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Optimal control of the bidomain system (II): Uniqueness and regularity theorems for weak solutions

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Karl Kunisch and Marcus Wagner

1. Introduction.

In this paper, we continue our investigation of optimal control problems for the bidomain equations. In a previous work [Kunisch/Wagner 11], control problems involving the monodomain approximation of the bidomain system have been considered. Well-posedness of the problem formulation was proved, and first-order optimality conditions were derived. Turning now to the investigation of optimal control of the *full bidomain system*, a thorough understanding of the solution concepts for the equations is essential. This is the focus of the present work while a detailed study of the arising control problem is left to an upcoming publication.

For a bounded domain $\Omega \subset \mathbb{R}^3$ and $T > 0$, the bidomain system reads as follows: [01)]

$$(\mathrm{B})_0 \quad \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \operatorname{div}\big(M_i \nabla \Phi_i\big) = I_i \quad \text{for almost all } (x,t) \in \Omega \times [0, T]; \tag{1.1}$$

$$\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) + \operatorname{div}\big(M_e \nabla \Phi_e\big) = -I_e \quad \text{for almost all } (x,t) \in \Omega \times [0, T]; \tag{1.2}$$

$$\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) = 0 \quad \text{for almost all } (x,t) \in \Omega \times [0, T]; \tag{1.3}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \nabla \Phi_i = 0 \quad \text{for all } (x,t) \in \partial\Omega \times [0, T]; \tag{1.4}$$

$$\mathfrak{n}^{\mathrm{T}} M_e \nabla \Phi_e = 0 \quad \text{for all } (x,t) \in \partial\Omega \times [0, T]; \tag{1.5}$$

$$\Phi_{tr}(x,0) = \Phi_i(x,0) - \Phi_e(x,0) = \Phi_0(x) \quad \text{and} \quad W(x,0) = W_0(x) \quad \text{for almost all } x \in \Omega. \tag{1.6}$$

Here Ω represents the domain occupied by the cardiac muscle, the variables Φ_i and Φ_e denote the intracellular and extracellular electric potentials, and $\Phi_{tr} = \Phi_i - \Phi_e$ is the transmembrane potential. The ionic current I_{ion} and the function G within the gating equation (1.3) will be specified according to the two-variable models discussed in Subsection 2.b) below. In these models, I_{ion} and G depend on Φ_{tr} as well as on a single gating variable W describing in a cumulative way the effects of the ion transport through the cell membranes. The anisotropic properties of the intracellular and the extracellular tissue parts are modeled by the conductivity tensors M_i and M_e. Finally, I_i and I_e describe the intracellular and extracellular stimulation currents, respectively.

As already mentioned, our aim is to analyze optimal control problems related to (1.1) – (1.6): (1.7)

$$(\mathrm{P}) \quad F(\Phi_{tr}, \Phi_e, W, I_e) = \int_0^T \int_\Omega r\big(x, t, \Phi_{tr}(x,t), \Phi_e(x,t), W(x,t)\big)\, dx\, dt + \frac{\mu}{2} \int_0^T \int_\Omega I_e(x,t)^2\, dx\, dt \longrightarrow \inf!$$

subject to the bidomain equations (1.1) – (1.6) in its weak formulation (see (2.7) – (2.10) below)

and the control restriction $\big| I_e(x,t) \big| \leqslant R$ for a. a. $(x,t) \in \Omega \times [0, T]$ (1.8)

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01) First considered in [Tung 78]. A detailed introduction may be found in [Sundnes/Lines/Cai/Nielsen/Mardal/Tveito 06], pp. 21 – 56.

where I_e is considered as control variable. [02] Since we do not impose any restriction on the temporal derivative of the controls, the formulation of (P) naturally calls for the analysis of a weak solution concept to the bidomain equations. [03] If an optimal control admits improved regularity properties then, as we will prove, the solutions of (2.7) – (2.10) gain additional regularity as well and become, in fact, strong solutions of (1.1) – (1.6) (Corollary 3.4.).

