega))} \le C\|u\|_{H^{2,1}(\Sigma_\infty)}.$$

Also, for the derivative we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{N}_{\widehat{\mathbf{A}}} M u(t) - \int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}} M u'(\tau)\,\mathrm{d}\tau\right) \\ &= \mathbf{N}_{\widehat{\mathbf{A}}} M u'(t) - \mathbf{N}_{\widehat{\mathbf{A}}} M u'(t) + \widehat{\mathbf{A}}\int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}} M u'(\tau)\,\mathrm{d}\tau \\ &= \widehat{\mathbf{A}}\int_0^t e^{-\widehat{\mathbf{A}}(t-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}} M u'(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Similar as in (26), we may use the fact that $u' \in L^2(\Sigma_\infty)$ to show that

$$\widehat{\mathbf{A}}\int_0^{\cdot} e^{-\widehat{\mathbf{A}}(\cdot-\tau)}\mathbf{N}_{\widehat{\mathbf{A}}} M u'(\tau)\,\mathrm{d}\tau \in H^{\frac{3}{4}-\frac{\mu}{2}}(0,\infty;\mathbf{L}^2(\Omega)).$$

So far we have shown that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} \in H^{\frac{3}{4}-\frac{\mu}{2}}(0,\infty;\mathbf{L}^2(\Omega))$. In other words, this means $\mathbf{y} \in H^{\frac{7}{4}-\frac{\mu}{2}}(0,\infty;\mathbf{L}^2(\Omega))$. The cases $0 < s < 1$ and $1 < s < 2$, respectively, follow by interpolation. ☐

**Lemma 2.4.** *Let* $\varepsilon \in \left(\frac{1}{2},1\right]$. *If* $f_1 \in L^2(0,\infty;H^{\varepsilon-1}(\Omega)), \mathbf{y}_0 \in H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)$, *then* (22) *has a unique solution* $\mathbf{y} \in L^2(0,\infty;\mathcal{D}(\widehat{\mathbf{A}}^\delta))$, *with* $\delta$ *as in* (15).

*Proof.* Due to the relation between $\mathbf{\Pi}$ and $\bar{\mathbf{\Pi}}$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{A}_{\mathbf{\Pi}}\mathbf{y} + \mathbf{f} = (\widehat{\mathbf{A}}^{-\delta}\mathbf{A}_{\bar{\mathbf{\Pi}}}\widehat{\mathbf{A}}^{\delta})\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0.$$

Hence, instead of (22), let us consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = \mathbf{A}_{\bar{\mathbf{\Pi}}}\mathbf{z} + \widehat{\mathbf{A}}^{\delta}\mathbf{f}, \quad \mathbf{z}(0) = \widehat{\mathbf{A}}^{\delta}\mathbf{y}_0,$$

where $\mathbf{z} = \widehat{\mathbf{A}}^\delta\mathbf{y}$. Due to (20) we know that $\bar{\mathbf{\Pi}}\bar{\mathbf{B}}M \in \mathcal{L}(U,\mathbf{Y})$. Moreover, as stated in (19), it holds that $\widehat{\mathbf{A}}^{-\bar{\gamma}}\bar{\mathbf{B}}M \in \mathcal{L}(U,\mathbf{Y}), \bar{\gamma} < 1$, implying also $M\bar{\mathbf{B}}^*(\widehat{\mathbf{A}}^*)^{-\bar{\gamma}} \in \mathcal{L}(\mathbf{Y},U)$. For all $\mathbf{y} \in \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\widehat{\mathbf{A}}^*)$ we now have

$$\left\|\bar{\mathbf{\Pi}}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)} = \left\|(\bar{\mathbf{\Pi}}\bar{\mathbf{B}}M)(M\bar{\mathbf{B}}^*(\widehat{\mathbf{A}}^*)^{-\bar{\gamma}})(\widehat{\mathbf{A}}^*)^{\bar{\gamma}}\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)} \le C\left\|(\widehat{\mathbf{A}}^*)^{\bar{\gamma}}\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)}.$$

Using [6, Proposition 5.1] it further holds that

$$\left\|(\widehat{\mathbf{A}}^*)^{\bar{\gamma}}\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)} \le C\left\|\widehat{\mathbf{A}}^*\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)}^{\bar{\gamma}}\|\mathbf{y}\|_{\mathbf{L}^2(\Omega)}^{1-\bar{\gamma}}$$

where the constant $C$ is independent of $\mathbf{y}$. Young's inequality with $p = \frac{1}{\bar{\gamma}}$ and $q = \frac{1}{1-\bar{\gamma}}$ for arbitrary $s > 0$ also yields

$$\left\|\widehat{\mathbf{A}}^*\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)}^{\bar{\gamma}}\|\mathbf{y}\|_{\mathbf{L}^2(\Omega)}^{1-\bar{\gamma}} \le \bar{\gamma}\varepsilon^{\frac{1}{\bar{\gamma}}}\left\|\widehat{\mathbf{A}}^*\mathbf{y}\right\|_{\mathbf{L}^2(\Omega)} + (1-\bar{\gamma})\varepsilon^{\frac{1}{\bar{\gamma}-1}}\|\mathbf{y}\|_{\mathbf{L}^2(\Omega)}.$$

Altogether, this means that $\bar{\mathbf{\Pi}}\bar{\mathbf{B}}M^2\bar{\mathbf{B}}^*$ is $\mathbf{A}^*$-bounded with relative bound zero and, thus,

$$\mathcal{D}(\mathbf{A}^*_{\bar{\mathbf{\Pi}}}) = \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\widehat{\mathbf{A}}^*) = \mathcal{D}(\widehat{\mathbf{A}}) = H^2_\nu(\Omega) \times L^2(\Omega).$$

Since $\varepsilon \in \left(\frac{1}{2}, 1\right]$, it holds that $H^{\varepsilon-1}(\Omega) = [\mathcal{D}(\widehat{\mathbf{A}}^{\frac{1-\varepsilon}{2}})]'$. Due to the fact that $\delta \in \left(\frac{1}{4}, \frac{1}{2}\right)$, we obtain

$$\widehat{\mathbf{A}}^\delta \mathbf{f} \in L^2(0,\infty;[\mathcal{D}(\widehat{\mathbf{A}}^{\frac{1-\varepsilon}{2}+\delta})]') \times L^2(Q_\infty) \subset L^2(0,\infty;[\mathcal{D}(\widehat{\mathbf{A}})]') = L^2(0,\infty;[\mathcal{D}(\widehat{\mathbf{A}}_{\mathbf{\Pi}})]').$$

Moreover, for $\delta \in \left(\frac{1}{4}, \frac{1}{2}\right)$ and $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times H^{1+\varepsilon}(\Omega)$, we clearly have that

$$\mathbf{z}(0) = \widehat{\mathbf{A}}^\delta \mathbf{y}_0 \in [\mathcal{D}(\mathbf{A}^*_{\bar{\mathbf{\Pi}}}), \mathbf{Y}]'_{\frac{1}{2}} = [H^1(\Omega) \times L^2(\Omega)]'.$$

Since $\mathbf{A}_{\bar{\mathbf{\Pi}}}$ is the infinitesimal generator of an analytic semigroup exponentially stable on $\mathbf{Y}$, we can use Theorem 2.2 with $\alpha = 1$ to show that the mapping

$$\mathbf{z} \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} - \mathbf{A}_{\bar{\mathbf{\Pi}}}\mathbf{z}, \mathbf{z}(0)\right)$$

$$L^2(0,\infty;\mathbf{Y}) \cap H^1(0,\infty;[\mathcal{D}(\mathbf{A}^*_{\bar{\mathbf{\Pi}}})]') \to L^2(0,\infty;[\mathcal{D}(\mathbf{A}^*_{\bar{\mathbf{\Pi}}})]') \times [\mathcal{D}(\mathbf{A}^*_{\bar{\mathbf{\Pi}}}), \mathbf{Y}]'_{\frac{1}{2}}$$

is an isomorphism. Now this in particular yields $\mathbf{y} = \widehat{\mathbf{A}}^{-\delta}\mathbf{z} \in L^2(0,\infty;\mathcal{D}(\widehat{\mathbf{A}}^\delta))$. □

With the previous Lemma, we obtain the following regularity result for the nonhomogeneous equation (22).

**Theorem 2.5.** *Let $\varepsilon \in \left(\frac{1}{2}, 1\right]$. If $f_1 \in L^2(0,\infty;H^{\varepsilon-1}(\Omega)), \mathbf{y}_0 \in H^\varepsilon(\Omega) \times H^{1+\varepsilon}(\Omega)$, then* (22) *has a unique solution*

$$\mathbf{y} \in \left(H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0,\infty);H^\varepsilon(\Omega))\right) \times H^1(0,\infty;H^{1+\varepsilon}(\Omega))$$

*satisfying*

$$\|\mathbf{y}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \le C_1 \left(\|\mathbf{y}_0\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)} + \|f_1\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))}\right).$$

*Proof.* We follow the strategy of the proof in [24, Theorem 4.1] and [7, Theorem 4.7]. Hence, let us consider the splitting $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_1 &= -\widehat{\mathbf{A}}\mathbf{y}_1 + \mathbf{f} + \lambda_0 \mathbf{y}, \quad \mathbf{y}_1(0) = \mathbf{y}_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_2 &= -\widehat{\mathbf{A}}\mathbf{y}_2 - \mathbf{B}M^2\mathbf{B}^*\mathbf{\Pi}\mathbf{y}, \quad \mathbf{y}_2(0) = 0, \end{aligned} \tag{27}$$

with $\mathbf{f}$ and $\mathbf{y}_0$ as above. From Lemma 2.4, we already know that $\mathbf{y} \in L^2(0,\infty;\mathcal{D}(\widehat{\mathbf{A}}^\delta))$. Even under the weaker regularity condition that $\mathbf{y} \in \mathbf{L}^2(Q_\infty)$, in the proof of [7, Theorem 4.7] we have already shown the existence of a unique solution

$$\mathbf{y}_1 \in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^1(0,\infty;H^{1+\varepsilon}(\Omega))$$

to (27). Thus, let us consider $\mathbf{y}_2$. From (18), we know that $M\mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), U)$. Hence, $M\mathbf{B}^*\mathbf{\Pi}\mathbf{y} \in L^2(\Sigma_\infty)$ and, by Lemma 2.3, it follows that $\mathbf{y}_2 \in \mathbf{H}^{\frac{3}{2}-\mu,\frac{3}{4}-\frac{\mu}{2}}(Q_\infty)$, for all $\mu > 0$. This also implies that $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in \mathbf{H}^{\frac{3}{2}-\mu,\frac{3}{4}-\frac{\mu}{2}}(Q_\infty)$. Again with (18) it follows that $(\widehat{\mathbf{A}}^*)^\gamma \mathbf{\Pi}\widehat{\mathbf{A}}^{-\delta} \in \mathcal{L}(\mathbf{Y})$. For $\gamma = \frac{1+s}{2}$ and $\delta = \frac{1}{2} - s$, with $s \in \left(\frac{1}{4}, \frac{1}{2}\right)$ this leads to

$$(\widehat{\mathbf{A}}^*)^{\frac{1+s}{2}}\mathbf{\Pi}(\widehat{\mathbf{A}}^{-(\frac{1}{2}-s)})(\widehat{\mathbf{A}}^{\frac{1}{2}-s})\mathbf{y} \in \mathbf{L}^2(Q_\infty).$$

Hence, we conclude that $\mathbf{\Pi y} \in L^2(0,\infty; H^{1+s}(\Omega)\times L^2(\Omega))$. According to Lemma 2.1 this shows that $M\mathbf{B}^*\mathbf{\Pi y} \in L^2(0,\infty; H^{\frac{1}{2}+s}(\Gamma))$. Furthermore, due to the intermediate derivatives theorem ([18, Theorem 4.1]) we have that

$$\mathbf{y} \in H^{\frac{1}{4}+\frac{s}{2}}(0,\infty; [\mathbf{H}^{\frac{3}{2}-\mu}(\Omega), \mathbf{L}^2(\Omega)]_{(\frac{1}{4}+\frac{s}{2})/(\frac{3}{4}-\frac{\mu}{2})}) = H^{\frac{1}{4}+\frac{s}{2}}(0,\infty; \mathbf{H}^{1-\mu-s}(\Omega)).$$

