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SFB-Report No. 2014-017 December 2014

A–8010 GRAZ, HEINRICHSTRASSE 36, AUSTRIA

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COMPENSATOR DESIGN FOR THE MONODOMAIN EQUATIONS

TOBIAS BREITEN[1] AND KARL KUNISCH[2]

Abstract. The problem of finite-dimensional compensator design for the monodomain equations is investigated. Exponential stabilizability and detectability of the linearized infinite-dimensional control system is studied. It is shown that the system is not exactly null-controllable but still can be exponentially stabilized by finite-rank input and output operators provided the desired stability margin is small enough. Based on existing results on model order reduction of infinite-dimensional systems, a finite-dimensional compensator is obtained by LQG-balanced truncation. The compensator is shown to be locally stabilizing for the infinite-dimensional nonlinear control system. Examples motivated by cardiophysiology are used to illustrate these results in a numerical setup.

1991 Mathematics Subject Classification. 35K57, 93B52, 93C20, 93D15.

The dates will be set by the publisher.

1. Introduction

Optimal treatment of human heart diseases is one of the central research topics in cardiac electrophysiology, see [34]. While certain problems can already be tackled by feedback-type control mechanisms, e.g., cardiac pacemakers, diseases such as cardiac arrhythmia still require the application of an external stimulus (defibrillation) to the heart issue. This circumstance has caused the study of feedback control methodologies from a mathematical point of view. The so-called monodomain equations represent a well-analyzed mathematical model of the electrical potential of the human heart, see, e.g., [23]. The dynamics are modeled by a parabolic reaction-diffusion equation that is coupled to a linear ordinary differential equation:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \Delta v - I_{\text{ion}}(v,w) + f + Bu, \quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial w}{\partial t} &= \gamma v - \delta w, \quad \text{in } \Omega \times (0,\infty),\\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \Gamma \times (0,\infty),\\
v(x,0) &= v_0(x) \text{ and } w(x,0) = w_0(x), \quad \text{in } \Omega,
\end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \in \{2,3\}$, denotes a bounded open set with smooth boundary $\Gamma = \partial\Omega$. Here, $v = v(x,t)$ describes the transmembrane electrical potential, $w = w(x,t)$ is a so-called gating variable, $f = f(x,t)$ is an

Keywords and phrases: compensator design, LQG-balanced truncation, monodomain equations, FitzHugh-Nagumo model

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external forcing term and $u = u(x,t) \in L^2(0,\infty;L^2(\omega_{\text{con}}))$, denotes the control. Here $\omega_{\text{con}} \subset \Omega$, is a nonempty open subset. As a model for the ionic current, we choose the FitzHugh-Nagumo model

$$I_{\text{ion}}(v,w) = av^3 - bv^2 + cv + dw,$$

where a, b, c, d are positive real constants. Many other choices for the model of the ionic current are of importance, see, e.g., [1, 22, 30]. We further assume that $\gamma, \delta > 0$ and denote by ν the outer unit normal to Γ. In applications it is of importance to model the heterogeneity of the issue and to replace Δv by $\operatorname{div}(a\nabla v)$. Our results can be extended to handle this case if a is positive definite and $W^{1,\infty}(\Omega)$. For fixed t, we define the linear operator $B \in \mathcal{L}(L^2(\omega_{\text{con}}), L^2(\Omega))$ as

$$(Bu)(x) = \begin{cases} u(x), & x \in \omega_{\text{con}}, \\ 0, & x \in \Omega \backslash \omega_{\text{con}}. \end{cases}$$

Its adjoint B^* is the restriction operator from Ω to ω_{con}. With slight abuse of notation, we also use B as operator from $L^2(0,\infty;L^2(\omega_{\text{con}})) \to L^2(0,\infty;L^2(\Omega))$ with $(Bu)(t) = (Bu(t))$ and analogously for B^*. In a realistic scenario, we do not have access to the whole state (v,w) but only to certain measurements. An exact modeling of the physical processes would involve the photon diffusion equation ([9]), rendering indirect (sparse) measurements on the boundary. Since the study of feedback or closed-loop techniques for the monodomain equations has attracted attention only recently, for now we assume some simplifications. In particular, we consider partial observations of the transmembrane potential in some nonempty open subsets $\omega_{\text{obs}} \subset \Omega$. Hence, for fixed t we define $C \in \mathcal{L}(L^2(\Omega), L^2(\omega_{\text{obs}}))$ as the restriction operator from Ω to ω_{obs}. In case $\omega_{\text{obs}} = \omega_{\text{con}}$, we have that $C = B^*$. Recent technological developments allow devices which act as controllers and observers essentially on the same subsets of the physiological domain, see [26].

For a given stationary but possibly unstable solution $(\bar{v}, \bar{w})$ of system (1), the goal of this paper is to design a finite-dimensional linear controller such that the original nonlinear system is locally stabilized around this stationary solution. We build our results upon the theory of infinite-dimensional control systems of the form

$$\begin{aligned} \dot{\vec{y}}(t) &= \mathcal{A}\vec{y}(t) + \mathcal{F}(\vec{y}) + \mathcal{B}u(t),\ \vec{y}(0) = \vec{y}_0, \\ y_{\text{obs}}(t) &= \mathcal{C}\vec{y}(t) + \mu(t), \end{aligned} \tag{2}$$

where $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup on a Hilbert space $\mathcal{Y}$, $\mathcal{B} \in \mathcal{L}(\mathcal{U},\mathcal{Y})$ and $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$ are bounded finite-rank input and output operators with $\mathcal{U} = \mathbb{R}^m$ and $\mathcal{Z} = \mathbb{R}^p$, respectively. Moreover, the nonlinear non-monotone operator $\mathcal{F}$ models the ionic current I_{ion}. For the measurement error we assume that $\mu \in L^2(0,\infty;\mathbb{R}^p)$.

We now consider a class of compensators of the form

$$\dot{z}(t) = \mathcal{A}_c z(t) + \mathcal{B}_c y_{\text{obs}}(t),\ z(0) = z_0, \quad u(t) = \mathcal{C}_c z(t), \tag{3}$$

and study the possibility of choosing $(\mathcal{A}_c, \mathcal{B}_c, \mathcal{C}_c)$ such that the *closed-loop system* characterized via (2) and (3) admits a unique solution that converges to zero. We first address local exponential stabilization in the case that (3) is infinite-dimensional. Subsequently it is shown that one can achieve local exponential stabilization even by finite-dimensional control systems. For linear infinite-dimensional control systems, the design of finite-dimensional stabilizing controller is well-known and has been studied in, e.g., [14,16,17,31]. Stabilization results for nonlinear infinite-dimensional systems based on compensator design of the underlying linearized system were analyzed in [29] for the case of the Burgers' equation, for example.

The structure of the paper is as follows. In Section 2 we reformulate system (1) as an abstract Cauchy problem of the form (2). We give an explicit expression for the spectrum of the linearized control system which we will use to show that the system cannot be null-controllable. For a stabilization margin $\beta < \delta$, we investigate the applicability of finite-rank input and output operators for local exponential stabilization. Based on classical results from linear quadratic Gaussian (LQG) control theory, in Section 3 we design a linear compensator for the nonlinear system. Section 4 recapitulates existing results from [15] that allow for the design of a finite-dimensional compensator by using the LQG-balanced truncation (BT) model reduction method. In Section 5, we study a concrete example resulting from a finite element discretization of (1). We verify stabilizability and detectability assumptions for the linearized system. Numerical simulations show the feasibility of the classical LQG as well as the LQG-BT approach. We end the paper with some conclusions in Section 6.

Notation

For $p \geq 1$ and $s \geq 0$, we denote by $L^p(\Omega)$ and $H^s(\Omega)$ the usual Lebesgue and Sobolev spaces. We define $H^s_\nu(\Omega) := \overline{\left\{ y \in \mathcal{D}(\Omega) \,\middle|\, \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}}$, where the closure is taken with respect to the $\|y\|_{H^s(\Omega)}, s \geq 0$, norm. For $s > \frac{3}{2}$, we have $H^s_\nu = \left\{ y \in H^s(\Omega) \,\middle|\, \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma \right\}$ and for $s \in [0, \frac{3}{2})$, we have $H^s_\nu(\Omega) = H^s(\Omega)$, see, e.g., [8, Section II-1]. Given a Hilbert space X, we denote with $L^2(0,\infty;X)$ (Bochner) square integrable functions on $(0,\infty)$ with values in X. For $Q_\infty = \Omega \times (0,\infty)$ and $r \geq 0, s \geq 0$, we will need the spaces

$$H^{r,s}(Q_\infty) = L^2(0,\infty;H^r(\Omega)) \cap H^s(0,\infty;L^2(\Omega)),$$

which are Hilbert spaces when endowed with the norm

$$\|u\|_{r,s} = \left(\int_0^\infty \|u(t)\|^2_{H^r(\Omega)} \mathrm{d}t + \|u\|^2_{H^s(0,\infty;L^2(\Omega))} \right)^{\frac{1}{2}}.$$

The space of bounded linear operators between two Hilbert spaces X and Y will be denoted by $\mathcal{L}(X,Y)$. For a closed densely defined linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$, the resolvent set and the spectrum $\mathcal{A}$ are denoted by $\rho(\mathcal{A})$ and $\sigma(\mathcal{A})$, respectively. Further $\sigma_p(\mathcal{A})$ denotes the point spectrum of the operator $\mathcal{A}$. The adjoint of $\mathcal{A}$ is denoted by $\mathcal{A}^*$. Finally $S(t)$ stands for the semigroup determined by its infinitesimal generator $\mathcal{A}$.

