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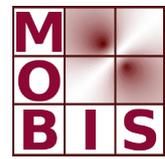
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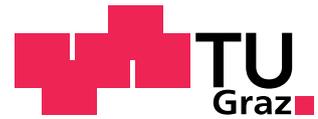
A smooth embedding domain method based on the penalty approach

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A smooth embedding domain method based on the penalty approach

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Abstract

It is well known that convergence of the fictitious domain formulation with boundary Lagrange multipliers is slow due to the lower global regularity of its solution. The paper presents a smoothed variant of this approach which is based on a formulation in the form of a state constraint optimal control problem. The convergence rate is increased as seen from a model example.

1   Introduction

Fictitious domain methods represent an efficient tool for the numerical realization of elliptic equations. The idea of such methods is simple: a boundary value problem defined in ω is replaced by a new one formulated in a domain Ω having a simple shape (a box, e.g.) and which contains ω in its interior. The new problem in Ω is defined in such a way that its solution when restricted to ω coincides with the solution of the original problem. One benefit of this approach is readily seen: we can use special (for ex. uniform) finite element partitions of Ω generating stiffness matrices with a particular structure which makes it possible to solve the resulting systems of linear algebraic equations by fast methods. A possible way to formulate the new problem in Ω is based on the use of boundary Lagrange multipliers. To illustrate the idea, we suppose that Dirichlet boundary conditions are prescribed on $\partial\omega$. This condition can be viewed as the constraint which will be treated by Lagrange multipliers defined on $\partial\omega$. Thus the new formulation in Ω involves two unknowns: the primal variable $u \in V$ and the corresponding Lagrange multiplier $\lambda \in \Lambda$ releasing the prescribed Dirichlet data on $\partial\omega$. For a discretization one can

use standard mixed finite element methods [2]. Let V_h, Λ_H be finite dimensional approximations of V and Λ, respectively. If partitions of Ω used for the construction of V_h do not respect the geometry of ω, yet another important fact is remarkable: the stiffness matrix A corresponding to V_h does not depend on ω. Thus it can be assembled and stored once for all. The only information on ω is encoded in the off-diagonal blocks of the resulting saddle-point matrix coupling the primal with dual variables. This property of A can be utilized in situations when the domain ω changes many times during computations (in shape optimization, free boundary problems, etc., for details see [7, 4]). It turned out however that this fictitious domain method converges slowly due to lack of global regularity of the solution u in Ω ([3]). The singularity of u is concentrated on $\partial\omega$. To improve the convergence rate an alternative approach was proposed in [6] which is based on approximate controllability arguments. A space of controls defined on a closed curve Γ lying in $\Omega \setminus \bar{\omega}$ was introduced which force the boundary condition on $\partial\omega$ to be satisfied with an arbitrarily small error measured in the $L^2(\partial\omega)$- norm. The solution to this new formulation still has a singularity in Ω but now located on Γ instead of $\partial\omega$. Thus in a vicinity of $\partial\omega$ the solution is smooth. This simple idea improved considerably the convergence rate. Unfortunately this new formulation and its discretization may have more than one solution, in general. Theoretically, uniqueness of the solution to the discretized problem is guaranteed provided that the distance between Γ and $\partial\omega$ is small enough. This paper presents a new formulation which overcomes this drawback since it has a unique solution to both, the continuous as well as the discrete setting. This formulation is given by a state constrained optimization problem in which the $L^2(\partial\omega)$-norm of the distance between the trace of the solution and the prescribed Dirichlet data bounded from above by a given positive constant ε represents the state constraint and the norm of the control variable serves as the cost functional. Taking a sequence of such optimization problems and letting ε tend to zero (i.e. satisfying the boundary condition on $\partial\omega$ with increasing accuracy) and simultaneously approaching Γ to $\partial\omega$ with an appropriate "velocity" depending on ε, we obtain in the limit the original (non-smooth) fictitious domain formulation with boundary Lagrange multipliers on $\partial\omega$.

The paper is organized as follows: Section 2 presents two variants of the fictitious domain formulation of non-homogeneous Dirichlet boundary value problems: the classical (non-smooth) one with Lagrange multipliers on the boundary and its smooth counterpart formulated as a state constraint optimal control problem. Section 3 contains main theoretical results. First, the stability with respect to variations of ω of the non-smooth variant of the fictitious domain approach is recalled. On the basis of this result we prove that under appropriate assumptions, the solutions to this new formulation are bounded. This enables us to pass to the limit in the

sequence of the smooth variants and to show that the respective limit is the solution to the original non-smooth setting. To realize the state constraint problem we use a penalty technique described in Section 4. Finally, in Section 5 we present numerical results of a model example which illustrate the efficiency of this new approach. Throughout the paper we use standard notation: $H^k(D)$, $k \geq 0$ integer, stands for the Sobolev space of functions which together with their generalized derivatives up to order k are square integrable in D, i.e. they belong to $L^2(D)$ $(H^0(D))$. The norm in $H^k(D)$ is denoted by $|| \cdot ||_{k,D}$ and the scalar product in $L^2(D)$ by $(\cdot, \cdot)_{0,D}$.

2 Formulation of a smooth embedding domain method

We consider a nonhomogeneous Dirichlet boundary value problem in a bounded domain $\omega_\gamma \subset \mathbb{R}^2$ with Lipschitz boundary $\gamma := \partial\omega_\gamma$:

$$\left. \begin{array}{rcl} -\triangle u = f & \text{in} & \omega_\gamma, \\ u = g & \text{on} & \gamma, \end{array} \right\} \tag{1}$$

where $f \in L^2_{loc}(\mathbb{R}^2)$ and $g \in H^{1/2}(\gamma)$ are given. Embedding ω_γ into a larger computational domain Ω such that $\bar\omega_\gamma \subset \Omega$, for example a rectangle (see Figure 1), the standard embedding domain approach replaces (1) by the saddle-point problem

$$\left. \begin{array}{l} Find \ (\hat{u}, \hat{\lambda}) \in H_0^1(\Omega) \times H^{-1/2}(\gamma) \ such \ that \\ (\nabla\hat{u}, \nabla v)_{0,\Omega} - \langle\hat{\lambda}, v\rangle_\gamma = (f, v)_{0,\Omega}, \quad v \in H_0^1(\Omega), \\ \langle\mu, \hat{u}\rangle_\gamma = \langle\mu, g\rangle_\gamma, \quad \mu \in H^{-1/2}(\gamma), \end{array} \right\} \tag{2}$$

where $H^{-1/2}(\gamma)$ is the dual space of $H^{1/2}(\gamma) := H^1(\Omega)_{\big|_\gamma}$ and $\langle , \rangle_\gamma$ stands for the duality pairing between $H^{-1/2}(\gamma)$ and $H^{1/2}(\gamma)$. The following result is well-known ([3]).

Proposition 1. *Problem* (2) *has a unique solution* $(\hat{u}, \hat{\lambda})$. *Moreover,* $\hat{u}_{\big|_{\omega_\gamma}}$ *is the solution of* (1) *and* $\hat{\lambda} = [\frac{\partial\hat{u}}{\partial\nu}]_\gamma$ *corresponds to the jump of the normal derivative of* $\hat{u}$ *across* γ.