With $\mu < s$ and $M\mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathcal{D}(\widehat{\mathbf{A}}^\delta), U)$, we obtain $M\mathbf{B}^*\mathbf{\Pi y} \in H^{\frac{1}{4}+\frac{s}{2}}(0,\infty; L^2(\Gamma))$. In summary, this shows that $M\mathbf{B}^*\mathbf{\Pi y} \in H^{\frac{1}{2}+s,\frac{1}{4}+\frac{s}{2}}(\Sigma_\infty)$. Again with Lemma 2.3 we arrive at $\mathbf{y}_2 \in \mathbf{H}^{2+s-\tilde{\mu},1+s-\frac{\tilde{\mu}}{2}}(Q_\infty)$ with $\tilde{\mu} > 0$ arbitrary. This however implies that $\mathbf{y}_2 \in \mathbf{H}^{2,1}(Q_\infty)$ and also

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in \mathbf{H}^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty).$$

Since the second equation in (22) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \iota y(t) - (\kappa - \sigma) z(t),$$

we further get $z \in H^1(0,\infty; H^{1+\varepsilon}(\Omega))$. Finally, with [18, Theorem 4.2] and its proof, we can show that

$$y \in C_b([0,\infty); [H^{1+\varepsilon}(\Omega), L^2(\Omega)]_{\frac{1}{2\frac{1+\varepsilon}{2}}}) = C_b([0,\infty); H^\varepsilon(\Omega)).$$

The proof is complete. □

2.3. **Results for the nonlinear system.** In [7] we have shown some Lipschitz estimates for nonlinearities of the form (2). In what follows, we extend these results to cover a broader range of parameters $\varepsilon$.

**Proposition 1.** *Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$. Assume that $f,g,h \in H^{1+\varepsilon}(\Omega), \varepsilon \in (0,1]$ ($\varepsilon \in \left[\frac{1}{2},1\right]$ for $n=3$). Then*

$$\|fg\|_{H^{\varepsilon-1}(\Omega)} \le C\|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}\|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)},$$
$$\|fgh\|_{H^{\varepsilon-1}(\Omega)} \le C\|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}\|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}\|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

*Proof.* Let us first assume that $\varepsilon \in \left(0,\frac{1}{2}\right)$ and $n=2$. Choosing $\lambda = \varepsilon - 1$ and $\mu = \omega = \frac{4}{3}$ in [13, Proposition B.1] we have that $\mu + \omega + \lambda = \frac{5}{3} + \varepsilon > 1$ and $2\lambda + \mu + \omega = \frac{2}{3} + 2\varepsilon > 0$. This shows the first inequality. For the second one, we note that with $\lambda = \varepsilon - 1, \mu = \frac{2}{3} - \varepsilon$ and $\omega = \frac{4}{3}$ it holds that $\lambda + \mu + \omega = 1$ as well as $2\lambda + \mu + \omega = \varepsilon > 0$. This yields

$$\|fgh\|_{H^{\varepsilon-1}(\Omega)} \le C\|fg\|_{H^{-\frac{1}{3}}(\Omega)}\|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Moreover, with $\lambda = -\frac{1}{3}$ and $\mu = \omega = \frac{2}{3} + \varepsilon$ in [13, Proposition B.1] we find that

$$\|fg\|_{H^{-\frac{1}{3}}(\Omega)} \le C\|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}\|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Next assume that $\varepsilon \in \left[\frac{1}{2},1\right]$ and $n \in \{2,3\}$. The first inequality follows exactly as before since it still holds that $\frac{5}{3} + \varepsilon > \frac{n}{2}$. For the second one, we choose $\lambda = \varepsilon - 1, \mu = \frac{7}{6} - \varepsilon$ and $\omega = \frac{4}{3}$ in [13, Proposition B.1]. This implies $\lambda + \mu + \omega = \frac{3}{2}$ and $2\lambda + \mu + \omega > 0$. Hence, we find that

$$\|fgh\|_{H^{\varepsilon-1}(\Omega)} \le C\|fg\|_{H^{\frac{1}{6}}(\Omega)}\|h\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

Moreover, with $\lambda = \frac{1}{6}$ and $\mu = \omega = \frac{1}{6} + \varepsilon$ in [13, Proposition B.1] this leads to

$$\|fg\|_{H^{\frac{1}{6}}(\Omega)} \le C\|f\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}\|g\|_{H^{\frac{1}{3}+\varepsilon}(\Omega)}.$$

This shows the second inequality. □

Proposition 1 allows to show the following results.

**Lemma 2.6.** *Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$. Assume that $\sigma \geq 0$ and $\varepsilon \in (0,1]$ ($\varepsilon \in \left[\frac{1}{2},1\right]$ for $n=3$). Then for $y_1,y_2 \in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)$, it holds that*

$$\left\|e^{-\sigma t}(y_1^2-y_2^2)\right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))} \leq C\left\|y_1-y_2\right\|_{1+\varepsilon,\frac{1+\varepsilon}{2}}\left(\|y_1\|_{1+\varepsilon,\frac{1+\varepsilon}{2}}+\|y_2\|_{1+\varepsilon,\frac{1+\varepsilon}{2}}\right),$$

$$\left\|e^{-2\sigma t}(y_1^3-y_2^3)\right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))} \leq C\left\|y_1-y_2\right\|_{1+\varepsilon,\frac{1+\varepsilon}{2}}\left(\|y_1\|^2_{1+\varepsilon,\frac{1+\varepsilon}{2}}+\|y_2\|^2_{1+\varepsilon,\frac{1+\varepsilon}{2}}\right).$$

*Proof.* Note first that by interpolation ([18, Chapter 1, Theorem 4.1]) we have that

$$y_1,y_2 \in H^{\frac{1}{3}}(0,\infty;[H^{1+\varepsilon}(\Omega),L^2(\Omega)]_{\frac{1}{3}/\frac{1+\varepsilon}{2}}) = H^{\frac{1}{3}}(0,\infty;H^{\frac{1}{3}+\varepsilon}(\Omega))$$
$$y_1,y_2 \in H^{\frac{1}{4}}(0,\infty;[H^{1+\varepsilon}(\Omega),L^2(\Omega)]_{\frac{1}{4}/\frac{1+\varepsilon}{2}}) = H^{\frac{1}{4}}(0,\infty;H^{\frac{1}{2}+\varepsilon}(\Omega)) \subset H^{\frac{1}{4}}(0,\infty;H^{\frac{1}{3}+\varepsilon}(\Omega)).$$

Moreover, by Sobolev embedding (see, e.g., [23, Theorem 7]), it follows that

$$y_1,y_2 \in L^4(0,\infty;H^{\frac{1}{3}+\varepsilon}(\Omega)) \cap L^6(0,\infty;H^{\frac{1}{3}+\varepsilon}(\Omega)).$$

The result now follows from the generalized Hölder inequality and the facts that

$$y_1^2-y_2^2 = (y_1-y_2)(y_1+y_2),$$
$$y_1^3-y_2^3 = (y_1-y_2)(y_1^2+2y_1y_2+y_2^2)$$

together with an application of Proposition 1. □

For the nonlinearity in (4), i.e.,

$$\mathbf{F}(\mathbf{y}) := \begin{pmatrix}(b-3a\bar{v})e^{-\sigma t}y^2 - ae^{-2\sigma t}y^3\\ 0\end{pmatrix},$$

we obtain the following local Lipschitz continuity property.

**Lemma 2.7.** *Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$. Assume that $\varepsilon \in (0,1]$ ($\varepsilon \in \left[\frac{1}{2},1\right]$ for $n=3$). If*

$$\mathbf{y}_1,\mathbf{y}_2 \in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^1(0,\infty;H^{1+\varepsilon}(\Omega)),$$

*then $\mathbf{F}$ is locally Lipschitz continuous from $H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^1(0,\infty;H^{1+\varepsilon}(\Omega))$ to $L^2(0,\infty;H^{\varepsilon-1}(\Omega)) \times L^2(Q_\infty)$. In particular, we have that*

$$\begin{aligned}&\|\mathbf{F}(\mathbf{y}_1)-\mathbf{F}(\mathbf{y}_2)\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)}\\ &\quad\leq C_2\Big(\|y_1\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}+\|y_2\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}\\ &\qquad\quad+\|y_1\|^2_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}+\|y_2\|^2_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}\Big)\|y_1-y_2\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}.\end{aligned}$$

*Proof.* By definition of $\mathbf{F}$, we have

$$\begin{aligned}&\|\mathbf{F}(\mathbf{y}_1)-\mathbf{F}(\mathbf{y}_2)\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)}\\ &\quad=\left\|(b-3a\bar{v})e^{-\sigma t}(y_1^2-y_2^2)-ae^{-2\sigma t}(y_1^3-y_2^3)\right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))}\\ &\quad\leq C_2\left(\left\|(b-3a\bar{v})e^{-\sigma t}(y_1^2-y_2^2)\right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))}+\left\|ae^{-2\sigma t}(y_1^3-y_2^3)\right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))}\right).\end{aligned}$$

Since $\bar{v} \in H^{\frac{n}{2}+s}(\Omega), s > 0$, [13, Proposition B.1], implies that

$$\|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)} \\ \leq C_2 \left( \left\| e^{-\sigma t}(y_1^2 - y_2^2) \right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))} + \left\| e^{-2\sigma t}(y_1^3 - y_2^3) \right\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))} \right).$$

The claim now follows with Lemma 2.6. □

Our final results concerns the local stabilization of the full nonlinear system.

**Theorem 2.8.** *Let $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$ and $\varepsilon \in \left(\frac{1}{2}, 1\right]$. Then there exist $\mu_0 > 0$ and a nondecreasing function $\eta$ from $\mathbb{R}^+$ into itself such that if $\mu \in (0, \mu_0)$ and $\|\mathbf{y}_0\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)} \leq \eta(\mu)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{A_\Pi y} + \mathbf{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

*admits a unique solution in the set*

$$D_\mu = \Big\{ \mathbf{y} \in \left( H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0,\infty); H^\varepsilon(\Omega)) \right) \times H^1(0,\infty; H^{1+\varepsilon}(\Omega)), \\ \|\mathbf{y}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \leq \mu \Big\}.$$

*Proof.* Following the arguments provided for similar results in [7, 22], let us show show that the mapping $\mathcal{M} \colon \mathbf{z} \mapsto \mathbf{y_z}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y_z} = \mathbf{A_\Pi y_z} + \mathbf{F}(\mathbf{z}), \ \mathbf{y_z}(0) = \mathbf{y}_0, \tag{28}$$

is a contraction in $D_\mu$. As in [7], define

$$\mu_0 = \frac{1}{2}\left( \sqrt{1 + \frac{1}{C_1 C_2}} - 1 \right) \quad \text{and} \quad \eta(\mu) = \frac{3}{4C_1}\mu,$$

with $C_1$ and $C_2$ as in Theorem 2.5 and Lemma 2.7, respectively.
*Step 1:* For $\mathbf{z} \in D_\mu$, from Lemma 2.7 we get

$$\|\mathbf{F}(\mathbf{z})\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)} \leq C_2(\mu + \mu^2)\mu.$$

Utilizing Theorem 2.5 we obtain that

$$\|\mathbf{y_z}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \\ \leq C_1 \left( \|\mathbf{y}_0\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)} + \|\mathbf{F}(\mathbf{z})\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)} \right) \\ \leq \frac{3}{4}\mu + C_1 C_2 (\mu + \mu^2)\mu.$$

Since $\mu \leq \mu_0 = \frac{1}{2}\left( \sqrt{1 + \frac{1}{C_1 C_2}} - 1 \right)$, we conclude that $\mu + \mu^2 \leq \frac{1}{4C_1 C_2}$. This implies

$$\|\mathbf{y_z}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \leq \mu.$$

Hence, $\mathcal{M}$ is mapping $D_\mu$ to itself.
*Step 2:* For $\mathbf{z}^{(1)} = \left(z_1^{(1)}, z_2^{(1)}\right), \mathbf{z}^{(2)} = \left(z_1^{(2)}, z_2^{(2)}\right) \in D_\mu$ we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}) = \mathbf{A_\Pi}(\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}) + \mathbf{F}(\mathbf{z}_1) - \mathbf{F}(\mathbf{z}_2), \quad \mathbf{y}_{\mathbf{z}_1}(0) - \mathbf{y}_{\mathbf{z}_2}(0) = 0.$$