2. Control-theoretic properties of the monodomain equations

In this section we provide some details on the formulation of (1) as an abstract Cauchy problem of the form (2). Following the setting in [11], let us assume that a possibly unstable stationary solution $(\bar{v}, \bar{w}) \in H^{\frac{n}{2}+s}(\Omega) \times L^\infty(\Omega), s > 0$ is given such that

$$\begin{aligned} 0 &= \Delta \bar{v} - I_{\text{ion}}(\bar{v}, \bar{w}) + f, \\ 0 &= \gamma \bar{v} - \delta \bar{w}, \end{aligned} \tag{4}$$

holds. We now define the state variable as the difference between (1) and (4), i.e.,

$$\vec{y} := (y_v, y_w) = (v - \bar{v}, w - \bar{w})$$

and obtain the infinite-dimensional control system

$$\dot{\vec{y}}(t) = \mathcal{A}\vec{y}(t) + \mathcal{F}(\vec{y}) + \mathcal{B}u(t), \quad \vec{y}(0) = \vec{y}_0,$$

where the operators are given as

$$\mathcal{A}\vec{y} = \begin{pmatrix} \Delta y_v - (3a\bar{v}^2 - 2b\bar{v} + c)y_v - dy_w \\ \gamma y_v - \delta y_w \end{pmatrix}, \quad \mathcal{F}(\vec{y}) = \begin{pmatrix} -ay_v^3 - (-b + 3a\bar{v})y_v^2 \\ 0 \end{pmatrix}, \quad \mathcal{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}. \tag{5}$$

As we have shown in [11], the linearized operator $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup on $\mathcal{Y} = L^2(\Omega) \times L^2(\Omega)$. Moreover, for $\beta < \delta$, the linearized control system

$$\dot{\vec{y}}(t) = \mathcal{A}\vec{y}(t) + \mathcal{B}u(t), \quad \vec{y}(0) = \vec{y}_0, \tag{6a}$$

is β-exponentially stabilizable by a feedback control of the form $u(t) = -\mathcal{B}^*\mathcal{P}\vec{y}(t)$, where $\mathcal{P}$ is the solution of an algebraic operator Riccati equation. In contrast to [11], we are interested in the situation where only parts of the system rather than the complete state can be observed. For this reason, we extend (6a) by including an output equation

$$y_{\mathrm{obs}}(t) = \mathcal{C}\vec{y}(t) + \mu(t). \tag{6b}$$

For the measurement error assume that $\mu \in L^2(0,\infty;\mathbb{R}^p)$ and, further, that $\mathcal{C} \in \mathcal{L}(\mathcal{Y},\mathbb{R}^p)$ is a finite-rank output operator such that for fixed t, we have

$$\mathcal{C}\vec{y}(t) = \left(\tfrac{1}{|\omega_1|}\int_{\omega_1} y_v(t), \dots, \tfrac{1}{|\omega_p|}\int_{\omega_p} y_v(t)\right) \in \mathbb{R}^p,$$

where $\omega_i \subset \Omega$ are non-empty open subsets.

2.1. The spectrum of $\mathcal{A}$

As has been pointed out in [10, 11], very little is known about controllability properties of system (6). In fact, so far it has only been shown that the system is approximately controllable, see [10]. Let us now focus on the precise location of the spectrum of the linearized operator $\mathcal{A}$. This will lead to some new insights into the properties of null controllability and stabilizability by finite-rank operators, respectively.

For what follows, we have to define the *essential spectrum* of a closed (possibly unbounded) operator $\mathcal{A}$. We follow [19, 21] and define

$$\sigma_{\mathrm{ess}}(\mathcal{A}) := \{\lambda \in \mathbb{C} | \mathcal{A} - \lambda \text{ is not Fredholm}\},$$

where an operator is called Fredholm if it is closed, its kernel is finite-dimensional and its range is finite co-dimensional. We further need the notion of relative boundedness and relative compactness as defined in [19].

Definition 2.1 (Relative boundedness/compactness)**.**
(i) Let $\mathcal{A}$ and $\mathcal{T}$ be operators with $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{T})$ and

$$\|\mathcal{T}u\| \le a\|u\| + b\|\mathcal{A}u\|, \quad u \in \mathcal{D}(\mathcal{A}),$$

where a, b are nonnegative constants. Then $\mathcal{T}$ is called relatively bounded with respect to $\mathcal{A}$ or simply $\mathcal{A}$-bounded. The greatest lower bound b_0 of all possible constants b will be called the $\mathcal{A}$-bound of $\mathcal{T}$.

(ii) Let $\mathcal{A}$ and $\mathcal{T}$ be operators with $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{T})$. Assume that for any sequence $u_n \in \mathcal{D}(\mathcal{A})$ with both $\{u_n\}$ and $\{\mathcal{A}u_n\}$ bounded, $\{\mathcal{T}u_n\}$ contains a convergent subsequence. Then $\mathcal{T}$ is called relatively compact with respect to $\mathcal{A}$ or simply $\mathcal{A}$-compact.

From [21], we have the following result on the perturbation of the essential spectrum of 2×2 block operator matrices as follows.

Theorem 2.2 ([21])**.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. For $i = 0, 1$, let $A_i : \mathcal{D}(\mathcal{A}_i) \subset \mathcal{H}_1 \to \mathcal{H}_1, B_i : \mathcal{D}(\mathcal{B}_i) \subset \mathcal{H}_2 \to \mathcal{H}_1, C_i : \mathcal{D}(\mathcal{C}_i) \subset \mathcal{H}_1 \to \mathcal{H}_2, D_i : \mathcal{D}(\mathcal{D}_i) \subset \mathcal{H}_2 \to \mathcal{H}_2$ be closed and (possibly) unbounded operators, and define*

$$\mathcal{A}_0 := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \quad \mathcal{A}_1 := \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

Suppose

(i) *that the domain inclusions* $\mathcal{D}(A_0) \subset \mathcal{D}(A_1), \mathcal{D}(A_0) \subset \mathcal{D}(C_i), i = 0, 1,$ $\mathcal{D}(D_0) \subset \mathcal{D}(D_1)$, *and* $\mathcal{D}(D_0) \subset \mathcal{D}(B_i), i = 0, 1,$ *all hold;*

(ii) *that* A_1 *and* C_1 *are* A_0*-compact,* B_1 *and* D_1 *are* D_0*-compact,* C_0 *is* A_0*-bounded with* A_0*-bound* b_C*, and* B_0 *is* D_0*-bounded with* D_0*-bound* b_B.

If $b_B b_C < 1$*, then* $\mathcal{A}_1$ *is* $\mathcal{A}_0$*-compact. If, in addition,* $\mathcal{A}_0$ *is closed, then* $\mathcal{A}_0+\mathcal{A}_1$ *is closed, and* $\sigma_{ess}(\mathcal{A}_0+\mathcal{A}_1) = \sigma_{ess}(\mathcal{A}_0)$.

With the previous result, the spectrum of the linearized monodomain operator can be described as follows.

Proposition 2.3. *The operator*

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega),$$
$$\mathcal{D}(\mathcal{A}) = \left\{ \vec{y} \in H^2(\Omega) \times L^2(\Omega) \,\middle|\, \frac{\partial y_v}{\partial \nu} = 0 \text{ on } \Gamma \right\},$$

with action

$$\mathcal{A}\vec{y} = \begin{pmatrix} \Delta y_v - (3a\bar{v}^2 - 2b\bar{v} + c)y_v - dy_w \\ \gamma y_v - \delta y_w \end{pmatrix},$$

is closed and $\sigma_{ess}(\mathcal{A}) = \sigma_c(\mathcal{A}) = \{-\delta\}$*. Moreover, its point spectrum is given as*

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda = -\frac{\delta - \mu}{2} \pm \frac{1}{2}\sqrt{(\delta + \mu)^2 - 4\gamma d},\ \mu \in \sigma_p(\Delta - (3a\bar{v}^2 - 2b\bar{v} + c)I) \right\}.$$

The corresponding eigenfunctions are complete in $L^2(\Omega) \times L^2(\Omega)$ *and are given as* $\vec{y} = (y_v, \frac{\gamma}{\delta+\lambda} y_v)$*, where,* y_v *are the eigenfunctions associated with the operator* $\Delta - (3a\bar{v}^2 - 2b\bar{v} + c)I$.

Proof. Using the splitting

$$\mathcal{A} = \underbrace{\begin{pmatrix} \Delta - (3a\bar{v}^2 - 2b\bar{v} + c) & -d \\ 0 & -\delta I \end{pmatrix}}_{\mathcal{A}_0} + \underbrace{\begin{pmatrix} 0 & 0 \\ \gamma I & 0 \end{pmatrix}}_{\mathcal{A}_1},$$

the assumptions in Theorem 2.2 (i) trivially hold. Moreover, for the boundedness constants we have $b_B b_C = 0$. By [12, Theorem B (ii)], we have the relative Δ-compactness of the constant multiplication operator γI and we can conclude that

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\mathcal{A}_0).$$

Since the self-adjoint operator $\Delta - (3a\bar{v}^2 - 2b\bar{v} + c)I$ has compact resolvent its spectrum is discrete, implying that $\sigma_{\text{ess}}(\mathcal{A}) = \{-\delta\}$. For the derivation of the point spectrum, we have to find λ and $\vec{y} \neq 0$ satisfying

$$\begin{pmatrix} \Delta y_v - (3a\bar{v}^2 - 2b\bar{v} + c)y_v - dy_w - \lambda y_v \\ \gamma y_v - \delta y_w - \lambda y_w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the second equation, we get $y_w = \frac{\gamma}{\delta+\lambda} y_v$, which, inserting into the first equation, leads to

$$\Delta y_v - (3a\bar{v}^2 - 2b\bar{v} + c)y_v - d\frac{\gamma}{\delta + \lambda} y_v - \lambda y_v = 0. \tag{7}$$

Assume now that y_v is an eigenfunction of $\Delta - (3a\bar{v}^2 - 2b\bar{v} + c)I$ for the eigenvalue μ. By multiplying (7) with $\delta + \lambda$, we thus find λ as a solution of the quadratic equation

$$\lambda^2 + \lambda(\delta - \mu) + d\gamma - \mu\delta = 0.$$

The statement follows by a simple computation of the roots for the latter equation. Completeness of $\{(y_v, \frac{\gamma}{\delta+\lambda} y_v)\}$ in $L^2(\Omega) \times L^2(\Omega)$ follows from completeness of $\{y_v\}$ in $L^2(\Omega)$ as eigenfunctions of $\Delta - (3a\bar{v}^2 - 2b\bar{v} + c)I$. $\square$

Remark 2.4. A similar derivation for $\mathcal{A}^*$ yields the same point spectrum with eigenfunctions $\vec{y} = (y_v, \frac{d}{-\delta-\lambda} y_v)$.

2.2. Lack of null controllability of the linearized system

Note that the spectrum of $\mathcal{A}$ has two accumulation points at $-\infty$ and $-\delta$. We are going to use this property to show that the system is not null-controllable, even when $B = I$, i.e. $\omega_{\text{con}} = \Omega$. First, we recall some concepts of infinite-dimensional control theory. We call a system β-stabilizable if it is exponentially stabilizable with rate $\beta > 0$.

Definition 2.5 ([35])**.** The pair $(\mathcal{A}, \mathcal{B})$ is optimizable if for every $y_0 \in \mathcal{Y}$, there exists a $u \in L^2([0, \infty), \mathcal{U})$ such that $y \in L^2([0, \infty), \mathcal{Y})$, where

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)\mathcal{B}u(\tau)\mathrm{d}\tau.$$

The pair $(\mathcal{A}, \mathcal{C})$ is estimatable if $(\mathcal{A}^*, \mathcal{C}^*)$ is optimizable.

Following [35], we have the following sufficient condition for the pair $(\mathcal{A}, \mathcal{C})$ to be estimatable.