Let us briefly discuss the results of the present paper in relation to those, which are available in the literature. In [Bourgault/Coudière/Pierre 09], global existence of weak solutions for the two-variable models from Subsection 2.b) was obtained. This result is cited below as Theorem 2.5. A sufficient condition for uniqueness based on eigenvalue stability of a matrix related to $\begin{pmatrix} c\,\partial I_{ion}/\partial\varphi & c\,\partial I_{ion}/\partial w \\ \partial G/\partial\varphi & \partial G/\partial w \end{pmatrix}$, however, could be proven only for the FitzHugh-Nagumo model. [04] Similar results can be found in [Boulakia/Fernández/Gerbeau/Zemzemi 08], namely a global existence theorem for weak solutions together with a uniqueness result requiring spectral conditions analogously to the former ones. [05] In [Bourgault/Coudière/Pierre 09], moreover, existence and uniqueness of strong solutions is verified locally in time. [06] Global existence and uniqueness of strong solutions has been proved in [Veneroni 09] for a generalized Luo-Rudy type model. [07]

In the present work, global existence and uniqueness of weak solutions for the models discussed in [Bourgault/Coudière/Pierre 09] is obtained (Theorem 2.8.). Consequently, the control-to-state mapping within (P) is well-posed. The uniqueness theorem is proved under assumptions, which are natural for the optimal control problem (P). Namely, in view of (1.8), we assume that the inhomogeneities I_i and I_e belong to $L^\infty\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]$. Instead of spectral conditions of the type discussed above, our uniqueness argument is based on an error estimate (Theorem 2.7.), which will become part of the proof of the necessary optimality conditions for the control problem (P) as well. In the case of controls admitting a weak time derivative, we obtain improved regularity of the solution to the bidomain equations by use of Veneroni's iterative procedure (Theorem 3.3. and Corollary 3.4.).

The paper is structured as follows: Section 2 contains the existence and uniqueness analysis of weak solutions. Higher regularity of these solutions is investigated in Section 3. The assumptions and results are presented at the beginning of each sections, while the rather technical proofs are given at their ends. Throughout we use standard function space notation, as summarized in [Kunisch/Wagner 11], pp. 3 and 29 f.

2. Existence and uniqueness of weak solutions for the bidomain system.

a) Equivalent formulations of the bidomain system; strong and weak solutions.

Introducing the transmembrane potential $\Phi_{tr} = \Phi_i - \Phi_e$, the bidomain system (1.1) – (1.6) can be equivalently stated in the following parabolic-elliptic form: [08]

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[02] For physiological reasons, the intracellular excitation I_i is set zero.

[03] The existence theory for (P) must be based on a weak solution notion as well, cf. [Kunisch/Wagner 11], p. 17 f., Theorem 3.3.

[04] [Bourgault/Coudière/Pierre 09], p. 479, Theorem 32, together with Subsect. 6.1., p. 479 f.

[05] [Boulakia/Fernández/Gerbeau/Zemzemi 08], p. 9, Theorem 2.2. Besides the models from Subsection 2.b), the authors consider the Mitchell-Schaeffer model.

[06] [Bourgault/Coudière/Pierre 09], p. 469, Theorem 20. The assumptions of this theorem are rather difficult to verify.

[07] [Veneroni 09], p. 854, Theorem 1.1.

[08] See [Bourgault/Coudière/Pierre 09], p. 459.

$$(\mathrm{B})_1 \quad \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \operatorname{div}\big(\, M_i \, \nabla \Phi_{tr} \,\big) - \operatorname{div}\big(\, M_i \, \nabla \Phi_e \,\big) \;=\; I_i \quad (\forall)\,(x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{2.1}$$

$$\operatorname{div}\big(\, M_i \, \nabla \Phi_{tr} \,\big) + \operatorname{div}\big(\,(M_i + M_e)\, \nabla \Phi_e \,\big) \;=\; -(I_i + I_e) \quad (\forall)\,(x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{2.2}$$

$$\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \;=\; 0 \quad (\forall)\,(x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{2.3}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \, \nabla \Phi_{tr} + \mathfrak{n}^{\mathrm{T}} M_i \, \nabla \Phi_e \;=\; 0 \quad \forall\,(x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{2.4}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \, \nabla \Phi_{tr} + \mathfrak{n}^{\mathrm{T}} (M_i + M_e)\, \nabla \Phi_e \;=\; 0 \quad \forall\,(x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{2.5}$$

$$\Phi_{tr}(x,0) \;=\; \Phi_0(x) \quad (\forall)\, x \in \Omega\,; \quad W(x,0) \;=\; W_0(x) \quad (\forall)\, x \in \Omega\,. \tag{2.6}$$