As a consequence, from Theorem 2.5 we know that

$$\|\mathbf{y}_{\mathbf{z}_1} - \mathbf{y}_{\mathbf{z}_2}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \leq C_1 \|\mathbf{F}(\mathbf{z}_1) - \mathbf{F}(\mathbf{z}_2)\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))\times L^2(Q_\infty)}.$$

Using Lemma 2.7, this yields

$$\begin{aligned}\|\mathbf{y}_{\mathbf{z}_1}-\mathbf{y}_{\mathbf{z}_2}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} &\le 2C_1C_2(\mu+\mu^2)\|z_1^{(1)}-z_1^{(2)}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}\\ &\le \frac{1}{2}\|z_1^{(1)}-z_1^{(2)}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)}.\end{aligned}$$

The mapping $\mathcal{M}$ is a contraction in $D_\mu$ and the proof is complete. □

3. **Dirichlet boundary control.** We now consider the linearization of (4) for $\theta = 0$. Thus, in this section we first focus on

$$\begin{aligned}\frac{\partial y}{\partial t} &= \Delta y + \alpha(x)y - dz \quad \text{in } \Omega\times(0,\infty),\\ \frac{\partial z}{\partial t} &= \iota y - (\kappa-\sigma)z \quad \text{in } \Omega\times(0,\infty),\\ y &= mu \quad \text{on } \Gamma\times(0,\infty),\\ y(x,0) &= y_0 \text{ and } z(x,0) = z_0 \quad \text{in } \Omega.\end{aligned} \tag{29}$$

Similar as before, let us introduce the operators $(\mathbf{A},\mathcal{D}(\mathbf{A}))$ and $(\mathbf{A}^*,\mathcal{D}(\mathbf{A}^*))$ with

$$\begin{aligned}\mathbf{A}\mathbf{y} &= \begin{pmatrix}\Delta y+\alpha(x)y-dz\\ \iota y-(\kappa-\sigma)z\end{pmatrix}, \quad \mathcal{D}(\mathbf{A}) = \left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times L^2(\Omega),\\ \mathbf{A}^*\mathbf{y} &= \begin{pmatrix}\Delta y+\alpha(x)y+\iota z\\ -dy-(\kappa-\sigma)z\end{pmatrix}, \quad \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\mathbf{A}).\end{aligned} \tag{30}$$

With $\lambda_0$ and $\widehat{\mathbf{A}}$ as in Section 2 let us define the Dirichlet map $\mathbf{D}_{\widehat{\mathbf{A}}}$ via $\mathbf{D}_{\widehat{\mathbf{A}}}u = \mathbf{y}$ iff

$$\begin{aligned}\lambda_0 y - \Delta y - \alpha(x)y + dz &= 0 \quad \text{in } \Omega, \qquad y = u \quad \text{on } \Gamma,\\ \lambda_0 z - \iota y + (\kappa-\sigma)z &= 0 \quad \text{in } \Omega.\end{aligned} \tag{31}$$

As before, by explicitly solving the second equation for $z$, and using well-known results from [17, Section 3.1] we conclude that

$$\mathbf{D}_{\widehat{\mathbf{A}}}\colon \text{continuous } H^s(\Gamma)\to \mathbf{H}^{s+\frac{1}{2}}(\Omega), \quad s\ge 0. \tag{32}$$

Consequently, for the linearized system we now obtain the following boundary control system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{B}Mu(t), \quad \mathbf{y}_0 = \begin{pmatrix}y_0\\ z_0\end{pmatrix}, \tag{33}$$

where $\mathbf{B} = \widehat{\mathbf{A}}\mathbf{D}_{\widehat{\mathbf{A}}} \in \mathcal{L}(L^2(\Gamma),[\mathcal{D}(\mathbf{A}^*)]')$ and $M$ is the multiplication operator associated with $m$.

**Lemma 3.1.** *For all* $\mathbf{p} = (p,q)\in\mathcal{D}(\mathbf{A}^*)$, *it holds that*

$$\mathbf{B}^*\mathbf{p} = \mathbf{D}_{\widehat{\mathbf{A}}}^*\widehat{\mathbf{A}}^*\mathbf{p} = -\left.\frac{\partial p}{\partial \nu}\right|_\Gamma. \tag{34}$$

*For* $\mathbf{p}\in H^{\frac{3}{2}+s}(\Omega)\times L^2(\Omega), s>0$, *it further holds*

$$\|\mathbf{B}^*\mathbf{p}\|_{H^s(\Gamma)} \le C\|\mathbf{p}\|_{H^{\frac{3}{2}+s}(\Omega)\times L^2(\Omega)}.$$

*Proof.* For $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*), u \in L^2(\Gamma)$ and $\mathbf{D}_{\widehat{\mathbf{A}}} u = \mathbf{y}$ we have

$$\begin{aligned}(\mathbf{B}u, \mathbf{p})_{\mathbf{L}^2(\Omega)} &= \left(\mathbf{D}_{\widehat{\mathbf{A}}} u, \widehat{\mathbf{A}}^* \mathbf{p}\right)_{\mathbf{L}^2(\Omega)} = \left(\mathbf{y}, \widehat{\mathbf{A}}^* \mathbf{p}\right)_{\mathbf{L}^2(\Omega)} \\ &= (y, \lambda_0 p - \Delta p - \alpha(x)p - \iota q)_{L^2(\Omega)} + (z, \lambda_0 q + (\kappa - \sigma)q + dp)_{L^2(\Omega)} \\ &= (\lambda_0 y - \Delta y - \alpha(x)y + dz, p)_{L^2(\Omega)} + (\lambda_0 z - \iota y + (\kappa - \sigma)z, q)_{L^2(\Omega)} \\ &\quad - \left(y, \frac{\partial p}{\partial \nu}\right)_{L^2(\Gamma)} + \left(\frac{\partial y}{\partial \nu}, p\right)_{L^2(\Gamma)}.\end{aligned}$$

By (31) and the fact that $\mathbf{p} \in \mathcal{D}(\mathbf{A}^*)$, we thus obtain

$$(\mathbf{B}u, \mathbf{p})_{\mathbf{L}^2(\Omega)} = \left(u, -\frac{\partial p}{\partial \nu}\right)_{L^2(\Gamma)} = (u, \mathbf{B}^* \mathbf{p})_{L^2(\Gamma)}.$$

For the estimate, see, e.g., [6, Part II, Chapter 3, Section 4]. □

Similar as before, let us note that

$$\widehat{\mathbf{A}}^{-\gamma} \mathbf{B} = \widehat{\mathbf{A}}^{1-\gamma} \mathbf{D}_{\widehat{\mathbf{A}}} \in \mathcal{L}(U, \mathbf{Y}) \quad \text{for } \gamma = \frac{3}{4} + \varepsilon, \quad \varepsilon > 0. \tag{35}$$

3.1. **Riccati theory for the linearized system.** Let us consider the following cost functional corresponding to (29)

$$\begin{aligned}\mathcal{J}(u, \mathbf{y}) &= \int_0^\infty \left(\|\mathbf{C}\mathbf{y}(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Gamma)}\right) \mathrm{d}t \\ &= \int_0^\infty \left(\|y(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Gamma)}\right) \mathrm{d}t.\end{aligned} \tag{36}$$

Obviously, for the observation operator $\mathbf{C}\mathbf{y}(t) = y(t)$ it holds that $\mathbf{C} \in \mathcal{L}(\mathbf{Y}, Z)$, where $Z = L^2(\Omega)$. As in the Neumann case, conditions (H1)-(H5) hold. From [17, Section 2.1] we now get the existence of a unique nonnegative self-adjoint solution $\mathbf{\Pi} = \mathbf{\Pi}^*$ to an algebraic Riccati equation. We further know ([17, Theorem 2.2.1]) that $(\widehat{\mathbf{A}}^*)^\nu \mathbf{\Pi} \in \mathcal{L}(\mathbf{Y})$ for any $0 \le \nu < 1$. As emphasized in [17] and investigated in [3, 21], in certain cases it is even possible to take $\nu = 1$ such that $\mathbf{\Pi} \in \mathcal{L}(\mathbf{Y}, \mathcal{D}(\widehat{\mathbf{A}}^*))$.

Moreover, for the case of the Navier-Stokes equations, in [21], it has been further shown that $\mathbf{\Pi} \in \mathcal{L}\left(H^\varepsilon(\Omega), H^{2+\varepsilon}(\Omega) \cap H^1_0(\Omega)\right), \varepsilon \in \left[0, \frac{1}{2}\right)$. With regard to the desired local stabilization result for the nonlinear system, the latter property of $\mathbf{\Pi}$ will be essential. As this property is not obvious a priori, in the following we briefly recapitulate the main steps and proofs from [21] and adapt them to our setting.

3.1.1. *A finite time horizon control problem.* We start with the finite time horizon counterpart of (29) and consider

$$\inf \left\{ J_T(t_0, \mathbf{y}, u) \,\middle|\, (\mathbf{y}, u) \text{ satisfies (38)}, u \in L^2(\Sigma_{t_0,T}) \right\} \tag{37}$$

where

$$\begin{aligned}&\frac{\partial y}{\partial t} = \Delta y + \alpha(x)y - dz \quad \text{in } Q_{t_0,T}, \\ &\frac{\partial z}{\partial t} = \iota y - (\kappa - \sigma)z \quad \text{in } Q_{t_0,T}, \\ &\quad y = mu \quad \text{on } \Sigma_{t_0,T}, \qquad \mathbf{y}(t_0) = \zeta = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad \text{in } \Omega\end{aligned} \tag{38}$$

and

$$J_T(t_0,\mathbf{y},u) = \frac{1}{2}\int_{t_0}^T \int_\Omega |y|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{t_0}^T \int_\Gamma |u|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

We now obtain the following characterization for optimality.

**Theorem 3.2.** *For all $t_0 \in [0,T]$ and all $\zeta \in \mathbf{L}^2(\Omega)$, problem (37) admits a unique solution $(\mathbf{y}_\zeta^{t_0}, u_\zeta^{t_0})$ and*

$$u_\zeta^{t_0} = -M\mathbf{B}^*\mathbf{p}_\zeta^{t_0} \quad \text{in } (t_0,T),$$

*where $\mathbf{p}_\zeta^{t_0}$ is the solution to the equation*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \mathbf{A}^*\mathbf{p} + \begin{pmatrix} y_\zeta^{t_0} \\ 0 \end{pmatrix} \quad \text{in } (t_0,T), \quad \mathbf{p}(T) = 0.$$

*Conversely the system*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} &= \mathbf{A}\mathbf{y} - \mathbf{B}M^2\mathbf{B}^*\mathbf{p} \quad \text{in } (t_0,T), \quad \mathbf{y}(t_0) = \zeta, \\ -\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} &= \mathbf{A}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{in } (t_0,T), \quad \mathbf{p}(T) = 0, \end{aligned} \tag{39}$$

*admits a unique solution $(\mathbf{y}_\zeta^{t_0}, \mathbf{p}_\zeta^{t_0}) \in \mathbf{L}^2(Q_{t_0,T}) \times \left(\mathbf{H}^{2,1}(Q_{t_0,T}) \cap L^2(t_0,T;\mathbf{H}_0^1(\Omega))\right)$.*