Proposition 2.6 ([35])**.** *If $(\mathcal{A}, \mathcal{C})$ is estimatable, then there exist $\alpha > 0$ and $m > 0$ such that, for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > -\alpha$,

$$\|(sI - \mathcal{A})z\| + \|\mathcal{C}z\| \geq m\|z\| \quad \forall z \in D(\mathcal{A}).$$

We now can show that the pair $(\mathcal{A}, \mathcal{B})$ in (6) cannot be null-controllable.

Theorem 2.7. *For $\beta \geq \delta$, system* (6) *is not β-stabilizable. In particular, the pair $(\mathcal{A}, \mathcal{B})$ is not exactly null-controllable.*

Proof. Assume that the system is β-exponentially stabilizable with $\beta \geq \delta$. This implies that the pair $(\mathcal{A}^* + \beta I, \mathcal{B}^*)$ is estimatable. Due to Proposition 2.6, there exist $\alpha > 0$ and $m > 0$ such that, for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -\alpha$,

$$\|(sI - \mathcal{A}^* - \beta I)\vec{y}\| + \|\mathcal{B}^*\vec{y}\| \geq m\|\vec{y}\| \quad \forall \vec{y} \in D(\mathcal{A}^*). \tag{8}$$

Now choose $\tilde{\lambda} \in \sigma_p(\mathcal{A}^* + \beta I)$ with corresponding eigenfunction $\tilde{y} = (y_v, \frac{d}{-\delta+\beta-\tilde{\lambda}} y_v)$. We then obtain that

$$\|(\tilde{\lambda}I - \mathcal{A}^* - \beta I)\tilde{y}\| + \|\mathcal{B}^*\tilde{y}\| = \|B^* y_v\|, \qquad m\|\tilde{y}\| = m\sqrt{1 + \left|\frac{d}{-\delta + \beta - \tilde{\lambda}}\right|^2}\, \|y_v\|.$$

Since the spectrum of $\mathcal{A}^* + \beta I$ has an accumulation point at $\varepsilon = -\delta + \beta \geq 0$, we can find $\tilde{\lambda}$ such that eventually $\mathrm{Re}(\tilde{\lambda}) > -\alpha$ as well as $m\sqrt{1 + \left|\frac{d}{\varepsilon - \tilde{\lambda}}\right|^2} > \|B^*\|$. This contradicts (8) and, hence, $(\mathcal{A}^* + \beta I, \mathcal{B}^*)$ is not estimatable and $(\mathcal{A} + \beta I, \mathcal{B})$ is not optimizable. Since stabilizability ([35, Remark 1]) as well as null-controllability imply optimizability (see e.g. [25, Theorem 3.14]), we conclude that $(\mathcal{A} + \beta I, \mathcal{B})$ can neither be null-controllable nor stabilizable. The latter means $(\mathcal{A}, \mathcal{B})$ is not β-stabilizable. Finally, null-controllability of $(\mathcal{A}+\beta I, \mathcal{B})$ is equivalent to null-controllability of $(\mathcal{A}, \mathcal{B})$, see e.g. [33, Remark 11.4], which proves the assertion.

$\square$

2.3. Stabilization by finite-rank input and output operators

While we have seen that the linearized system is only β-exponentially stabilizable as long as $\beta < \delta$, for these values of β, we now investigate stabilizability by finite-rank input and output operators, respectively. Hence, we assume that $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^m, \mathcal{Y})$ and, for fixed t, it has the form

$$\mathcal{B}u(t) = \begin{bmatrix} \sum_{i=1}^m \chi_{|\tilde{\omega}_i} u_i(t) \\ 0 \end{bmatrix},$$

where $\chi_{|\tilde{\omega}_i}$ denotes the characteristic function for the set $\tilde{\omega}_i \subset \Omega$.

Following [17], in order to study exponential stabilizability of $(\mathcal{A}, \mathcal{B})$ and exponential detectability of $(\mathcal{A}, \mathcal{C})$, for $\alpha \in \mathbb{R}$ we decompose the spectrum of $\mathcal{A}$ into two disjoint sets:

$$\sigma_\alpha^+(\mathcal{A}) := \sigma(\mathcal{A}) \cap \overline{\mathbb{C}_\alpha^+}; \quad \mathbb{C}_\alpha^+ = \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) > \alpha\},$$
$$\sigma_\alpha^-(\mathcal{A}) := \sigma(\mathcal{A}) \cap \mathbb{C}_\alpha^-; \quad \mathbb{C}_\alpha^- = \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) < \alpha\}.$$

As in [17, 19, 28], our goal is to decompose the state space $\mathcal{Y}$ into a stable part $\mathcal{Y}_-$ and an unstable part $\mathcal{Y}_+$. Provided that the dimension of $\mathcal{Y}_+$ is finite, stabilizability and detectability can then be checked by means of well known finite-dimensional criteria.

Definition 2.8 ([17])**.** $\mathcal{A}$ satisfies the spectrum decomposition assumption at α if $\sigma_\alpha^+(\mathcal{A})$ is bounded and separated from $\sigma_\alpha^-(\mathcal{A})$ in such a way that a rectifiable, simple, closed curve, Γ_α, can be drawn so as to enclose an open set containing $\sigma_\alpha^+(\mathcal{A})$ in its interior and $\sigma_\alpha^-(\mathcal{A})$ in its exterior.

We now define the spectral projection

$$P_\alpha \vec{y} = \frac{1}{2\pi\imath} \int_{\Gamma_\alpha} (\lambda I - \mathcal{A})^{-1} \vec{y} \, \mathrm{d}\lambda$$

and decompose our system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ into the sum of the two subsystems $(\mathcal{A}_\alpha^+, \mathcal{B}_\alpha^+, \mathcal{C}_\alpha^+)$ and $(\mathcal{A}_\alpha^-, \mathcal{B}_\alpha^-, \mathcal{C}_\alpha^-)$, where $\mathcal{A}_\alpha^+ = P_\alpha \mathcal{A}, \mathcal{A}_\alpha^- = (I - P_\alpha)\mathcal{A}, \mathcal{B}_\alpha^+ = P_\alpha \mathcal{B}, \mathcal{B}_\alpha^- = (I - P_\alpha)\mathcal{B}, \mathcal{C}_\alpha^+ = \mathcal{C} P_\alpha$ and $\mathcal{C}_\alpha^- = \mathcal{C}(I - P_\alpha)$. Since this approach is quite common in the literature, for more details, we refer to, e.g., [17, 19, 28].

For our purposes, it is important to note that, by Proposition 2.3, $\mathcal{A}$ indeed satisfies the spectrum decomposition assumption at $-\beta$ for $\beta < \delta$. Moreover, since the only accumulation points of $\sigma(\mathcal{A})$ are $-\infty$ and $-\delta$, we conclude that $\sigma_{-\beta}^+(\mathcal{A})$ is finite. For the rest of the paper, we make the following two assumptions:

(A1) $\mathrm{ran}(sI - \mathcal{A}) + \mathrm{ran}(\mathcal{B}) = \mathcal{Y}$, for $s \in \overline{\mathbb{C}_{-\beta}^+}$.
(A2) $\ker(sI - \mathcal{A}) \cap \ker(\mathcal{C}) = \{0\}$, for $s \in \overline{\mathbb{C}_{-\beta}^+}$.

With Theorem [17, Theorem 5.2.11], we find that the linearized system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is β-exponentially stabilizable by finite-rank input operators as long as $\beta < \delta$. Similarly it is exponentially detectable for finite-rank output operators. In Section 5, we show that assumptions (A1) and (A2) are actually fulfilled for our numerical examples.

3. LQG compensator design

For the linearized system (6), let us define the quadratic cost functional

$$J(\vec{y}_0; u) = \int_0^\infty \|y_{\mathrm{obs}}(t)\|_{\mathbb{R}^p}^2 + \langle u(t), Ru(t) \rangle_{\mathbb{R}^m} \, \mathrm{d}t, \tag{9}$$

with weight matrix $R = R^T \succ 0$ and $y_{\text{obs}}(t)$ as in (6b). Under the assumptions (A1) and (A2), it is well-known ([17, Theorem 6.2.7]) that the optimal control minimizing (9) is determined by the unique nonnegative solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{L}(\mathcal{Y})$ to the algebraic operator Riccati equation

$$\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{C}^*\mathcal{C} - \mathcal{P}\mathcal{B}R^{-1}\mathcal{B}^*\mathcal{P} = 0. \tag{10}$$

Similarly, we have the existence of a unique solution $\mathcal{Q} = \mathcal{Q}^* \succeq 0$ to

$$\mathcal{A}\mathcal{Q} + \mathcal{Q}\mathcal{A}^* + \mathcal{B}\mathcal{B}^* - \mathcal{Q}\mathcal{C}^*\mathcal{C}\mathcal{Q} = 0. \tag{11}$$

Denoting $\mathcal{K} = -R^{-1}\mathcal{B}^*\mathcal{P} \in \mathcal{L}(\mathcal{Y}, \mathbb{R}^m)$ and $\mathcal{H} = -\mathcal{Q}\mathcal{C}^* \in \mathcal{L}(\mathbb{R}^p, \mathcal{Y})$ it moreover follows that $\mathcal{A}+\mathcal{B}\mathcal{K}$ and $\mathcal{A}+\mathcal{H}\mathcal{C}$ are infinitesimal generators of exponentially stable semigroups. Hence, with [17, Theorem 5.3.3] we obtain an exponentially stabilizing compensator of the form

$$\begin{aligned} \dot{\vec{y}}(t) &= \mathcal{A}\vec{y}(t) + \mathcal{B}\mathcal{K}\vec{z}(t), \\ \dot{\vec{z}}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K} + \mathcal{H}\mathcal{C})\vec{z}(t) - \mathcal{H}y_{\text{obs}}(t). \end{aligned} \tag{12}$$

