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A pointwise characterization of the subdifferential of the total variation functional*

K. Bredies[†] M. Holler[‡]

Abstract

We derive a new, pointwise characterization of the subdifferential of the total variation (TV) functional: We show that $u^* \in \partial \operatorname{TV}(u)$ if and only if $u^* = -\operatorname{div} g$ with a vector field g such that $\|g\|_\infty \leq 1$, $|\mathrm{D}u|$ – almost every point is a Lebesgue point of g and $g = \sigma_u$, the density function of the measure $\mathrm{D}u$. This provides an analytic justification for the formal expression $-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \in \partial \operatorname{TV}(u)$ and significantly extends the standard characterization of $\partial \operatorname{TV}$ in terms of the normal trace of g, frequently used, for example, to characterize the total variation flow.

Keywords. Total variation, subdifferential characterization, Lebesgue points, precise representative.

AMS subject classifications. 49K20, 46G05, 35A15.

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1 Introduction

The aim of this paper is to derive a new, pointwise characterization of the subdifferential of the TV functional in Lebesgue spaces. This characterization analytically justifies the formal expression

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \in \partial\,\mathrm{TV}(u),$$

for $u \in \mathrm{BV}(\Omega)$, where $\frac{\nabla u}{|\nabla u|} = \sigma_u \in L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$ is the unique density function such that $\mathrm{D}u = \sigma_u|\mathrm{D}u|$. In particular, we show that if $-\operatorname{div} g \in \partial\,\mathrm{TV}(u)$ for a vector field g with $\|g\|_\infty \leq 1$, then g indeed is sufficiently regular to obtain $g = \sigma_u$, pointwise almost everywhere in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$.

Regarding TV as functional on $L^p(\Omega)$, $1 < p \leq d/(d-1)$, with $\Omega \subset \mathbb{R}^d$, it is well known that, if $u^* \in \partial\,\mathrm{TV}(u)$, there exist a vector field $g \in L^q(\Omega, \mathbb{R}^d)$, $q = p/(p-1)$, with $\|g\|_\infty \leq 1$ such that

$$-\operatorname{div} g = u^* \quad \text{and} \quad \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g.$$

Formally applying a Gauss-Green formula to the integral equation one obtains $g = \sigma_u$, almost everywhere with respect to $|\mathrm{D}u|$. However, a priori, g is only defined almost everywhere with respect to the Lebesgue measure, thus one has to give meaning to g as integrable function with respect to $|\mathrm{D}u|$.

A commonly used solution to this is to apply a trace operator as in [7] to such vector fields: There, Anzellotti introduces a normal trace $\theta(g, \mathrm{D}u) \in L^1(\Omega; |\mathrm{D}u|)$ for vector fields $g \in W^q(\operatorname{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$ (see Section 2) that allows the following characterization: $u^* \in \partial\,\mathrm{TV}(u)$ if and only if, there exists $g \in W^q_0(\operatorname{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$ and

$$\theta(g, \mathrm{D}u) = 1 \quad \text{in } L^1(\Omega; |\mathrm{D}u|).$$

This approach is commonly used to characterize the total variation flow, as for example in [3, 4, 5, 6, 8, 9, 11]. Note that, while we are able to obtain a much stronger regularity for such vector fields in the case that $-\operatorname{div} g \in \partial\,\mathrm{TV}(u)$, the approach of Anzellotti applies to any $g \in W^q(\operatorname{div};\Omega) \cap L^\infty(\Omega, \mathbb{R}^d)$ independent of the subdifferential.

In [12], for dimension 2 and 3, a result has been obtained that allows to deduce the following: With $g \in W^q_0(\operatorname{div};\Omega)$, $\|g\|_\infty \leq 1$, $u \in \mathrm{BV}(\Omega)$ and χ_E the characteristic function of a level set of u, if $-\operatorname{div} g \in \partial\,\mathrm{TV}(\chi_E)$, then any point of the reduced boundary of E is a Lebesgue point of g. Moreover, g coincides with the measure theoretic inward normal of the level set at any such point. This strongly relates the regularity of g with the regularity of the boundary of the level sets of u. In particular it implies that, if $-\operatorname{div} g \in \partial\,\mathrm{TV}(u)$, every point of the reduced boundary of each level set of u of finite perimeter is also a Lebesgue point of g. But, for an arbitrary $\mathrm{BV}(\Omega)$ function u, the set of all points that are in the reduced boundary of at least some level set of u does still not cover $|\mathrm{D}u|$ almost every point of the domain Ω.

We will show that the set $\partial\,\mathrm{TV}(u)$ can be described as: $u^* \in \partial\,\mathrm{TV}(u)$ if and only if, there exists $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$, $|\mathrm{D}u|$ – almost every $x \in \Omega$ is a Lebesgue point of g and

$$g = \sigma_u \quad \text{in } L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|).$$

The outline of the paper is as follows: In the second section, we give some preliminary results about functions of bounded variation, introduce a straightforward generalization of the space $H(\mathrm{div})$ and state an approximation result. The third section is the main section. There, we first repeat the term of normal trace introduced in [7], which can be used as motivation for our main result and is still the standard notion for characterizing the subdifferential of TV. Then, we show how such a characterization can be extended to a pointwise characterization in terms of Lebesgue points and precise representatives. In the fourth section we address some topics where this characterization of the TV subdifferential can be applied: We use it to reformulate well known results, such as a characterization of the total variation flow, a characterization of Cheeger sets and optimality conditions for mathematical imaging problems. In the last section we give a conclusion.

2 Preliminaries

This section is devoted to introduce notation and basic results. After some preliminary definitions, we start with a short introduction to functions of bounded variation. For further information and proofs we refer to [2, 22, 16]. For convenience, we always assume $\Omega \subset \mathbb{R}^d$ to be a bounded Lipschitz domain. Further, throughout this work, we often denote $\int_\Omega \phi$ or $\int_\Omega \phi \,\mathrm{d}x$ instead of $\int_\Omega \phi(x)\,\mathrm{d}x$ for the Lebesgue integral of a measurable function ϕ, when the usage of the Lebesgue measure, denoted by $\mathcal{L}^d$, and the integration variable are clear from the context. For a set $E \subset \mathbb{R}^d$, $|E|$ always denotes the Lebesgue measure of E. When writing $|x|$ for $x \in \mathbb{R}^d$ we refer to the Euclidean norm.

We use a standard notation for continuously differentiable-, compactly supported- or integrable functions. Continuously differentiable functions on a closed set are defined as follows:

Definition 1 (Continuous functions on a closed set)**.** *Given a domain $A \subsetneq \mathbb{R}^d$ and $m \in \mathbb{N}$, we define*

$$C(\overline{A}, \mathbb{R}^m) = \{\phi : \overline{A} \to \mathbb{R}^m \,|\, \phi \text{ is uniformly continuous on } A\},$$

$$C^k(\overline{A}, \mathbb{R}^m) = \{\phi : \overline{A} \to \mathbb{R}^m \,|\, \mathrm{D}^\alpha \phi \in C(\overline{A}, \mathbb{R}^m) \text{ for all } |\alpha| \leq k\}$$

and

$$C^\infty(\overline{A}, \mathbb{R}^m) = \bigcap_{k\in\mathbb{N}} C^k(\overline{A}, \mathbb{R}^m).$$

By virtue of Whitney's Extension Theorem [21, Theorem 1], for bounded domains, $\phi \in C(\overline{A}, \mathbb{R}^m)$ is equivalent to ϕ being the restriction of a function in $C_c(\mathbb{R}^d, \mathbb{R}^m)$. This also applies to $C^k(\overline{A}, \mathbb{R}^m)$ and $C^\infty(\overline{A}, \mathbb{R}^m)$ with $C_c^k(\mathbb{R}^d, \mathbb{R}^m)$ and $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$, respectively.

Definition 2 (Finite Radon measure)**.** *Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra generated by the open subsets of Ω. We say that a function $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$, for $m \in \mathbb{N}$, is a finite $\mathbb{R}^m$-valued Radon measure if $\mu(\emptyset) = 0$ and μ is σ-additive. We denote by $\mathcal{M}(\Omega, \mathbb{R}^m)$ the space of all finite $\mathbb{R}^m$-Radon measures on Ω. Further we denote by $|\mu|$ the variation of $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, defined by*

$$|\mu|(E) = \sup \left\{ \sum_{i=0}^{\infty} |\mu(E_i)| \,\middle|\, E_i \in \mathcal{B}(\Omega),\ i \geq 0,\ \textit{pairwise disjoint}, E = \bigcup_{i=0}^{\infty} E_i \right\},$$

for $E \in \mathcal{B}(\Omega)$.

Definition 3 (Functions of bounded variation)**.** *We say that a function $u \in L^1(\Omega)$ is of bounded variation, if there exists a finite $\mathbb{R}^d$-valued Radon measure, denoted by $Du = (D_1 u, ..., D_d u)$, such that for all $i \in \{1, ..., d\}$, $D_i u$ represents the distributional derivative of u with respect to the ith coordinate, i.e., we have*

$$\int_\Omega u \partial_i \phi = -\int_\Omega \phi \, \mathrm{d}D_i u \quad \textit{for all } \phi \in C_c^\infty(\Omega).$$

By $\mathrm{BV}(\Omega)$ we denote the space of all functions $u \in L^1(\Omega)$ of bounded variation.

Definition 4 (Total variation)**.** *For $u \in L^1(\Omega)$, we define the functional $\mathrm{TV} : L^1(\Omega) \to \overline{\mathbb{R}}$ as*

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \phi \,\middle|\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\}$$

where we set $\mathrm{TV}(u) = \infty$ if the set is unbounded from above. We call $\mathrm{TV}(u)$ the total variation of u.

Proposition 1. *The functional $\mathrm{TV} : L^1(\Omega) \to \overline{\mathbb{R}}$ is convex and lower semicontinuous with respect to L^1-convergence. For $u \in L^1(\Omega)$ we have that*

$$u \in \mathrm{BV}(\Omega) \textit{ if and only if } \mathrm{TV}(u) < \infty.$$

In addition, the total variation of u coincides with the variation of the measure Du, i.e., $\mathrm{TV}(u) = |Du|(\Omega)$. Further,

$$\|u\|_{\mathrm{BV}} := \|u\|_{L^1} + \mathrm{TV}(u)$$

defines a norm on $\mathrm{BV}(\Omega)$ and endowed with this norm, $\mathrm{BV}(\Omega)$ is a Banach space.

Remark 1. *For $u \in \mathrm{BV}(\Omega)$ such that $\mathrm{D}u \in \mathcal{M}(\Omega, \mathbb{R}^d)$, we henceforth denote by $\sigma_u \in L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$ the unique density function such that $\mathrm{D}u = \sigma_u |\mathrm{D}u|$.*

Definition 5 (Strict Convergence)**.** *For $(u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathrm{BV}(\Omega)$, $n \in \mathbb{N}$, and $u \in \mathrm{BV}(\Omega)$ we say that $(u_n)_{n\in\mathbb{N}}$ strictly converges to u if*

$$\|u_n - u\|_{L^1} \to 0 \text{ and } \mathrm{TV}(u_n) \to \mathrm{TV}(u)$$

as $n \to \infty$.

Definition 6 (Lebesgue Point)**.** *Let $f \in L^p(\Omega, \mathbb{R}^d)$, $1 \leq p < \infty$. We say that $x \in \Omega$ is a Lebesgue point of f if*

$$\lim_{r\to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y) - f(x)| \,\mathrm{d}y \to 0$$

as $n \to \infty$. Note that $|B_r(x)|$ denotes the Lebesgue measure of the ball with radius r around $x \in \Omega$.

Remark 2. *Remember that for any $f \in L^p(\Omega)$, $1 \leq p < \infty$, $\mathcal{L}^d$ – almost every $x \in \Omega$ is a Lebesgue point of f (see [16, Corollary 1.7.1]).*

Definition 7 (Precise Representative)**.** *Let $f \in L^p(\Omega, \mathbb{R}^d)$, $1 \leq p < \infty$. The precise representative of f is given by*

$$f^*(x) = \begin{cases} \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \,\mathrm{d}y & \text{if this limit exists} \\ 0 & \text{else.} \end{cases}$$

Remark 3. *Note that x being a Lebesgue point is a stronger assertion than assuming the limit in the definition of the precise representative to exist. Hence, by Remark 2, the precise representative f^* equals f $\mathcal{L}^d$ – almost everywhere.*

Next we recall some standard notations and facts from convex analysis. For proofs and further introduction we refer to [15].