*Proof.* We follow the arguments provided in [21, Theorem 3.1]. Existence and uniqueness of a solution $(\mathbf{y}_\zeta^{t_0}, u_\zeta^{t_0})$ to (37) are well-known, see, e.g., [25, Theorem 2.16]. Consider $u,v \in L^2(\Sigma_{t_0,T})$ and introduce the reduced cost functional

$$J(u) = J_T(t_0,\mathbf{y}_u,u),$$

where $\mathbf{y}_u$ solves (38). It now holds that

$$J'(u)v = \int_{Q_{t_0,T}} y_u\,\varphi + \int_{\Sigma_{t_0,T}} u\,v, \tag{40}$$

with $\mathbf{\Phi} = (\varphi,\psi)$ being the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi} = \mathbf{A}\mathbf{\Phi} + \mathbf{B}Mv \quad \text{in } (t_0,T), \quad \mathbf{\Phi}(t_0) = 0. \tag{41}$$

Consider the solution $\mathbf{p}$ to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \mathbf{A}^*\mathbf{p} + \begin{pmatrix} y_u \\ 0 \end{pmatrix} \quad \text{in } (0,T), \quad \mathbf{p}(T) = 0. \tag{42}$$

From [6, Chapter 2, Proposition 3.7] we conclude that $\mathbf{p} \in \left(H^{2,1}(Q_{t_0,T}) \cap L^2(t_0,T;H_0^1(\Omega))\right) \times L^2(Q_{t_0,T})$. By integration of the ODE part we also have that $\mathbf{p} \in \mathbf{H}^{2,1}(Q_{t_0,T}) \cap L^2(t_0,T;\mathbf{H}_0^1(\Omega))$. With Lemma 3.1, we now get $\mathbf{B}^*\mathbf{p} \in L^2(\Sigma_{t_0,T})$. Note that we have

$$\begin{aligned} \int_{Q_{t_0,T}} y_u\,\varphi &= -\int_{t_0}^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p},\mathbf{\Phi}\right)_{\mathbf{L}^2(\Omega)} \mathrm{d}t - \int_{t_0}^T (\mathbf{A}^*\mathbf{p},\mathbf{\Phi})_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t \\ &= \int_{t_0}^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi} - \mathbf{A}\mathbf{\Phi}, \mathbf{p} \right\rangle_{[\mathcal{D}(\mathbf{A}^*)]',\mathcal{D}(\mathbf{A}^*)} = \int_{t_0}^T \langle \mathbf{B}Mv,\mathbf{p}\rangle_{[\mathcal{D}(\mathbf{A}^*)]',\mathcal{D}(\mathbf{A}^*)} \\ &= \int_{t_0}^T (v, M\mathbf{B}^*\mathbf{p})_{L^2(\Gamma)}\,. \end{aligned}$$

Combining this with (40), we obtain that

$$J'(u)v = \int_{t_0}^{T} (u + M\mathbf{B}^*\mathbf{p}, v)_{L^2(\Gamma)}.$$

Consequently, if $(\mathbf{y}_\zeta^{t_0}, u_\zeta^{t_0})$ solves (37), we have $J'(u_\zeta^{t_0}) = 0$ and, thus, $u_\zeta^{t_0} = -M\mathbf{B}^*\mathbf{p}_\zeta^{t_0}$. In particular, $(\mathbf{y}_\zeta^{t_0}, \mathbf{p}_\zeta^{t_0})$ is a solution of system (39).

Conversely, if $(\bar{\mathbf{y}}, \bar{\mathbf{p}})$ is a solution of (39) by setting $\bar{u} = -M\mathbf{B}^*\bar{\mathbf{p}}$ we obtain $J'(\bar{u}) = 0$. This however means that $\bar{u} = u_\zeta^{t_0}$ and, hence, $\bar{\mathbf{y}} = \mathbf{y}_\zeta^{t_0}$ and $\bar{\mathbf{p}} = \mathbf{p}_\zeta^{t_0}$. □

We can further show the following improved regularity result for the solution of the optimality system.

**Theorem 3.3.** *The solution* $(\mathbf{y}_\zeta^{t_0}, \mathbf{p}_\zeta^{t_0})$ *to system* (39) *satisfies*

$$\begin{aligned} \mathbf{y}_\zeta^{t_0} &\in H^{1,\frac{1}{2}}(Q_{t_0,T}) \times H^1(t_0,T;L^2(\Omega)) \\ \mathbf{p}_\zeta^{t_0} &\in \mathbf{H}^{3,\frac{3}{2}}(Q_{t_0,T}) \cap L^2(t_0,T;\mathbf{H}_0^1(\Omega)). \end{aligned}$$

*In particular,* $\mathbf{p}_\zeta^{t_0} \in C([t_0,T];\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))$.

*Proof.* From Theorem 3.2 we know that $\mathbf{p}_\zeta^{t_0} \in L^2(t_0,T;H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(Q_{t_0,T})$. Due to Lemma 3.1, this implies that $u_\zeta^{t_0} = -M\mathbf{B}^*\mathbf{p}_\zeta^{t_0} \in L^2(\Sigma_{t_0,T})$. Consider then the splitting $\mathbf{y}_\zeta^{t_0} = \mathbf{y}_{\zeta,1}^{t_0} + \mathbf{y}_{\zeta,2}^{t_0}$, where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_{\zeta,1}^{t_0} &= -\widehat{\mathbf{A}}\mathbf{y}_{\zeta,1}^{t_0} + \lambda_0 \mathbf{y}_\zeta^{t_0} && \text{in } (t_0,T), \quad \mathbf{y}_{\zeta,1}^{t_0}(t_0) = \zeta, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_{\zeta,2}^{t_0} &= -\widehat{\mathbf{A}}\mathbf{y}_{\zeta,2}^{t_0} + \mathbf{B}Mu_\zeta^{t_0} && \text{in } (t_0,T), \quad \mathbf{y}_{\zeta,2}^{t_0}(t_0) = 0. \end{aligned}$$

Since we have $\mathbf{y}_\zeta^{t_0} \in \mathbf{L}^2(Q_{t_0,T})$, we can use [6, Chapter 2, Theorem 2.2] with $\alpha = \frac{1}{2}$ in order to obtain that

$$\mathbf{y}_{\zeta,1}^{t_0} \in L^2(t_0,T;H_0^1(\Omega) \times L^2(\Omega)) \cap H^1(t_0,T;H^{-1}(\Omega) \times L^2(\Omega)).$$

By interpolation ([18, Theorem 4.1]), we particularly find that

$$\mathbf{y}_{\zeta,1}^{t_0} \in H^{1,\frac{1}{2}}(Q_{t_0,T}) \times L^2(Q_{t_0,T}).$$

With Lemma 3.5 below, we further have

$$\mathbf{y}_{\zeta,2}^{t_0} \in H^{\frac{1}{2}-\varepsilon,\frac{1}{4}-\frac{\varepsilon}{2}}(Q_{t_0,T}) \times L^2(Q_{t_0,T}).$$

Hence, we also get

$$\mathbf{y}_\zeta^{t_0} \in H^{\frac{1}{2}-\varepsilon,\frac{1}{4}-\frac{\varepsilon}{2}}(Q_{t_0,T}) \times L^2(Q_{t_0,T}).$$

Following Lemma A.1 with $s = \frac{1}{2} - \varepsilon$, it holds that $\mathbf{p}_\zeta^{t_0} \in \mathbf{H}^{\frac{5}{2}-\varepsilon,\frac{5}{4}-\frac{\varepsilon}{2}}(Q_{t_0,T})$. With [18, Chapter 4, Theorem 2.1], we arrive at $u_\zeta^{t_0} = -M\mathbf{B}^*\mathbf{p}_\zeta^{t_0} \in H^{1-\varepsilon,\frac{1}{2}-\frac{\varepsilon}{2}}(\Sigma_{t_0,T})$. Again with Lemma 3.5 it follows that $\mathbf{y}_{\zeta,2}^{t_0} \in \mathbf{H}^{\frac{3}{2}-2\varepsilon,\frac{3}{4}-\varepsilon}(Q_{t_0,T})$. Hence, we have $\mathbf{y}_\zeta^{t_0} \in H^{1,\frac{1}{2}}(Q_{t_0,T}) \times L^2(Q_{t_0,T})$. By Lemma A.1 this implies $\mathbf{p}_\zeta^{t_0} \in H^{3,\frac{3}{2}}(Q_{t_0,T})$. Finally, using [18, Chapter 1, Theorem 4.2] we obtain that $\mathbf{p}_\zeta^{t_0} \in C([t_0,T];(H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega))$. □

3.1.2. *An infinite time horizon control problem.* Let us now come back to the stabilization problem for the monodomain equations and consider

$$\inf\left\{ J(\mathbf{y},u) \,\middle|\, (\mathbf{y},u) \text{ satisfies (29)}, u \in L^2(\Sigma_\infty) \right\} \tag{43}$$

where $J(\mathbf{y},u)$ is as in (36). With exactly the same arguments as in [21] one can show the following results.

**Theorem 3.4.** *For all* $\mathbf{y}_0 \in \mathbf{L}^2(\Omega)$, *problem* (43) *admits a unique solution* $(\mathbf{y}_{\mathbf{y}_0}, u_{\mathbf{y}_0})$. *There exists* $\mathbf{\Pi} \in \mathcal{L}(\mathbf{L}^2(\Omega))$, *obeying* $\mathbf{\Pi} = \mathbf{\Pi}^*$, *such that the optimal cost is given by* $J(\mathbf{y}_{\mathbf{y}_0}, u_{\mathbf{y}_0}) = \frac{1}{2}(\mathbf{\Pi}\mathbf{y}_0, \mathbf{y}_0)_{\mathbf{L}^2(\Omega)}$.

Before we obtain the main results of this section, we give the analogue of Lemma 2.3.

**Lemma 3.5.** *If* $u \in H^{s,\frac{s}{2}}(\Sigma_\infty)$ *with* $0 \le s < 1$, *then the mild solution to the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\widehat{\mathbf{A}}\mathbf{y} + \mathbf{B}Mu \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = 0, \tag{44}$$

*satisfies*

$$\|\mathbf{y}\|_{\mathbf{H}^{s+\frac{1}{2}-\mu,\frac{s}{2}+\frac{1}{4}-\frac{\mu}{2}}(Q_\infty)} \le C\|u\|_{H^{s,\frac{s}{2}}(\Sigma_\infty)}, \quad \mu > 0.$$

*Furthermore, if* $u \in H^{s,\frac{s}{2}}(\Sigma_\infty)$ *with* $1 < s \le 2$ *and* $u(0) = 0$ *then the same estimate holds true.*

*Proof.* The assertion follows by similar arguments we have used in the proof of Lemma 2.3. □

As in [21], we obtain the following optimality system in the infinite horizon case.

**Theorem 3.6.** *For every* $\mathbf{y}_0 \in \mathbf{L}^2(\Omega)$, *the system*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{A}\mathbf{y} - \mathbf{B}M^2\mathbf{B}^*\mathbf{p} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0, \\ -&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \mathbf{A}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{in } (0,\infty), \quad \mathbf{p}(\infty) = 0, \\ &\mathbf{p}(t) = \mathbf{\Pi}\mathbf{y}(t) \quad \text{for all } t \in (0,\infty), \end{aligned} \tag{45}$$

*admits a unique solution* $(\mathbf{y},\mathbf{p})$ *in* $\mathbf{L}^2(Q_\infty) \times H^{2,1}(Q_\infty) \times L^2(Q_\infty)$. *This solution belongs to* $C_b([0,\infty);\mathbf{L}^2(\Omega)) \cap (H^{1,\frac{1}{2}}(Q_\infty) \times H^{\frac{3}{2}}(0,\infty;L^2(\Omega))) \times H^{3,\frac{3}{2}}(Q_\infty) \times H^{3,\frac{5}{2}}(Q_\infty)$.