3.1. **Local stabilization of the nonlinear system**

Following [29, 32], let us consider the effect of the feedback law obtained by the linearized system when applied to the full nonlinear system. We have the nonlinear closed-loop system

$$\begin{aligned} \dot{\vec{y}}(t) &= \mathcal{A}\vec{y}(t) + \mathcal{B}\mathcal{K}\vec{z}(t) + \mathcal{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0, \\ \dot{\vec{z}}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K} + \mathcal{H}\mathcal{C})\vec{z}(t) - \mathcal{H}y_{\text{obs}}(t), \quad \vec{z}(0) = \vec{z}_0, \end{aligned} \tag{13}$$

with $\mathcal{F}$ denoting the nonlinearity from (5). In order to obtain the (local) existence of a unique solution to (13), we need to consider the nonhomogeneous equation

$$\begin{aligned} \dot{\vec{y}}(t) &= \mathcal{A}\vec{y}(t) + \mathcal{B}\mathcal{K}\vec{z}(t) + \vec{f}(t), \quad \vec{y}(0) = \vec{y}_0, \\ \dot{\vec{z}}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K} + \mathcal{H}\mathcal{C})\vec{z}(t) - \mathcal{H}y_{\text{obs}}(t), \quad \vec{z}(0) = \vec{z}_0, \end{aligned} \tag{14}$$

where $\vec{f} = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. With similar arguments as in [11], we have the following result.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ ($\Omega \subset \mathbb{R}^3$) with $\varepsilon \in (\frac{1}{2}, 1]$ ($\varepsilon = 1$). If $f_1 \in L^2(0,\infty; H^{\varepsilon-1}(\Omega))$, $\mu \in L^2(0,\infty;\mathbb{R}^p)$, $\vec{y}_0 \in H^\varepsilon(\Omega) \times H^{1+\varepsilon}(\Omega)$ and $\vec{z}_0 \in H^\varepsilon(\Omega) \times H^\varepsilon(\Omega)$, then (14) has a unique solution*

$$\begin{aligned} (\vec{y}, \vec{z}) \in &\left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)\right) \times \left(C_b([0,\infty); H^{1+\varepsilon}(\Omega)) \cap H^1(0,\infty; H^{1+\varepsilon}(\Omega))\right) \\ &\times \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)\right) \times \left(H^1(0,\infty; L^2(\Omega))\right) \end{aligned}$$

satisfying

$$\begin{aligned} &\|(\vec{y}, \vec{z})\|_{\left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)\right) \times (H^1(0,\infty;H^{1+\varepsilon}(\Omega))) \times \left(H^{1+\varepsilon, \frac{1+\varepsilon}{2}}(Q_\infty)\right) \times (H^1(0,\infty;L^2(\Omega)))} \\ &\quad \le C_2 \left(\|\vec{y}_0\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)} + \|\vec{z}_0\|_{H^\varepsilon(\Omega)\times H^\varepsilon(\Omega)} + \|f_1\|_{L^2(0,\infty;H^{\varepsilon-1}(\Omega))} + \|\mu\|_{L^2(0,\infty;\mathbb{R}^p)} \right). \end{aligned}$$

Proof. The proof follows to a large extent the arguments provided for analogous results obtained in, e.g., [11,32]. However, below we briefly recapitulate the most important steps. The analysis of the involved interpolation spaces ([11, Theorem 4.7, Step 1]) can be transferred to our setting. The role of the operator $\mathcal{A}$ in [11] is now taken by

$$\mathcal{A}_{\text{ex}} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{bmatrix}.$$

As mentioned above, the operator

$$\mathcal{A}_{\text{LQG}} = \begin{bmatrix} \mathcal{A} & \mathcal{BK} \\ -\mathcal{HC} & \mathcal{A}+\mathcal{BK}+\mathcal{HC} \end{bmatrix}$$

is the infinitesimal generator of an exponentially stable semigroup. It moreover is a bounded perturbation of $\mathcal{A}_{ex}$ and thus the semigroup is analytic. With [32, Lemma 4.2] we first conclude that $(\vec{y},\vec{z}) \in L^2(Q_\infty) \times L^2(Q_\infty) \times L^2(Q_\infty) \times L^2(Q_\infty)$. Based on the splitting $\vec{y} = \vec{y}_1 + \vec{y}_2$ and $\vec{z} = \vec{z}_1 + \vec{z}_2$, where

$$\begin{aligned}
\dot{\vec{y}}_1 &= (\mathcal{A} - \lambda_0 I)\vec{y}_1 + \vec{f} + \lambda_0 \vec{y}, && \vec{y}_1(0) = \vec{y}_0, \\
\dot{\vec{z}}_1 &= (\mathcal{A} - \lambda_0 I)\vec{z}_1 - \mathcal{H}\mu + \lambda_0 \vec{z}, && \vec{z}_1(0) = \vec{z}_0, \\
\dot{\vec{y}}_2 &= (\mathcal{A} - \lambda_0 I)\vec{y}_2 + \mathcal{BK}\vec{z}, && \vec{y}_2(0) = 0, \\
\dot{\vec{z}}_2 &= (\mathcal{A} - \lambda_0 I)\vec{z}_2 + (\mathcal{BK}+\mathcal{HC})\vec{z} - \mathcal{HC}\vec{y}, && \vec{z}_2(0) = 0,
\end{aligned}$$

we conclude that

$$\vec{y},\vec{z} \in L^2(0,\infty; H_\nu^{1+\varepsilon}(\Omega) \times L^2(\Omega)) \cap H^1(0,\infty; H^{\varepsilon-1}(\Omega) \times L^2(\Omega)).$$

As we did in step 4 in [11], we can utilize the special structure of $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$. Note that we assumed the control to be restricted to the PDE variable. Hence, by using the explicit solution formula for y_w and $y_w(0) \in H^{1+\varepsilon}(\Omega)$, we even get that $y_w \in C_b([0,\infty); H^{1+\varepsilon}(\Omega)) \cap H^1(0,\infty; H^{1+\varepsilon}(\Omega))$.

□

Remark 3.2. Note that we cannot obtain more regularity for z_w due to the properties of the operator $\mathcal{HC}$ which involves $\mathcal{Q} \in \mathcal{L}(\mathcal{Y})$.

Combining the local Lipschitz continuity properties ([11, Lemma 4.4/4.5]) of the nonlinear operator $\mathcal{F}$, a fixed point argument can be employed in order to show local asymptotic stability of (13).

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2 (\Omega \subset \mathbb{R}^3)$ and $\varepsilon \in (\frac{1}{2}, 1]$ $(\varepsilon = 1)$. Then there exist $C_0 < 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $C \in (0, C_0)$ and*

$$\|(\vec{y}_0,\vec{z}_0)\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)\times H^\varepsilon(\Omega)\times H^\varepsilon(\Omega)} + \|\mu\|_{L^2(0,\infty;\mathbb{R}^p)} \le \eta(C),$$

then (13) admits a unique solution satisfying

$$\|(\vec{y},\vec{z})\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))\times H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;L^2(\Omega))} \le C.$$

Having established local asymptotic stability, we now extend the results to the case of exponential stabilization.

3.2. Local exponential stabilization

Instead of (13), assume that the following closed-loop system is given

$$\begin{aligned}
\dot{\tilde{y}}(t) &= (\mathcal{A} + \beta I)\tilde{y}(t) + \mathcal{B}\tilde{\mathcal{K}}\tilde{z}(t) + \tilde{\mathcal{F}}(\tilde{y}(t)), \quad \tilde{y}(0) = \tilde{y}_0, \\
\dot{\tilde{z}}(t) &= (\mathcal{A} + \mathcal{B}\tilde{\mathcal{K}} + \tilde{\mathcal{H}}\mathcal{C} + \beta I)\tilde{z}(t) - \tilde{\mathcal{H}}\tilde{y}_{\text{obs}}(t), \quad \tilde{z}(0) = \tilde{z}_0,
\end{aligned} \tag{15}$$

where

$$\tilde F(\tilde y(t)) = \begin{pmatrix} -e^{-2\beta t} a \tilde y_v^3 - e^{-\beta t}(-b+3a\bar v)\tilde y_v^2 \\ 0 \end{pmatrix}, \quad \tilde y_{\text{obs}}(t) = \mathcal{C}\tilde y(t) + \tilde\mu(t), \quad \tilde\mu(t) = e^{\beta t}\mu(t)$$

and the feedback and observer gains $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{H}}$ are implicitly determined by the shifted operator Riccati equations

$$\begin{aligned}(\mathcal{A}^* + \beta I)\tilde{\mathcal{P}} + \tilde{\mathcal{P}}(\mathcal{A}+\beta I) + \mathcal{C}^*\mathcal{C} - \tilde{\mathcal{P}}\mathcal{B}R^{-1}\mathcal{B}^*\tilde{\mathcal{P}} = 0,\\ (\mathcal{A}+\beta I)\tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}(\mathcal{A}^*+\beta I) + \mathcal{B}\mathcal{B}^* - \tilde{\mathcal{Q}}\mathcal{C}^*\mathcal{C}\tilde{\mathcal{Q}} = 0.\end{aligned}$$

As we already indicated in [11], as long as $\beta < \delta$, all previous arguments can be utilized to show local asymptotic stability of (15). In particular, this implies local exponential stability of (13). We summarize the precise result as follows.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^2 (\Omega \subset \mathbb{R}^3)$ and $\varepsilon \in (\frac{1}{2}, 1]$ $(\varepsilon = 1)$. Then for $\beta < \delta$ there exist $C_0 < 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $C \in (0, C_0)$ and*

$$\|(\vec y_0, \vec z_0)\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)\times H^\varepsilon(\Omega)\times H^\varepsilon(\Omega)} + \|e^{\beta t}\mu\|_{L^2(0,\infty;\mathbb{R}^p)} \le \eta(C),$$

then (13) admits a unique solution satisfying

$$\|(\vec y, \vec z)\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))\times H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;L^2(\Omega))} \le Ce^{-\beta t}.$$

With regard to practical applicability of a locally exponentially stabilizing compensator, the question of finite-dimensional compensator arises. Below we address this problem by using the method of LQG-balanced truncation.

4. Finite-dimensional and reduced order compensators

The practical realization of the LQG approach is still impeded by the infinite-dimensionality of the system. An obvious first step consists in introducing a finite-dimensional approximation based on finite elements, for example. The resulting system, however, is still high dimensional. For this reason we propose the use of reduced order compensator dynamics. To start with the FEM approximation, consider the finite-dimensional control system of the form

$$\dot y^n(t) = A^n y^n(t) + B^n u(t), \quad y^n_{\text{obs}}(t) = C^n y^n(t), \tag{16}$$

with (A^n, B^n, C^n) to be defined below. Proceeding in this manner, the next step might consist in designing a stabilizing compensator for the finite-dimensional system. However, the approximation step has to be done carefully such that the finite-dimensional compensator is still stabilizing for the original infinite-dimensional system. With this in mind, we have to specify a uniform structure preserving approximation. The following properties have been shown to be essential when designing a robust finite-dimensional controller, see [15].

(A3) Y^n is a sequence of finite-dimensional subspaces of $\mathcal{Y}$ and Π^n is the orthogonal projection of $\mathcal{Y}$ onto Y^n such that

$$\begin{aligned}\Pi^n\vec y \to \vec y \ \text{ as } n\to\infty \ \text{ for all } \vec y \in \mathcal{Y},\\ B^n = \Pi^n\mathcal{B}, \quad C^n = \mathcal{C}_{|Y^n};\end{aligned}$$

(A4) $A^n \in \mathcal{L}(Y^n)$ and for each $\vec y \in \mathcal{Y}$ there holds
 (i) $e^{A^n t}\Pi^n\vec y \to S(t)\vec y$,
 (ii) $(e^{A^n t})^*\Pi^n\vec y \to S(t)^*\vec y$
uniformly in t on bounded intervals as $n\to\infty$.