Although the extended solution $\hat{u}$ is smooth on $\omega_\gamma \cup (\Omega \setminus \bar\omega_\gamma)$ it is not smooth across γ. In view of Proposition 1 this is due to the fact that in general the non-zero Lagrange multiplier $\hat{\lambda}$ is concentrated on γ. This gives rise to a loss of accuracy of numerical methods in a neighborhood of γ. To overcome this inherent difficulty we
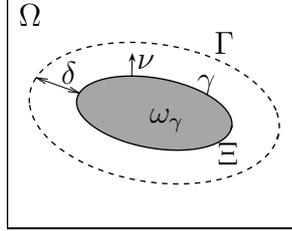
Figure 1: Geometry.

introduce an artificial control boundary $\Gamma \subset \Omega \setminus \bar{\omega}_\gamma$ (see Figure 1) and formulate the control problem:

$$\left.\begin{array}{l} \textit{Find } \mu \in H^{-1/2}(\Gamma) \textit{ such that } u := u(\mu) \in H_0^1(\Omega) \textit{ solves} \\ (\nabla u, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \mu, v \rangle_\Gamma, \quad v \in H_0^1(\Omega), \\ u = g \qquad \textit{on } \gamma, \end{array}\right\} \tag{3}$$

where $\langle \, , \rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. However this control problem has in general no solution: indeed, since μ is defined on Γ the solution $u(\mu)$ of (3) is smooth in a neighborhood of γ by elliptic interior regularity results which is the desired effect. Therefore, $u(\mu)$ does not satisfy the boundary condition $u(\mu) = g \in H^{1/2}(\gamma)$ on γ in general. But it can be satisfied with an arbitrary accuracy due to the following elliptic approximate controllability result (see [6]).

Proposition 2. *Define the linear mapping* $\phi \colon H^{-1/2}(\Gamma) \to H^{1/2}(\gamma)$ *by* $\phi(\mu) = u(\mu)|_\gamma$, *where* $u(\mu)$ *is the solution of* (3). *Then the range* $\phi(H^{-1/2}(\Gamma))$ *is dense in* $H^{1/2}(\gamma)$.

Remark 1. *As a consequence, the range of* ϕ *is also dense in* $L^2(\gamma)$.

For given $\Gamma \subset \Omega \setminus \bar{\omega}_\gamma$ and $\varepsilon > 0$ we define the set of admissible controls:

$$C(\Gamma, \varepsilon) = \{ \mu \in H^{-1/2}(\Gamma) \colon \|u(\mu) - g\|_{0,\gamma} \leq \varepsilon, \ u(\mu) \text{ satisfies } (3) \}. \tag{4}$$

Proposition 3. *The set* $C(\Gamma, \varepsilon)$ *is non-empty, convex and weakly closed for every* Γ *and* $\varepsilon > 0$.

Proof. Proposition 2 ensures that $C(\Gamma, \varepsilon)$ is non-empty. The rest of the assertion is straightforward. $\square$

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Proposition 3 motivates the following auxiliary minimization problem:

$$\min_{\mu \in C(\Gamma,\varepsilon)} \frac{1}{2}\|\mu\|^2_{-1/2,\Gamma}. \qquad\qquad (\mathbb{P})_{\Gamma,\varepsilon}$$

We remark that $(\mathbb{P})_{\Gamma,\varepsilon}$ is a *state constrained* optimal control problem with μ as the control variable, the state equation (3) and $\|u(\mu) - g\|_{0,\gamma} \leq \varepsilon$ representing the state constraint. From Proposition 3 we obtain the following existence result.

Proposition 4. *Problem* $(\mathbb{P})_{\Gamma,\varepsilon}$ *has a unique solution for any* Γ *and any* $\varepsilon > 0$.

Let $\{\varepsilon_n\}$ be an arbitrary sequence of positive numbers tending to zero and $\mu_n^* \in C(\Gamma, \varepsilon_n)$ be the corresponding solutions of $(\mathbb{P})_{\Gamma,\varepsilon_n}$. Then the sequence $\{\mu_n^*\}$ is in general unbounded in $H^{-1/2}(\Gamma)$. Indeed, if it was bounded one could argue the existence of $\mu^* \in C(\Gamma, 0)$, i.e. there would exist a control μ^* such that $u(\mu^*) = g$ holds a.e. on γ which in general is not possible.

In order to obtain a sequence of controls which is bounded as $\varepsilon_n \to 0$ we shall in what follows replace the fixed control boundary Γ by a sequence $\{\Gamma_n\}$ such that $\Gamma_n \to \gamma$ (in an appropriate way) as $\varepsilon_n \to 0$.

3 Convergence analysis

So far Γ was fixed. Now we shall allow Γ to vary in a set of admissible boundaries specified below. Furthermore, let $G \in H_0^1(\Omega)$ denote an extension of $g \in H^{1/2}(\gamma)$ to Ω. Instead of (1) in ω_γ we shall consider a family of problems in ω_Γ:

$$\left.\begin{array}{rll} -\triangle u_\Gamma = f & \text{in} & \omega_\Gamma, \\ u_\Gamma = G & \text{on} & \Gamma := \partial\omega_\Gamma \end{array}\right\}$$

whose fictitious domain formulation reads as follows:

$$\left.\begin{array}{l} \textit{Find } (\hat{u}_\Gamma, \hat{\lambda}_\Gamma) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma) \textit{ such that} \\ (\nabla\hat{u}_\Gamma, \nabla v)_{0,\Omega} = (f,v)_{0,\Omega} + \langle\hat{\lambda}_\Gamma, v\rangle_\Gamma \quad v \in H_0^1(\Omega), \\ \langle\mu, \hat{u}_\Gamma\rangle_\Gamma = \langle\mu, G\rangle_\Gamma \quad \mu \in H^{-1/2}(\Gamma). \end{array}\right\} \qquad (\hat{\mathcal{P}}(\Gamma))$$

The dependence of the solution $(\hat{u}_\Gamma, \hat{\lambda}_\Gamma)$ to $(\hat{\mathcal{P}}(\Gamma))$ on Γ has been investigated in [5]. Since the techniques of [5] will be used in our subsequent analysis we recall some of the basic results. There we considered domains ω_Γ whose boundaries Γ are parametrized by curves $\xi : [0, 2\pi] \mapsto \mathbb{R}^2$ taken from a set of admissible parametrizations $\mathcal{U}_{ad}$. If Γ is parametrized by ξ we shall write ω_ξ instead of ω_Γ.

The set $\mathcal{U}_{ad}$ was characterized by the following assumptions:

($A1$) $\xi \in C^2_{2\pi}$ (i.e. ξ is 2π-periodic and twice differentiable in $[0, 2\pi]$);

($A2$) there exist positive constants α, ξ_1, ξ_2 independent of ξ such that

$$|\dot{\xi}(t)| \geq \alpha, \quad t \in [0, 2\pi],$$

$$|\dot{\xi}|_\infty \leq \xi_1, \quad |\ddot{\xi}|_\infty \leq \xi_2;$$

($A3$) ξ represents a positively oriented curve;

($A4$) $\overline{\omega}_\xi \subset \Omega$;

($A5$) there exists a positive constant $d > 0$ independent of ξ such that

$$\text{dist}(\partial\Omega, \partial\omega_\xi) \geq d;$$

($A6$) there exists a constant h which does not depend on ξ such that for all $t \in [0, 2\pi]$ there are two open discs B_i and B_o of radius h such that $B_i \subset \omega_\xi$, $B_o \subset \Omega \setminus \overline{\omega}_\xi$ and $\xi(t) \in \overline{B}_o \cap \overline{B}_i$.