*Proof.* The existence of a unique solution $(\mathbf{y},\mathbf{p}) \in \mathbf{L}^2(Q_\infty) \times H^{2,1}(Q_\infty) \times L^2(Q_\infty)$ can be shown as in [21, Lemma 4.2]. For showing the improved regularity let us consider the following splitting

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_1 = -\widehat{\mathbf{A}}\mathbf{y}_1 + \lambda_0\mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{y}_1(0) = \mathbf{y}_0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_2 = -\widehat{\mathbf{A}}\mathbf{y}_2 - \mathbf{B}M^2\mathbf{B}^*\mathbf{p} \quad \text{in } (0,\infty), \quad \mathbf{y}_2(0) = 0, \\ -&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\widehat{\mathbf{A}}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} + \lambda_0\mathbf{p} \quad \text{in } (0,\infty), \quad \mathbf{p}(\infty) = 0. \end{aligned}$$

Since $\mathbf{y} \in \mathbf{L}^2(Q_\infty)$ and $\mathbf{y}_0 \in \mathbf{L}^2(\Omega)$, we can apply [6, Part II, Chapter 3, Theorem 2.2] with $\alpha = \frac{1}{2}$ in order to obtain $\mathbf{y}_1 \in L^2(0,\infty;H_0^1(\Omega)) \cap H^1(0,\infty;H^{-1}(\Omega)) \times$

$H^1(0,\infty;L^2(\Omega))$. By interpolation [18, Chapter 1, Theorem 4.1/4.2] we particularly have $\mathbf{y}_1 \in C_b([0,\infty);\mathbf{L}^2(\Omega)) \cap \left(H^{1,\frac{1}{2}}(Q_\infty) \times H^1(0,\infty;L^2(\Omega))\right)$. We already know that $\mathbf{p} \in H^{2,1}(Q_\infty) \times L^2(Q_\infty)$ such that, from [18, Chapter 4, Theorem 2.1], we conclude that $u = -M\mathbf{B}^*\mathbf{p} \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_\infty)$. With Lemma 3.5 we get $\mathbf{y}_2 \in \mathbf{H}^{1-\varepsilon,\frac{1-\varepsilon}{2}}(Q_\infty)$ and also $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in H^{1-\varepsilon,\frac{1-\varepsilon}{2}}(Q_\infty) \times H^{\frac{1-\varepsilon}{2}}(0,\infty;L^2(\Omega))$. After an explicit integration of the ODE part of $\mathbf{p}$ we moreover find

$$q(t) = -\int_t^\infty d e^{(\tau-t)(\kappa-\sigma)} p(\tau)\,\mathrm{d}\tau. \tag{46}$$

With $p \in H^{2,1}(Q_\infty)$ we therefore have $q \in H^{2,1}(Q_\infty)$ as well. This clearly implies that

$$\begin{pmatrix} y \\ 0 \end{pmatrix} + \lambda_0 \mathbf{p} \in \mathbf{H}^{1-\varepsilon,\frac{1-\varepsilon}{2}}(Q_\infty)$$

and an application of Lemma A.3 shows that $\mathbf{p} \in H^{3-\varepsilon,\frac{3-\varepsilon}{2}}(Q_\infty) \times H^{1-\varepsilon,\frac{3-\varepsilon}{2}}(Q_\infty)$. Since this implies $u \in H^{\frac{3}{2}-\varepsilon,\frac{3}{4}-\frac{\varepsilon}{2}}(\Sigma_\infty)$ again with Lemma 3.5 and the splitting $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, we find $\mathbf{y} \in C_b([0,\infty);\mathbf{L}^2(\Omega)) \cap \left(H^{1,\frac{1}{2}}(Q_\infty) \times H^{\frac{1}{2}}(0,\infty;L^2(\Omega))\right)$. Lemma A.3 thus gives us $\mathbf{p} \in H^{3,\frac{3}{2}}(Q_\infty) \times H^{1,\frac{3}{2}}(Q_\infty)$. Again due to (46), it holds that $q \in L^2(0,\infty;H^3(\Omega))$. Moreover, we can exploit that

$$-\frac{\mathrm{d}}{\mathrm{d}t} q = -dp - (\kappa - \sigma) q$$

and, hence, $\frac{\mathrm{d}}{\mathrm{d}t} q \in H^{\frac{3}{2}}(0,\infty;L^2(\Omega))$, i.e. $q \in H^{\frac{5}{2}}(0,\infty;L^2(\Omega))$. □

Our main result concerns the (improved) regularity of the optimal solution depending on the smoothness of the initial data.

**Corollary 1.** *If* $\mathbf{y}_0 \in H^\varepsilon(\Omega) \times L^2(\Omega)$ *where* $\varepsilon \in [0,\frac{1}{2})$, *then the solution* $(\mathbf{y},\mathbf{p})$ *of* (45) *satisfies*

$$\begin{aligned}(\mathbf{y},\mathbf{p}) \in\ & H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0,\infty;L^2(\Omega)) \\ & \times H^{3+\varepsilon,\frac{3+\varepsilon}{2}}(Q_\infty) \times \left(H^1(0,\infty;H^{3+\varepsilon}(\Omega)) \cap H^{\frac{5+\varepsilon}{2}}(0,\infty;L^2(\Omega))\right).\end{aligned}$$

*Proof.* Consider again the splitting

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_1 &= -\widehat{\mathbf{A}}\mathbf{y}_1 + \lambda_0 \mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{y}_1(0) = \mathbf{y}_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_2 &= -\widehat{\mathbf{A}}\mathbf{y}_2 - \mathbf{B}M^2\mathbf{B}^*\mathbf{p} \quad \text{in } (0,\infty), \quad \mathbf{y}_2(0) = 0, \\ -\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} &= -\widehat{\mathbf{A}}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} + \lambda_0 \mathbf{p} \quad \text{in } (0,\infty), \quad \mathbf{p}(\infty) = 0.\end{aligned}$$

Let us first focus on $\mathbf{y}_1$. Since $\varepsilon \in [0,\frac{1}{2})$, with [17, Section 3A], for the fractional powers of $\widehat{\mathbf{A}}$ we obtain

$$\begin{aligned}[\mathcal{D}(\widehat{\mathbf{A}}),\mathbf{Y}]_{\frac{1-\varepsilon}{2}} &= \mathcal{D}(\widehat{\mathbf{A}}^{\frac{1+\varepsilon}{2}}) = H_0^{1+\varepsilon}(\Omega) \times L^2(\Omega), \\ [\mathcal{D}(\widehat{\mathbf{A}}^*),\mathbf{Y}]'_{\frac{1+\varepsilon}{2}} &= [\mathcal{D}((\widehat{\mathbf{A}}^*)^{\frac{1-\varepsilon}{2}})]' = H^{\varepsilon-1}(\Omega) \times L^2(\Omega).\end{aligned}$$

With [18, Chapter 1, Theorem 12.3], we conclude that

$$\left[[\mathcal{D}(\widehat{\mathbf{A}}),\mathbf{Y}]_{\frac{1-\varepsilon}{2}},[\mathcal{D}(\widehat{\mathbf{A}}^*),\mathbf{Y}]'_{\frac{1+\varepsilon}{2}}\right]_{\frac{1}{2}}=H^{\varepsilon}(\Omega)\times L^2(\Omega).$$

Since $\mathbf{y}\in L^2(Q_\infty)$, we can apply Theorem 2.2 with $\alpha=\frac{1-\varepsilon}{2}$ and obtain the existence of a unique solution $\mathbf{y}_1$ with

$$\mathbf{y}_1\in L^2(0,\infty;H_0^{1+\varepsilon}(\Omega)\times L^2(\Omega))\cap H^1(0,\infty;H^{\varepsilon-1}(\Omega)\times L^2(\Omega)).$$

By interpolation ([18, Chapter 1, Theorem 4.1]) we particularly find

$$y_1\in H^{\frac{1+\varepsilon}{2}}\left(0,\infty;[H_0^{1+\varepsilon}(\Omega),H^{\varepsilon-1}(\Omega)]_{\frac{1+\varepsilon}{2}}\right)=H^{\frac{1+\varepsilon}{2}}\left(0,\infty;L^2(\Omega)\right).$$

Moreover, for the ODE part we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1=\iota y_1-(\kappa-\sigma)z_1\in H^{\frac{1+\varepsilon}{2}}(0,\infty;L^2(\Omega)).$$

This implies $z_1\in H^{\frac{3+\varepsilon}{2}}(0,\infty;L^2(\Omega))$ and, thus, $\mathbf{y}_1\in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^{\frac{3+\varepsilon}{2}}(0,\infty;L^2(\Omega))$. From Theorem 3.6 we conclude that $\mathbf{p}\in H^{3,\frac{3}{2}}(Q_\infty)\times H^{3,\frac{5}{2}}(Q_\infty)$. With [18, Chapter 4, Theorem 2.1] it follows that $u=-M\mathbf{B}^*\mathbf{p}\in H^{\frac{3}{2},\frac{3}{4}}(\Sigma_\infty)$. In particular, we have $u\in H^{1-\mu,\frac{1-\mu}{2}}(\Sigma_\infty)$, for any $\mu>0$. Lemma 3.5 then yields $\mathbf{y}_2\in\mathbf{H}^{\frac{3}{2}-\tilde{\mu},\frac{3}{4}-\frac{\tilde{\mu}}{2}}(Q_\infty)$. Since this holds for any $\tilde{\mu}>0$, we get $\mathbf{y}\in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^{\frac{3}{4}-\tilde{\mu}}(0,\infty;L^2(\Omega)),\varepsilon\in[0,\frac{1}{2})$. Similar as before, for the ODE part of $\mathbf{y}$ we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}z=\iota y-(\kappa-\sigma)z\in H^{\frac{1+\varepsilon}{2}}(0,\infty;L^2(\Omega))$$

such that $z\in H^{\frac{3+\varepsilon}{2}}(0,\infty;L^2(\Omega))$. Moreover, it holds that

$$\begin{pmatrix}y\\0\end{pmatrix}+\lambda_0\mathbf{p}\in\mathbf{H}^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty).$$

Thus we can apply Lemma A.3 and obtain that $\mathbf{p}\in H^{3+\varepsilon,\frac{3+\varepsilon}{2}}(Q_\infty)\times H^{1+\varepsilon,\frac{3+\varepsilon}{2}}(Q_\infty)$. Integration of the ODE part of $\mathbf{p}$ now yields

$$q(\cdot)=-\int_{\cdot}^{\infty}de^{(\tau-\cdot)(\kappa-\sigma)}p(\tau)\,\mathrm{d}\tau\in H^1(0,\infty;H^{3+\varepsilon}(\Omega)).$$

Finally, using $-\frac{\mathrm{d}}{\mathrm{d}t}q=-dp-(\kappa-\sigma)q$ we also arrive at $q\in H^{\frac{5+\varepsilon}{2}}(0,\infty;L^2(\Omega)$. The proof is complete. $\square$

Let us point out that with [18, Chapter 1, Theorem 4.2] Theorem 3.6 implies that $\mathbf{p}(0)\in\left(H^{2+\varepsilon}(\Omega)\cap H_0^1(\Omega)\right)\times H^3(\Omega),\varepsilon\in[0,\frac{1}{2})$. Since $\mathbf{p}(0)=\mathbf{\Pi}\mathbf{y}_0$ we obtain that $\mathbf{\Pi}\colon H^{\varepsilon}(\Omega)\times L^2(\Omega)\to\left(H^{2+\varepsilon}(\Omega)\cap H_0^1(\Omega)\right)\times H^3(\Omega)$. Moreover, using that $\mathbf{\Pi}\in\mathcal{L}(\mathbf{L}^2(\Omega))$, it follows that $\mathbf{\Pi}\colon H^{\varepsilon}(\Omega)\times L^2(\Omega)\to\left(H^{2+\varepsilon}(\Omega)\cap H_0^1(\Omega)\right)\times H^3(\Omega)$ is closed, and hence bounded. Equivalently, we conclude that $\mathbf{B}^*\mathbf{\Pi}\in\mathcal{L}\left(H^{\varepsilon}(\Omega)\times L^2(\Omega),H^{\frac{1}{2}+\varepsilon}(\Gamma)\right),\varepsilon\in[0,\frac{1}{2})$.

3.2. **Results for the nonlinear system.** As in the case of Neumann boundary conditions, we first have to study a nonhomogeneous equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}=\mathbf{A}_{\mathbf{\Pi}}\mathbf{y}+\mathbf{f},\quad\mathbf{y}(0)=\mathbf{y}_0,\tag{47}$$

with $\mathbf{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. For the proof of the result below, we utilize the fact that

$$\mathcal{D}(\mathbf{A}_{\mathbf{\Pi}}^*) = \mathcal{D}(\mathbf{A}^*) = \mathcal{D}(\widehat{\mathbf{A}}^*)$$

which follows by arguments similar to those used in the proof of Lemma 2.4.