Definition 8 (Convex conjugate and subdifferential)**.** *For a normed vector space V and a function $F : V \to \overline{\mathbb{R}}$ we define its convex conjugate, or Legendre-Fenchel transform, denoted by $F^* : V^* \to \overline{\mathbb{R}}$, as*

$$F^*(u^*) = \sup_{v\in V} \langle v, u^* \rangle_{V,V^*} - F(v).$$

Further F is said to be subdifferentiable at $u \in V$ if $F(u)$ is finite and there exists $u^* \in V^*$ such that

$$\langle v - u, u^* \rangle_{V,V^*} + F(u) \leq F(v)$$

for all $v \in V$. The element $u^* \in V^*$ is then called a subgradient of F at u and the set of all subgradients at u is denoted by $\partial F(u)$.

Definition 9 (Convex indicator functional)**.** *For a normed vector space V and $U \subset V$ a convex set, we denote by $\mathcal{I}_U : V \to \overline{\mathbb{R}}$ the convex indicator functional of U, defined by*

$$\mathcal{I}_U(u) = \begin{cases} 0 & \text{if } u \in U, \\ \infty & \text{else.} \end{cases}$$

Next we define the space $W^q(\operatorname{div};\Omega)$, which is fundamental for the characterization of the TV subdifferential.

Definition 10 (The space $W^q(\operatorname{div};\Omega)$)**.** *Let $1 \leq q < \infty$ and $g \in L^q(\Omega, \mathbb{R}^d)$. We say that $\operatorname{div} g \in L^q(\Omega)$ if there exists $w \in L^q(\Omega)$ such that for all $v \in C_c^\infty(\Omega)$*

$$\int_\Omega \nabla v \cdot g = -\int_\Omega vw.$$

Furthermore we define

$$W^q(\operatorname{div};\Omega) = \left\{ g \in L^q(\Omega, \mathbb{R}^d) \mid \operatorname{div} g \in L^q(\Omega) \right\}$$

with the norm $\|g\|^q_{W^q(\operatorname{div})} := \|g\|^q_{L^q} + \|\operatorname{div} g\|^q_{L^q}$.

Remark 4. *Density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$ implies that, if there exists $w \in L^q(\Omega)$ as above, it is unique. Hence it makes sense to write $\operatorname{div} g = w$. By completeness of $L^q(\Omega)$ and $L^q(\Omega, \mathbb{R}^d)$ it follows that $W^q(\operatorname{div};\Omega)$ is a Banach space when equipped with $\|\cdot\|_{W^q(\operatorname{div})}$.*

Remark 5. *Note that $W^q(\operatorname{div};\Omega)$ is just a straightforward generalization of the well known space $H(\operatorname{div};\Omega)$. Also classical results like density of $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ and existence of a normal trace on $\partial\Omega$ can be derived for $W^q(\operatorname{div};\Omega)$ as straightforward generalizations of the proofs given for example in [17, Chapter 1].*

Definition 11 (The space $W_0^q(\operatorname{div};\Omega)$)**.** *For $1 \leq q < \infty$, we define*

$$W_0^q(\operatorname{div};\Omega) = \overline{C_c^\infty(\Omega, \mathbb{R}^d)}^{\|\cdot\|_{W^q(\operatorname{div})}}.$$

Remark 6. *By density it follows that, for $g \in W_0^q \operatorname{div};\Omega)$, we have*

$$\int_\Omega \nabla v g = -\int_\Omega v \operatorname{div} g$$

for all $v \in C^\infty(\overline{\Omega})$.

The following approximation result will be needed frequently:

Proposition 2. *For $1 \leq q < \infty$ and $g \in W^q(\operatorname{div};\Omega)$, there exists a sequence of vector fields $(g_n)_{n \geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ such that*

1. *$\|g_n - g\|_{W^q(\operatorname{div})} \to 0$ as $n \to \infty$,*

2. $\|g_n\|_\infty \le \|g\|_\infty$ *for each* $n \in \mathbb{N}$*, if* $\|g\|_\infty < \infty$*,*

3. $g_n(x) \to g(x)$ *for every Lebesgue point* $x \in \Omega$ *of* g*,*

4. $\|g_n - g\|_{\infty,\overline{\Omega}} \to 0$ *as* $n \to \infty$*, if, additionally,* $g \in C(\overline{\Omega}, \mathbb{R}^d)$*.*

A proof can be found in the Appendix. If a sequence $(g_n)_n$ in $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converges to $g \in W^q(\mathrm{div};\Omega)$ such that 1.) - 4.) of the proposition above holds, we henceforth write

$$g_n \overset{\sim}{\to} g.$$

At last, we remember the concept of mollification, that will be frequently used in this work.

Definition 12 (Mollifier Kernel)**.** *A function* $\eta : \mathbb{R}^d \to \mathbb{R}$ *is called mollifier kernel if*

1. $\eta \in C_c^\infty(\mathbb{R}^d)$

2. $\int_{\mathbb{R}^d} \eta = 1$

3. $\eta(x) \ge 0$

4. $\eta(-x) = \eta(x)$

5. $\mathrm{supp}(\eta) \subset \overline{B_1(0)}$

For a mollifier kernel η*, we define*

$$\eta_\epsilon(x) := \frac{1}{\epsilon^d}\eta(\frac{x}{\epsilon}).$$

Remark 7. *Given a mollifier kernel* $\eta \in C_c^\infty(\mathbb{R}^d)$*, a smooth approximation of any* $f \in L^1_{loc}(\Omega)$ *can then be defined on any compact subset of* Ω *by*

$$f^\epsilon(x) = \int\limits_\Omega \eta_\epsilon(x-y) f(y)\,\mathrm{d}y.$$

This is a standard way of approximating integrable functions and we refer to [16, Section 4.2] for some properties.

Remark 8. *Denoting* $f(x) := \frac{1}{|B_1(0)|}\chi_{B_1(0)}(x)$*, with* χ_V *the characteristic function of a set* $V \subset \mathbb{R}^d$*, for each* $\lambda > 0$ *there exists a mollifier kernel* $\eta^\lambda : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\|\eta^\lambda - f\|_{L^1(\mathbb{R}^d)} < \lambda.$$

Proof. Such a kernel can be constructed by cutting off f close to $\partial B_1(0)$, using mollification and dividing the resulting function by its mean value. □

3 Subdifferential of TV

In order to describe the subdifferential of the TV functional, for $u \in \mathrm{BV}(\Omega)$, we have to give a meaning to $W^q(\mathrm{div};\Omega)$ vector fields as functions in $L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$. As already mentioned in the introduction, one way of doing so is to define a trace operator.

3.1 The normal trace

We revisit the normal trace introduced in [7]. We do so by defining it for $W^q(\mathrm{div};\Omega)$ vector fields as a closed operator. In this subsection, if not restricted further, let always be $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain, $1 \le q < \infty$, $p = \frac{q}{q-1}$ if $q \neq 1$ or $p = \infty$ else.

Proposition 3. *Set $\tilde{D}_N := W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d)$. Then, with $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ fixed, for any $z \in \tilde{D}_N$ there exists a function $\theta(z,\mathrm{D}u) \in L^1(\Omega;|\mathrm{D}u|)$ such that*

$$\int_\Omega \theta(z,\mathrm{D}u)\psi \,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \,\mathrm{div}(z\psi)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\Omega)$.

Proof. For $z \in \tilde{D}_N$ we define

$$\begin{aligned} L_z : C_c^\infty(\Omega) &\to \mathbb{R} \\ \psi &\mapsto -\int_\Omega u\,\mathrm{div}(z\psi)\,\mathrm{d}x \end{aligned}$$

and show that L_z can be extended to a linear, continuous operator from $C_0(\Omega)$ to $\mathbb{R}$.

It is clear that L_z is well-defined and linear, hence by definition of $C_0(\Omega)$ as closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_\infty$, it suffices to show that L_z is continuous with respect to $\|\cdot\|_\infty$. With $\psi \in C_c^\infty(\Omega)$ and $(z_n)_{n\ge 0} \subset C^\infty(\overline{\Omega},\mathbb{R}^d)$ converging to z as in Proposition 2, we estimate

$$\begin{aligned} |L_z(\psi)| &= \lim_{n\to\infty}\left| -\int_\Omega u\,\mathrm{div}(z_n\psi)\,\mathrm{d}x\right| = \lim_{n\to\infty}\left|\int_\Omega z_n\psi\,\mathrm{d}\mathrm{D}u\right| \\ &\le \|z\|_\infty \int_\Omega |\psi|\,\mathrm{d}|\mathrm{D}u| \le \|z\|_\infty\|\psi\|_\infty|\mathrm{D}u|(\Omega), \end{aligned}$$

where we used that $\|z_n - z\|_{W^q(\mathrm{div})} \to 0$ as $n \to \infty$ and that $\|z_n\|_\infty \le \|z\|_\infty$ for each $n \in \mathbb{N}$.

Thus, for any $z \in W^q(\mathrm{div};\Omega) \cap L^\infty(\Omega,\mathbb{R}^d)$, we have that $L_z \in C_0(\Omega)^* = \mathcal{M}(\Omega)$ and we can write $(z,\mathrm{D}u)$ for the Radon measure associated with L_z. Performing the above calculations for $\psi \in C_c^\infty(A)$ with any open $A \subset \Omega$ yields

$|L_z(\psi)| \leq \|z\|_\infty \|\psi\|_\infty |\mathrm{D}u|(A)$. Thus it follows that $(z, \mathrm{D}u) \ll |\mathrm{D}u|$ and hence by the Radon-Nikodym theorem there exists $\theta(z, \mathrm{D}u) \in L^1(\Omega; |\mathrm{D}u|)$ such that $(z, \mathrm{D}u) = \theta(z, \mathrm{D}u)|\mathrm{D}u|$. □

With that we can define the normal trace operator and prove additional properties:

Proposition 4 (Normal trace operator)**.** *With $\tilde{D}_N$ as in Proposition 3 and $u \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$ fixed, the operator*

$$\begin{aligned} \widetilde{T_N} : \tilde{D}_N \subset W^q(\mathrm{div}; \Omega) \quad &\rightarrow \quad L^1(\Omega; |\mathrm{D}u|) \\ z \quad &\mapsto \quad \theta(z, \mathrm{D}u) \end{aligned}$$

with $\theta(z, \mathrm{D}u)$ the density function of the measure $(z, \mathrm{D}u)$ with respect to $|\mathrm{D}u|$ as above, is well-defined and closeable. Further, with $T_N : D_N \rightarrow L^1(\Omega; |\mathrm{D}u|)$ denoting the closure of $\widetilde{T_N}$ defined on $D_N \subset W^q(\mathrm{div}; \Omega)$, we have that, for $z \in D_N$,

$$\|T_N z\|_\infty \leq \|z\|_\infty$$

whenever $z \in L^\infty(\Omega, \mathbb{R}^d)$ and, for $\phi \in C(\overline{\Omega}, \mathbb{R}^d) \cap W^q(\mathrm{div}; \Omega)$, that

$$T_N \phi = \phi \cdot \sigma_u \in L^1(\Omega; |\mathrm{D}u|).$$

Proof. Well-definition is clear since the representation of L_z as a measure and also its density function with respect to $|\mathrm{D}u|$ is unique. Let now $(z_n)_{n\geq 0}, (\tilde{z}_n)_{n\geq 0} \subset \tilde{D}_N$ be two sequences converging to z in $W^q(\mathrm{div}; \Omega)$ and suppose that $\widetilde{T_N} z_n \rightarrow h$ and $\widetilde{T_N} \tilde{z}_n \rightarrow \tilde{h}$ with $h, \tilde{h} \in L^1(\Omega; |\mathrm{D}u|)$. With $\psi \in C_c^\infty(\Omega)$ we can write, using $\lim\limits_{n\rightarrow\infty} \mathrm{div}(z_n\psi) = \mathrm{div}(z\psi) = \lim\limits_{n\rightarrow\infty} \mathrm{div}(\tilde{z}_n\psi)$ in $L^q(\Omega)$,

$$\begin{aligned} \int_\Omega h\psi \,\mathrm{d}|\mathrm{D}u| \quad &= \quad \lim_{n\rightarrow\infty} \int_\Omega (\widetilde{T_N} z_n)\psi \,\mathrm{d}|\mathrm{D}u| = \lim_{n\rightarrow\infty} -\int_\Omega u \,\mathrm{div}(z_n\psi)\,\mathrm{d}x \\ &= \quad \lim_{n\rightarrow\infty} -\int_\Omega u \,\mathrm{div}(\tilde{z}_n\psi)\,\mathrm{d}x = \lim_{n\rightarrow\infty} \int_\Omega (\widetilde{T_N} \tilde{z}_n)\psi \,\mathrm{d}|\mathrm{D}u| \\ &= \quad \int_\Omega \tilde{h}\psi \,\mathrm{d}|\mathrm{D}u| \end{aligned}$$

and thus, by density, $h = \tilde{h}$ and, consequently, $\widetilde{T_N}$ is closeable. The assertion $\|T_N z\|_\infty \leq \|z\|_\infty$ for $z \in D_N$ follows from $\left|\int_A \theta(z, \mathrm{D}u)\,\mathrm{d}|\mathrm{D}u|\right| \leq \|z\|_\infty |\mathrm{D}u|(A)$, for all $A \subset \Omega$ measurable, in the case that $\|z\|_\infty < \infty$, since then $z \in \tilde{D}_N$. If $\|z\|_\infty = \infty$, the inequality is trivially satisfied.