(A5) (A^n, B^n) is uniformly exponentially stabilizable; i.e., there exists a uniformly bounded sequence of operators $K^n \in \mathcal{L}(Y^n, \mathbb{R}^m)$ such that

$$\left\| e^{(A^n - B^n K^n)t} \Pi^n \vec{y} \right\|_{\mathcal{Y}} \leq M_1 e^{-\alpha_1 t} \|\vec{y}\|_{\mathcal{Y}}$$

for some positive constants $M_1 \geq 1$ and α_1;

(A6) (A^n, C^n) is uniformly exponentially detectable; i.e., there exists a uniformly bounded sequence of operators $H^n \in \mathcal{L}(\mathbb{R}^p, Y^n)$ such that

$$\left\| e^{(A^n - H^n C^n)t} \Pi^n \vec{y} \right\|_{\mathcal{Y}} \leq M_2 e^{-\alpha_2 t} \|\vec{y}\|_{\mathcal{Y}}$$

for some positive constants $M_2 \geq 1$ and α_2.

The applicability of FEM discretizations has been discuseed in [3]. We can therefore construct an LQG-based compensator for the linearized finite-dimensional system (16) by solving the two finite-dimensional Riccati equations

$$\begin{aligned} (A^n)^T P^n + P^n A^n + C^n (C^n)^T - P^n B^n R^{-1} (B^n)^T P^n = 0, \\ A^n Q^n + Q^n (A^n)^T + B^n (B^n)^T - Q^n (C^n)^T C^n Q^n = 0, \end{aligned} \tag{17}$$

for the unknowns P^n and Q^n.

Unfortunately, finite-element discretizations often result in finite-dimensional but very large scale systems making the computation of P^n and Q^n a formidable task on its own. Moreover, the simulation time for the resulting n-dimensional LQG compensator

$$\dot{z}^n(t) = (A^n + B^n K^n + H^n C^n) z^n(t) - H^n C^n y_{\mathrm{obs}}(t), \quad u(t) = K^n z^n(t), \tag{18}$$

where $K^n = -R^{-1}(B^n)^T P^n$ and $H^n = -Q^n (C^n)^T$ might exceed reasonable time scales due to the large system dimension. We are thus interested in constructing an r-dimensional reduced compensator

$$\dot{z}_r^n(t) = A_r^n z_r^n(t) + B_r^n y_{\mathrm{obs}}(t), \quad u(t) = C_r^n z_r^n(t), \tag{19}$$

such that $r \ll n$. There are two issues that have to be addressed to obtain (19). First, how does one compute the reduced compensator? Second, how to ensure that it is indeed stabilizing the infinite-dimensional system? Both question have been investigated in [15] and we briefly recapitulate the necessary concepts presented therein.

The design procedure from [15] basically consists of a model order reduction step, followed by a compensator design step for the already reduced system. For finite-dimensional linear control systems, the topic of model order reduction can be considered well understood. For a detailed overview of different reduction techniques, we refer to [2]. Many of these finite-dimensional concepts have been generalized to the infinite-dimensional setting, see e.g. [5,15,18,24]. Since we allow the linearized system (6a) to be (possibly) unstable, the method of balanced truncation is in general not applicable. Instead, we want to construct a reduced system approximating (6a) by the LQG-balanced truncation method.

For this, we first have to define an important class of infinite-dimensional systems.

Definition 4.1 ([15])**.** A system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a Pritchard-Salamon system with respect to the Hilbert spaces V, W if the following hold:

(i) $W \hookrightarrow V$.

(ii) $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on V which restricts to a strongly continuous semigroup on W.

(iii) $\mathcal{B} \in \mathcal{L}(\mathbb{C}^m, V)$ and there exist t_1, α such that

$$\int_0^{t_1} S(t_1-\tau)\mathcal{B}u(\tau)\mathrm{d}t \in W \quad \text{and}$$
$$\left\| \int_0^{t_1} S(t_1-\tau)\mathcal{B}u(\tau)\mathrm{d}t \right\|_W \le \alpha \|u\|_{L^2(0,t_1;\mathbb{C}^m)} \quad \text{for all } u \in L^2(0,t_1;\mathbb{C}^m).$$

(iv) $\mathcal{C} \in \mathcal{L}(W, \mathbb{C}^p)$ and there exist $t_2, \beta > 0$ such that

$$\|\mathcal{C}S(\cdot)z\|_{L^2(0,t_2;\mathbb{C}^p)} \le \beta \|z\|_V \quad \text{for all } z \in W.$$

The central idea behind LQG-balanced truncation is to balance certain system Gramians in the following sense.

Definition 4.2. A Pritchard-Salamon system $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is called an LQG-balanced realization of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ if there exist bounded, self-adjoint, nonnegative solutions $\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}$ to its control and filter Riccati equations

$$\begin{aligned} \tilde{\mathcal{A}}^*\tilde{\mathcal{P}} + \tilde{\mathcal{P}}\tilde{\mathcal{A}} - \tilde{\mathcal{P}}\tilde{\mathcal{B}}\tilde{\mathcal{B}}^*\tilde{\mathcal{P}} + \tilde{\mathcal{C}}^*\tilde{\mathcal{C}} = 0, \\ \tilde{\mathcal{A}}\tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}\tilde{\mathcal{A}}^* - \tilde{\mathcal{Q}}\tilde{\mathcal{C}}^*\tilde{\mathcal{C}}\tilde{\mathcal{Q}} + \tilde{\mathcal{B}}\tilde{\mathcal{B}}^* = 0. \end{aligned} \tag{20}$$

such that $\tilde{\mathcal{P}} = \tilde{\mathcal{Q}} = \Lambda$ is a diagonal operator.

For exponentially stabilizable systems with finite-rank input and output-operators, [15, Theorem 4.8] shows that there always exists an LQG-balanced realization $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Moreover, partitioning the transformed LQG-balanced system as follows

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A}_r & * \\ * & * \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} \tilde{B}_r \\ * \end{bmatrix}, \quad \tilde{\mathcal{C}} = \begin{bmatrix} \tilde{C}_r & * \end{bmatrix} \tag{21}$$

yields a reduced order system that converges to the infinite-dimensional system in the gap topology. Here, $\tilde{A}_r, \tilde{B}_r$ and $\tilde{C}_r$ are operators that allow for a matrix representation of dimension r.

Theorem 4.3 ([15])**.** *The transfer functions G_r of the LQG-balanced truncations $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ converge in the gap topology as $r \to \infty$ to G, the transfer function of $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.*

Remark 4.4. Convergence in the gap topology has to be understood in terms of normalized left-coprime factorizations of the system $G(s)$ in the form of $G(s) = M(s)^{-1}N(s)$. As in [15, 17], this means that instead of an exponentially stabilizable and detectable linear system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, we consider the exponentially stabilized system

$$(\mathcal{A} - \mathcal{P}\mathcal{C}^*\mathcal{C}, [\mathcal{B}, -\mathcal{P}\mathcal{C}^*], \mathcal{C}, [0, I]).$$

The latter system has the transfer function $[N(s), M(s)]$ with

$$N(s) = \mathcal{C}(sI - \mathcal{A} + \mathcal{P}\mathcal{C}^*\mathcal{C})^{-1}\mathcal{B}, \quad M(s) = I - \mathcal{C}(sI - \mathcal{A} + \mathcal{P}\mathcal{C}^*)^{-1}\mathcal{P}\mathcal{C}^*$$

and $M(s), N(s)$ further satisfy a Bezout identity and a normalization property. For more details on left-coprime factorizations, see [17, Chapter 7]. The convergence result from Theorem 4.3 can be interpreted as convergence of $\|[N_r, M_r] - [N, M]\|_\infty$.

A realization of the above procedure requires the solutions of the operator Riccati equations (20) in order to perform the balancing step. Numerically, this is achieved by a discretization of the system such that assumptions (A3)-(A6) are fulfilled. We then have to solve the finite-dimensional Riccati equations (17) and compute an LQG-balanced reduced-order system. For the precise computation of the required projection matrices, we refer to [6], where a similar approach has been studied for the linearized Navier-Stokes equations.

4.1. Robust finite-dimensional controller design

After having obtained the finite-dimensional reduced-order system $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$, we have to find a controller (A_r^n, B_r^n, C_r^n) such that the closed-loop system

$$\begin{aligned}\dot{\tilde{y}}_r(t) &= \tilde{A}_r\tilde{y}_r(t) + \tilde{B}_r C_r^n z_r^n(t),\\ \dot{z}_r^n(t) &= A_r^n z_r^n(t) + B_r^n\tilde{C}_r\tilde{y}_r(t),\end{aligned}$$

is (robustly) stabilized. Again, for finite-rank input and output operators as considered here, in [15], it was shown that this can be done in such a way that (A_r^n, B_r^n, C_r^n) is stabilizing $(\mathcal{A},\mathcal{B},\mathcal{C})$ as well. The precise result is as follows.

Theorem 4.5 ([15])**.** *Under the assumptions (A3)-(A6), given a positive $\varepsilon < \sqrt{1-\sigma_1^2}$, we can always find two integers $r \ll n$ such that there is a controller (A_r^n, B_r^n, C_r^n) that stabilizes the system $(\mathcal{A},\mathcal{B},\mathcal{C})$ with a robustness margin with respect to left-coprime factor perturbations of $\varepsilon - \sqrt{1-\sigma_1^2}$.*

Note that in [15], stabilizing is meant in the sense of input-output stability as in [17, Definition 9.1.2]. For the special situation considered here, this also implies exponential stability of the closed-loop semigroup generated by its infinitesimal generator

$$\mathcal{A}_{cl} = \begin{bmatrix} \mathcal{A} & \mathcal{B}C_r^n \\ \mathcal{C}B_r^n & A_r^n \end{bmatrix}.$$

Moreover, an explicit numerical procedure for computing (A_r^n, B_r^n, C_r^n) is given, see [15, Section 4.5.1].