**Theorem 3.7.** *Let* $\varepsilon \in \left(0, \frac{1}{2}\right)$. *If* $f_1 \in L^2(0,\infty; H^{\varepsilon-1}(\Omega)), \mathbf{y}_0 \in H^{\varepsilon}(\Omega) \times H^{1+\varepsilon}(\Omega)$, *then* (22) *has a unique solution*

$$\mathbf{y} \in \left(H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0,\infty); H^{\varepsilon}(\Omega))\right) \times H^1(0,\infty; H^{1+\varepsilon}(\Omega))$$

*satisfying*

$$\|\mathbf{y}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \le C_2 \left(\|\mathbf{y}_0\|_{H^{\varepsilon}(\Omega)\times H^{1+\varepsilon}(\Omega)} + \|f_1\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))}\right).$$

*Proof.* By assumption, it holds that

$$\begin{aligned} \mathbf{y}_0 &\in H^{\varepsilon}(\Omega) \times H^{1+\varepsilon}(\Omega) \subset H^{-1}(\Omega) \times L^2(\Omega) = [\mathbf{Y}, [\mathcal{D}(\mathbf{A}_{\mathbf{\Pi}}^*)]']_{\frac{1}{2}}, \\ \mathbf{f} &\in L^2(0,\infty; H^{\varepsilon-1}(\Omega)) \subset L^2\left(0,\infty; [\mathcal{D}(\mathbf{A}_{\mathbf{\Pi}}^*)]'\right). \end{aligned}$$

Since the semigroup generated by $\mathbf{A}_{\mathbf{\Pi}}$ is exponentially stable on $\mathbf{Y}$, setting $\alpha = 1$ in Theorem 2.2 yields a unique solution $\mathbf{y}$ such that

$$\mathbf{y} \in L^2(0,\infty; \mathbf{Y}) \cap H^1(0,\infty; [\mathcal{D}(\mathbf{A}_{\mathbf{\Pi}}^*)]').$$

As in the proof of Theorem 2.5, we next utilize the splitting $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_1 &= -\widehat{\mathbf{A}}\mathbf{y}_1 + \mathbf{f} + \lambda_0\mathbf{y}, \quad \mathbf{y}_1(0) = \mathbf{y}_0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_2 &= -\widehat{\mathbf{A}}\mathbf{y}_2 - \mathbf{B}M^2\mathbf{B}^*\mathbf{\Pi}\mathbf{y}, \quad \mathbf{y}_2(0) = 0. \end{aligned} \tag{48}$$

The regularity result for $\mathbf{y}_1$ follows with exactly those arguments used previously for Corollary 1. We thus obtain that

$$\mathbf{y}_1 \in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{3+\varepsilon}{2}}(0,\infty; L^2(\Omega)). \tag{49}$$

Let us therefore focus on $\mathbf{y}_2$. We have shown that $\mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{\frac{1}{2}}(\Gamma))$. For $\mathbf{y} \in L^2(0,\infty;\mathbf{Y})$ this clearly gives $M\mathbf{B}^*\mathbf{\Pi}\mathbf{y} \in L^2(\Sigma_\infty)$. Lemma 3.5 therefore implies that $\mathbf{y}_2 \in \mathbf{H}^{\frac{1}{2}-\mu,\frac{1}{4}-\frac{\mu}{2}}(Q_\infty)$ which shows that

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in H^{\frac{1}{2}-\mu,\frac{1}{4}-\frac{\mu}{2}}(Q_\infty) \times H^{\frac{1}{4}-\frac{\mu}{2}}(0,\infty; L^2(\Omega)),$$

for any $\mu > 0$. Again, due to $\mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{\frac{1}{2}}(\Gamma))$ it particularly holds that

$$M\mathbf{B}^*\mathbf{\Pi}\mathbf{y} \in L^2(0,\infty; H^{\frac{1}{2}-\mu}(\Gamma)) \cap H^{\frac{1}{4}-\frac{\mu}{2}}(0,\infty; L^2(\Gamma)).$$

From Lemma 3.5 we now get $\mathbf{y}_2 \in \mathbf{H}^{1-\tilde{\mu},\frac{1}{2}-\frac{\tilde{\mu}}{2}}(Q_\infty)$ for any $\tilde{\mu} > 0$. Again, due to (49) we conclude that

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in H^{1-\tilde{\mu},\frac{1}{2}-\frac{\tilde{\mu}}{2}}(Q_\infty) \times H^{\frac{1}{2}-\frac{\tilde{\mu}}{2}}(0,\infty; L^2(\Omega)).$$

Making use of the fact that we additionally know $\mathbf{B}^*\mathbf{\Pi} \in \mathcal{L}(H^{\frac{1}{2}-\tilde{\mu}}(\Omega)\times L^2(\Omega), H^{1-\tilde{\mu}}(\Gamma))$ we obtain that

$$M\mathbf{B}^*\mathbf{\Pi}\mathbf{y} \in L^2(0,\infty; H^{1-\tilde{\mu}}(\Gamma)) \cap H^{\frac{1}{2}-\frac{\tilde{\mu}}{2}}(0,\infty; L^2(\Gamma)).$$

A final application of Lemma 3.5 then shows that $\mathbf{y}_2 \in \mathbf{H}^{\frac{3}{2}-\hat{\mu},\frac{3}{4}-\frac{\hat{\mu}}{2}}(Q_\infty)$. Since this holds for arbitrary $\hat{\mu} > 0$, we find that

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 \in H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \times H^{\frac{1+\varepsilon}{2}}(0,\infty;L^2(\Omega)).$$

The regularity of the ODE part $z$ of $\mathbf{y}$ again follows by explicit integration. □

As for the Neumann case, we then have the following local stabilization result for the nonlinear system.

**Theorem 3.8.** *Let $\Omega \subset \mathbb{R}^2$ and $\varepsilon \in \left(0,\frac{1}{2}\right)$. Then there exist $\mu_0 > 0$ and a nondecreasing function $\eta$ from $\mathbb{R}^+$ into itself such that if $\mu \in (0,\mu_0)$ and $\|\mathbf{y}_0\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)} \leq \eta(\mu)$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{A}_{\mathbf{\Pi}}\mathbf{y} + \mathbf{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

*admits a unique solution in the set*

$$D_\mu = \Big\{ \mathbf{y} \in \Big( H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty) \cap C_b([0,\infty);H^\varepsilon(\Omega)) \Big) \times H^1(0,\infty;H^{1+\varepsilon}(\Omega)),$$
$$\|\mathbf{y}\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))} \leq \mu \Big\}.$$

**Remark 1.** For the results in Theorem 2.8 and Theorem 3.8 we assumed that $U = L^2(\Gamma)$. However, in practical applications it is often more appropriate to utilize *finite dimensional controllers*. In this case, one typically assumes that the control function $u$ is separable, i.e., $u(x,t) = \sum_{i=1}^{\ell} h_i(x)u_i(t)$, where $h_i$ are shape functions and $\ell$ denotes the dimension of the control space $U = \mathbb{R}^\ell$. An essential tool for showing stabilizability by finite dimensional controllers then is the decomposition of the spectrum into an infinite dimensional stable part and a finite dimensional unstable part, see, e.g., [10, Chapter 5]. In essence, the stabilizability problem then reduces to its finite dimensional counterpart such that, given shape functions $h_i, i = 1,\ldots,\ell$, one may utilize the Hautus test for stabilizability. Let us emphasize that due to the structure of the monodomain equations, in [8], we have explicitly specified the spectrum in terms of the spectrum of the PDE part. In particular, as long as the desired stabilization rate is limited by the stability of the ODE, the unstable part of the system is discrete. This way, one may apply the exact same arguments to extend the results in Theorem 2.8 and Theorem 3.8 to the case of finite dimensional controllers.

## 4. **Numerical examples.**

For the numerical validation of the theory presented in this manuscript, we study the following version of the monodomain equations

$$\begin{aligned} \frac{\partial v}{\partial t} &= \alpha\Delta v - av^3 + bv^2 - cv - dw \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial w}{\partial t} &= \iota v - \kappa w \quad \text{in } \Omega \times (0,\infty), \\ \theta\frac{\partial v}{\partial \nu} &= (\theta - 1)v + m\tilde{u} + g \quad \text{on } \Gamma \times (0,\infty), \\ v(x,0) &= \bar{v} + y_0 \text{ and } w(x,0) = \bar{w} + z_0 \quad \text{in } \Omega, \end{aligned} \tag{50}$$

where $\Omega = (0,1)\times(0,1)$ and all other parameters are to be specified below. A finite element discretization is obtained by the MATLAB® PDE toolbox. All results correspond to a $64 \times 64$ regular grid with $n = 2 \cdot 4225 = 8450$ degrees of freedom.

Given a stationary solution of the form (3), the finite dimensional approximation of (4) with $\sigma = 0$ is given by

$$\mathbf{E}_n\dot{\mathbf{y}}_n = \mathbf{A}_n\mathbf{y}_n + \mathbf{F}_n(\mathbf{y}_n) + \mathbf{B}_n u, \tag{51}$$

where $\mathbf{A}_n, \mathbf{E}_n \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times \ell}$. The nonlinearity $\mathbf{F}_n \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined elementwise according to the nonlinearity in (50). For the discrete control operator, $\ell$ denotes the dimension of the finite dimensional control space (cf. Remark 1). By $\mathbf{C}_n \in \mathbb{R}^{p\times n}$ we denote the finite dimensional approximation of the output operator appearing in (13) and (36), respectively. In both Neumann and Dirichlet case, the multiplication operator $m$ is taken from [24]. The closed loop system is obtained by solving the following generalized algebraic matrix Riccati equation

$$\mathbf{A}_n^T\mathbf{\Pi}_n\mathbf{E}_n + \mathbf{E}_n^T\mathbf{\Pi}_n\mathbf{A}_n - \mathbf{E}^T\mathbf{\Pi}_n\mathbf{B}_n\mathbf{B}_n^T\mathbf{\Pi}_n\mathbf{E}_n + \mathbf{C}_n^T\mathbf{C}_n = 0.$$

For this, a Kleinman-Newton iteration as described in, e.g., [5, 9, 15], has been used. As a stabilizing initial guess $\mathbf{\Pi}_n^{(0)}$ we solved a $k$-dimensional algebraic Bernoulli equation corresponding to the unstable subspace (of dimension $k$) of the matrix pencil $(\mathbf{A}_n, \mathbf{E}_n)$. Each Lyapunov equation arising within the Kleinman-Newton method has been explicitly solved by the MATLAB® function `lyap`.

All simulations are generated on an Intel®Xeon(R) CPU E31270 @ 3.40 GHz x 8, 16 GB RAM, Ubuntu Linux 14.04, MATLAB® Version 8.0.0.783 (R2012b) 64-bit (glnxa64). The solutions of the ODE systems are always obtained by the MATLAB® routine `ode23`.