In order to show that $T_N\phi = \phi \cdot \sigma_u$ for $\phi \in C(\overline{\Omega}, \mathbb{R}^d) \cap W^q(\mathrm{div}; \Omega)$ first note that $\phi \in \tilde{D}_N$. Thus, $T_N\phi$ is defined and we can use that, due to continuity of ϕ, the approximating vector fields $(\phi_n)_{n\geq 0}$ as in Proposition 2 converge uniformly to ϕ and write, again for $\psi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega (T_N\phi)\psi \,\mathrm{d}|\mathrm{D}u| &= -\int_\Omega u \operatorname{div}(\phi\psi)\,\mathrm{d}x = \lim_{n\to\infty} -\int_\Omega u \operatorname{div}(\phi_n\psi)\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_\Omega \phi_n\psi \,\mathrm{dD}u = \int_\Omega (\phi\cdot\sigma_u)\psi \,\mathrm{d}|\mathrm{D}u|. \qquad \square\end{aligned}$$

Remark 9. *Note that by similar arguments one could also show that $\widehat{T_N} : X(\Omega) := W^q(\operatorname{div};\Omega)\cap L^\infty(\Omega,\mathbb{R}^d) \to L^1(\Omega;|\mathrm{D}u|)$ is continuous, when X is equipped with the norm $\|z\|_X := \|z\|_\infty + \|\operatorname{div} z\|_{L^q}$.*

We therefore have a suitable notion of normal trace for a dense subset of $W^q(\operatorname{div};\Omega)$. The closedness of the operator T_N can be interpreted as follows: If $z \in W^q(\operatorname{div};\Omega)\cap L^\infty(\Omega,\mathbb{R}^d)$ is sufficiently regular in the sense that the normal trace of its approximating vector fields as in Proposition 2 converges to some $h \in L^1(\Omega;|\mathrm{D}u|)$ with respect to $\|\cdot\|_{L^1}$ (which is satisfied for example if z_n converges pointwise $|\mathrm{D}u|$-almost everywhere), then $T_N z = h = \lim_{n\to\infty}(z_n\cdot\sigma_u)$.

In [7, Theorem 1.9] it was also shown that, for $u \in \mathrm{BV}(\Omega)\cap L^p(\Omega)$ and $g \in W^q(\operatorname{div};\Omega)\cap L^\infty(\Omega,\mathbb{R}^d)$, denoting by $\theta(g,\mathrm{D}u)$ the normal trace of g as in Proposition 4, the following Gauss-Green formula holds:

$$\int_\Omega u \operatorname{div} g\,\mathrm{d}x + \int_\Omega \theta(g,\mathrm{D}u)\,|\mathrm{D}u| = \int_{\partial\Omega} [g\cdot\nu] u^\Omega \,\mathrm{d}\mathcal{H}^{d-1},$$

where $[g\cdot\nu] \in L^\infty(\partial\Omega;\mathcal{H}^{d-1})$ and $u^\Omega \in L^1(\partial\Omega;\mathcal{H}^{d-1})$ denote the boundary trace functions of g and u, respectively.

With Proposition 5 below, this implies the following characterization of the subdifferential of TV as function on $L^p(\Omega)$, $1 < p \le d/(d-1)$: For $u \in \mathrm{BV}(\Omega)\cap L^p(\Omega)$, $u^* \in L^q(\Omega)$,

$$u^* \in \partial\,\mathrm{TV}(u)$$

if and only if there exits $g \in W_0^q(\operatorname{div};\Omega)$ with $\|g\|_\infty \le 1$ and

$$\theta(g,\mathrm{D}u) = 1, \quad |\mathrm{D}u| - \text{almost everywhere}.$$

Given that $\theta(g,\mathrm{D}u) = g\cdot\sigma_u$ if g is sufficiently regular, one may ask if this is true for any $-\operatorname{div} g \in \partial\,\mathrm{TV}(u)$. This would allow to replace the condition $\theta(g,\mathrm{D}u) = 1$, which only provides information on g in direction σ_u, by $g = \sigma_u$. The aim of the following section is to show that this can indeed be done.

3.2 Pointwise subdifferential characterization

We will now obtain a pointwise characterization of the subdifferential of the TV functional. To this aim, we give a meaning to $W_0^q(\operatorname{div};\Omega)$ vector fields with $\operatorname{div} g \in \partial\,\mathrm{TV}(u)$, $\|g\|_\infty \le 1$, as functions in $L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$ by showing that such a vector field admits a Lebesgue point almost everywhere with respect to the $|\mathrm{D}u|$ measure.

In this subsection, we always assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $d \geq 2$, $1 < p \leq \frac{d}{d-1}$ and $q = \frac{p}{p-1}$. We define

$$\mathrm{TV} : L^p(\Omega) \to \overline{\mathbb{R}},$$

as

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \phi \;\middle|\; \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_\infty \leq 1 \right\}$$

where TV may also attain the value ∞.

First remember a well known result, which provides a characterization by using an integral equation.

Proposition 5 (Integral characterization)**.** *Let $u \in L^p(\Omega)$ and $u^* \in L^q(\Omega)$. Then $u^* \in \partial \mathrm{TV}(u)$ if and only if*

$$\begin{cases} u \in \mathrm{BV}(\Omega) \text{ and there exists } g \in W_0^q(\mathrm{div};\Omega) \\ \text{with } \|g\|_\infty \leq 1 \text{ such that } u^* = -\operatorname{div} g \text{ and} \\ \displaystyle\int_\Omega \mathbf{1} \,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g. \end{cases}$$

Proof. For the sake of completeness, we elaborate on the proof: Denoting by $C = \big\{\operatorname{div}\phi \,|\, \phi \in C_c^\infty(\Omega,\mathbb{R}^d),\ \|\phi\|_\infty \leq 1\big\}$, we have

$$\mathrm{TV}(u) = \mathcal{I}_C^*(u),$$

where $\mathcal{I}_C^*$ denotes the polar of $\mathcal{I}_C$ [15, Definition I.4.1], and, consequently, see [15, Example I.4.3],

$$\mathrm{TV}^*(u^*) = \mathcal{I}_C^{**}(u^*) = \mathcal{I}_{\overline{C}}(u^*)$$

where the closure of C is taken with respect to the L^q norm. Using the equivalence [15, Proposition I.5.1]

$$u^* \in \partial \mathrm{TV}(u) \quad \Leftrightarrow \quad \mathrm{TV}(u) + \mathrm{TV}^*(u^*) = (u, u^*)_{L^p, L^q},$$

it therefore suffices to show that

$$\overline{C} = \{\operatorname{div} g \,|\, g \in W_0^q(\mathrm{div},\Omega),\ \|g\|_\infty \leq 1\} =: K$$

to obtain the desired assertion. Since clearly $C \subset K$, it is sufficient for $\overline{C} \subset K$ to show that K is closed with respect to the L^q norm. For this purpose take $(g_n)_{n\geq 0} \subset W_0^q(\mathrm{div};\Omega)$ with $\|g_n\|_\infty \leq 1$ such that

$$\operatorname{div} g_n \to h \text{ in } L^q(\Omega) \text{ as } n \to \infty.$$

By boundedness of $(g_n)_{n\geq 0}$ there exists a subsequence $(g_{n_i})_{i\geq 0}$ weakly converging to some $g \in L^q(\Omega, \mathbb{R}^d)$. Now for any $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega g \cdot \nabla\phi = \lim_{i\to\infty} \int_\Omega g_{n_i} \cdot \nabla\phi = \lim_{i\to\infty} -\int_\Omega \operatorname{div}(g_{n_i})\phi = -\int_\Omega h\phi,$$

from which follows that $g \in W^q(\operatorname{div};\Omega)$ and $\operatorname{div} g = h$. To show that $\|g\|_\infty \leq 1$ and $g \in W_0^q(\operatorname{div};\Omega)$ note that the set

$$\{(f, \operatorname{div} f) \mid f \in W_0^q(\operatorname{div};\Omega),\ \|f\|_\infty \leq 1\} \subset L^q(\Omega, \mathbb{R}^{d+1})$$

forms a convex and closed – and therefore weakly closed – subset of $L^q(\Omega, \mathbb{R}^{d+1})$ [15, Section I.1.2]. Since the sequence $((g_{n_i}, \operatorname{div} g_{n_i}))_{i\geq 0}$ is contained in this set and converges weakly in $L^q(\Omega, \mathbb{R}^{d+1})$ to $(g, \operatorname{div} g)$, we have $g \in W_0^q(\operatorname{div};\Omega)$ and $\|g\|_\infty \leq 1$, hence $\operatorname{div} g \in K$. For $K \subset \overline{C}$ it suffices to show that, for any $g \in W_0^q(\operatorname{div};\Omega)$ with $\|g\|_\infty \leq 1$ fixed, we have for all $v \in L^p(\Omega)$ that

$$\int_\Omega v \operatorname{div} g \leq \mathrm{TV}(v)$$

since this implies $\mathrm{TV}^*(\operatorname{div} g) = \mathcal{I}_{\overline{C}}(\operatorname{div} g) = 0$. Now for such a $v \in L^p(\Omega)$ we can assume that $v \in \mathrm{BV}(\Omega)$ since in the other case the inequality is trivially satisfied. Thus we can take a sequence $(v_n)_{n\geq 0} \subset C^\infty(\overline{\Omega})$ strictly converging to v [2, Theorem 3.9], for which we can also assume that $v_n \to v$ with respect to $\|\cdot\|_{L^p}$. Using Remark 6 it follows

$$\begin{aligned}\int_\Omega v \operatorname{div} g &= \lim_{n\to\infty} \int_\Omega v_n \operatorname{div} g = \lim_{n\to\infty} -\int_\Omega \nabla v_n \cdot g \\ &\leq \lim_{n\to\infty} \int_\Omega |\nabla v_n||g| \leq \lim_{n\to\infty} \mathrm{TV}(v_n) = \mathrm{TV}(v). \square\end{aligned}$$

Remark 10. *Note that in the last part of the proof of Proposition 5 we have in particular shown that for any $g \in W_0^q(\operatorname{div};\Omega)$ with $\|g\|_\infty \leq 1$ and any $v \in L^p(\Omega)$, the inequality*

$$\int_\Omega v \operatorname{div} g \leq \mathrm{TV}(v)$$

holds.

A consequence of Proposition 5 is the following pointwise characterization of g, whose proof will follow immediately after proving Proposition 6 below.

Corollary 1. *With the assumptions of Proposition 5 and $u^* = -\operatorname{div} g \in \partial\,\mathrm{TV}(u)$ with $\|g\|_\infty \leq 1$ it follows that*

$$g = \sigma_u \quad |\mathrm{D}^a u| \text{ – almost everywhere,}$$

where $|\mathrm{D}u| = |\mathrm{D}^a u| + |\mathrm{D}^s u|$ with $|\mathrm{D}^a u|$ and $|\mathrm{D}^s u|$ the absolute continuous and singular part of $|\mathrm{D}u|$ with respect to $\mathcal{L}^d$, respectively.

To show the above corollary we will use that $|\mathrm{D}^a u|$ – almost every $x \in \Omega$ is a Lebesgue point of g. This gives already a hint of how we will obtain the main subdifferential characterization result: Provided that we can show that even $|\mathrm{D}u|$ – almost every $x \in \Omega$ is a Lebesgue point of g, the desired equality will follow $|\mathrm{D}u|$ – almost everywhere for the precise representative of g.