4.2. Local exponential stabilization

Finally, we consider the reduced r-dimensional controller from Theorem 4.5 when used as a compensator for the nonlinear system

$$\begin{aligned}\dot{\vec{y}}(t) &= \mathcal{A}\vec{y}(t) + \mathcal{B}C_r^n z_r^n(t) + \mathcal{F}(\vec{y}(t)), \quad \vec{y}(0) = \vec{y}_0,\\ \dot{z}_r^n(t) &= A_r^n z_r^n(t) + B_r^n y_{\mathrm{obs}}(t), \quad z_r^n(0) = z_{r,0}^n,\end{aligned} \tag{22}$$

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^2 (\Omega \subset \mathbb{R}^3)$ and $\varepsilon \in (\frac{1}{2}, 1]$ $(\varepsilon = 1)$. Then for $\beta < \delta$ there exist $C_0 < 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $C \in (0, C_0)$ and*

$$\|(\vec{y}_0, z_{r,0}^n)\|_{H^\varepsilon(\Omega)\times H^{1+\varepsilon}(\Omega)\times\mathbb{R}^r} + \|e^{\beta t}\mu\|_{L^2(0,\infty;\mathbb{R}^p)} \le \eta(C),$$

then (22) *admits a unique solution in the set*

$$\begin{aligned}\Big\{(\vec{y}, z_r^n) \in \Big(H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\Big) &\times \big(C_b([0,\infty); H^{1+\varepsilon}(\Omega)) \cap H^1(0,\infty; H^{1+\varepsilon}(\Omega))\big)\\ &\times \big(C_b([0,\infty);\mathbb{R}^r) \cap H^1(0,\infty;\mathbb{R}^r)\big)\\ &\Big| \|(\vec{y}, z_r^n)\|_{H^{1+\varepsilon,\frac{1+\varepsilon}{2}}(Q_\infty)\times H^1(0,\infty;H^{1+\varepsilon}(\Omega))\times H^1(0,\infty;\mathbb{R}^r)} \le Ce^{-\beta t}\Big\}.\end{aligned}$$

Proof. The arguments required to show the assertion follow the lines of those used for similar results in, e.g., [11, 27, 32]. The main idea is based on a contraction principle and again requires local Lipschitz continuity of the nonlinear operator $\mathcal{F}$. $\square$

5. Numerical examples

In our numerical examples, we consider the following version of the monodomain equations

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \alpha \Delta v - I_{\text{ion}}(v,w) + Bu, && \text{in } \Omega \times (0,T), \\
\frac{\partial w}{\partial t} &= \frac{\eta_2}{v_{pk}} v - \eta_2 \eta_3 w, && \text{in } \Omega \times (0,T), \\
\frac{\partial v}{\partial \nu} &= 0, && \text{on } \Gamma \times (0,T), \\
v(x,0) &= v_0(x) \text{ and } w(x,0) = w_0(x), && \text{in } \Omega,
\end{aligned} \tag{23}$$

where $\Omega = (0,1) \times (0,1)$. For the ionic current we use the FitzHugh-Nagumo model

$$I_{\text{ion}}(v,w) = \eta_0 v + \eta_1 w - \eta_0 \left(\frac{1}{v_{th}} + \frac{1}{v_{pk}} \right) v^2 + \frac{\eta_0}{v_{th} v_{pk}} v^3.$$

The parameters are $\alpha = 0.0015, \eta_0 = 1.614, \eta_1 = 215.6, \eta_2 = 0.015, \eta_3 = 1, v_{th} = 13$ and $v_{pk} = 100$. The control operator is chosen as

$$Bu(t) = \sum_{i=1}^{8} \chi_{|\tilde{\omega}_i} u_i(t), \tag{24}$$

with control domains

$$\begin{aligned}
\tilde{\omega}_1 &= \left\{ (x_1,x_2) \in \left[\frac{5}{32}, \frac{9}{32} \right] \times \left[\frac{4}{32}, \frac{8}{32} \right] \right\}, \quad & \tilde{\omega}_2 &= \left\{ (x_1,x_2) \in \left[\frac{4}{32}, \frac{8}{32} \right] \times \left[\frac{23}{32}, \frac{27}{32} \right] \right\}, \\
\tilde{\omega}_3 &= \left\{ (x_1,x_2) \in \left[\frac{3}{32}, \frac{7}{32} \right] \times \left[\frac{14}{32}, \frac{18}{32} \right] \right\}, \quad & \tilde{\omega}_4 &= \left\{ (x_1,x_2) \in \left[\frac{24}{32}, \frac{28}{32} \right] \times \left[\frac{14}{32}, \frac{18}{32} \right] \right\}, \\
\tilde{\omega}_5 &= \left\{ (x_1,x_2) \in \left[\frac{14}{32}, \frac{18}{32} \right] \times \left[\frac{4}{32}, \frac{8}{32} \right] \right\}, \quad & \tilde{\omega}_6 &= \left\{ (x_1,x_2) \in \left[\frac{14}{32}, \frac{18}{32} \right] \times \left[\frac{24}{32}, \frac{28}{32} \right] \right\}, \\
\tilde{\omega}_7 &= \left\{ (x_1,x_2) \in \left[\frac{24}{32}, \frac{28}{32} \right] \times \left[\frac{4}{32}, \frac{8}{32} \right] \right\}, \quad & \tilde{\omega}_8 &= \left\{ (x_1,x_2) \in \left[\frac{25}{32}, \frac{29}{32} \right] \times \left[\frac{24}{32}, \frac{28}{32} \right] \right\}.
\end{aligned}$$

For system (23) we use the observation operator

$$v_{\text{obs}}(t) = \mathcal{C}(v(t), w(t)) = Cv(t) := \left(\tfrac{1}{|\omega_1|} \int_{\omega_1} v(t), \ldots, \tfrac{1}{|\omega_8|} \int_{\omega_8} v(t) \right) \in \mathbb{R}^8, \tag{25}$$

where ω_i are squares covering $\tilde{\omega}_i$ with an extra $\frac{1}{32}$ margin all around. The control and observation configurations are also shown in Figure 1. We point out that using observation domains on or next to the control domains highly improved the stabilization results compared to using non overlapping control and observation domains. For the classical LQG approach we were also able to stabilize all of the presented scenarios even with disjoint control and observation domains.

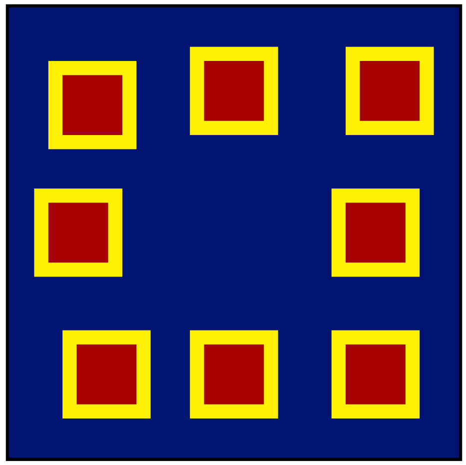


Figure 1. Control (red) and observation (yellow) domains.

5.1. **Stabilizability and detectability properties**

For the underlying ODE of (23), consider a stationary solution to

$$\begin{aligned} 0 &= \eta_0 v + \eta_1 w - \eta_0 \left(\frac{1}{v_{th}} + \frac{1}{v_{pk}} \right) v^2 + \frac{\eta_0}{v_{th} v_{pk}} v^3, \\ 0 &= \frac{\eta_2}{v_{pk}} v - \eta_2 \eta_3 w. \end{aligned} \tag{26}$$

One easily verifies that one solution $(\bar{v}, \bar{w})$ to (26) is given as

$$\bar{v} = \frac{v_{pk} + v_{th}}{2} + \frac{1}{2} \sqrt{(v_{pk} - v_{th})^2 - 4 \frac{\eta_1 v_{th}}{\eta_0 \eta_3}} \approx 68.98, \quad \bar{w} \approx 0.0698. \tag{27}$$

For the stabilization concepts from Theorem 3.4 and Theorem 4.6 we are particularly interested in unstable stationary solutions. In order to investigate stability of $(\bar{v}, \bar{w})$, we thus need the spectrum of $\mathcal{A}$ as in (5). According to Proposition 2.3, we can analytically specify $\sigma_p(\mathcal{A})$ by means of the point spectrum of

$$A_v := \alpha \Delta - 3 \frac{\eta_0}{v_{th} v_{pk}} \bar{v}^2 I + 2\eta_0 \left(\frac{1}{v_{th}} + \frac{1}{v_{pk}} \right) \bar{v} I - \eta_0 I.$$

Since $\bar{v}$ is constant, we conclude that

$$\sigma_p(A_v) = \left\{ \mu \in \mathbb{C} \,\middle|\, \mu = -\alpha \pi^2 (i^2 + j^2) - 3 \frac{\eta_0}{v_{th} v_{pk}} \bar{v}^2 + 2\eta_0 \left(\frac{1}{v_{th}} + \frac{1}{v_{pk}} \right) \bar{v} - \eta_0, \ i, j \in \mathbb{N}_0 \right\}.$$

Considering now the rightmost eigenvalue

$$\bar{\mu} = -3 \frac{\eta_0}{v_{th} v_{pk}} \bar{v}^2 + 2\eta_0 \left(\frac{1}{v_{th}} + \frac{1}{v_{pk}} \right) \bar{v} - \eta_0,$$

by Proposition 2.3, the associated eigenvalues $\bar{\lambda}_\pm$ of $\mathcal{A}$ are

$$\bar{\lambda}_\pm = -\frac{\delta - \bar{\mu}}{2} \pm \frac{1}{2} \sqrt{(\delta + \bar{\mu})^2 - 4\gamma d} \approx 0.0019 \pm 0.1790\imath.$$

The stationary solution $(\bar{v}, \bar{w})$ thus is unstable. These two eigenvalues are the only ones with positive real part. As a consequence, stabilizability of (23) can be investigated by checking assumptions (A1) and (A2) for $\bar{\lambda}_\pm$.

Proposition 5.1. *The pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}$ as in (24) is stabilizable.*

Proof. Instead of stabilizability of $(\mathcal{A}, \mathcal{B})$, we are going to show detectability of $(\mathcal{A}^*, \mathcal{B}^*)$. Since the eigenvectors of A_v are of the form $y(x_1, x_2) = \cos(i\pi x_1)\cos(j\pi x_2)$, by Remark 2.4, the eigenvectors associated with the unstable eigenvalues $\bar{\lambda}_\pm$ are $\vec{z}_\pm = \left(1, \frac{d}{-\delta - \bar{\lambda}_\pm} 1\right)$. Hence, we obtain that

$$\mathcal{B}^* \vec{z}_\pm = \left(\int_{\tilde{\omega}_1} 1 \, \mathrm{d}x, \ldots, \int_{\tilde{\omega}_8} 1 \, \mathrm{d}x\right) = \left(\frac{1}{64}, \ldots, \frac{1}{64}\right) \neq 0$$

showing that $\ker(\bar{\lambda}_\pm I - \mathcal{A}^*) \cap \ker(\mathcal{B}^*) = 0$.

□

Proposition 5.2. *The pair $(\mathcal{A}, \mathcal{C})$ with $\mathcal{C}$ as in (24) is detectable.*

Proof. Similar as in Proposition (5.1), for the eigenvectors associated with $\bar{\lambda}_\pm$, it holds that

$$\mathcal{C} \vec{z}_\pm = \left(\frac{1}{|\omega_1|} \int_{\omega_1} 1 \, \mathrm{d}x, \ldots, \frac{1}{|\omega_8|} \int_{\omega_8} 1 \, \mathrm{d}x\right) = (1, \ldots, 1) \neq 0$$

showing that $\ker(\bar{\lambda}_\pm I - \mathcal{A}) \cap \ker(\mathcal{C}) = 0$. □

5.2. Discretization and numerical solution of AREs

All simulations are generated on an Intel®Xeon(R) CPU E31270 @ 3.40 GHz x 8, 16 GB RAM, Ubuntu Linux 14.04, MATLAB® Version 8.0.0.783 (R2012b) 64-bit (glnxa64). The solutions of the ODE systems are always obtained by the MATLAB routine `ode45`.