4.1. **Neumann boundary control.** We start with the Neumann case and thus consider (50) with $\theta = 1$. We further set $\alpha = 0.0015, a = 0.0012, b = 0.1403, c = 1.6140, d = 215.6, \iota = 0.00015, \kappa = 0.015$ and $g = 0$. The control and observation domains are shown in Figure 1a. In particular, we have $\mathbf{B}_n = \mathbf{C}_n^T \in \mathbb{R}^{n\times 12}$. Based on the underlying nonlinear system

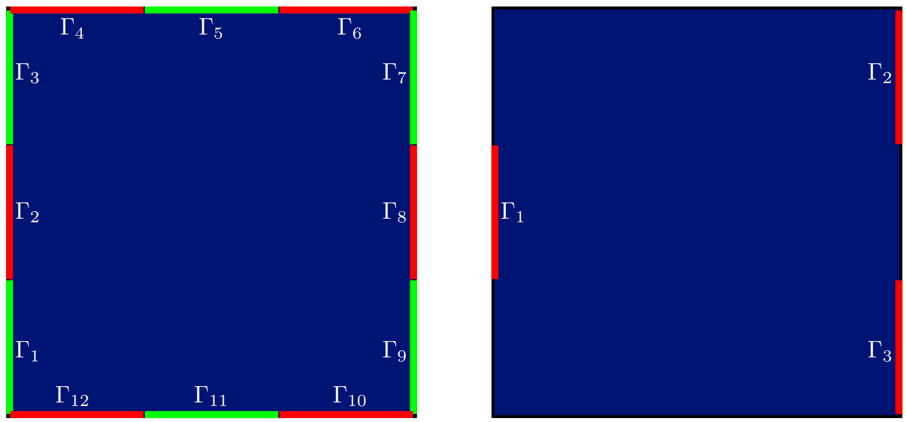
$$\begin{aligned} 0 &= -av^3 + bv^2 - cv - dw \\ 0 &= \iota v - \kappa w \end{aligned}$$

we obtain three constant (in space and time) stationary solutions to (50) which can be numerically computed as

$$(\bar{v}_1, \bar{w}_1) = (0, 0), \quad (\bar{v}_2, \bar{w}_2) \approx (44.0222, 0.4402), \quad (\bar{v}_3, \bar{w}_3) \approx (68.9778, 0.6898).$$

In our experiments, we chose $(\bar{v}_3, \bar{w}_3)$ for which the corresponding linearized system exhibited an unstable subspace of dimension 2.

*Perturbation around constant stationary state.* In Figure 2, numerical results for an initial value of the form $(v_0, w_0) = (\bar{v}_3 + \xi_v, \bar{w}_3 + \xi_w)$ with $\xi_v = 0.3 \cdot \texttt{randn(n,1)}$ and $\xi_w = 0.003 \cdot \texttt{randn(n,1)}$ are given. As is seen in Figure 2a the uncontrolled system remains close to the unstable stationary for a certain period of time before it slowly starts to decrease in magnitude. Finally, for $t = 1000$, the system approaches the stable stationary solution $(\bar{v}_1, \bar{w}_1)$ from below. On the other hand, Figure 2b demonstrates the successful local stabilization of the nonlinear system. The different dynamical behavior between uncontrolled and controlled system is even more evident from the results shown in Figure 2d. Here, the temporal evolution of the $L^2(\Omega)$ error between computed and desired state is visualized. Again we emphasize that the uncontrolled system remains close to the desired state first. Figure 2c shows the control law for some of the control domains specified in Figure




(A) The Neumann case. (B) The Dirichlet case.

FIGURE 1. Control setup.

1a. In particular, we see that the magnitude of the feedback control is relatively small when compared to the actual state.

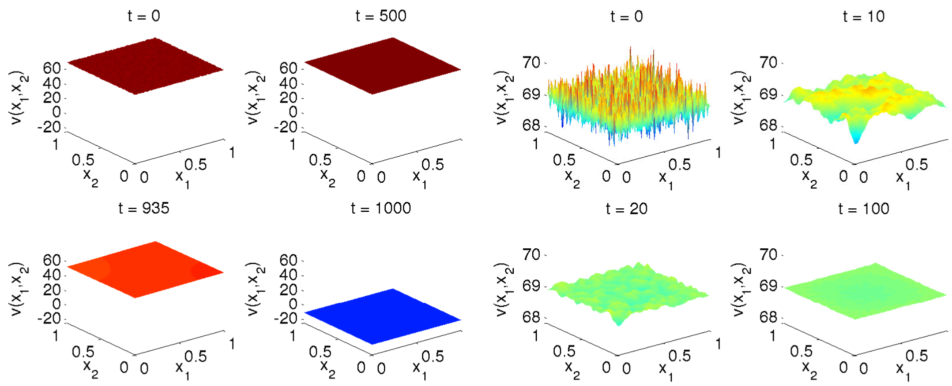
*Stabilization of a reentry wave.* As a second test case, we study the feedback law when the initial value $(v_0, w_0)$ is chosen such that it causes a reentry pattern (as arising in context of fibrillation processes). To be more precise, in Figure 3a we illustrate the wave-like dynamics appearing for the uncontrolled system. Snapshots of the evolution for the closed loop system are provided in Figure 3b. We emphasize that, as we have also seen in the first example, when the control is switched off close to the stationary solution $(\bar{v}_3, \bar{w}_3)$, the system will converge to the stable resting state $(\bar{v}_1, \bar{w}_1)$. This way, the feedback law implicitly allows to perform a defibrillation process by first controlling the system to the unstable stationary state. From there the uncontrolled dynamics converges to the origin. Again, a more detailed comparison between controlled and uncontrolled dynamics can be obtained in terms of the $L^2(\Omega)$ error, see Figure 3d. For a precise pattern of the control laws, we refer to 2c.

4.2. **Dirichlet boundary control.** Next, we consider (50) with $\theta = 0$. For the parameters, we take $\alpha = 0.0015, a = 0.012, b = 0.1395, c = 1.6050, d = 215.6, \iota = 0.00015$ and $\kappa = 0.015$. Again, we use the underlying ODE system to compute three constant (in space and time) stationary solutions as

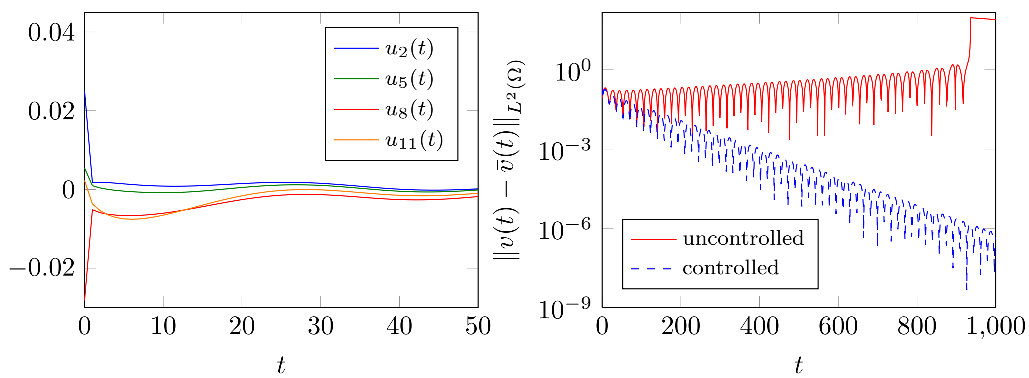
$$(\bar{v}_1, \bar{w}_1) = (0, 0), \quad (\bar{v}_2, \bar{w}_2) \approx (44.4187, 0.4442), \quad (\bar{v}_3, \bar{w}_3) \approx (68.5813, 0.6858).$$

In order to ensure that $(\bar{v}_3, \bar{w}_3)$ is also a stationary solution of the PDE system, in (50) we assume an inhomogeneous Dirichlet boundary condition $g = \bar{v}_3$. The resulting linearized system has 6 eigenvalues with positive real part. In contrast to the Neumann case, we only take three controls that are visualized in Figure 1b.

*Perturbation around constant stationary state.* The results shown in Figure 4 correspond to a similar setup as in the Neumann case. For both, uncontrolled and controlled system, the initial value was taken as a perturbation $(0.3\ \texttt{randn(n,1)})$ of the unstable stationary solution $(\bar{v}_3, \bar{w}_3)$. As a consequence, the uncontrolled system shows an oscillatory behavior, see Figure 4a. On the other hand, the feedback




(A) Evolution of $v(x,t)$ (uncontrolled). (B) Evolution of $v(x,t)$ (controlled).

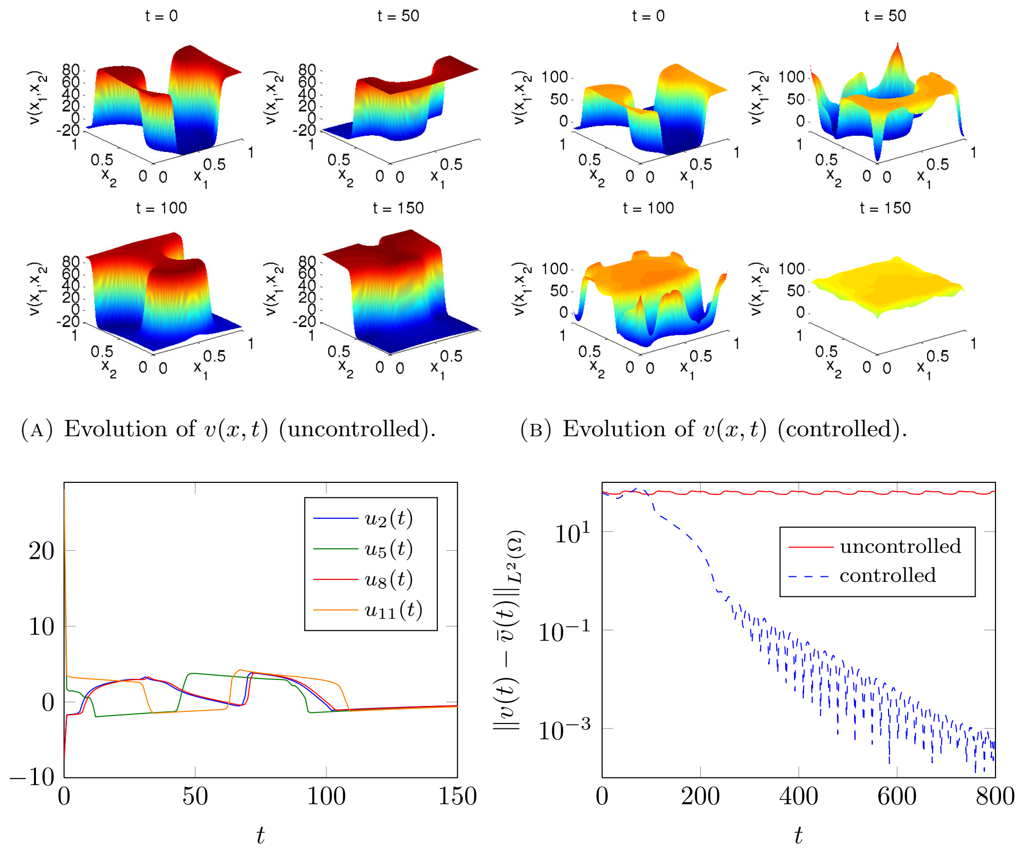



(C) Control action. (D) Evolution of $L_2$-error.

FIGURE 2. Stabilization of perturbed initial state.

law computed from the solution of the algebraic Riccati equation allows to locally stabilize the nonlinear system, see Figure 4b. For a more detailed comparison, we refer to the temporal evolution of the $L^2(\Omega)$ error that is provided in Figure 4d. The performance of the three boundary control patches is visualized in Figure 4c, also underlining the convergence of the closed loop system.

5. **Conclusions.** We have studied boundary feedback control problems for a class of nonlinear reaction diffusion systems of PDE-ODE type. A particular emphasis was on the so-called monodomain equations. For the linearized system, we have investigated the use of classical Riccati-based feedback controllers. In case of Dirichlet boundary conditions, we have derived a smoothing property (known in case of the Navier-Stokes equations) of the solution of the algebraic operator Riccati equation. Based on some new Lipschitz estimates for the cubic non monotone nonlinearity, we could show that the feedback law derived for the linearized system locally stabilizes the nonlinear system as well. While in the case of Neumann boundary conditions, both the two and three dimensional case could be handled, in the Dirichlet case we restricted ourselves to the two dimensional setting. Numerical examples for a




(A) Evolution of $v(x,t)$ (uncontrolled). (B) Evolution of $v(x,t)$ (controlled).

(C) Control action. (D) Evolution of $L_2$-error.

FIGURE 3. Stabilization of a reentry wave.

FEM discretization of the monodomain equations illustrated the main theoretical findings.