The following weak formulation corresponds to the system (2.1) – (2.6): [09]

$$(\mathrm{B})_2 \quad \int_\Omega \Big(\frac{\partial \Phi_{tr}}{\partial t} \cdot \psi + \nabla \psi^{\mathrm{T}} M_i (\, \nabla \Phi_{tr} + \nabla \Phi_e \,) + I_{ion}(\Phi_{tr}, W)\, \psi \Big)\, dx \;=\; \int_\Omega I_i \, \psi \, dx \quad \forall\, \psi \in W^{1,2}(\Omega)\,; \tag{2.7}$$

$$\int_\Omega \Big(\nabla \psi^{\mathrm{T}} M_i \nabla \Phi_{tr} + \nabla \psi^{\mathrm{T}} (M_i + M_e)\, \nabla \Phi_e \Big)\, dx \;=\; \int_\Omega \Big(\, I_i + I_e \Big)\, \psi \, dx \tag{2.8}$$

$$\forall\, \psi \in W^{1,2}(\Omega) \;\text{ with }\; \int_\Omega \psi(x)\, dx = 0\,;$$

$$\int_\Omega \Big(\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \Big)\, \psi \, dx \;=\; 0 \quad \forall\, \psi \in L^2(\Omega)\,; \tag{2.9}$$

$$\Phi_{tr}(x,0) \;=\; \Phi_0(x) \quad (\forall)\, x \in \Omega\,; \quad W(x,0) \;=\; W_0(x) \quad (\forall)\, x \in \Omega\,. \tag{2.10}$$

The corresponding notions of strong and weak solutions will be clarified by the following definition.

Definition 2.1. *1)* **(Strong solution of the bidomain system** $(\mathrm{B})_1$ **)** [10] *A triple* (Φ_{tr}, Φ_e, W) *is called a strong solution of the bidomain system* (2.1) – (2.6) *on* $[\,0\,,\,T\,]$ *iff the functions* Φ_{tr}, Φ_e *and* W *satisfy the equations* (2.1) – (2.3) *a. e. on* $\Omega \times [\,0\,,\,T\,]$ *as well as the initial and boundary conditions* (2.4) – (2.6) *on* $\partial\Omega \times [\,0\,,\,T\,]$*, respectively. Moreover, the functions belong to the spaces*

$$\Phi_{tr} \in L^2\big[\,(\,0\,,\,T\,)\,,\, W^{2,2}(\Omega)\,\big] \;\cap\; W^{1,2}\big[\,(\,0\,,\,T\,)\,,\, L^2(\Omega)\,\big]\,; \tag{2.11}$$

$$\Phi_e \in L^2\big[\,(\,0\,,\,T\,)\,,\, W^{2,2}(\Omega)\,\big]\,; \tag{2.12}$$

$$W \in C^0\big[\,[\,0\,,\,T\,]\,,\, L^2(\Omega)\,\big] \;\cap\; W^{1,2}\big[\,(\,0\,,\,T\,)\,,\, L^2(\Omega)\,\big]\,, \tag{2.13}$$

and for all $t \in [\,0\,,\,T\,]$*, it holds that* $\int_\Omega \Phi_e(x,t)\, dx \;=\; 0.$

2) **(Weak solution of the bidomain system** $(\mathrm{B})_2$ **)** [11] *A triple* (Φ_{tr}, Φ_e, W) *is called a weak solution of the bidomain system* (2.7) – (2.10) *on* $[\,0\,,\,T\,]$ *iff the functions* Φ_{tr}, Φ_e *and* W *satisfy equations* (2.7) – (2.9) *and obey the initial conditions* (2.10)*. Moreover, the functions belong to the spaces*

$$\Phi_{tr} \in C^0\big[\,[\,0\,,\,T\,]\,,\, L^2(\Omega)\,\big] \;\cap\; L^2\big[\,(\,0\,,\,T\,)\,,\, W^{1,2}(\Omega)\,\big] \;\cap\; L^p(\Omega_T) \quad \textit{with}\;\; 2 \leqslant p \leqslant 6\,; \tag{2.14}$$

$$\Phi_e \in L^2\big[\,(\,0\,,\,T\,)\,,\, W^{1,2}(\Omega)\,\big]\,; \tag{2.15}$$

$$W \in C^0\big[\,[\,0\,,\,T\,]\,,\, L^2(\Omega)\,\big]\,, \tag{2.16}$$

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09) Following [Bourgault/Coudière/Pierre 09] , p. 472, Remark 27, the occuring distributional time derivatives can be identified as functions $\partial\Phi_{tr}/\partial t$, $\partial W/\partial t \in L^2\big[\,(\,0\,,\,T\,)\,,\, \big(\, W^{1,2}(\Omega)\,\big)^*\,\big]$.