The FEM discretization used in the numerical experiments was obtained by the PDE toolbox from MATLAB. All results correspond to a 64×64 regular grid with $n = 2 \cdot 4225 = 8450$ degrees of freedom.

Due to the mass matrix M obtained from the FEM discretization, the generalized Riccati equations are of the form

$$(A^n)^T P^n M^n + (M^n)^T P^n A^n + (C^n)^T C^n - (M^n)^T P^n B^n R^{-1} (B^n)^T P^n M^n = 0,$$
$$A^n Q^n (M^n)^T + M^n Q^n (A^n)^T + B^n (B^n)^T - M^n Q^n (C^n)^T (C^n) Q^n (M^n)^T = 0,$$

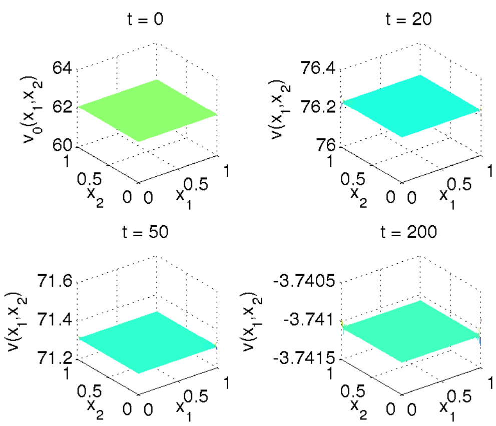
with $A^n, M^n = (M^n)^T \in \mathbb{R}^{8450 \times 8450}$, $B^n, (C^n)^T \in \mathbb{R}^{8450 \times 8}$ and $R \in \mathbb{R}^{8 \times 8}$. Since we were not able to compute the solutions P^n and Q^n with the MATLAB routine `care`, we used a Kleinman-Newton iteration as described in, e.g., [4, 7, 13, 20]. The Lyapunov equations arising in every Newton step where solved by an appropriate low rank implementation of the BiCGstab method.

5.3. LQG compensator design

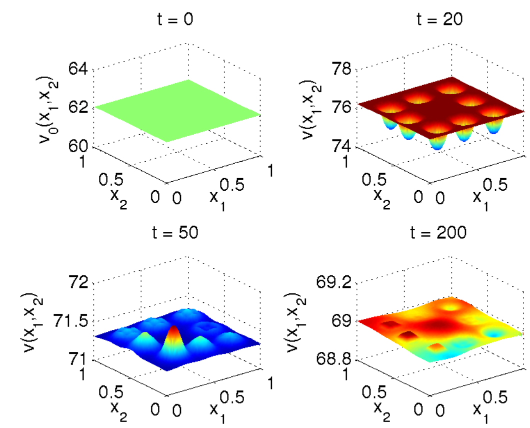
We first report on the results obtained for the classical LQG approach from Section 3. We chose the weight matrix $R = 10^{-4} I$. In accordance with Theorem 3.4, the desired decay rate was set to $\beta = 0.01$. For the shifted linearized system from Subsection 3.2, we have 6 unstable eigenvalues. Similar as in Proposition 5.1 and 5.2, one can show exponential stabilizability of the pairs $(\mathcal{A} + \beta I, \mathcal{B})$ and $(\mathcal{A} + \beta I, \mathcal{C})$, respectively.

5.3.1. *Stabilization of perturbed initial state*

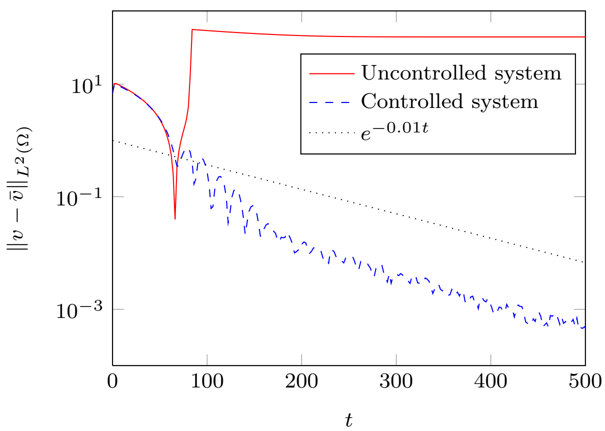
In the first example, we initialize the system with a constant perturbation around the unstable stationary solution (27), see Figure 2a at $t = 0$. This perturbed initial state is given as $(v_0, w_0) = 0.9 \cdot (\bar{v}, \bar{w})$. As a consequence of the Neumann boundary conditions, the state remains constant in space as time evolves. The transmembrane potential $v(t)$ of the uncontrolled system first is increased up to a value $v(t) \approx 80$ before it



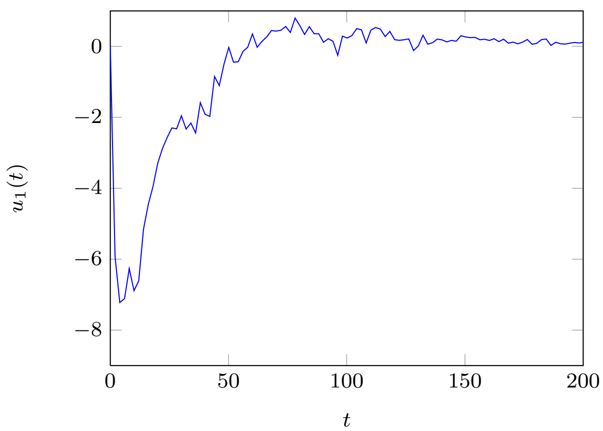
(A) Evolution of $v(x,t)$ (uncontrolled).



(B) Evolution of $v(x,t)$ (controlled).



(C) Comparison of $L^2(\Omega)$-norm evolutions.



(D) Feedback control $u_1(t)$.

FIGURE 2. Stabilization of perturbed initial state by LQG compensator.

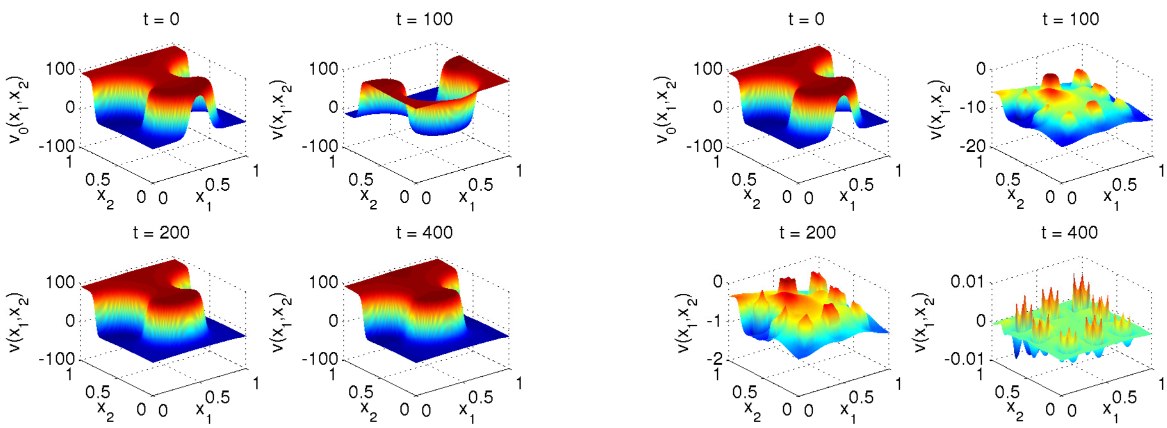
quickly becomes negative. From here, it then approaches the (stable) zero state. For the LQG approach, we initialize the state of the Kalman estimator with $z_{v,0} = z_{w,0} = 0$. We further incorporate a noisy observation of the form

$$v_{\text{obs}}(t) = C^n v(t) + e^{-0.011t}\texttt{randn}(8,1),$$

where `randn()` denote normally distributed numbers generated with MATLAB. Similarly to the uncontrolled system, the state $v(t)$ is first increased above the unstable stationary solution $\bar{v}$. As can be seen in Figure 2b, from here the control then drives it to $\bar{v}$. The described behavior of uncontrolled and controlled system are also reflected in Figure 2c which shows the deviation of $v(t)$ from $\bar{v}$. For the evolution of the discrete $L^2(\Omega)$-norm, the M-inner product

$$\|v(t)-\bar{v}\|_M = \sqrt{\langle v(t)-\bar{v}, M(v(t)-\bar{v})\rangle}$$

is used. As shown in the figure, the error of the uncontrolled system converges to the value $\|\bar{v}\|_{L^2(\Omega)}$ while the error of the controlled system approaches zero. For numerical evidence of the decay rate, we compare the error with the function $e^{-0.01t}$. Figure 2 shows the feedback control for the control domain ω_1. Interestingly, despite the choice of $R = 10^{-4}I$, the amplitude of $u_1(t)$ is only moderate.



(A) Evolution of $v(x,t)$ (uncontrolled).

(B) Evolution of $v(x,t)$ (controlled).

FIGURE 3. Termination of a reentry wave by LQG compensator.

5.3.2. *Termination of a reentry wave*

The second example deals with the termination of a reentry wave. As already mentioned in the introduction, these reentry phenomena model fibrillation processes of the heart. As a consequence, one is typically interested in terminating the wave, i.e., controlling the state to zero. In contrast to the first example, we thus consider $(\bar{v}, \bar{w}) = 0$. It now easily follows that the corresponding linearized system is stable. However, the system is not globally stable. As seen in Figure 3a, certain initial configurations require the use of external controls for stabilization. While the results from Theorem 3.4 do not guarantee an extension of the radius of attraction, the performance of feedback-based control laws is still of interest. As in the previous example, the Riccati solutions are obtained for the parameters $R = 10^{-4}I$ and $\beta = 0.01$. Figure 2b shows the dynamics of the corresponding closed-loop system. The control first forces the transmembrane potential below zero and from there it steers it to zero.