A first step towards this pointwise characterization of the subdifferential of the TV functional is to obtain a characterization in terms of approximating sequences:

Proposition 6. *With the assumptions of Proposition 5 we have that* $u^* \in \partial\,\mathrm{TV}(u)$ *if and only if*

$$1) \begin{cases} u \in \mathrm{BV}(\Omega) \text{ and there exists } g \in W_0^q(\mathrm{div};\Omega) \\ \text{with } \|g\|_\infty \leq 1 \text{ such that } u^* = -\operatorname{div} g \text{ and} \end{cases}$$

$$2) \begin{cases} \text{for any sequence } (g_n)_n \text{ in } C^\infty(\overline{\Omega}, \mathbb{R}^d), \\ g_n \overset{\sim}{\to} g \text{ implies } \|\sigma_u - g_n\|_{L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)} \to 0 \text{ as } n \to \infty. \end{cases}$$

Proof. Let $u^* \in \partial\,\mathrm{TV}(u)$. Using Proposition 5, with $g \in W_0^q(\mathrm{div},\Omega)$ provided there, it suffices to show that, for $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ such that $g_n \overset{\sim}{\to} g$ it follows

$$\|\sigma_u - g_n\|_{L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)} \to 0.$$

Testing the zero extension of u, denoted by $w \in \mathrm{BV}(\mathbb{R}^d)$, with $(g_n)_{n\geq 0}$ extended to be in $C^1(\mathbb{R}^d, \mathbb{R}^d)$ yields, by virtue of [2, Corollary 3.89],

$$\begin{aligned} \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| &= -\int_\Omega u \operatorname{div} g\,\mathrm{d}x = \lim_{n\to\infty} -\int_\Omega u \operatorname{div} g_n\,\mathrm{d}x \\ &= \lim_{n\to\infty} -\int_{\mathbb{R}^d} w \operatorname{div} g_n\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^d} g_n\,\mathrm{d}\mathrm{D}w \\ &= \lim_{n\to\infty} \left(\int_\Omega g_n \cdot \sigma_u\,\mathrm{d}|\mathrm{D}u| + \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega\,\mathrm{d}\mathcal{H}^{d-1} \right) \end{aligned} \qquad (1)$$

where, $u^\Omega \in L^1(\partial\Omega; \mathcal{H}^{d-1})$ denotes the trace of u on $\partial\Omega$ and ν_Ω is the generalized inner unit normal vector of $\partial\Omega$. Next, we like to show that the boundary term vanishes as $n \to \infty$. By density of $C^\infty(\overline{\Omega})$ in $\mathrm{BV}(\Omega)$ and continuity of the trace operator for BV functions with respect to strict convergence (see [2, Theorem 3.88]), for arbitrary $\epsilon > 0$, there exists $\phi_\epsilon \in C^\infty(\overline{\Omega})$ such that $\|u^\Omega - \phi_\epsilon^\Omega\|_{L^1(\partial\Omega)} < \epsilon$. By the standard Gauss-Green theorem we can write

$$\int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon\,\mathrm{d}\mathcal{H}^{d-1} = -\int_\Omega \mathrm{div}(g_n)\phi_\epsilon\,\mathrm{d}x - \int_\Omega g_n \cdot \nabla\phi_\epsilon\,\mathrm{d}x$$

and taking the limit as $n \to \infty$ we get, by $g_n \to g$ in $W^q(\mathrm{div};\Omega)$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \, \mathrm{d}\mathcal{H}^{d-1} &= \lim_{n\to\infty} \left(-\int_\Omega \mathrm{div}(g_n)\phi_\epsilon \, \mathrm{d}x - \int_\Omega g_n \cdot \nabla\phi_\epsilon \, \mathrm{d}x \right) \\ &= -\int_\Omega \mathrm{div}(g)\phi_\epsilon \, \mathrm{d}x - \int_\Omega g \cdot \nabla\phi_\epsilon \, \mathrm{d}x = 0.\end{aligned}$$

For $n \in \mathbb{N}$ we thus have, since $\|g_n\|_\infty \leq \|g\|_\infty$,

$$\begin{aligned}\left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} \right| &= \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)(u^\Omega - \phi_\epsilon) + (g_n \cdot \nu_\Omega)\phi_\epsilon \, \mathrm{d}\mathcal{H}^{d-1} \right| \\ &\leq \|g_n\|_\infty \|u^\Omega - \phi_\epsilon\|_{L^1(\partial\Omega)} + \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \, \mathrm{d}\mathcal{H}^{d-1} \right| \\ &\leq \epsilon + \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega)\phi_\epsilon \, \mathrm{d}\mathcal{H}^{d-1} \right|.\end{aligned}$$

Hence

$$\limsup_n \left| \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} \right| \leq \epsilon$$

and, since ϵ was chosen arbitrarily,

$$\lim_{n\to\infty} \int_{\partial\Omega} (g_n \cdot \nu_\Omega) u^\Omega \, \mathrm{d}\mathcal{H}^{d-1} = 0.$$

Together with equation (1) this implies

$$\int_\Omega \mathbf{1} \, \mathrm{d}|\mathrm{D}u| = \lim_{n\to\infty} \int_\Omega g_n \cdot \sigma_u \, \mathrm{d}|\mathrm{D}u|. \tag{2}$$

Using that $|g_n(x)| \leq 1$ for all $x \in \Omega$ and $|\sigma_u(x)| = 1$, $|\mathrm{D}u|-$almost everywhere, we estimate $1 - (g_n \cdot \sigma_u)$:

$$\begin{aligned}1 - (g_n \cdot \sigma_u) &= \frac{1}{2}|\sigma_u|^2 - (g_n \cdot \sigma_u) + \frac{1}{2}|g_n|^2 + \frac{1}{2}|\sigma_u|^2 - \frac{1}{2}|g_n|^2 \\ &= \frac{1}{2}|\sigma_u - g_n|^2 + \frac{1}{2}|\sigma_u|^2 - \frac{1}{2}|g_n|^2 \\ &\geq \frac{1}{2}|\sigma_u - g_n|^2 \quad |\mathrm{D}u| - \text{almost everywhere.}\end{aligned}$$

Hence we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega |\sigma_u - g_n|\,\mathrm{d}|\mathrm{D}u| &\leq \left(|\mathrm{D}u|(\Omega)\lim_{n\to\infty}\int_\Omega |\sigma_u - g_n|^2\,\mathrm{d}|\mathrm{D}u|\right)^{\frac{1}{2}} \\ &\leq \left(2|\mathrm{D}u|(\Omega)\lim_{n\to\infty}\int_\Omega 1-(g_n\cdot\sigma_u)\,\mathrm{d}|\mathrm{D}u|\right)^{\frac{1}{2}} = 0\end{aligned}$$

from which the assertion follows.

In order to show the converse implication, we assume now that $u \in \mathrm{BV}(\Omega)$ and that there exists $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that $u^* = -\operatorname{div} g$ and $\sigma_u = g$ in $L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$. Using Proposition 5, it is sufficient to show that

$$\int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u\operatorname{div} g\,\mathrm{d}x.$$

Taking $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega},\mathbb{R}^d)$ the approximating sequence as in Proposition 2, we have, analogously to the above, that

$$\int_{\partial\Omega} (g_n\cdot\nu_\Omega)u^\Omega\,\mathrm{d}\mathcal{H}^{d-1} \to 0$$

as $n\to\infty$ and, consequently, as $\lim_{n\to\infty} g_n = \sigma_u$ in $L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$,

$$\begin{aligned}\int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| &= \int_\Omega (\sigma_u\cdot\sigma_u)\,\mathrm{d}|\mathrm{D}u| \\ &= \lim_{n\to\infty}\int_\Omega (g_n\cdot\sigma_u)\,\mathrm{d}|\mathrm{D}u| \\ &= \lim_{n\to\infty}\left(-\int_\Omega \operatorname{div}(g_n)u\,\mathrm{d}x - \int_{\partial\Omega}(g_n\cdot\nu_\Omega)u^\Omega\,\mathrm{d}\mathcal{H}^{d-1}\right) \\ &= -\int_\Omega \operatorname{div}(g)u\,\mathrm{d}x. \qquad\square\end{aligned}$$

Proof of Corollary 1. Take $(g_n)_n$ again to be a sequence in $C^\infty(\overline{\Omega},\mathbb{R}^d)$ such that $g_n \overset{\sim}{\to} g \in W_0^q(\mathrm{div};\Omega)$ as $n\to\infty$. Then, using Equation 2, the fact that $g_n(x)\to g(x)$ for every Lebesgue point $x\in\Omega$, and the dominated convergence theorem,

$$\begin{aligned}\int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| &= \lim_{n\to\infty}\left(\int_\Omega g_n\cdot\sigma_u\,\mathrm{d}|\mathrm{D}^a u| + \int_\Omega g_n\cdot\sigma_u\,\mathrm{d}|\mathrm{D}^s u|\right) \\ &= \int_\Omega g\cdot\sigma_u\,\mathrm{d}|\mathrm{D}^a u| + \lim_{n\to\infty}\int_\Omega g_n\cdot\sigma_u\,\mathrm{d}|\mathrm{D}^s u|.\end{aligned}$$

Now if there is a set $A \subset \Omega$ with $|\mathrm{D}^a u|(A) \neq 0$ such that $g \neq \sigma_u$, the right hand side of the above equation can be estimated according to

$$\int_\Omega g \cdot \sigma_u \,\mathrm{d}|\mathrm{D}^a u| + \lim_{n\to\infty} \int_\Omega g_n \cdot \sigma_u \,\mathrm{d}|\mathrm{D}^s u| < \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}^a u| + \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}^s u|$$
$$= \int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u|,$$

which is a contradiction. Thus $g = \sigma_u$, $|\mathrm{D}^a u|$ – almost everywhere. □

Remark 11. *Note that with the proof of Proposition 6, we have in particular shown that assertion 2) of this proposition is already satisfied if $g \in W_0^q(\mathrm{div};\Omega)$ satisfies $\|g\|_\infty \leq 1$ and $\int_\Omega \mathbf{1}\,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g$.*

The following result is the main step towards showing the pointwise subdifferential characterization. It provides the Lebesgue-point property for any $g \in W_0^q(\mathrm{div};\Omega)$ with $\operatorname{div} g \in \partial\,\mathrm{TV}(u)$ and $\|g\|_\infty \leq 1$. The main theorem will then follow immediately.

For convenience, we introduce the following notation: Given two positive real numbers α, ρ, we define

$$\chi_\alpha(x) := \frac{1}{|B_\alpha(0)|}\chi_{B_\alpha(0)}\left(\frac{x}{\alpha}\right)$$

and

$$\Omega_\rho := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \rho\}.$$

Proposition 7. *Let $u \in L^p(\Omega)$ and $g \in W_0^q(\mathrm{div};\Omega)$ with $-\operatorname{div} g \in \partial\,\mathrm{TV}(u)$ and $\|g\|_\infty \leq 1$. Then, the precise representative of g, denoted by*

$$g^*(x) = \begin{cases} \lim_{\epsilon\to 0} \int_\Omega \chi_\epsilon(x-y) g(y)\,\mathrm{d}y & \textit{if this limit exists,} \\ 0 & \textit{else,} \end{cases}$$

coincides with σ_u $|\mathrm{D}u|$ – almost everywhere. Moreover,

$$|\mathrm{D}u| \textit{ – almost every point in } \Omega \textit{ is a Lebesgue point of } g.$$

In view of Corollary 1, the statement means that also $|\mathrm{D}^s u|$ – almost every $x \in \Omega$ is a Lebesgue point of x.

The proof of the proposition will be divided into three steps.

Step 1. *For any $\rho > 0$, any positive sequence $(\alpha_n)_n$ converging to zero and any $\delta > 0$, there exists a set $C \subset \Omega$ such that*

$$|\mathrm{D}u|(C) = 0$$

and for all $x \in \Omega_\rho \setminus C$ there exists $n_0 \geq 0$ such that for all $n \geq n_0$

$$\left| \int_\Omega \chi_{\alpha_n}(x-y) g(y)\,\mathrm{d}y - \sigma_u(x) \right| \leq \delta.$$

Proof. Let $\rho > 0$, $(\alpha_n)_n$ and $\delta > 0$ be given. At first, using Remark 8, we choose $\lambda_0 = \delta/3$ and get

$$\int_{\mathbb{R}^d} |\eta^{\lambda_0}(y) - \chi_1(y)| \,\mathrm{d}y \leq \frac{\delta}{3}.$$

Now according to Remark 12 in the Appendix, we can choose a sequence $(g_n)_n$ such that $g_n \overset{\sim}{\to} g$ as $n \to \infty$, and there exists $n_1 \geq 0$ such that, for all $n \geq n_1$, $x \in \Omega_\rho$,

$$g_n(x) = \int_{\mathbb{R}^d} \eta^{\lambda_0}_{\alpha_n}(x-y) g(y) \,\mathrm{d}y.$$

In particular, the support of $y \mapsto \eta^{\lambda_0}_{\alpha_n}(x-y)$ is contained in Ω. By definition of the convergence $\overset{\sim}{\to}$, there exists a set $A \subset \Omega$ with $|A| = 0$ and

$$g_n(x) \to g(x) \quad \text{for all} \quad x \in \Omega \setminus A \text{ as } n \to \infty.$$

Also, since $-\operatorname{div} g \in \partial \operatorname{TV}(u)$ implies $\int_\Omega \mathbf{1} \,\mathrm{d}|\mathrm{D}u| = -\int_\Omega u \operatorname{div} g$, we can use Remark 11 to get that $(g_n)_n$ converges to σ_u in $L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u|)$. In particular we can choose $B \subset \Omega$ with $|\mathrm{D}u|(B) = 0$ and a subsequence $(n_i)_i$ such that

$$g_{n_i}(x) \to \sigma_u(x) \quad \text{for all} \quad x \in \Omega \setminus B \text{ as } i \to \infty.$$

Defining $\overline{g}$ pointwise for $x \in \Omega$ as

$$\overline{g}(x) := \begin{cases} \lim_{i\to\infty} g_{n_i}(x) & \text{whenever the limit exists,} \\ 0 & \text{else,} \end{cases}$$

we get that $\overline{g}$ is measurable with respect to the sum measure $\mu := |\mathrm{D}u| + \mathcal{L}^d$ by being the pointwise limit of continuous functions up to the set $(A \cap B)$ with $\mu(A \cap B) \leq |\mathrm{D}u|(B) + |A| = 0$. Also, $\|\overline{g}\|_{\infty,\mu} \leq 1$ since $\|g_{n_i}\|_\infty \leq 1$ for all $i \in \mathbb{N}$.