It was shown that ($A6$) guarantees the existence of a cut tubular neighborhood D_ξ of $\Gamma_\xi := \partial\omega_\xi$ given by

$$D_\xi = \{\xi(t) + h\eta\nu(t), \ \eta \in (-1, 1), \quad t \in (0, 2\pi)\}$$

for some $h > 0$ independent of $\xi \in \mathcal{U}_{ad}$. Above $\nu(t)$ denotes the outer normal unit vector to Γ_ξ at the point $\xi(t)$. For $Q = (0, 2\pi) \times (-1, 1)$ the mapping $S^h_\xi \colon \bar{Q} \to \bar{D}_\xi$

$$S^h_\xi(t, \eta) = \xi(t) + h\eta\nu(t), \quad (t, \eta) \in \bar{Q} \tag{5}$$

defines a diffeomorphism from Q onto D_ξ. Replacing $v \in L^2(\Gamma_\xi)$ by $v \circ S^h_\xi(\cdot, 0)$ one obtains a 2π-periodic function in $L^2(0, 2\pi)$. This made it possible to compare functions defined on different boundaries Γ_ξ, $\xi \in \mathcal{U}_{ad}$. More specifically, we defined the space $L^2_{2\pi}$ as the closure of the space of continuous, $2\pi-$periodic functions and introduced the subspace

$$H^{1/2}_{2\pi} = \{\varphi \in L^2_{2\pi}| \ \|\varphi\|_{1/2, 2\pi} < \infty\},$$

where

$$\|\varphi\|_{1/2, 2\pi} := (|\varphi|^2_{L^2_{2\pi}} + \int_0^{2\pi}\int_0^{2\pi} \frac{|\varphi(s) - \varphi(t)|^2}{|\sin((t-s)/2)|^2} \, dt \, ds)^{1/2}.$$

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Using the method of mappings we constructed the trace operators $\mathcal{T}_\xi : H_0^1(\Omega) \to H_{2\pi}^{1/2}$ which transforms traces on Γ_ξ into periodic functions from $H_{2\pi}^{1/2}$. It was shown that the family $\{\mathcal{T}_\xi\}_{\xi \in \mathcal{U}_{ad}}$ is uniformly bounded. Using $\mathcal{T}_\xi$ one can relate the spaces $H^{-1/2}(\Gamma_\xi)$ and $H_{2\pi}^{-1/2}$ (the dual of $H_{2\pi}^{1/2}$). Indeed, with any $\mu \in H^{-1/2}(\Gamma_\xi)$ we associate $\tilde{\mu} \in H_{2\pi}^{-1/2}$ by means of

$$\langle \tilde{\mu}, \mathcal{T}_\xi v \rangle_{2\pi} := \langle \mu, v \rangle_{\Gamma_\xi} \quad v \in H_0^1(\Omega).$$

Thus problem $(\hat{\mathcal{P}}(\Gamma))$ can be reformulated as follows:

$$\left. \begin{array}{l} Find \ (\hat{u}_\xi, \tilde{\lambda}_\xi) \in H_0^1(\Omega) \times H_{2\pi}^{-1/2} \ such \ that \\ (\nabla \hat{u}_\xi, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\lambda}_\xi, \mathcal{T}_\xi v \rangle_{2\pi} \quad v \in H_0^1(\Omega), \\ \langle \tilde{\mu}, \mathcal{T}_\xi \hat{u}_\xi \rangle_{2\pi} = \langle \tilde{\mu}, \mathcal{T}_\xi G \rangle_{2\pi} \quad \tilde{\mu} \in H_{2\pi}^{-1/2}, \end{array} \right\} \qquad (\tilde{\mathcal{P}}(\xi))$$

where $\xi \in \mathcal{U}_{ad}$. In [5] we proved the following result (the proof can be readily adapted to cover inhomogeneous boundary conditions too).

Theorem 1. *Let $\xi_n \to \xi$ in $C^1([0, 2\pi], \mathbb{R}^2)$, $\xi_n, \xi \in \mathcal{U}_{ad}$ and let $(\hat{u}_n, \tilde{\lambda}_n) \in H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ be the solution of $(\tilde{\mathcal{P}}(\xi_n))$. Then*

$$\begin{array}{l} \hat{u}_n \to \hat{u}_\xi \ in \ H_0^1(\Omega) \\ \tilde{\lambda}_n \rightharpoonup \tilde{\lambda}_\xi \ (weakly) \ in \ H_{2\pi}^{-1/2}, \ n \to \infty \end{array} \qquad (6)$$

and $(\hat{u}_\xi, \tilde{\lambda}_\xi)$ is the unique solution of $(\tilde{\mathcal{P}}(\xi))$. In addition, if $\xi_n \to \xi$ in $C^2([0, 2\pi], \mathbb{R}^2)$ then $\{\tilde{\lambda}_n\}$ converges strongly to $\tilde{\lambda}_\xi$ in $H_{2\pi}^{-1/2}$.

Now we return to problem $(\mathbb{P})_{\Gamma, \varepsilon}$. Subsequently we shall assume that the boundary γ of ω_γ is parameterized by a (fixed) function $\xi \in \mathcal{U}_{ad} \cap C^3([0, 2\pi]; \mathbb{R}^2)$. The control boundary Γ is constructed by a shift of γ along the normal direction. Thus Γ will be parameterized by

$$\tilde{\xi}(t) = \xi(t) + \delta\nu(t), \quad t \in [0, 2\pi], \qquad (7)$$

for some $\delta \in (0, h)$, where h is specified in $(\mathcal{A}6)$. Note that

$$\tilde{\xi} = S_\xi^\delta(\cdot, 1),$$

where S_ξ^δ is the diffeomorphism defined by (5) with $h := \delta$. Defining

$$\mathcal{U}_{ad,\xi} = \{\tilde{\xi} = \xi + \delta\nu, 0 < \delta < h\}$$

and taking δ sufficiently small one can verify $(\mathcal{A}1)$ - $(\mathcal{A}6)$ but with different constants $0 < \tilde{\alpha}, \tilde{\xi}_1, \tilde{\xi}_2$, and $\tilde{d}$ depending on ξ in $(\mathcal{A}2)$ and $(\mathcal{A}5)$. Hence Theorem 1 remains valid if $\mathcal{U}_{ad}$ is replaced by $\mathcal{U}_{ad,\xi}$.

Remark 2. *The additional regularity of ξ can be avoided regularizing the normal field ν in the definition of the diffeomorphism S_ξ^δ and in (7). For further details we refer to [1].*

Proposition 5. *Let $v \in H_0^1(\Omega)$ and $\Gamma \subset \Omega \backslash \bar{\omega}_\gamma$ be parametrized by $\tilde{\xi}(t) = \xi(t) + \delta \nu(t)$, $t \in [0, 2\pi]$, for some $\xi \in \mathcal{U}_{ad}$ and δ sufficiently small. Then*

$$\|v \circ \tilde{\xi} - v \circ \xi\|_{0,2\pi} \leq \sqrt{\frac{2}{\alpha}} \|\nabla v\|_{0,\Omega} \sqrt{\delta}. \tag{8}$$

In addition, if $v|_\Gamma = 0$ one has

$$\|v\|_{0,\gamma} \leq \sqrt{\frac{2\xi_1}{\alpha}} \|\nabla v\|_{0,\Omega} \sqrt{\delta} \tag{9}$$

with the constants α and ξ_1 from $(A2)$.