10) Slightly modified from [Bourgault/Coudière/Pierre 09] , p. 469, Definition 18.

11) [Bourgault/Coudière/Pierre 09] , p. 472, Definition 26.

and for almost all $t \in [\,0\,,\,T\,]$*, it holds that* $\displaystyle\int_\Omega \Phi_e(x,t)\,dx \;=\; 0$*. Note that* $\Omega_T = \Omega \times [\,0\,,\,T\,]$*.*

Assumptions 2.2. (Basic assumptions on the data in $(\mathrm{B})_1$ **and** $(\mathrm{B})_2$**)**

1) $\Omega \subset \mathbb{R}^3$ is a bounded strongly Lipschitz domain.

2) $M_i,\, M_e:\ \mathrm{cl}\,(\Omega) \to \mathbb{R}^{3\times 3}$ are symmetric, positive definite matrix functions with $L^\infty(\Omega)$-coefficients, obeying uniform ellipticity conditions:

$$0 \leqslant \mu_1\,\|\,\xi\,\|^2 \leqslant \xi^{\mathrm{T}} M_i(x)\,\xi \leqslant \mu_2\,\|\,\xi\,\|^2 \;\text{ and }\; 0 \leqslant \mu_1\,\|\,\xi\,\|^2 \leqslant \xi^{\mathrm{T}} M_e(x)\,\xi \leqslant \mu_2\,\|\,\xi\,\|^2 \quad \forall\,\xi \in \mathbb{R}^3 \;\; \forall\, x \in \Omega \tag{2.17}$$

with $\mu_1,\, \mu_2 > 0$.

3) I_{ion} and G are affine-linear with respect to W with

$$I_{ion}(\varphi, w) = F_1(\varphi) + F_2(\varphi)\,w \;\;\text{ and }\;\; G(\varphi, w) = G_1(\varphi) + g_2\,w \tag{2.18}$$

with continuous functions $F_1,\, F_2,\, G_1:\ \mathbb{R} \to \mathbb{R}$ and $g_2 \in \mathbb{R}$.

4) The functions F_1, F_2 and G_1 obey the following growth conditions: For all $\varphi \in \mathbb{R}$, it holds that

$$|\,F_1(\varphi)\,| \;\leqslant\; c_1 + c_2\,|\,\varphi\,|^{p-1}\,; \tag{2.19}$$

$$|\,F_2(\varphi)\,| \;\leqslant\; c_3 + c_4\,|\,\varphi\,|^{p/2-1}\,; \tag{2.20}$$

$$|\,G_1(\varphi)\,| \;\leqslant\; c_5 + c_6\,|\,\varphi\,|^{p/2} \tag{2.21}$$

with nonnegative constants $c_1,\,\dots\,,\,c_6 \geqslant 0$ and some $2 \leqslant p \leqslant 6$. Further, for all $\varphi,\, w \in \mathbb{R}$, it holds that

$$a_0\,|\,\varphi\,|^p - b_0\,\big(\,\varrho\,|\,\varphi\,|^2 + |\,w\,|^2\,\big) - c_0 \;\leqslant\; \varrho\,\big(\,F_1(\varphi) + F_2(\varphi)\,w\,\big)\cdot\varphi + \big(\,G_1(\varphi) + g_2\,w\,\big)\cdot w \tag{2.22}$$

with constants $a_0 > 0$, $\varrho > 0$, $b_0,\, c_0 \geqslant 0$ and $2 \leqslant p \leqslant 6$ as above.

5) The initial values Φ_0, W_0 belong to the space $L^2(\Omega)$.