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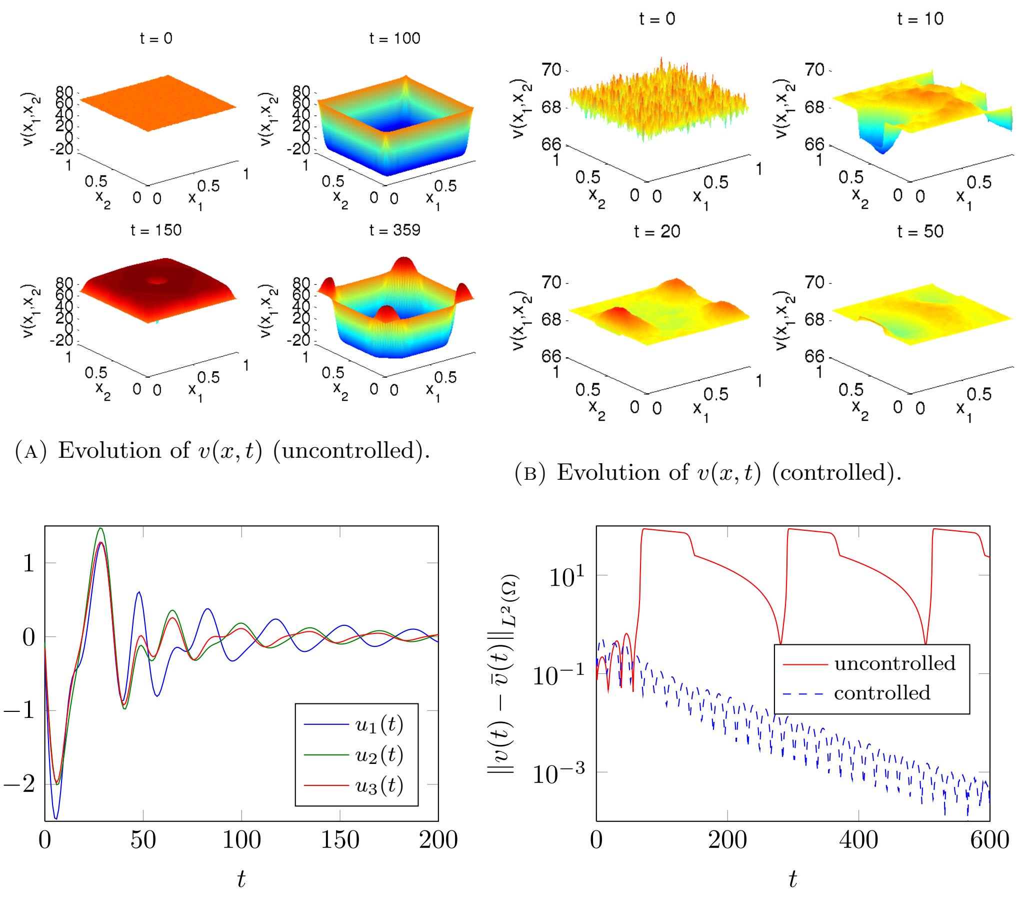
## Appendix A. Some auxiliary results.

**Lemma A.1.** *For all* $y \in H^{s,\frac{s}{2}}(Q_T)$ *with* $0 \le s < \frac{3}{2}$, *the solution to the equation*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\widehat{\mathbf{A}}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} \quad \text{in } (t_0,T), \quad \mathbf{p}(T) = 0,$$

*satisfies*

$$\|\mathbf{p}\|_{\mathbf{H}^{s+2,\frac{s}{2}+1}(Q_{t_0,T})} \le C\|y\|_{H^{s,\frac{s}{2}}(Q_{t_0,T})}.$$




(A) Evolution of $v(x,t)$ (uncontrolled).

(B) Evolution of $v(x,t)$ (controlled).

(C) Control action.

(D) Evolution of $L_2$-error.

FIGURE 4. Stabilization of perturbed initial state.

*Proof.* For $s = 0$ the assertion follows from [6, Chapter 2, Proposition 3.7] and subsequent integration of the ODE variable. Assume now that $s = 2$ and consider

$$\widehat{\mathbf{A}}^* \mathbf{p} = \widehat{\mathbf{A}}^* \int_t^T e^{-(\tau - t)\widehat{\mathbf{A}}^*} \begin{pmatrix} y(\tau) \\ 0 \end{pmatrix} \mathrm{d}\tau.$$

Since $y \in H^{2,1}(Q_{t_0,T})$, we also have

$$\begin{aligned} \widehat{\mathbf{A}}^* \mathbf{p} &= -e^{-(\tau - t)\widehat{\mathbf{A}}^*} \begin{pmatrix} y(\tau) \\ 0 \end{pmatrix} \Bigg|_t^T + \int_t^T e^{-(\tau - t)\widehat{\mathbf{A}}^*} \begin{pmatrix} y'(\tau) \\ 0 \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} y(t) \\ 0 \end{pmatrix} - e^{-(T-t)\widehat{\mathbf{A}}^*} \begin{pmatrix} y(T) \\ 0 \end{pmatrix} + \int_t^T e^{-(\tau - t)\widehat{\mathbf{A}}^*} \begin{pmatrix} y'(\tau) \\ 0 \end{pmatrix} \mathrm{d}\tau. \end{aligned}$$

Note that the last two terms define the mild solution $\mathbf{z}$ to the problem

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = -\widehat{\mathbf{A}}^* \mathbf{z} + \begin{pmatrix} y' \\ 0 \end{pmatrix} \quad \text{in } (t_0, T), \quad \mathbf{z}(T) = -\begin{pmatrix} y(T) \\ 0 \end{pmatrix}.$$

Since $y' \in L^2(Q_{t_0,T})$, if we assume that $y(T) \in H_0^1(\Omega)$ with [6, Chapter 2, Proposition 3.7] we conclude that $\mathbf{z} \in H^{2,1}(Q_{t_0,T}) \times H^1(Q_{t_0,T})$. Moreover, by integration of the ODE part, we also get $\mathbf{z} \in \mathbf{H}^{2,1}(Q_{t_0,T})$. Hence, it follows that $\widehat{\mathbf{A}}^*\mathbf{p} \in \mathbf{H}^{2,1}(Q_{t_0,T})$. With Lemma A.2 and subsequent integration of the ODE part, we get

$$\|\mathbf{p}\|_{L^2(t_0,T;\mathbf{H}^4(\Omega))} \leq \|y\|_{H^{2,1}(Q_{t_0,T})}.$$

The temporal regularity follows from the fact that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\widehat{\mathbf{A}}^*\mathbf{p} + \begin{pmatrix} y \\ 0 \end{pmatrix} \in H^1(t_0,T;\mathbf{L}^2(\Omega)).$$

For $0 < s < 2$, the result now follows by interpolation. For $s > \frac{3}{2}$ we additionally require that $\mathbf{y}(T) \in H_0^{s-1}(\Omega)$. This compatibility condition however is not needed for $s < \frac{3}{2}$, see also [18, Chapter 4, Section 6]. □

**Lemma A.2.** *For all* $\mathbf{y} \in \mathbf{H}^2(\Omega)$, *the solution* $\mathbf{p} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ *to the stationary equation* $\widehat{\mathbf{A}}^*\mathbf{p} = \mathbf{y}$ *obeys*

$$\|\mathbf{p}\|_{H^4(\Omega)\times H^2(\Omega)} \leq C\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)}.$$

*Proof.* Solving the second equation for $q$, we obtain $q = \frac{dp+z}{\lambda_0+\kappa-\sigma}$. Hence, the first equation reads

$$\lambda_0 p - \Delta p = y + \alpha(x)p - \iota\frac{dp+z}{\lambda_0+\kappa-\sigma}.$$

Recall that $\bar{v} \in H^2(\Omega)$ as well as $\alpha = -3a\bar{v}^2 + 2b\bar{v} - c$. Hence, we have $\alpha p \in H^2(\Omega)$ and, thus, $y + \alpha p - \iota\frac{dp+z}{\lambda_0+\kappa-\sigma} \in H^2(\Omega)$ which implies that $p \in H^4(\Omega)$. Since $q = \frac{dp+z}{\lambda_0+\kappa-\sigma}$ the claim is shown. □

**Lemma A.3.** *If* $\mathbf{y} \in \mathbf{H}^{s,\frac{s}{2}}(Q_\infty)$ *with* $0 \leq s \leq 2$, *then the solution* $\mathbf{p}$ *to the equation*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\widehat{\mathbf{A}}^*\mathbf{p} + \mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{p}(\infty) = 0,$$

*satisfies*

$$\|\mathbf{p}\|_{H^{s+2,\frac{s}{2}+1}(Q_\infty)\times H^{s,\frac{s}{2}+1}(Q_\infty)} \leq C\|\mathbf{y}\|_{H^{s,\frac{s}{2}}(Q_\infty)\times H^{\frac{s}{2}}(0,\infty;L^2(\Omega))}.$$

*Proof.* We begin with the case $s = 0$. From [6, Chapter 2, Proposition 3.7] we immediately obtain that

$$\mathbf{p} \in H^{2,1}(Q_\infty) \times H^1(0,\infty;L^2(\Omega)).$$

Consider now the case $s = 2$. It holds that

$$\begin{aligned} \widehat{\mathbf{A}}^*\mathbf{p}(t) &= \int_t^\infty \widehat{\mathbf{A}}^* e^{-(\tau-t)\widehat{\mathbf{A}}^*}\mathbf{y}(\tau)\,\mathrm{d}\tau \\ &= \int_t^\infty e^{-(\tau-t)\widehat{\mathbf{A}}^*}\mathbf{y}'(\tau)\,\mathrm{d}\tau + \mathbf{y}(t) - \lim_{\tau\to\infty} e^{-(\tau-t)\widehat{\mathbf{A}}^*}\mathbf{y}(\tau). \end{aligned}$$

Due to the exponential stability of the semigroup $e^{-\widehat{\mathbf{A}}^*\cdot}$ and the fact that $\mathbf{y} \in C_b([0,\infty);\mathbf{L}^2(\Omega))$, the last term vanishes. Since $\mathbf{y}' \in \mathbf{L}^2(Q_\infty)$, the arguments for $s = 0$ apply to the first term and we obtain

$$\left\|\int_t^\infty e^{-(\tau-t)\widehat{\mathbf{A}}^*}\mathbf{y}'(\tau)\,\mathrm{d}\tau\right\|_{H^{2,1}(Q_\infty)\times H^1(0,\infty;L^2(\Omega))} \leq C\|\mathbf{y}'\|_{\mathbf{L}^2(Q_\infty)}.$$

Hence, this implies that $\tilde{\mathbf{p}} := (\tilde{p}, \tilde{q}) = \widehat{\mathbf{A}}^*\mathbf{p} \in L^2(0,\infty; H^2(\Omega) \times L^2(\Omega))$. Note that an explicit integration of the ODE part of $\mathbf{p}$ also yields

$$q(t) = \int_t^\infty e^{-(\tau - t)(\lambda_0 + \kappa - \sigma)}(z - dp)\,\mathrm{d}\tau \in L^2(0,\infty; H^2(\Omega)).$$

This particularly shows that $\tilde{q} = -dp - (\lambda_0 + \kappa - \sigma)q \in L^2(0,\infty; H^2(\Omega))$. Hence, we obtain

$$\left\|\widehat{\mathbf{A}}^*\mathbf{p}\right\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} \le C\left(\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} + \|\mathbf{y}'\|_{\mathbf{L}^2(Q_\infty)}\right) \le C\|\mathbf{y}\|_{\mathbf{H}^{2,1}(Q_\infty)}.$$

From Lemma A.2 it follows that

$$\|\mathbf{p}\|_{L^2(0,\infty;H^4(\Omega)\times H^2(\Omega))} \le C\|\widehat{\mathbf{A}}^*\mathbf{p}\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))}$$

and, consequently,

$$\|\mathbf{p}\|_{L^2(0,\infty;H^4(\Omega)\times H^2(\Omega))} \le C\|\mathbf{y}\|_{\mathbf{H}^{2,1}(Q_\infty)}.$$

We further find

$$\|\mathbf{p}'\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} \le C\left(\|\widehat{\mathbf{A}}^*\mathbf{p}\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} + \|\mathbf{y}\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))}\right) \le C\|\mathbf{y}\|_{\mathbf{H}^{2,1}(Q_\infty)}.$$

The case $0 < s < 2$ can be obtained by interpolation. □

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*E-mail address*: tobias.breiten@uni-graz.at
*E-mail address*: karl.kunisch@uni-graz.at