Proof. It suffices to consider $v \in C^1(\bar{\Omega})$. We obtain:

$$v(\tilde{\xi}(t)) - v(\xi(t)) = \delta \int_0^1 \nabla v(\xi(t) + \delta \eta \nu(t)) \cdot \nu(t)\, d\eta = \delta \int_0^1 (\nabla v)(S_\xi^\delta(t, \eta)) \cdot \nu(t)\, d\eta.$$

From $(A2)$ it follows that

$$|\det DS_\xi^\delta(t, \eta)| \geq \delta(\|\dot{\xi}\| - \delta\|\dot{\nu}\|) \geq \delta \frac{\alpha}{2}$$

holds for h sufficiently small. Therefore

$$\|v \circ \tilde{\xi} - v \circ \xi\|_{0,2\pi}^2 \leq \delta^2 \int_0^{2\pi} \int_0^1 |(\nabla v) \circ S_\xi^\delta|^2\, d\eta dt$$

$$\leq \delta \frac{2}{\alpha} \int_0^{2\pi} \int_0^1 |(\nabla v) \circ S_\xi^\delta|^2 |\det DS_\xi^\delta(t, \eta)|\, d\eta dt = \delta \frac{2}{\alpha} \int_{D_\xi} |\nabla v|^2 dx \leq \delta \frac{2}{\alpha} \|\nabla v\|_{0,\Omega}^2.$$

If $v|_\Gamma = 0$, equivalently if $v \circ \tilde{\xi} = 0$, the result follows from

$$\|v \circ \xi\|_{0,2\pi} \geq \frac{1}{\sqrt{\xi_1}} \|v\|_{0,\gamma}$$

making use of $(A2)$ again.

$\square$

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Using the spaces $H_{2\pi}^{1/2}$ and $H_{2\pi}^{-1/2}$ the state equation in (3) is written as follows:

> *Given $\tilde{\mu} \in H_{2\pi}^{-1/2}$, find $u_{\tilde{\xi}} \in H_0^1(\Omega)$ such that*
> $$(\nabla u_{\tilde{\xi}}, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\mu}, \mathcal{T}_{\tilde{\xi}} v \rangle_{2\pi}, \quad v \in H_0^1(\Omega), \tag{10}$$
> *with $\tilde{\xi}$ defined by (7).*

Then the minimization problem $(\mathbb{P})_{\Gamma,\varepsilon}$ can be reformulated as

$$\left. \min_{\tilde{\mu} \in \tilde{C}(\tilde{\xi},\varepsilon)} \frac{1}{2} \|\tilde{\mu}\|_{-1/2,2\pi}^2, \right\} \tag{$\tilde{\mathbb{P}})_{\tilde{\xi},\varepsilon}$}$$

where

$$\tilde{C}(\tilde{\xi}, \varepsilon) = \{\tilde{\mu} \in H_{2\pi}^{-1/2} : \|u_{\tilde{\xi}} - G\|_{0,\gamma} \le \varepsilon, \ u_{\tilde{\xi}} \text{ satisfies } (10)\}.$$

Next we shall consider sequences $\varepsilon_n \to 0^+$ and $\delta_n \to 0^+$ and establish an a priori bound for the corresponding optimal pairs $(u_n^*, \tilde{\lambda}_n^*)$ of $(\tilde{\mathbb{P}})_{\tilde{\xi}_n,\varepsilon_n}$, where $\tilde{\xi}_n$ refers to the parametrization (7) with δ replaced by δ_n.

Proposition 6. *Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ converge to zero and let the boundaries Γ_n and γ be parameterized by $\tilde{\xi}_n$ and $\xi \in \mathcal{U}_{ad} \cap C^3([0,2\pi];\mathbb{R}^2)$, respectively. If δ_n satisfies*

$$\sqrt{\delta_n}\,\hat{c} \le \varepsilon_n, \quad n \in \mathbb{N} \tag{11}$$

(with a constant $\hat{c}$ specified in (13)), the sequence of optimal pairs $\{(u_n^*, \tilde{\lambda}_n^*)\}$ of $(\tilde{\mathbb{P}})_{\tilde{\xi}_n,\varepsilon_n}$, $n \to \infty$ is bounded in $H_0^1(\Omega) \times H_{2\pi}^{-1/2}$.

Proof. Let $\{(\hat{u}_n, \tilde{\lambda}_n)\}$ denote the sequence of the solutions to the fictitious domain problems $(\tilde{\mathcal{P}}(\tilde{\xi}_n))$, $n \in \mathbb{N}$. Since $\tilde{\xi}_n$ converges to ξ in $C^1([0,2\pi],\mathbb{R}^2)$, Theorem 1 implies the existence of a constant $c_\lambda > 0$ such that

$$\|\tilde{\lambda}_n\|_{-1/2,2\pi} \le c_\lambda, \qquad n \in \mathbb{N}.$$

Since $\hat{u}_n - G = 0$ on Γ_n, we infer from (9)

$$\|\hat{u}_n - g\|_{0,\gamma} \le \sqrt{\frac{2\xi_1}{\alpha}} \|\nabla(\hat{u}_n - G)\|_{0,\Omega} \sqrt{\delta_n}.$$

Combined with the estimate

$$\|\nabla \hat{u}_n\|_{0,\Omega} \le c_1(\|f\|_{0,\Omega} + \|\nabla G\|_{0,\Omega}), \quad n \in \mathbb{N},$$

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we find that

$$\|\hat{u}_n - g\|_{0,\gamma} \leq \hat{c}\sqrt{\delta_n}, \tag{12}$$

where

$$\hat{c} = \sqrt{\frac{2\xi_1}{\alpha}}(c_1\|f\|_{0,\Omega} + (1 + c_1)\|\nabla G\|_{0,\Omega}). \tag{13}$$

Then (11) entails

$$\|\hat{u}_n - g\|_{0,\gamma} \leq \varepsilon_n,$$

hence $\tilde{\lambda}_n \in \tilde{C}(\tilde{\xi}_n, \varepsilon_n)$. From the definition of $(\tilde{\mathbb{P}}_{\tilde{\xi}_n,\varepsilon_n})$ we have

$$\|\tilde{\lambda}_n^*\|_{-1/2,2\pi} \leq \|\tilde{\lambda}_n\|_{-1/2,2\pi} \leq c_\lambda \quad n \in \mathbb{N}. \tag{14}$$

Let u_n^* be the solution of the state equation corresponding to $\tilde{\lambda}_n^*$:

$$(\nabla u_n^*, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\lambda}_n^*, \mathcal{T}_{\tilde{\xi}_n} v\rangle_{2\pi} \quad v \in H_0^1(\Omega).$$

Inserting $v := u_n^*$, using (14) and the uniform boundedness of $(\mathcal{T}_{\tilde{\xi}_n})$ (see [5], Lemma 4) one finds that the sequence $\{u_n^*\}$ is bounded in $H_0^1(\Omega)$ as well. □

Now we can prove the following convergence result.