6) I_i and I_e belong to the space $L^2\big[\,(\,0\,,\,T\,)\,,\,\big(\,W^{1,2}(\Omega)\,\big)^*\,\big]$ and satisfy the compatibility condition

$$\int_\Omega \Big(\,I_i(x,t) + I_e(x,t)\,\Big)\,dx \;=\; 0 \quad (\forall)\,t \in [\,0\,,\,T\,]\,. \tag{2.23}$$

Under these assumptions, the weak bidomain system (2.7) – (2.10) allows a further reformulation, which is crucial for establishing existence and uniqueness results.

Theorem 2.3. (The reduced bidomain system and the bidomain bilinear form)

1)[12] *Under the Assumptions 2.2., 1) – 6), a triple* (Φ_{tr}, Φ_e, W) *forms a weak solution of* $(\mathrm{B})_2$ *on the interval* $[\,0\,,\,T\,]$ *for the initial values* Φ_0*,* $W_0 \in L^2(\Omega)$ *iff it satisfies the reduced bidomain system*

$$(\mathrm{B})_3 \quad \frac{d}{dt}\langle\,\Phi_{tr}(t)\,,\,\psi\,\rangle + A\big(\,\Phi_{tr}(t)\,,\,\psi\,\big) + \int_\Omega I_{ion}(\Phi_{tr}(t), W(t))\,\psi\,dx \;=\; \langle\,S(t)\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega)\,; \tag{2.24}$$

$$\frac{d}{dt}\langle\,W(t)\,,\,\psi\,\rangle + \int_\Omega G(\Phi_{tr}(t), W(t))\,\psi\,dx \;=\; 0 \quad \forall\,\psi \in L^2(\Omega)\,; \tag{2.25}$$

$$\Phi_{tr}(x,0) \;=\; \Phi_0(x) \quad (\forall)\,x \in \Omega\,; \quad W(x,0) \;=\; W_0(x) \quad (\forall)\,x \in \Omega \tag{2.26}$$

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12) [Bourgault/Coudière/Pierre 09], p. 473, Lemma 28, together with p. 464, Definition 5.

on $[\,0\,,\,T\,]$ *in distributional sense. Here the bidomain bilinear form* $A\colon\ W^{1,2}(\Omega)\times W^{1,2}(\Omega)\to\mathbb{R}$ *is defined through*

$$A(\psi_1\,,\,\psi_2)\;=\;\int_\Omega \nabla\psi_1^{\mathrm{T}} M_i\,\nabla\psi_2\,dx+\int_\Omega \nabla\widetilde{\psi}_e^{\mathrm{T}} M_i\,\nabla\psi_2\,dx \tag{2.27}$$

where $\widetilde{\psi}_e\in W^{1,2}(\Omega)$ *is the uniquely determined solution of the variational equation*

$$\int_\Omega \nabla\widetilde{\psi}_e^{\mathrm{T}}(M_i+M_e)\,\nabla\psi\,dx\;=\;-\int_\Omega \nabla\psi_1^{\mathrm{T}} M_i\,\nabla\psi\,dx\quad\forall\,\psi\in W^{1,2}(\Omega)\,. \tag{2.28}$$

Accordingly, the linear functionals $S(t)\in\big(W^{1,2}(\Omega)\big)^*$ *are defined through*

$$\langle\,S(t)\,,\,\psi\,\rangle\;=\;\langle\,I_i(t)\,,\,\psi\,\rangle-\int_\Omega \nabla\overline{\psi}_e^{\mathrm{T}} M_i\,\nabla\psi\,dx \tag{2.29}$$

where $\overline{\psi}_e\in W^{1,2}(\Omega)$ *is the uniquely determined solution of the variational equation*

$$\int_\Omega \nabla\overline{\psi}_e^{\mathrm{T}}(M_i+M_e)\,\nabla\psi\,dx\;=\;\langle\,I_i(t)+I_e(t)\,,\,\psi\,\rangle\;\;\forall\,\psi\in W^{1,2}(\Omega)\text{ with }\int_\Omega\overline{\psi}_e(x,t)\,dx=0\;\;(\forall)\,t\in(\,0\,,\,T\,)\,. \tag{2.30}$$

2) [13] *The bilinear form* A *is symmetric, continuous and coercive with*

$$\beta\,\|\,\psi\,\|^2_{W^{1,2}(\Omega)}\;\leqslant\;A(\psi\,,\,\psi)+\beta\,\|\,\psi\,\|^2_{L^2(\Omega)}\quad\forall\,\psi\in W^{1,2}(\Omega)\quad\text{and} \tag{2.31}$$

$$\big|\,A(\psi_1\,,\,\psi_2)\,\big|\;\leqslant\;\gamma\cdot\|\,\psi_1\,\|_{W^{1,2}(\Omega)}\cdot\|\,\psi_2\,\|_{W^{1,2}(\Omega)}\quad\forall\,\psi_1\,,\;\psi_2\in W^{1,2}(\Omega) \tag{2.32}$$

where $\beta,\ \gamma>0$.

b) The models for the ionic current.