Now by [2, Theorem 1.45], for any $m \in \mathbb{N}_0$, we can choose $\phi_m : \Omega \to \mathbb{R}^d$ continuous with $\|\phi_m\|_{\infty,\mu} \leq 1$ and

$$\mu(M_m) < 2^{-m},$$

where $M_m := \{x \in \Omega \,|\, \phi_m(x) \neq \overline{g}(x)\}$. We then define

$$M = \limsup_{i\to\infty} M_i = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} M_j.$$

It follows that

$$\mu(M) \leq \mu(\bigcup_{j=i}^{\infty} M_j) \leq \sum_{j=i}^{\infty} \mu(M_j) \leq 2^{-i} \sum_{j=0}^{\infty} 2^{-j} \leq 2^{1-i} \quad \text{for all } i \in \mathbb{N}$$

and thus $\mu(M) = 0$.

We define $C = B \cup M$ and pick $x \in \Omega_\rho \setminus C$ arbitrary. First note that $|\mathrm{D}u|(C) = 0$ and, since $x \notin B$, $\overline{g}(x) = \sigma_u(x)$. Also, it follows that $x \notin \bigcup_{j=i_0}^{\infty} M_j$ for some $i_0 \geq 0$ and thus $x \notin M_m$ for all $m \geq i_0$. Thus we can choose $m_0 \in \mathbb{N}$ such that $x \notin M_{m_0}$ and

$$\mu(M_{m_0}) < \delta \left(6\|\eta^{\lambda_0}\|_\infty\right)^{-1}.$$

Note that this implies that $\sigma_u(x) = \overline{g}(x) = \phi_{m_0}(x)$. Furthermore, since $\overline{g} = g$, $\mathcal{L}^d$ – almost everywhere,

$$\begin{aligned}\left|\int_\Omega \eta^{\lambda_0}_{\alpha_n}(x-y)g(y)\,\mathrm{d}y - \sigma_u(x)\right| \leq{}& \left|\int_\Omega \eta^{\lambda_0}_{\alpha_n}(x-y)\phi_{m_0}(y)\,\mathrm{d}y - \phi_{m_0}(x)\right| \\ &+ \left|\int_\Omega \eta^{\lambda_0}_{\alpha_n}(x-y)\left[\overline{g}(y) - \phi_{m_0}(y)\right]\mathrm{d}y\right|.\end{aligned} \tag{3}$$

Choosing $n \geq n_1$ and applying a change of variables, the second term of the right hand side can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^d} |\eta^{\lambda_0}_{\alpha_n}(x-y)[\overline{g}(y) - \phi_{m_0}(y)]|\,\mathrm{d}y &\leq \int_{B_1(0)} \eta^{\lambda_0}(z)|\overline{g}(x+\alpha_n z) - \phi_{m_0}(x+\alpha_n z)|\,\mathrm{d}z \\ &\leq \|\eta^{\lambda_0}\|_\infty \int_\Omega |\overline{g}(z) - \phi_{m_0}(z)|\,\mathrm{d}z \\ &\leq 2\|\eta^{\lambda_0}\|_\infty \mu(M_{m_0}) \leq \frac{\delta}{3}.\end{aligned} \tag{4}$$

The last estimates follow from $\phi_{m_0} = \overline{g}$ on $\Omega \setminus M_{m_0}$, $\|g\|_\infty \leq 1$, $\|\phi_{m_0}\|_\infty \leq 1$ and the choice of M_{m_0}.

Now since ϕ_{m_0} is continuous, the mollified sequence $\int_\Omega \eta^{\lambda_0}_{\alpha_n}(z-y)\phi_{m_0}(y)\,\mathrm{d}y$ converges to $\phi_{m_0}(z)$ for any $z \in \Omega$, thus we can choose $n_0 \geq n_1$ such that for all $n \geq n_0$

$$\left|\int_\Omega \eta^{\lambda_0}_{\alpha_n}(x-y)\phi_{m_0}(y)\,\mathrm{d}y - \phi_{m_0}(x)\right| < \frac{\delta}{3}. \tag{5}$$

In total, we get

$$\begin{aligned}\left|\int_\Omega \chi_{\alpha_n}(x-y)g(y)\,\mathrm{d}y - \sigma_u(x)\right| &\leq \int_\Omega |\chi_{\alpha_n}(x-y)g(y) - \eta^{\lambda_0}_{\alpha_n}(x-y)g(y)|\,\mathrm{d}y \\ &\quad + \left|\int_\Omega \eta^{\lambda_0}_{\alpha_n}(x-y)g(y)\,\mathrm{d}y - \sigma_u(x)\right| \\ &\leq \int_{B_1(0)} \left|\left[\chi_1(z) - \eta^{\lambda_0}(z)\right]g(x+\alpha_n z)\right|\,\mathrm{d}z + \frac{2\delta}{3} \leq \delta. \square\end{aligned}$$

Step 2. *There exists a set $E \subset \Omega$ such that*

$$|\mathrm{D}u|(E) = 0$$

and for any $x \in \Omega \setminus E$

$$\lim_{\epsilon \to 0} \left(\int_{\Omega} \chi_\epsilon(x-y) g(y) \,\mathrm{d}y \right) = \sigma_u(x).$$

Proof. At first we fix some $\rho > 0$. Choosing the sequence $(\frac{1}{m})_{m\in\mathbb{N}}$, according to Step 1, for each $i \in \mathbb{N}$, we can choose $C_i \subset \Omega$ with $|\mathrm{D}u|(C_i) = 0$ such that for all $x \in \Omega_\rho \setminus C_i$ there exists $m_0 \geq 0$ such that for all $m \geq m_0$

$$\left| \int_{\Omega} \chi_{1/m}(x-y) g(y) \,\mathrm{d}y - \sigma_u(x) \right| < \frac{1}{i}.$$

Taking the union of all such C_i we obtain a set $D \subset \Omega$ with $|\mathrm{D}u|(D) = 0$ and, for any $x \in \Omega_\rho \setminus D$,

$$\lim_{m \to \infty} \left(\int_{\Omega} \chi_{1/m}(x-y) g(y) \,\mathrm{d}y \right) = \sigma_u(x).$$

Take now $x \in \Omega_\rho \setminus D$ and choose an arbitrary positive sequence $(\epsilon_n)_n$ converging to zero. For $n \in \mathbb{N}$ fixed we can assume $\epsilon_n \leq 1$ and take $m \in \mathbb{N}$ such that $\frac{1}{m} \geq \epsilon_n > \frac{1}{m+1}$. We estimate

$$\begin{aligned} \left| \int_{\Omega} \chi_{\epsilon_n}(x-y) g(y) \,\mathrm{d}y - \sigma_u(x) \right| &\leq \left| \int_{B_\epsilon(0)} \chi_{\epsilon_n}(x-y) g(y) - \chi_{1/m}(x-y) g(y) \,\mathrm{d}y \right| \\ &\quad + \left| \int_{\Omega} \chi_{1/m}(x-y) g(y) \,\mathrm{d}y - \sigma_u(x) \right|. \end{aligned} \tag{6}$$

Since $\|g\|_\infty \leq 1$, the first expression on the right hand side of (6) can further be estimated by

$$\begin{aligned} \int_{B_{\epsilon_n}(0)} \left| \frac{1}{|B_{\epsilon_n}|} - \frac{1}{|B_{1/m}|} \right| + \int_{B_{1/m}(0) \setminus B_{\epsilon_n}(0)} \frac{1}{|B_{1/m}|} &\leq 1 - \frac{|B_{\epsilon_n}|}{|B_{1/m}|} + \frac{|B_{1/m}| - |B_{\epsilon_n}|}{|B_{1/m}|} \\ &\leq 2 \left(1 - \frac{m^d}{(m+1)^d} \right). \end{aligned} \tag{7}$$

Let now $\delta > 0$ be given. Hence we can choose $n_0 \geq 0$ such that, for all $m \geq n_0$,

$$2\left(1 - \frac{m^d}{(m+1)^d}\right) < \frac{\delta}{2}. \tag{8}$$

Also, we can choose $n_1 \geq n_0$ such that for all $m \geq n_1$

$$\left| \int_\Omega \chi_{1/m}(x-y) g(y)\, \mathrm{d}y - \sigma_u(x) \right| \leq \frac{\delta}{2}. \tag{9}$$

Take $n_2 \geq 0$ such that for all $n \geq n_2$, $|\epsilon_n| < \frac{1}{n_1}$. For arbitrary $n \geq n_2$ fixed, we choose $m \geq n_1$ such that

$$\frac{1}{m} \geq \epsilon_n > \frac{1}{m+1}.$$

Using the estimates (6), (7), (8) and (9), we get

$$\left| \int_\Omega \chi_{\epsilon_n}(x-y) g(y)\, \mathrm{d}y - \sigma_u(x) \right| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and the desired convergence for $x \in \Omega_\rho \setminus D$ follows.

Consequently, for any $i \in \mathbb{N}$ we can choose a set D_i with $|\mathrm{D}u|(D_i) = 0$ and for any $x \in \Omega_{(1/i)} \setminus D_i$,

$$\lim_{\epsilon \to 0} \left(\int_\Omega \chi_\epsilon(x-y) g(y)\, \mathrm{d}y \right) = \sigma_u(x).$$

Taking $E \subset \Omega$ to be the union of all such D_i, the assertion follows. □

Step 3. $|\mathrm{D}u|$ *– almost every point* $x \in \Omega$ *is a Lebesgue point of* g.

Proof. Remember that $|\sigma_u(x)| = 1$, $|\mathrm{D}u|$ – almost everywhere in Ω. We choose $E \subset \Omega$ as in the previous step and $F \subset \Omega$ with $|\mathrm{D}u|(F) = 0$ and $|\sigma_u(x)| = 1$ on $\Omega \setminus F$. It follows that $|\mathrm{D}u|(E \cup F) = 0$ and for $x \in \Omega \setminus (E \cup F)$ we can estimate, for $\mathcal{L}^d$ – almost every $y \in \Omega$,

$$\begin{aligned} \frac{1}{2}|g(y) - \sigma_u(x)|^2 &\leq \frac{1}{2}|g(y) - \sigma_u(x)|^2 + \frac{1}{2}|\sigma_u(x)|^2 - \frac{1}{2}|g(y)|^2 \\ &\leq -(g(y), \sigma_u(x))_{\mathbb{R}^d} + |\sigma_u(x)|^2 = (\sigma_u(x) - g(y), \sigma_u(x))_{\mathbb{R}^d}. \end{aligned}$$

Taking the mean value over $B_\epsilon(x)$ for $\epsilon > 0$ sufficiently small we get

$$\frac{1}{2|B_\epsilon(x)|} \int_{B_\epsilon(x)} |g(y) - \sigma_u(x)|^2\, \mathrm{d}y \leq \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} (\sigma_u(x) - g(y), \sigma_u(x))_{\mathbb{R}^d}\, \mathrm{d}y$$

By Step 2 the limit on the right hand side exists and equals zero, thus

$$\frac{1}{|B_\epsilon(x)|}\int\limits_{B_\epsilon(x)} |g(y)-\sigma_u(x)|\,\mathrm{d}y \le \frac{1}{|B_\epsilon(x)|}\sqrt{\int\limits_{B_\epsilon(x)} |g(y)-\sigma_u(x)|^2\,\mathrm{d}y}\sqrt{|B_\epsilon(x)|}$$
$$= \sqrt{2}\sqrt{\frac{1}{2|B_\epsilon(x)|}\int\limits_{B_\epsilon(x)} |g(y)-\sigma_u(x)|^2\,\mathrm{d}y} \to 0.$$

as $\epsilon \to 0$ and the assertion follows. □

The characterization of the subdifferential $\partial\,\mathrm{TV}(u)$ is now given as follows.