Theorem 2. *Let all the assumptions of Proposition 6 be satisfied. Then the optimal pairs $(u_n^*, \tilde{\lambda}_n^*)$ of $(\tilde{\mathbb{P}})_{\tilde{\xi}_n,\varepsilon_n}$ converge strongly in $H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ to the unique solution $(\hat{u}_\xi, \tilde{\lambda}_\xi)$ of the fictitious domain problem $(\tilde{\mathcal{P}}(\xi))$.*

Proof. By Proposition 6 there is a subsequence (denoted again by $\{(u_n^*, \tilde{\lambda}_n^*)\}$), which converges weakly in $H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ to some $(\hat{u}_\xi, \tilde{\lambda}_\xi)$. Since $(u_n^*, \tilde{\lambda}_n^*) \in H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ satisfies

$$(\nabla u_n^*, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\lambda}_n^*, \mathcal{T}_{\tilde{\xi}_n} v\rangle_{2\pi} \quad v \in H_0^1(\Omega) \tag{15}$$

and $\mathcal{T}_{\tilde{\xi}_n}$ tends strongly to $\mathcal{T}_\xi$ (see [5], Lemma 9) passing to the limit in (15) we deduce

$$(\nabla \hat{u}_\xi, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\lambda}_\xi, \mathcal{T}_\xi v\rangle_{2\pi} \quad v \in H_0^1(\Omega).$$

In view of $\|u_n^* - g\|_{0,\gamma} \leq \varepsilon_n$ and strong convergence of u_n^* to $\hat{u}_\xi$ in $L^2(\gamma)$ we furthermore obtain

$$\|\hat{u}_\xi - g\|_{0,\gamma} = \lim_{n\to\infty} \|u_n^* - g\|_{0,\gamma} = 0.$$

This shows that the pair $(\hat{u}_\xi, \tilde{\lambda}_\xi)$ solves $(\tilde{\mathcal{P}}(\xi))$. Since this solution is unique, the original sequence $\{(u_n^*, \tilde{\lambda}_n^*)\}$ converges weakly to $(\hat{u}_\xi, \tilde{\lambda}_\xi)$. Taking into account that $\tilde{\xi}_n \to \xi$ in $C^2([0, 2\pi], \mathbb{R}^2)$, Theorem 1 implies

$$\begin{aligned}
\|\tilde{\lambda}_\xi\|_{-1/2,2\pi} &\leq \liminf_{n\to\infty} \|\tilde{\lambda}_n^*\|_{-1/2,2\pi} \leq \limsup_{n\to\infty} \|\tilde{\lambda}_n^*\|_{-1/2,2\pi} \\
&\leq \lim_{n\to\infty} \|\tilde{\lambda}_n\|_{-1/2,2\pi} = \|\tilde{\lambda}_\xi\|_{-1/2,2\pi}.
\end{aligned}$$

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Therefore $\lim_{n\to\infty}\|\tilde{\lambda}_n^*\|_{-1/2,2\pi} = \|\tilde{\lambda}_\xi\|_{-1/2,2\pi}$ which ensures strong convergence of $\tilde{\lambda}_n^*$ to $\tilde{\lambda}_\xi$ in $H_{2\pi}^{-1/2}$.

We are now able to prove strong convergence of $\hat{u}_n^*$ to $\hat{u}_\xi$ in $H_0^1(\Omega)$. In fact, the difference $\hat{u}_n^* - \hat{u}_n$ satisfies (recall that $(\hat{u}_n, \tilde{\lambda}_n)$ is the solution of the fictitious domain problem $(\mathcal{P}(\tilde{\xi}_n))$)

$$(\nabla(\hat{u}_n^* - \hat{u}_n), \nabla v)_{0,\Omega} = \langle \tilde{\lambda}_n^* - \tilde{\lambda}_n, \mathcal{T}_{\tilde{\xi}_n} v\rangle_{2\pi}, \qquad v \in H_0^1(\Omega),$$

so that

$$\|\hat{u}_n^* - \hat{u}_n\|_{H_0^1(\Omega)} \le c\|\tilde{\lambda}_n^* - \tilde{\lambda}_n\|_{-1/2,2\pi}\|\mathcal{T}_{\tilde{\xi}_n}\|$$

with a constant $c > 0$ which does not depend on n. This shows $\hat{u}_n^* - \hat{u}_n \to 0$ strongly in $H_0^1(\Omega)$ using the boundedness of the sequence $\{\mathcal{T}_{\tilde{\xi}_n}\}$. Now, the convergence result follows from the triangle inequality

$$\|\hat{u}_n^* - \hat{u}_\xi\|_{H_0^1(\Omega)} \le \|\hat{u}_n^* - \hat{u}_n\|_{H_0^1(\Omega)} + \|\hat{u}_n - \hat{u}_\xi\|_{H_0^1(\Omega)}$$

and Theorem 1. $\qquad\square$

4 A penalty approach

We consider at first a fixed control boundary Γ. The state constraint $\|u - g\|_{0,\gamma} \le \varepsilon$ in problem $(\mathbb{P})_{\Gamma,\varepsilon}$ is numerically realized by a penalty approach. We introduce for $k > 0$ the augmented cost functional

$$J_k(u, \lambda) = \frac{1}{2}\|\lambda\|_{-1/2,\Gamma}^2 + \frac{k}{2}\|u - g\|_{0,\gamma}^2,$$

and define the new problem

$$\left.\begin{array}{l} \min J_k(u, \lambda) \ subject\ to \\ (u, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma), \\ (\nabla u, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \lambda, v\rangle_\Gamma, \quad v \in H_0^1(\Omega), \end{array}\right\} \qquad (\mathbb{P})_{\Gamma,k}$$

which has a unique solution denoted by (w_k, σ_k). In view of (12) one obtains by an argument adapted from [9] the estimate

$$J_k(w_k, \sigma_k) \le J_k(\hat{u}_\Gamma, \hat{\lambda}_\Gamma) \le \frac{1}{2}\|\hat{\lambda}_\Gamma\|_{-1/2,\Gamma}^2 + \frac{k}{2}\hat{c}^2\delta,$$

where $(\hat{u}_\Gamma, \hat{\lambda}_\Gamma)$ is the solution of $(\hat{\mathcal{P}}(\Gamma))$. This implies

$$\frac{k}{2}\|w_k - g\|_{0,\gamma}^2 \le \frac{1}{2}\|\hat{\lambda}_\Gamma\|_{-1/2,\Gamma}^2 + \frac{k}{2}\hat{c}^2\delta$$

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so that

$$\|w_k - g\|_{0,\gamma}^2 \leq \frac{\|\hat{\lambda}_\Gamma\|_{-1/2,\Gamma}^2}{k} + \hat{c}^2 \delta.$$