5.4. Reduced compensator design

We now turn to the concepts investigated in Section 4. For successful stabilization we had to introduce weights into the control and observation operators which were chosen as

$$Bu(t) = \sum_{i=1}^{8} \frac{1}{|\tilde{\omega}_i|} \chi_{|\tilde{\omega}_i} u_i(t), \quad v_{\mathrm{obs}}(t) = 50 \cdot \left(\tfrac{1}{|\omega_1|} \int_{\omega_1} v(t), \dots, \tfrac{1}{|\omega_8|} \int_{\omega_8} v(t) \right).$$

Since in our case $\frac{1}{|\tilde{\omega}_i|} > 1$, proceeding in this way we expect the control to have more influence. Similarly, deviations from the desired output experience higher penalization. Note that the assertions in Proposition 5.1 and Proposition 5.2 are not affected by these modifications. We briefly summarize the method from [15] in order to explain how the reduced compensator is actually obtained and. In a first step, we solve the algebraic Riccati equations

$$\begin{aligned}
(A^n)^T P^n M^n + (M^n)^T P^n A^n + (C^n)^T C^n - (M^n)^T P^n B^n (B^n)^T P^n M^n = 0, \\
A^n Q^n (M^n)^T + M^n Q^n (A^n)^T + B^n (B^n)^T - M^n Q^n (C^n)^T (C^n) Q^n (M^n)^T = 0.
\end{aligned}$$

Due to the Kleinman-Newton scheme and the low rank implementation of BiCGstab, we obtain numerical approximations of the form $P^n \approx UU^T$ and $Q^n \approx LL^T$, where $U \in \mathbb{R}^{2n \times q}$ and $L \in \mathbb{R}^{2n \times \ell}$ are appropriate

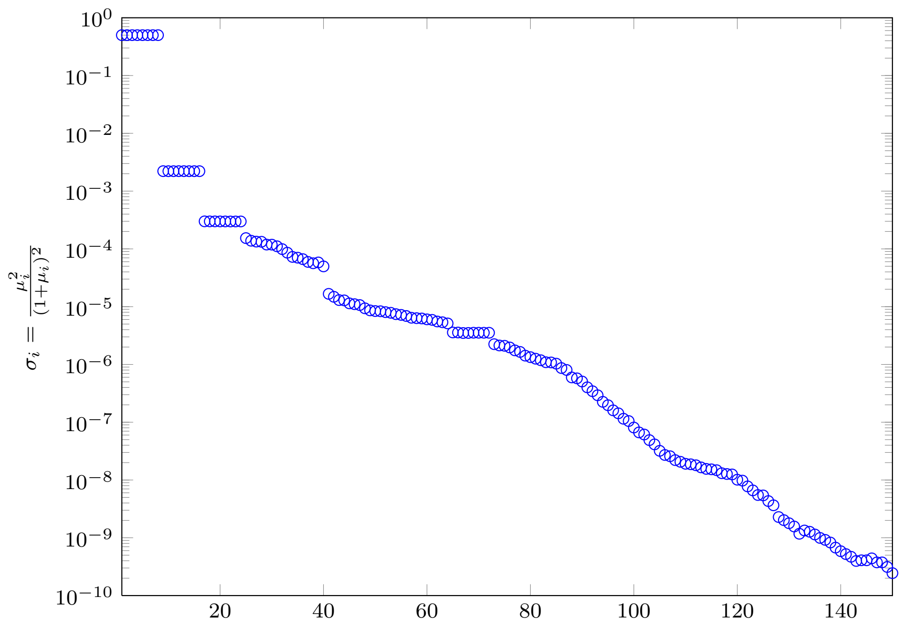


FIGURE 4. Decay of the first 150 Hankel singular values.

low rank factors. For the computation of the LQG reduced model, we first compute the Hankel singular values $\sigma_i = \sqrt{\frac{\mu_i^2}{1+\mu_i^2}}$, where μ_i^2 denote the eigenvalues of the matrix $P^n M^n Q^n$. Based on the decay of the first 150 Hankel singular values (see Figure 4), for the dimension of the reduced model we choose $r = 40$. The required projection matrices to generate $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ as in (21) are given as $W_r = UZ_r S_r^{-\frac{1}{2}}$ and $V_r = LY_r S_r^{-\frac{1}{2}}$ where Z_r, S_r, Y_r are obtained as the first r components of the singular value decomposition of $U^T M^n L = ZSY^T \approx Z_r S_r Y_r^T$. Following [15, Section 4.5.1] and [17, Theorem 9.4.16], the transfer function of the final reduced compensator is given as

$$G_r^n(s) = \frac{1}{\varepsilon^2 - 1} \tilde{B}_r \Lambda_r (sI - \hat{A}_r)^{-1} F_r \tilde{C}_r^T,$$

with

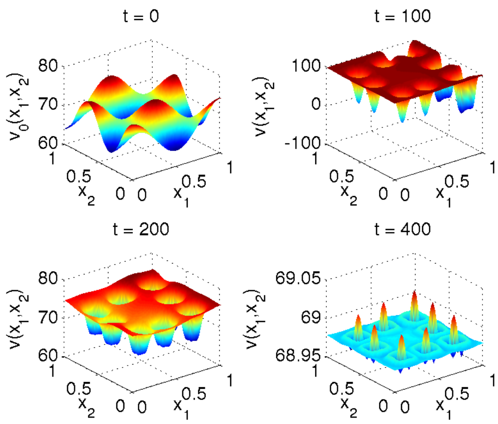
$$\Lambda_r = \operatorname{diag}\{\mu_i\}_{i=1}^r, \quad F_r = \operatorname{diag}\left\{\frac{(1-\varepsilon^2)\mu_i}{1-\varepsilon^2-\varepsilon^2\mu_i^2}\right\}_{i=1}^r, \quad \hat{A}_r = \tilde{A}_r - \tilde{B}_r \tilde{B}_r^T \Lambda_r + \frac{1}{\varepsilon^2-1} F_r \tilde{C}_r^T \tilde{C}_r.$$

For the robustness margin, we choose $\varepsilon = \frac{1}{2\sqrt{1+\mu_1^2}}$. Again we consider the unstable stationary solution $(\bar{v}, \bar{w})$ from (27). As a first example, assume that the following perturbed initial state is given:

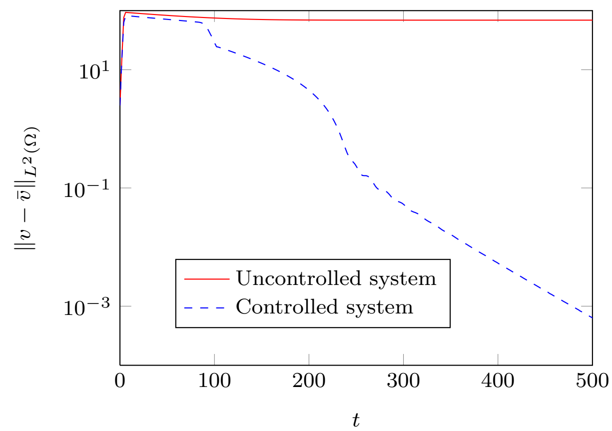
$$v_0(x_1, x_2) = \bar{v} + 5\cos(2\pi x_1)\cos(3\pi x_2), \quad w_0(x_1, x_2) = \bar{w} + \frac{5}{100}\cos(2\pi x_1)\cos(3\pi x_2).$$

5.4.1. *Stabilization of perturbed initial state*

We implement the reduced compensator with the same noisy observation as for the classical LQG approach. In Figure 5a we show the evolution of $v(x,t)$ for the closed-loop system at different time steps. The dynamics can be summarized as follows. First the transmembrane potential collapses and briefly remains below zero. It

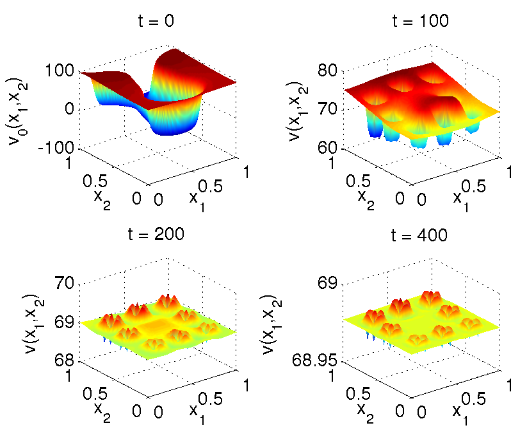


(A) Evolution of $v(x,t)$ (controlled).

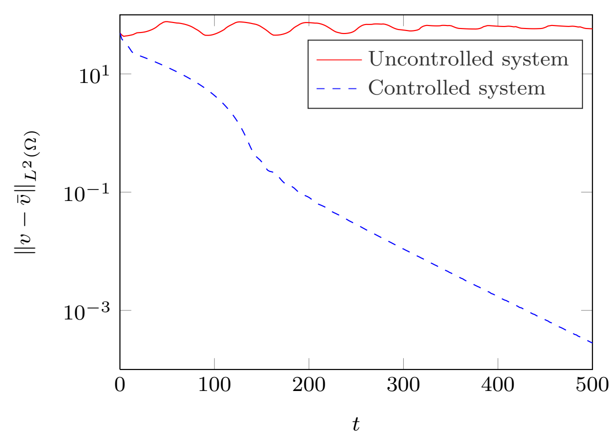


(B) Comparison of $L^2(\Omega)$-norm evolutions.

FIGURE 5. Stabilization of perturbation around stationary solution.



(A) Evolution of $v(x,t)$ (controlled).



(B) Comparison of $L^2(\Omega)$-norm evolutions.

FIGURE 6. Stabilization of a reentry wave.

it then pushed back close to the peak value $v_{pk} = 100$ from where it is then controlled to $\bar{v}$. Without control, the system converges to the stable zero state from below. This is seen in Figure 5b where the error for the uncontrolled system almost stagnates after very few time steps. On the other hand, the controlled system again converges exponentially to the unstable solution. Comparing the results from Figure 2c and Figure 5b for the $L^2(\Omega)$-error of the controlled system, we note that the reduced order compensator appears to be insensitive to the noisy observations.

5.4.2. *Stabilization of a reentry wave*

The final example is a reentry wave which, in contrast to the LQG approach, should be controlled to the unstable solution $(\bar{v}, \bar{w})$. The results shown in Figure 6 are qualitatively similar to those obtained for the perturbed initial state.

6. Conclusions

We have studied compensator design for a coupled PDE-ODE reaction diffusion system that arises in the context of cardiac electrophysiology. For the monodomain equations with the FitzHugh-Nagumo model, an explicit characterization of the spectrum of the linearized system allowed for some new controllability results. In particular, due to a finite accumulation point of the spectrum, we have seen that the system is not exactly null controllable. On the other hand, for stabilizability margins below a certain value (determined by the ODE), the system is locally exponentially stabilizable by finite-rank input and output operators. Besides the classical LQG approach this opened up the possibility to use reduced order compensators obtained by the method of LQG-balanced truncation. Numerical examples for both approaches illustrate the applicability of the theoretical results.

Since our results so far do not allow for a concrete specification of the radius of attraction for the compensators, we want to further study the use of other types of feedback mechanisms. Moreover, extending the method of LQG-balanced truncation to allow for weight matrices seems to deserve closer attention.

The authors gratefully acknowledge the Austrian Science Fund (FWF) for financial support under SFB F32, "Mathematical Optimization and Applications in Biomedical Sciences."

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