Theorem 1. *Let $u \in L^p(\Omega)$, $u^* \in L^q(\Omega)$. Then*

$$u^* \in \partial\,\mathrm{TV}(u) \quad \textit{if and only if } u \in \mathrm{BV}(\Omega) \textit{ and } u^* = -\operatorname{div} g \textit{ with}$$

1. $\operatorname{div} g \in W_0^q(\mathrm{div};\Omega)$, $\|g\|_\infty \le 1$
2. $|\mathrm{D}u|$ *– almost every point in* Ω *is a Lebesgue point of* g
3. $\sigma_u = g \quad |\mathrm{D}u|$ *– almoste everywhere in* Ω.

Proof. Suppose that $u^* \in \partial\,\mathrm{TV}(u)$. Then $u \in \mathrm{BV}(\Omega)$ and existence of $g \in W_0^q(\mathrm{div};\Omega)$ satisfying all assertions follows from Propositions 5 and 7. Conversely, suppose $u \in \mathrm{BV}(\Omega)$ and $u^* = \operatorname{div} g$ with g satisfying 1) - 3). Take any sequence $(g_n)_n$ in $C^\infty(\overline{\Omega},\mathbb{R}^d)$ such that $g_n \overset{\sim}{\to} g$. Since $(g_n)_n$ converges to g in every Lebesgue point, and since $\sigma_u = g \quad |\mathrm{D}u|$ – almost everywhere, we can apply the dominated convergence theorem to get

$$\lim_{n\to\infty}\int\limits_{\Omega} |g_n(x)-\sigma_u(x)|\,\mathrm{d}|\mathrm{D}u| = \int\limits_{\Omega} \lim_{n\to\infty} |g_n(x)-\sigma_u(x)|\,\mathrm{d}|\mathrm{D}u| = 0.$$

By Proposition 6 the result follows. □

Note that, for g as in Theorem 1, we do not distinguish between g and its precise representative. In particular, we regard g at the same time as a function in $W_0^q(\mathrm{div};\Omega)$ and $L^1(\Omega,\mathbb{R}^d;|\mathrm{D}u|)$.

At last, let us further specify the expression $g = \sigma_u$. This can be done using the decomposition of $\mathrm{D}u$ into an absolute continuous part with respect to the Lebesgue measure, a Cantor part and a jump part, denoted by $\mathrm{D}^a u$, $\mathrm{D}^c u$ and $\mathrm{D}^j u$, respectively [2, Section 3.9]. The absolute continuous part can further be written as $\mathrm{D}^a u = \nabla u\,\mathrm{d}\mathcal{L}^2$ and the jump part as

$$\mathrm{D}^j u = (u^+(x) - u^-(x))\nu_u\,\mathrm{d}\mathcal{H}^{d-1}|_{S_u}$$

where $(u^+(x), u^-(x), \nu_u(x))$ represents uniquely, up to a change of sign, the jump at $x \in J_u$, with J_u and S_u denoting the jump set and the discontinuity set, respectively (see [2, Definition 3.67]). Since the measures $\mathrm{D}^a u$, $\mathrm{D}^c u$ and $\mathrm{D}^j u$ are mutually singular and $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$, the following result follows from Theorem 1.

Proposition 8. *Let the assumptions of Proposition 5 be satisfied. For $u \in L^p(\Omega)$ and $u^* \in L^q(\Omega)$ we have that $u^* \in \partial\,\mathrm{TV}(u)$ if and only if $u \in \mathrm{BV}(\Omega)$ and there exists $g \in W_0^q(\mathrm{div};\Omega)$ with $\|g\|_\infty \le 1$ such that $u^* = -\operatorname{div} g$ and*

$$\begin{aligned} g &= \frac{\nabla u}{|\nabla u|} && \mathcal{L}^d \text{ – almost everywhere on } \Omega \setminus \{x : \nabla u(x) = 0\}, \\ g &= \frac{u^+(x) - u^-(x)}{|(u^+(x) - u^-(x))|}\nu_u && \mathcal{H}^1 \text{ – almost everywhere on } S_u, \\ g &= \sigma_{C_u} && |\mathrm{D}^c u| \text{ – almost everywhere,} \end{aligned}$$

where σ_{C_u} is the density function of $\mathrm{D}^c u$ with respect to $|\mathrm{D}^c u|$.

4 Applications

In this section we present some applications where the pointwise subdifferential characterization of the previous section can be used to extend known results involving the subdifferential of the TV functional. Ω is again always assumed to be a bounded Lipschitz domain. For simplicity, we now restrict ourselves to the two dimensional setting, i.e. $\Omega \subset \mathbb{R}^2$, and use the more common notation $H(\mathrm{div};\Omega)$ for the space $W^2(\mathrm{div};\Omega)$.

As already mentioned in the introduction, the term of normal trace for $H(\mathrm{div};\Omega)$ functions is frequently used to describe the total variational flow, i.e., the solution of the formal equation [3, 4]

$$(\mathcal{P}_F) \begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}\right) & \text{in} \quad (0,\infty) \times \Omega \\ u(0,\cdot) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases}$$

Defining the functional $\mathrm{TV} : L^2(\Omega) \to \overline{\mathbb{R}}$, this corresponds to the evolution problem

$$(\mathcal{P}) \begin{cases} \frac{\partial u(t)}{\partial t} + \partial\,\mathrm{TV}(u(t)) \ni 0 & \text{for} \quad t \in (0,\infty) \\ u(0) = u_0 & \text{in} \quad L^2(\Omega) \end{cases}$$

which appears in the steepest descent method to minimize the TV functional.

A solution to $(\mathcal{P})$ is a continuous function $u : [0,\infty) \to L^2(\Omega)$ with $u(0) = u_0$, which is absolutely continuous on $[a,b]$ for each $0 < a < b$, and hence differentiable almost everywhere, with $\frac{\partial u}{\partial t} \in L^1((a,b), L^2(\Omega))$ and $-\frac{\partial u(t)}{\partial t} \in \partial\,\mathrm{TV}(u(t))$ for almost every $t \in (0,\infty)$.

Using this notation, one gets the following existence result:

Proposition 9. *Let $u_0 \in L^2(\Omega)$. Then there exists a unique solution to $(\mathcal{P})$.*

Proof. Using [15, Corollary I.6.2] it follows that the closure of the domain of $\partial\,\mathrm{TV}$ is already $L^2(\Omega)$ and thus the result follows from [19, Corollary IV.3.2] □

Using Theorem 1 we can now provide an equivalent characterization of a solution to $(\mathcal{P})$. For the proof, we need some properties for the solution which are stated in a lemma.

Lemma 1. *Consider $\partial\,\mathrm{TV}$ as a maximal monotone operator on $L^2(\Omega)$ and denote by*

$$A_0(u) = \operatorname*{arg\,min}_{v\in\partial\,\mathrm{TV}(u)} \|v\|_{L^2}$$

the minimal section *of $\partial\,\mathrm{TV}$.*

If $u_0 \in dom(\partial\,\mathrm{TV})$, then the solution u of $(\mathcal{P})$ satisfies:

(i) $u : [0,\infty) \to L^2(\Omega)$ is right-differentiable with right-derivative D^+u solving

$$D^+u(t) + A_0\big(u(t)\big) = 0 \qquad \text{for all } t \geq 0,$$

(ii) $A_0 \circ u : [0,\infty) \to L^2(\Omega)$, $(A_0 \circ u)(t) = A_0\big(u(t)\big)$ is right-continuous with $t \mapsto \|A_0\big(u(t)\big)\|_{L^2}$ non-increasing,

Proof. The items (i) and (ii) follow directly from [19, Proposition IV.3.1] applied to $\partial\,\mathrm{TV}$. □

The characterization of the total variation flow then reads as follows.

Proposition 10. *A continuous function $u : [0,\infty) \to L^2(\Omega)$ is a solution to $(\mathcal{P})$ if and only if*

(i) u is absolutely continuous on $[a,b]$ for each $0 < a < b$ with derivative $\frac{\partial u}{\partial t} \in L^1((a,b);L^2(\Omega))$,

(ii) $u(t) \in \mathrm{BV}(\Omega)$ for each $t > 0$, $u(0) = u_0$,

(iii) there exists $g \in L^\infty((0,\infty)\times\Omega,\mathbb{R}^d)$ with $\|g\|_\infty \leq 1$ and

(iv) $g : (0,\infty) \to H_0(\mathrm{div};\Omega)$ is measurable with $\frac{\partial u(t)}{\partial t} = \mathrm{div}\, g(t)$ as well as

$$g(t) = \sigma_u(t) \quad \text{in} \quad L^1(\Omega,\mathbb{R}^2;|\mathrm{D}u(t)|)$$

for almost every $t \in (0,\infty)$.

Proof. First note that without loss of generality, we can assume that $u_0 \in \mathrm{dom}(\partial\,\mathrm{TV})$: From [19, Proposition IV.3.2] follows that for each $t_0 > 0$, the translated solution $t \mapsto u(t+t_0)$ solves $(\mathcal{P})$ with initial value $u(t_0) \in \mathrm{dom}(\partial\,\mathrm{TV})$. Consequently, if the claimed statements are true on each $[t_0,\infty)$, then also on $(0,\infty)$.

Choose $L > 0$. We will now approximate u on $[0,L)$ as well as $\frac{\partial u}{\partial t}$ by piecewise constant functions as follows. Denote by $0 = t_0 < t_1 < \ldots < t_K = L$ a partition of $[0,L)$. For $t \in [0,L)$ denote by $k(t) = \min\,\{k' : t_{k'} > t\}$ as well as $\tau(t) = t_{k(t)} - t_{k(t)-1}$. For each $\varepsilon > 0$ we can now choose, due to the uniform continuity of u on $[0,L]$, a partition which satisfies

$$\|u(t) - u(t_{k(t)})\|_{L^2} < \varepsilon.$$

for all $t \in [0, L)$. It is moreover possible to achieve that these partitions are nested which implies that $t_{k(t)} \to t$, $\tau(t) \to 0$ as $\varepsilon \to 0$, both monotonically decreasing. Then, the function

$$u^\varepsilon : [0, L) \to L^2(\Omega), \qquad u^\varepsilon(t) = u(t_{k(t)})$$

obviously converges to u in $L^\infty((0, L), L^2(\Omega))$. Likewise, the function

$$(u^\varepsilon)' : [0, L) \to L^2(\Omega), \qquad (u^\varepsilon)'(t) = -A_0\bigl(u(t_{k(t)})\bigr)$$

satisfies, on the one hand, $-(u^\varepsilon)'(t) \in \partial\,\mathrm{TV}\bigl(u^\varepsilon(t)\bigr)$ for $t \in [0, L)$ by definition of A_0, see Lemma 1. On the other hand, for $t \in [0, L)$, we have $t_{k(t)} \to t$ monotonically decreasing, which implies by the right continuity of $t \mapsto A_0\bigl(u(t)\bigr)$, see Lemma 1, that

$$\lim_{\varepsilon \to 0} (u^\varepsilon)'(t) = -A_0\bigl(u(t)\bigr) \qquad \text{in } L^2(\Omega).$$

Also $\|(u^\varepsilon)'(t)\|_2 \leq \|A_0(u_0)\|_2$, again by Lemma 1, so there exists an integrable majorant and by Lebesgue's theorem, $\lim_{\varepsilon \to 0} (u^\varepsilon)' = -A_0 \circ u$ in $L^2((0, L), L^2(\Omega))$. However, Lemma 1 yields $-A_0 \circ u = D^+ u$, so $(u^\varepsilon)'$ is indeed approximating $\frac{\partial u}{\partial t}$.

As each u^ε, $(u^\varepsilon)'$ is constant on the finitely many intervals $[t_{k(t)-1}, t_{k(t)})$ and $-(u^\varepsilon)'(t) \in \partial\,\mathrm{TV}\bigl(u^\varepsilon(t)\bigr)$, we can choose a vector field g according to Proposition 5 on each of these intervals. Composing these g yields a measurable $g^\varepsilon \in L^2((0, L); H_0(\mathrm{div}, \Omega))$, $\|g^\varepsilon\|_\infty \leq 1$ in $L^\infty((0, L) \times \Omega, \mathbb{R}^d)$ and such that $(u^\varepsilon)' = \mathrm{div}\, g^\varepsilon$ in the weak sense. Moreover,

$$\int_0^L \int_\Omega \mathbf{1} \,\mathrm{d}|\mathrm{D}u^\varepsilon(t)| \,\mathrm{d}t = -\int_0^L \int_\Omega u^\varepsilon \,\mathrm{div}\, g^\varepsilon \,\mathrm{d}x \,\mathrm{d}t. \tag{10}$$

Now, $\{g^\varepsilon\}$ is bounded in $L^2((0, L), H_0(\mathrm{div}, \Omega))$, hence there exists a weakly convergent subsequence (not relabeled) and a limit g with $\|g\|_\infty \leq 1$ in $L^\infty((0, L) \times \Omega, \mathbb{R}^d)$. In particular, as $(u^\varepsilon)' = \mathrm{div}\, g^\varepsilon$, we have $\mathrm{div}\, g^\varepsilon \to \frac{\partial u}{\partial t}$ in $L^2((0, L), L^2(\Omega))$. By weak closedness of the divergence operator, also $\mathrm{div}\, g = \frac{\partial u}{\partial t}$.