Choosing

$$k \geq \frac{\|\hat{\lambda}_\Gamma\|_{-1/2,\Gamma}^2}{\hat{c}^2 \delta}$$

we obtain

$$\|w_k - g\|_{0,\gamma} \leq \sqrt{2}\hat{c}\sqrt{\delta}. \tag{16}$$

In order to ensure at least weak convergence of the optimal pairs (w_k, σ_k), $k \to \infty$ we again introduce a sequence of the control boundaries Γ_n parametrized by (7) with δ replaced by δ_n. We select for each Γ_n a penalty parameter $k_n > 0$ such that

$$\frac{\|\hat{\lambda}_n\|_{-1/2,\Gamma_n}^2}{\hat{c}^2} \leq \delta_n k_n \leq \kappa, \tag{17}$$

where $(\hat{u}_n, \hat{\lambda}_n) \in H_0^1(\Omega) \times H^{1/2}(\Gamma_n)$ is the solution of the fictitious domain formulation $(\hat{\mathcal{P}}(\Gamma_n))$ and $\kappa > 0$ is fixed. Such a choice of κ is possible due to the boundedness of $\{\hat{\lambda}_n\}$. Then the sequence $\{J_{k_n}(w_{k_n}, \sigma_{k_n})\}$ is bounded by

$$J_{k_n}(w_{k_n}, \sigma_{k_n}) \leq \frac{1}{2}\|\hat{\lambda}_n\|_{-1/2,\Gamma_n}^2 + \frac{1}{2}\kappa\hat{c}^2.$$

which yields the boundedness of σ_{k_n} in $H^{-1/2}(\Gamma_n)$, $n \in \mathbb{N}$. Let $\tilde{\sigma}_{k_n} \in H_{2\pi}^{-1/2}$ stand for the functional σ_{k_n} interpreted on $H_{2\pi}^{1/2}$. As in the proof of Theorem 2 one concludes the existence of a subsequence of $\{(w_{k_n}, \tilde{\sigma}_{k_n})\}$ which converges weakly in $H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ to a pair $(w, \tilde{\sigma})$ such that

$$(\nabla w, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \langle \tilde{\sigma}, \mathcal{T}_\xi v \rangle_{2\pi}, \qquad v \in H_0^1(\Omega).$$

In view of (16) we also have

$$\|w_{k_n} - g\|_{0,\gamma} \leq \sqrt{2}\hat{c}\sqrt{\delta_n}$$

so that w satisfies the original boundary condition $w = g$ on γ. Hence $(w, \tilde{\sigma})$ is a solution of the fictitious domain problem $(\tilde{\mathcal{P}}(\xi))$. Consequently, $(w, \tilde{\sigma}) = (\hat{u}_\xi, \tilde{\lambda}_\xi)$ must hold. In addition, the (whole) sequence $\{(\hat{w}_{k_n}, \tilde{\sigma}_{k_n})\}$ tends weakly in $H_0^1(\Omega) \times H_{2\pi}^{-1/2}$ to $(\hat{u}_\xi, \tilde{\lambda}_\xi)$.

We now turn to the first order necessary and sufficient optimality condition for problem $(\mathbb{P})_{\Gamma,k}$ for a fixed control boundary Γ. Let $S \in \mathcal{L}(H^{-1/2}(\Gamma), L^2(\gamma))$ denote

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the Poincare-Steklov like operator $S\lambda = \tau_\gamma u$ where $u \in H_0^1(\Omega)$ is the unique solution of

$$(\nabla u, \nabla v)_{0,\Omega} = \langle \lambda, \tau_\Gamma v \rangle_\Gamma, \quad v \in H_0^1(\Omega),$$

where τ_γ, τ_Γ are the trace mappings from $H_0^1(\Omega)$ into $L^2(\gamma)$ and $H^{1/2}(\Gamma)$, respectively. If $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ denotes the isomorphism induced by the inner product in $H_0^1(\Omega)$, one can represent S as

$$S = \tau_\gamma A^{-1} \tau_\Gamma^*.$$

Thus one can formulate the equivalent problem in terms of λ:

$$\min_{\lambda \in H^{-1/2}(\Gamma)} \tilde{J}_k(\lambda) = \frac{1}{2}\|\lambda\|_{-1/2,\Gamma}^2 + \frac{k}{2}\|S\lambda - d\|_{0,\gamma}^2 \tag{18}$$

with

$$d = g - \tau_\gamma A^{-1} f.$$

We endow $H^{1/2}(\Gamma)$ with the norm

$$\|\varphi\|_{1/2,\Gamma} = \inf\{\|\nabla v\|_{0,\Omega} \colon \tau_\Gamma v = \varphi,\, v \in H_0^1(\Omega)\}, \quad \varphi \in H^{1/2}(\Gamma),$$

which is induced by the inner product

$$(\varphi, \psi)_{1/2,\Gamma} = (\nabla y_\varphi, \nabla y_\psi)_{0,\Omega}, \qquad \varphi, \psi \in H^{1/2}(\Gamma).$$

Here y_φ (and analogously y_ψ) is the first component of the solution $(y_\varphi, \lambda_\varphi) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma)$ to the saddle-point problem

$$\begin{aligned}
(\nabla y_\varphi, \nabla v)_{0,\Omega} - \langle \lambda_\varphi, \tau_\Gamma v \rangle_\Gamma &= 0, \qquad v \in H_0^1(\Omega), \\
\langle \mu, \tau_\Gamma y_\varphi \rangle_\Gamma &= \langle \mu, \varphi \rangle_\Gamma, \quad \mu \in H^{-1/2}(\Gamma).
\end{aligned} \tag{19}$$

Thus the Riesz mapping $R \colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is given by

$$\langle R\varphi, \psi \rangle_\Gamma = (\nabla y_\varphi, \nabla y_\psi)_{0,\Omega}.$$

From this one deduces that the Riesz representation $\varphi_\lambda \in H^{1/2}(\Gamma)$ of $\lambda \in H^{-1/2}(\Gamma)$ is $\varphi_\lambda = \tau_\Gamma y_{R^{-1}\lambda}$, where $y_{R^{-1}\lambda}$ is the unique solution of

$$(\nabla y_{R^{-1}\lambda}, \nabla v)_{0,\Omega} = \langle \lambda, \tau_\Gamma v \rangle_\Gamma, \qquad v \in H_0^1(\Omega)$$

which implies

$$R^{-1}\lambda = \tau_\Gamma A^{-1} \tau_\Gamma^* \lambda.$$

Consequently, the norm in $H^{-1/2}(\Gamma)$ can be realized by

$$\|\lambda\|_{-1/2,\Gamma} = \|\nabla y_{R^{-1}\lambda}\|_{0,\Omega}$$

and the inner product in $H^{-1/2}(\Gamma)$ by

$$(\lambda, \varrho)_{-1/2,\Gamma} = (R^{-1}\lambda, R^{-1}\varrho)_{1/2,\Gamma} = (\nabla y_{R^{-1}\lambda}, \nabla y_{R^{-1}\varrho})_{0,\Omega}, \qquad \lambda, \varrho \in H^{-1/2}(\Gamma).$$

Thus the derivative of the reduced cost functional J_k is given by

$$\langle \tilde{J}'_k(\lambda), \varrho \rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)} = (\lambda, \varrho)_{-1/2,\Gamma} + k(S\lambda - d, S\varrho)_{0,\gamma},$$

which leads to the optimality condition

$$(R^{-1} + kS^*S)\lambda = kS^*d \tag{20}$$

in $H^{1/2}(\Gamma)$. We note that one evaluation of $(R^{-1} + kS^*S)\lambda$ requires the solution of two elliptic equations. Indeed, solving the system

$$\begin{aligned} Ay &= \tau_\Gamma^* \lambda, \\ Az &= \tau_\gamma^* \tau_\gamma y \end{aligned} \tag{21}$$

one finds that

$$(R^{-1} + kS^*S)\lambda = \tau_\Gamma y + k\tau_\Gamma z.$$

We remark that to solve the equations in (21) is numerically inexpensive.