Finally, taking the limits in (10) yields

$$\int_0^L \int_\Omega \mathbf{1} \,\mathrm{d}|\mathrm{D}u| \,\mathrm{d}t \leq \liminf_{\varepsilon \to 0} \int_0^L \int_\Omega \mathbf{1} \,\mathrm{d}|\mathrm{D}u^\varepsilon| \,\mathrm{d}t = -\int_0^L \int_\Omega u \,\mathrm{div}\, g \,\mathrm{d}x \,\mathrm{d}t.$$

On the other hand, as for almost every $t \in (0, L)$, $g \in H_0(\mathrm{div}; \Omega)$ and $\|g(t)\|_\infty \leq 1$, according to Remark 10 it follows that $-\int_\Omega u(t) \,\mathrm{div}\, g(t) \leq \mathrm{TV}\bigl(u(t)\bigr)$. Hence, the above is only possible if $-\int_\Omega u(t) \,\mathrm{div}\, g(t) = \mathrm{TV}\bigl(u(t)\bigr)$ for almost every $t \in (0, L)$. This yields $-\mathrm{div}\, g(t) \in \partial\,\mathrm{TV}(u(t))$ for almost every $t \in (0, L)$ and by Proposition 7,

$$g(t) = \sigma_u(t) \quad \text{in} \quad L^1(\Omega, \mathbb{R}^d; |\mathrm{D}u(t)|) \qquad \text{for almost every} \quad t \in (0, L).$$

Conversely, if we now assume that $u : [0, \infty) \to L^2(\Omega)$ satisfies i - iv, in order to establish that u is a solution to $(\mathcal{P})$ it is left to show that $-\frac{\partial u(t)}{\partial t} \in \partial\,\mathrm{TV}(u(t))$

for almost every $t \in (0,\infty)$. But since at almost every $t \in (0,\infty)$ we have, for $g \in L^\infty((0,\infty) \times \Omega, \mathbb{R}^d)$ as in iii, that $g(t) \in H_0(\operatorname{div};\Omega)$, $\|g(t)\|_\infty \le 1$, $\frac{\partial u(t)}{\partial t} = \operatorname{div} g(t)$ and $g(t) = \sigma_u(t)$, this follows as immediate consequence of Theorem 1. □

In a related context, a Cheeger set [13, 18] of a bounded set G of finite perimeter [2, Section 3.3] is defined to be the minimizer of

$$\min_{A \subset \overline{G}} \frac{|\partial A|}{|A|}. \tag{11}$$

Defining the constant

$$\lambda_G = \frac{|\partial G|}{|G|},$$

a sufficient condition for G to be a Cheeger set of itself, or in other words to be calibrable, is that $v := \chi_G$ satisfies the equation [8, Lemma 3]

$$-\operatorname{div}(\sigma_v) = \lambda_G v \quad \text{on } \mathbb{R}^2, \tag{12}$$

i.e. there exists a vector field $\xi \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $\|\xi\|_\infty \le 1$,

$$-\operatorname{div}\xi = \lambda_G v \quad \text{on } \mathbb{R}^2$$

and

$$\int_{\mathbb{R}^2} \theta(\xi, \mathrm{D}v)\, \mathrm{d}|\mathrm{D}v| = \int_{\mathbb{R}^2} \mathbf{1}\, \mathrm{d}|\mathrm{D}v|.$$

This condition is further equivalent to [8, Theorem 4]:

1. G is convex.
2. ∂G is of class $C^{1,1}$.
3. It holds
$$\operatorname*{ess\,sup}_p \kappa_{\partial G}(p) \le \frac{P(G)}{|G|},$$

where $\kappa_{\partial G}$ is the curvature of ∂G. Using the pointwise subdifferential characterization, we can provide the following sufficient condition for G being calibrable:

Proposition 11. *Let $G \subset \mathbb{R}^2$ be a bounded set of finite perimeter. Then $v = \chi_G \in \mathrm{BV}(\mathbb{R}^2)$ satisfies condition (12) if there exists a bounded Lipschitz domain K such that $\overline{G} \subset K$ and $\xi \in H_0(\operatorname{div};K)$ with $\|\xi\|_\infty \le 1$ such that*

$$-\operatorname{div}\xi = \lambda_G v \quad \text{on } K$$

and

$$\xi = \nu_G \quad \mathcal{H}^1 - \text{almost everywhere on } \mathcal{F}G,$$

where $\mathcal{F}G$ is the reduced boundary, i.e. the set of all points $x \in \operatorname{supp}|\mathrm{D}\chi_G|$ such that the limit

$$\nu_G(x) := \lim_{\rho \to 0^+} \frac{\mathrm{D}\chi_G(B_\rho(x))}{|\mathrm{D}\chi_G(B_\rho(x))|}$$

exists.

Proof. The proof is straightforward: Using that $|\mathrm{D}\chi_G| = \mathcal{H}^1|_{\mathcal{F}G}$ and that $\mathrm{D}\chi_G = \nu_G|\mathrm{D}\chi_G|$ [2, Section 3.5] it follows that

$$\int_K |\mathrm{D}v| = \int_K \xi \cdot \nu_G \,\mathrm{d}|\mathrm{D}v| = \int_K \theta(\xi, \mathrm{D}v)\,\mathrm{d}|\mathrm{D}v|.$$

From this and the fact that $\xi \in H_0(\mathrm{div}; K)$ it follows that its extension by 0 to the whole $\mathbb{R}^2$ is contained in $H(G; \mathbb{R}^2)$ and satisfies condition (12). □

The subdifferential characterization can of course also be used to formulate optimality conditions for optimization problems appearing in mathematical imaging. A typical problem formulation would be

$$\min_{u \in L^2(\Omega)} \mathrm{TV}(u) + F(u), \tag{13}$$

where TV plays the role of a regularization term and $F : L^2(\Omega) \to \overline{\mathbb{R}}$ reflects data fidelity. Under weak assumptions on F we can derive the following general optimality condition:

Proposition 12. *Suppose that $F : L^2(\Omega) \to \overline{\mathbb{R}}$ is such that $\partial(\mathrm{TV} + F) = \partial\,\mathrm{TV} + \partial F$. Then we have that $u \in L^2(\Omega)$ solves* (13) *if and only if there exists $g \in H_0(\mathrm{div}; \Omega)$ such that $\|g\|_\infty \leq 1$,*

$$\mathrm{div}\, g \in \partial F(u)$$

and

$$g = \sigma_u \quad \textit{in} \quad L^1(\Omega, \mathbb{R}^2; |\mathrm{D}u|)$$

Proof. This follows immediately from $\partial(\mathrm{TV} + F) = \partial\,\mathrm{TV} + \partial F$ and the characterization of $\partial\,\mathrm{TV}$ in Theorem 1. □

In [20], a problem of this type, but with a generalized regularization term was considered. Existence and a characterization of solutions to

$$\min_{u \in \mathrm{BV}} \int_\Omega \varphi(|\mathrm{D}u|) + \int_\Omega |Ku - u_0|^2$$

was shown, a problem which appears in denoising, deblurring or zooming of digital images.. For the characterization of optimal solutions, again the term $g \cdot \sigma_u$, with $g \in H(\mathrm{div}; \Omega)$, was associated to a measure and then, following [14], it was split into a measure corresponding the absolute continuous part of $\mathrm{D}u$ with respect to the Lebesgue measure and a singular part. By applying Propositions 8 and 12, we can now get a characterization of solutions similar to [20, Proposition 4.1], but in terms of $L^1(\Omega, \mathbb{R}^2; |\mathrm{D}u|)$ functions, for the special case that φ is the identity:

Proposition 13. *Let $u_0 \in L^2(\Omega)$ and $K : L^2(\Omega) \to L^2(\Omega)$ a continuous, linear operator. Then, $u \in L^2(\Omega)$ is a solution to*

$$\min_{u \in \mathrm{BV}} \int_\Omega |\mathrm{D}u| + \int_\Omega |Ku - u_0|^2$$

if and only if $u \in \mathrm{BV}(\Omega)$ and there exists $g \in H_0(\mathrm{div};\Omega)$ with $\|g\|_\infty \leq 1$ such that

$$2K^*(Ku - u_0) = \operatorname{div} g$$

and

$$\begin{aligned} g &= \frac{\nabla u}{|\nabla u|} && \mathcal{L}^2 \text{ – almost everywhere on } \Omega \setminus \{x : \nabla u(x) = 0\} \\ g &= \frac{u^+(x) - u^-(x)}{|(u^+(x) - u^-(x))|}\nu_u && \mathcal{H}^1 \text{ – almost everywhere on } S_u \\ g &= \sigma_{C_u} && |\mathrm{D}^c u| \text{ – almost everywhere,} \end{aligned}$$

where $u^+, u^-, \nu_u, S_u, C_u, \nabla u$ and $|\mathrm{D}^c u|$ are defined as in Proposition 8 and its preceding paragraph.

Proof. By continuity of $F(u) = \int_\Omega |Ku - u_0|^2$ it follows that $\partial(\mathrm{TV} + F) = \partial\,\mathrm{TV} + \partial F$ and we can apply Proposition 12. The characterization follows then by Proposition 8 and the fact that $\partial F(v) = \{2K^*(Ku - u_0)\}$ for any $v \in L^2(\Omega)$. □

The general formulation of an imaging problem as in (13) also applies, for example, to the minimization problem presented in [10]: There, as part of an infinite dimensional modeling of an improved JPEG reconstruction process, one solves

$$\min_{u \in L^2(\Omega)} \mathrm{TV}(u) + \mathcal{I}_U(u) \tag{14}$$

where $U = \{u \in L^2(\Omega) \,|\, Au \in J_n \text{ for all } n \in \mathbb{N}\}$, $A : L^2(\Omega) \to \ell^2$ is a linear basis transformation operator and $(J_n)_{n\in\mathbb{N}} = ([l_n, r_n])_{n\in\mathbb{N}}$ a given data set. Under some additional assumptions, a necessary and sufficient condition for u being a minimizer of (14) is stated in [10, Theorem 5]. Using the pointwise subdifferential characterization, this condition can now be extended as follows:

Proposition 14. *With the assumptions of [10, Theorem 5], the function $u \in L^2(\Omega)$ is a minimizer of (14) if and only if $u \in \mathrm{BV}(\Omega) \cap U$ and there exists $g \in H_0(\mathrm{div};\Omega)$ satisfying*

1. $\|g\|_\infty \leq 1$,
2. $g = \sigma_u$, $|\mathrm{D}u|$ *– almost everywhere,*
3. $$\begin{cases} (\operatorname{div} g, a_n)_{L^2} \geq 0 & \text{if } (Au)_n = r_n \neq l_n, \\ (\operatorname{div} g, a_n)_{L^2} \leq 0 & \text{if } (Au)_n = l_n \neq r_n, \\ (\operatorname{div} g, a_n)_{L^2} = 0 & \text{if } (Au)_n \in \overset{\circ}{J}_n, \end{cases} \qquad \forall n \in \mathbb{N}.$$

5 Conclusion

We have achieved a pointwise characterization of the subdifferential of the total variation functional when considered as a functional from $L^p(\Omega)$ to the extended reals. This characterization gives an analytical justification for the notation

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \in \partial \operatorname{TV}(u),$$

frequently used in mathematical imaging problems related to TV minimization. We further have shown that the characterization can be used in several applications, for example, to characterize the total variational flow.

A An approximation result

Since existence of a suitable approximating sequence for $W^q(\operatorname{div};\Omega)$-vector fields is frequently used in this work, we give here an example of how to construct such a sequence. For Ω a bounded Lipschitz domain, $1 \leq q < \infty$ and $g \in W^q(\operatorname{div};\Omega)$, we have to show existence of $(g_n)_{n\geq 0} \subset C^\infty(\overline{\Omega}, \mathbb{R}^d)$ satisfying:

1. $\|g_n - g\|_{W^q(\operatorname{div})} \to 0 \quad \text{as} \quad n \to \infty$,
2. $\|g_n\|_\infty \leq \|g\|_\infty$ for each $n \in \mathbb{N}$ if $g \in L^\infty(\Omega, \mathbb{R}^d) \cap W^q(\operatorname{div};\Omega)$,
3. $g_n(x) \to g(x)$ for every Lebesgue point $x \in \Omega$ of g,
4. $\|g_n - g\|_{\infty,\overline{\Omega}} \to 0$ as $n \to \infty$, if, additionally, $g \in C(\overline{\Omega}, \mathbb{R}^d)$.