5   Numerical examples

In the following, we address two aspects of the smooth embedding method based on the penalty approach introduced in the previous section. Firstly, we illustrate a higher accuracy of this approach compared to the classical fictitious domain method [3]. Secondly, we demonstrate its computational efficiency when solving large scale problems (up to 16 millions unknowns). We shall consider the model example

$$\left. \begin{aligned} -\triangle u &= f \quad \text{in} \quad \omega, \\ u &= g \quad \text{on} \quad \gamma = \partial\omega. \end{aligned} \right\} \tag{$\mathcal{P}(\omega)$}$$

The domain ω is represented by the interior of the ellipse

$$\omega = \{(x,y) \in \mathbb{R}^2 \,|\, \frac{(x-0.5)^2}{0.4^2} + \frac{(y-0.5)^2}{0.2^2} < 1\}$$

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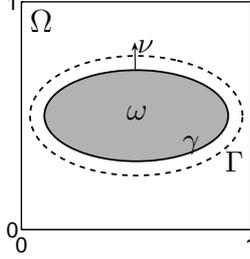
Figure 2: Geometry of ω and Ω.

embedded in the unit square $\Omega = (0,1) \times (0,1)$ (see Figure 2). The right hand-sides f and g in $(\mathcal{P}(\omega))$ are $f = -\Delta\hat{u}$ in $\mathbb{R}^2$ and $g = \hat{u}|_\gamma$, where

$$\hat{u}(x,y) = e^y \sin(\pi(x-y)), \ (x,y) \in \mathbb{R}^2.$$

Thus exact solution u_{ex} of $(\mathcal{P}(\omega))$ is given by $\hat{u}\big|_\omega$.

To discretize the space $H_0^1(\Omega)$ we use the subspace V_h of all continuous piecewise bilinear functions vanishing on $\partial\Omega$ constructed over a *uniform* partition of Ω into squares of size h. The spaces $H^{-1/2}(\gamma)$ and $H^{-1/2}(\Gamma)$ are approximated by the subspaces Λ_{H_γ} and Λ_{H_Γ} of functions which are piecewise constant on partitions $\mathcal{T}_{H_\gamma}$, $\mathcal{T}_{H_\Gamma}$ of polygonal approximations of γ and Γ, respectively. We emphasize that the meshnorms H_γ, H_Γ of $\mathcal{T}_{H_\gamma}$ and $\mathcal{T}_{H_\Gamma}$, respectively, may be chosen independently in contrast to the approach in [6].

The resulting algebraic formulation of (20) is given by

$$(\mathbf{R}^{-1} + k\mathbf{S}^T\mathbf{S})\boldsymbol{\lambda} = k\mathbf{S}^T\mathbf{d}, \tag{22}$$

where

$$\mathbf{R}^{-1} = \mathbf{B}_\Gamma\mathbf{A}^{-1}\mathbf{B}_\Gamma^T, \quad \mathbf{S} = \mathbf{B}_\gamma\mathbf{A}^{-1}\mathbf{B}_\Gamma^T,$$

and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the stiffness matrix, $\mathbf{B}_\Gamma \in \mathbb{R}^{m_\Gamma \times n}$ and $\mathbf{B}_\gamma \in \mathbb{R}^{m_\gamma \times n}$ are the matrices representing the discrete trace mappings on Γ and γ, respectively, $n = \dim V_h$, $m_\Gamma = \dim \Lambda_{H_\Gamma}$, and $m_\gamma = \dim \Lambda_{H_\gamma}$. Finally, $\mathbf{d} = \mathbf{g} - \mathbf{B}_\gamma\mathbf{A}^{-1}\mathbf{f} \in \mathbb{R}^{m_\gamma}$, where $\mathbf{f} \in \mathbb{R}^n$ denotes the discrete load vector and $\mathbf{g} \in \mathbb{R}^{m_\gamma}$ is a piecewise constant approximation of g. More details about the discretization may be found in [6].

The algebraic system (22) is solved by the conjugate gradient method with the relative stopping tolerance $\epsilon = h^2$ and zero as the initial guess. Analogously to the continuous setting one multiplication of the system matrix with the vector $\boldsymbol{\lambda}$ in (22)

requires the solution of the following two systems:

$$\begin{aligned}
\mathbf{Ay} &= \mathbf{B}_\Gamma^T \boldsymbol{\lambda}, \\
\mathbf{Az} &= \mathbf{B}_\gamma^T \mathbf{B}_\gamma \mathbf{y},
\end{aligned} \tag{23}$$

so that

$$(\mathbf{R}^{-1} + k\mathbf{S}^T\mathbf{S})\boldsymbol{\lambda} = \mathbf{B}_\Gamma \mathbf{y} + k\mathbf{B}_\Gamma \mathbf{z}.$$

Finally, to solve the system (23) at a low cost we replace the space $H_0^1(\Omega)$ by $H_{per}^1(\Omega)$, i.e. the subspace of functions in $H^1(\Omega)$ which are periodic. The theoretical results of the previous sections remain valid also in this case. Then $V_h \subset H_{per}^1(\Omega)$ will be the space of all continuous piecewise bilinear functions having the same traces on opposite sides of Ω. The advantage of this choice consists in the fact that the resulting stiffness matrix has a circulant structure which enables the use of the highly efficient Poisson like solver ([8], [6]) based on the discrete Fourier transform.

In the first numerical experiment we compare the classical fictitious domain approach $(\hat{\mathcal{P}}(\Gamma))$ with our smooth embedding domain method. We approximate at first the solution u_{ex} of $(\mathcal{P}(\omega))$ by the finite element approximations of $\hat{u}_\Gamma|_\omega$ where $\hat{u}_\Gamma$ is the first component of the solution to $(\hat{\mathcal{P}}(\Gamma))$ for different control boundaries Γ. The Dirichlet boundary condition for $\hat{u}_\Gamma$ on Γ is determined by $\hat{u}_\Gamma \circ \tilde{\xi} = g \circ \xi$ where ξ and $\tilde{\xi}$ are the parametrizations of γ and Γ, respectively. In Table 1 we report the resulting relative errors of the finite element solution $\hat{u}_h$ corresponding to the fixed meshsize $h = 1/256$ in the following norms

$$\mathrm{Err}_{L^2(\omega)} = \frac{\|\hat{u}_h - u_{ex}\|_{L^2(\omega)}}{\|u_{ex}\|_{L^2(\omega)}}, \quad \mathrm{Err}_{H^1(\omega)} = \frac{\|\hat{u}_h - u_{ex}\|_{H^1(\omega)}}{\|u_{ex}\|_{H^1(\omega)}}, \quad \mathrm{Err}_{L^2(\gamma)} = \frac{\|\hat{u}_h - u_{ex}\|_{L^2(\gamma)}}{\|u_{ex}\|_{L^2(\gamma)}}$$

decreasing the distance δ between the boundary Γ and the original boundary γ from $\delta = 8h$ to $\delta = 0$. We also record $n = \dim V_h$, $m_\Gamma = \dim \Lambda_{H_\Gamma}$, $m_\gamma = \dim \Lambda_{H_\gamma}$, the number of CG iterations and the computing time.