Proof. The proof follows basic ideas presented in [16, Theorem 4.2.3] for a density proof for Sobolev functions. We make use of the Lipschitz property of $\partial\Omega$: For $x \in \partial\Omega$, take $r > 0$ and $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ Lipschitz continuous, such that – upon rotating and relabeling the coordinate axes if necessary – we have

$$\Omega \cap Q_r(x) = \{y \in \mathbb{R}^d \,|\, \gamma(y_1, \ldots, y_{d-1}) < y_d\} \cap Q_r(x) \tag{15}$$

where $Q_r(x) = \{y \in \mathbb{R}^d \,|\, |y_i - x_i| < r \,,\, i = 1, .., d\}$. Now for fixed $x \in \partial\Omega$, we define $Q = Q_r(x)$ and $Q' = Q_{\frac{r}{2}}(x)$. In the first step, we suppose that

$$\operatorname{spt}(g) := \overline{\{y \in \Omega : g(y) \neq 0\}} \subset Q'$$

and show that there exist vector fields $g_\epsilon \in C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging, as $\epsilon \to 0$, to g – in $W^q(\operatorname{div};\Omega)$, pointwise in every Lebesgue point $y \in \Omega$ and uniformly on $\overline{\Omega}$ if additionally $g \in C(\overline{\Omega}, \mathbb{R}^d)$ – and satisfying the boundedness property 2).

Choose $\alpha = \operatorname{Lip}(\gamma) + 2$ fixed and $0 < \epsilon < \frac{r}{2(\alpha+1)}$ arbitrarily. It follows then by straightforward estimations that, for any $y \in \overline{\Omega \cap Q'}$, with $y^\epsilon = y + \epsilon\alpha e_d$, where e_d is the dth coordinate vector according to (15), we have $\overline{B_\epsilon(y^\epsilon)} \subset \Omega \cap Q$.

Now with $\eta : \mathbb{R}^d \to \mathbb{R}$ a standard mollifier kernel supported in the unit ball, we define

$$\eta_\epsilon(y) = \frac{1}{\epsilon^d}\eta\left(\frac{y}{\epsilon}\right).$$

Using that $\overline{B_\epsilon(y^\epsilon)} \subset \Omega \cap Q$, for $y \in \overline{\Omega \cap Q'}$, it follows that the support of the functions

$$x \mapsto \eta_\epsilon(y + \epsilon\alpha e_d - x)$$

is contained in $\Omega \cap Q$. Thus, for $1 \leq j \leq d$, the functions $g^j_\epsilon : \overline{\Omega \cap Q'} \to \mathbb{R}$,

$$\begin{aligned} g^j_\epsilon(y) &= \int_{\mathbb{R}^d} \eta_\epsilon(y + \epsilon\alpha e_d - x) g^j(x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \eta_\epsilon(y - z) g^j(z + \epsilon\alpha e_d)\, \mathrm{d}z = \left(\eta_\epsilon * g^j_{S_\epsilon}\right)(y), \end{aligned} \tag{16}$$

where

$$g^j_{S_\epsilon}(y) := g^j(y + \epsilon\alpha e_d)$$

denotes the composition of g^j with a translation operator, are well defined. Using standard results, given for example in [1, Section 2.12 and Proposition 2.14], it follows that $g^j_\epsilon \in C^\infty(\overline{\Omega \cap Q'})$ and, extending by 0 outside of $\overline{\Omega \cap Q'}$, that

$$\begin{aligned} \|g^j_\epsilon - g^j\|_{L^q(\Omega\cap Q')} &\leq \|\eta_\epsilon * g^j_{S_\epsilon} - \eta_\epsilon * g^j\|_{L^q(\mathbb{R}^d)} + \|\eta_\epsilon * g^j - g^j\|_{L^q(\mathbb{R}^d)} \\ &\leq \|\eta_\epsilon\|_{L^1(\mathbb{R}^d)}\|g^j_{S_\epsilon} - g^j\|_{L^q(\mathbb{R}^d)} + \|\eta_\epsilon * g^j - g^j\|_{L^q(\mathbb{R}^d)} \to 0 \end{aligned}$$

as $\epsilon \to 0$. By equivalence of norms in $\mathbb{R}^d$ it thus follows that the vector valued functions $g_\epsilon = (g^1_\epsilon, \ldots, g^d_\epsilon)$ are contained in $C^\infty(\overline{\Omega \cap Q'})$ and that $\|g_\epsilon - g\|_{L^q(\Omega\cap Q')} \to 0$ as $\epsilon \to 0$. Since, for $i \in \{1 \ldots d\}$,

$$\partial_i(\eta_\epsilon * g^j_{S_\epsilon}) = \partial_i\eta_\epsilon * g^j_{S_\epsilon},$$

we have, for $y \in \overline{\Omega \cap Q'}$, that

$$\begin{aligned} \operatorname{div} g_\epsilon(y) &= \int_{\mathbb{R}^d} \nabla_y(\eta_\epsilon(y - x)) \cdot g_{S_\epsilon}(x)\, \mathrm{d}x \\ &= \int_{\Omega\cap Q} \nabla_y(\eta_\epsilon(y + \epsilon\alpha e_d - z)) \cdot g(z)\, \mathrm{d}z \\ &= -\int_{\Omega\cap Q} \nabla_z(\eta_\epsilon(y + \epsilon\alpha e_d - z)) \cdot g(z)\, \mathrm{d}z \\ &= \int_{\Omega\cap Q} (\eta_\epsilon(y + \epsilon\alpha e_d - z)) \operatorname{div} g(z)\, \mathrm{d}z \\ &= \int_{\mathbb{R}^d} (\eta_\epsilon(y + \epsilon\alpha e_d - z)) \operatorname{div} g(z)\, \mathrm{d}z, \end{aligned}$$

where we used that $x \mapsto \eta_\epsilon(y + \epsilon\alpha e_d - x) \in C_c^\infty(\Omega \cap Q)$ and the weak definition of div. An argumentation analogous to the above thus yields $\|\operatorname{div} g_\epsilon - \operatorname{div} g\|_{L^q(\Omega\cap Q')} \to 0$ as $\epsilon \to 0$. Now let $y \in \Omega \cap Q'$ be a Lebesgue point of g. Again by equivalence of norms it suffices to show that $g_\epsilon^j(y) \to g^j(y)$ for y being a Lebesgue point of g^j, $1 \le j \le d$. With $\epsilon > 0$ sufficiently small such that, with $t := 1 + \alpha$, we have $B_{\epsilon t}(y) \subset \Omega \cap Q$ we can estimate

$$\begin{aligned}
|g_\epsilon^j(y) - g^j(y)| &= \left| \frac{1}{\epsilon^d} \int\limits_{\mathbb{R}^d} \eta\Big(\frac{y-w}{\epsilon}\Big)\big(g^j(w + \epsilon\alpha e_n) - g^j(y)\big)\, \mathrm{d}w \right| \\
&\le C(d) \frac{1}{|B_\epsilon(y)|} \int\limits_{B_\epsilon(y)} |g^j(w + \epsilon\alpha e_n) - g^j(y)|\, \mathrm{d}w \\
&= C(d) \frac{1}{|B_\epsilon(y)|} \int\limits_{B_\epsilon(y+\epsilon\alpha e_n)} |g^j(w) - g^j(y)|\, \mathrm{d}w \\
&\le \tilde{C}(d) \frac{1}{|B_{\epsilon t}(y)|} \int\limits_{B_{\epsilon t}(y)} |g^j(w) - g^j(y)|\, \mathrm{d}w,
\end{aligned}$$

with $C(d), \tilde{C}(d) > 0$ constants depending only on d. Now since y was assumed to be a Lebesgue point of g^j, the desired convergence follows.

Now, additionally suppose that $g \in C(\overline{\Omega}, \mathbb{R}^d)$. Note that $\epsilon > 0$ can also be chosen such that with $\tau = \alpha + 1$, $B_{\epsilon t} \subset \Omega \cap Q$ for all $y \in \overline{\Omega \cap Q'}$, so the above implies

$$\begin{aligned}
|g_\epsilon^j(y) - g^j(y)| &\le \tilde{C}(d) \frac{1}{|B_{\epsilon t}(y)|} \int\limits_{B_{\epsilon t}(y)\cap\Omega\cap Q} |g^j(w) - g^j(y)|\, \mathrm{d}w \\
&\le \tilde{C}(d) \sup_{w \in \overline{B_{\epsilon t}(y)\cap\Omega\cap Q}} \big(|g^j(w) - g^j(y)|\big).
\end{aligned}$$

By uniform continuity of g in the compact set $\overline{\Omega}$ it follows that $\|g_\epsilon^j - g^j\|_{\infty,\overline{\Omega\cap Q}}$ – and thus also $\|g_\epsilon - g\|_{\infty,\overline{\Omega\cap Q'}}$ – converges to zero as $\epsilon \to 0$.

Next we estimate the sup-norm of g_ϵ: Suppose $\|g\|_\infty \le C$. For $y \in \overline{\Omega \cap Q'}$ we then have:

$$\begin{aligned}
|g_\epsilon(y)|^2 &= \frac{1}{\epsilon^{2d}} \sum_{i=1}^d \left(\int\limits_{\Omega\cap Q} \sqrt{\eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)} \sqrt{\eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)} g^i(w)\, \mathrm{d}w \right)^2 \\
&\le \frac{1}{\epsilon^{2d}} \left(\int\limits_{\Omega\cap Q} \eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big) \sum_{i=1}^d g^i(w)^2\, \mathrm{d}w \right) \cdot \\
&\quad \left(\int\limits_{\Omega\cap Q} \eta\Big(\frac{y-w}{\epsilon} + \alpha e_n\Big)\, \mathrm{d}w \right) \\
&\le C^2.
\end{aligned}$$

At last, since $\mathrm{spt}(g) \subset Q'$ it follows that $\mathrm{spt}(g_\epsilon) \subset Q'$ for sufficiently small ϵ and thus we can extend it by 0 to the rest of $\overline{\Omega}$. Note that the convergence of g_ϵ to g – in $W^q(\Omega, \mathrm{div})$, in every Lebesgue point $y \in \Omega \setminus Q'$ and uniformly on $\overline{\Omega}$ in the case that additionally $g \in C(\overline{\Omega}, \mathbb{R}^d)$ – and also the uniform boundedness on all of $\overline{\Omega}$ are trivially satisfied.

In the second step we make use of the previous calculations to get an approximation to g without additional assumptions: Since $\partial\Omega$ is compact, there exist finitely many cubes $Q'_i = Q_{\frac{r_i}{2}}(x_i)$, $1 \le i \le M$ as above, which cover $\partial\Omega$. Let $(\zeta_i)_{0\le i\le M}$ be C^∞-functions, such that

$$\begin{cases} 0 \le \zeta_i \le 1 \quad \mathrm{spt}(\zeta_i) \subset Q'_i \quad \text{for } 1 \le i \le M, \\ 0 \le \zeta_0 \le 1 \quad \mathrm{spt}(\zeta_0) \subset \Omega, \\ \displaystyle\sum_{i=0}^{M} \zeta_i \equiv 1 \quad \text{on } \Omega. \end{cases}$$

As shown above, for $g\zeta_i$, $1 \le i \le M$ we can construct vector fields $g_{\epsilon,i} \in C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to $g\zeta_i$ in the desired sense. Using any standard mollifier kernel η, we get that also

$$g_{\epsilon,0}(x) := \int_\Omega \eta_\epsilon(x-y)g(y)\zeta_0(y) \tag{17}$$

converges to $g\zeta_0$ in the desired sense. Setting

$$g_\epsilon = \sum_{i=0}^{M} g_{\epsilon,i}$$

we finally obtain vector fields in $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ converging to g in $W^q(\mathrm{div};\Omega)$ as $\epsilon \to 0$ and, as one can check easily, satisfying also the additional boundedness and convergence properties 2), 3), 4). □

Remark 12. *Given any $g \in W^q(\mathrm{div};\Omega)$ and $\rho > 0$, the above construction allows to get an approximating sequence $(g_\epsilon)_\epsilon$ satisfying the properties 1) to 4) such that additionally, for ϵ sufficiently small and any $x \in \Omega_\rho = \{x \in \Omega \mid dist(x, \partial\Omega) > \rho\}$,*

$$g_\epsilon(x) = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)g(y)$$

with η an arbitrary mollifier kernel.

Indeed, in the construction of $(g_\epsilon)_\epsilon$ we can choose the cubes Q'_i covering $\partial\Omega$ sufficiently small, and hence get for the cut-off functions $(\zeta_i)_i$ that

$$\sum_{i=0}^{M} \zeta_i(y) = \zeta_0(y) = 1$$

for $y \in \Omega_{\rho-\epsilon}$. Then, from the definition of $g_{\epsilon,0}$ in Equation (17), the assertion follows.

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