From Theorem 1 we know that if Γ is close to γ (in the appropriate sense) then also the solution $(\hat{u}_\Gamma, \hat{\lambda}_\Gamma)$ of $(\hat{\mathcal{P}}(\Gamma))$ is close to the solution $(\hat{u}_\gamma, \hat{\lambda}_\gamma)$ of $(\hat{\mathcal{P}}(\gamma))$. It is not surprising that the best result is obtained for $\delta = 0$, i.e. $\gamma \equiv \Gamma$, as seen from Table 1. This can be explained observing that $\hat{u}|_\omega$ solves the original problem $(\mathcal{P}(\omega))$ and that in particular $\hat{u} = g$ is satisfied exactly on γ whereas the function $\hat{u}_\Gamma$ satisfies the same boundary condition on Γ but not on γ. The longer the distance of Γ from γ is the worse solution one obtains. On the other hand if the smooth variant of the fictitious domain method is used the behaviour is opposite: results are better for increasing distances of Γ from γ. Table 2 illustrates that the Dirichlet condition on γ is satisfied with a higher accuracy than in the previous case while

the singularity of the solution to the state problem which is concentrated on Γ is sufficiently far away from γ.

It has been already mentioned that the approach in [6] is based on the assumption that $\dim \Lambda_{H_\Gamma} = \dim \Lambda_{H_\gamma}$. This restriction is not needed in the new approach any more. Hence, the number of controls on Γ may be smaller than the number of arcs used to discretize γ. This can improve the conditioning of the system matrix when $m_\Gamma < m_\gamma$. In Tables 3 and 4 we report similar data as before for two different choices of the penalty parameter k. The distance of the control boundary to the original boundary is $\delta = 8h$, we set $m_\gamma = 2m_\Gamma$ and decrease the meshsize h from $1/128$ to $1/4096$. For $k = 100/h^2$ one observes the optimal convergence rate in $L^2(\omega)$ and $H^1(\omega)$. However, the number of CG iterations and accordingly the computing time increase with $n = \dim V_h$. This drawback can be significantly reduced by the nested iterations technique which uses a hierarchy of nested grids. The solution of the smooth embedding domain method on a coarse level is used as an initial guess for the CG method on the next finer level (see [6] for more details). The results for this case are reported in Table 5.

Table 1: Classical FD approach applied to problems defined in ω_Γ

δ	$n/(m_\Gamma = m_\gamma)$	Iters.	S.time[s]	Err$_{L^2(\omega)}$	Err$_{H^1(\omega)}$	Err$_{L^2(\gamma)}$
$8h$	66049/62	12	0.4836	1.2463e-001	1.9095e-001	1.3455e-001
$4h$	66049/62	13	0.4836	6.7877e-002	1.7496e-001	7.8189e-002
$2h$	66049/62	13	0.5616	3.7895e-002	1.9069e-001	5.6803e-002
$1h$	66049/62	12	0.4992	1.9722e-002	1.5933e-001	3.7727e-002
$0h$	66049/62	11	0.4368	2.4426e-003	4.9886e-002	2.2412e-003

Table 2: Smooth penalty approach ($k = 10^2/h^2$)

δ	$n/(m_\Gamma = m_\gamma)$	Iters.	S.time[s]	Err$_{L^2(\omega)}$	Err$_{H^1(\omega)}$	Err$_{L^2(\gamma)}$
$8h$	66049/62	25	0.9516	1.3087e-005	1.1775e-003	1.8968e-005
$4h$	66049/62	14	0.5148	2.0114e-005	1.5453e-003	7.8423e-005
$2h$	66049/62	13	0.4992	4.8910e-005	2.9809e-003	2.3736e-004
$1h$	66049/62	11	0.4368	1.4074e-004	7.5131e-003	6.4170e-004
$0h$	66049/62	11	0.4524	2.4426e-003	4.9886e-002	2.2412e-003

Table 3: Smooth penalty approach with $k = 1/h^2$

Step h	$n/m_\Gamma/m_\gamma$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/18/36	12	0.15	1.7625e-02	6.7895e-02	2.8024e-02
1/256	66049/31/62	19	0.67	4.4524e-03	2.4064e-02	6.3279e-03
1/512	263169/55/110	28	4.84	1.5542e-03	1.3141e-02	2.0537e-03
1/1024	1050625/99/198	52	34.17	6.2905e-04	8.2158e-03	8.0627e-04
1/2048	4198401/180/360	82	261.3	2.7170e-04	5.3690e-03	3.4317e-04
1/4096	16785409/331/662	170	2634.7	1.2181e-04	3.5870e-03	1.5273e-04
Convergence rates:				1.408	0.8109	1.473

Table 4: Smooth penalty approach with $k = 10^2/h^2$

Step h	$n/m_\Gamma/m_\gamma$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/18/36	12	0.15	1.9692e-03	2.3229e-02	5.3687e-03
1/256	66049/31/62	19	0.73	1.4821e-04	4.6439e-03	5.1751e-04
1/512	263169/55/110	30	5.33	3.0126e-05	1.9723e-03	1.2875e-04
1/1024	1050625/99/198	50	32.67	9.5058e-06	1.0753e-03	4.5825e-05
1/2048	4198401/180/360	85	282.0	3.1593e-06	6.0307e-04	1.4711e-05
1/4096	16785409/331/662	170	2622.0	1.2982e-06	3.8004e-04	5.7447e-06
Convergence rates:				2.033	1.125	1.893

Table 5: Smooth penalty approach with $k = 10^2/h^2$ with nested iterations

Step h	$n/m_\Gamma/m_\gamma$	Iters.	S.time[s]	$\text{Err}_{L^2(\omega)}$	$\text{Err}_{H^1(\omega)}$	$\text{Err}_{L^2(\gamma)}$
1/128	16641/18/36	12	0.22	1.9629e-03	2.3141e-02	5.3546e-03
1/256	66049/32/64	22	0.94	1.3408e-04	4.3896e-03	4.7949e-04
1/512	263169/56/112	27	5.10	2.7062e-05	1.8580e-03	1.1596e-04
1/1024	1050625/100/200	32	30.67	8.1246e-06	9.8913e-04	3.8310e-05
1/2048	4198401/180/360	36	146.4	3.0539e-06	5.9225e-04	1.3877e-05
1/4096	16785409/330/660	45	958.7	1.2377e-06	3.6990e-04	4.9212e-06
Convergence rates:				2.036	1.126	1.925

6   Conclusions

To improve the convergence rate of the standard fictitious domain method with boundary Lagrange multipliers we proposed a smooth variant of this approach given by state constrained optimal control problems. The control variables are represented by forces concentrated on a closed curve Γ surrounding the original domain ω. By means of them we control the behavior of the solution u to the state problem. Our goal was to find a control with the smallest norm such that the error between u and the prescribed Dirichlet data on γ (measured in the $L^2(\gamma)$-norm) is below a prescribed tolerance ε. To treat this state constraint numerically we used the penalty approach. The respective necessary and sufficient optimality condition leads to a linear equation in the space of controls. Its discretization by finite elements was used for finding the (unique) discrete optimal solution. The improvement of the convergence rate in comparison to the standard fictitious domain method is illustrated with a model example.

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