For the ionic current I_{ion} and the function G within the gating equation, we will consider the following three models:

a) The Rogers-McCulloch model. [14]

$$I_{ion}(\varphi,w)\;=\;b\cdot\varphi\,(\varphi-a)\,(\varphi-1)+\varphi\cdot w\;=\;b\,\varphi^3-(a+1)\,b\,\varphi^2+a\,b\,\varphi+\varphi\,w\,; \tag{2.33}$$

$$G(\varphi,w)\;=\;\varepsilon\,w-\varepsilon\,\kappa\,\varphi \tag{2.34}$$

with $0<a<1$, $b>0$, $\kappa>0$ and $\varepsilon>0$. Consequently, the gating variable obeys the linear ODE

$$\frac{\partial W}{\partial t}+\varepsilon\,W\;=\;\varepsilon\,\kappa\,\Phi_{tr}\,. \tag{2.35}$$

b) The FitzHugh-Nagumo model. [15]

$$I_{ion}(\varphi,w)\;=\;\varphi\,(\varphi-a)\,(\varphi-1)+w\;=\;\varphi^3-(a+1)\,\varphi^2+a\,\varphi+w\,; \tag{2.36}$$

$$G(\varphi,w)\;=\;\varepsilon\,w-\varepsilon\,\kappa\,\varphi \tag{2.37}$$

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13) [Bourgault/Coudière/Pierre 09], p. 464, Theorem 6.

14) [Rogers/McCulloch 94].

15) [FitzHugh 61], together with [Nagumo/Arimoto/Yoshizawa 62].

with $0 < a < 1$, $\kappa > 0$ and $\varepsilon > 0$. Consequently, the gating variable obeys the same linear ODE (2.35) as before.

c) The linearized Aliev-Panfilov model. [16)]

$$I_{ion}(\varphi, w) = b \cdot \varphi\,(\varphi - a)\,(\varphi - 1) + \varphi \cdot w = b\,\varphi^3 - (a+1)\,b\,\varphi^2 + a\,b\,\varphi + \varphi\,w\,; \tag{2.38}$$
$$G(\varphi, w) = \varepsilon\,w - \varepsilon\,\kappa\,\big(\,(a+1)\,\varphi - \varphi^2\,\big) \tag{2.39}$$

with $0 < a < 1$, $b > 0$, $\kappa > 0$ and $\varepsilon > 0$. The linear ODE for the gating variable is

$$\frac{\partial W}{\partial t} + \varepsilon\,W = \varepsilon\,\kappa\,\big(\,(a+1)\,\Phi_{tr} - {\Phi_{tr}}^2\,\big)\,. \tag{2.40}$$

Proposition 2.4. (Analytical properties of the ionic current models) [17)] *The Rogers-McCulloch, FitzHugh-Nagumo and the linearized Aliev-Panfilov model satisfy Assumptions 2.2., 3) and 4) with* $p = 4$.

c) Existence of weak solutions.

The following Theorems 2.5. and 2.6. were obtained in [Bourgault/Coudière/Pierre 09] and [Nagaiah/Kunisch/Plank 09] .

Theorem 2.5. (Existence of weak solutions) [18)] *Assume that the data within* $(\mathrm{B})_2$ *obey Assumptions 2.2., 1) – 6) with* $2 \leqslant p \leqslant 6$. *Then for arbitrary initial values* Φ_0, $W_0 \in L^2(\Omega)$, *the bidomain system* $(\mathrm{B})_2$ *admits on* $[\,0\,,\,T\,]$ *at least one weak solution* (Φ_{tr}, Φ_e, W) *in the sense of Definition 2.1., 2), where p is taken from Assumption 2.2., 4).*

Consequently, for any of the three models described in Subsection 2.b), weak solutions with $p = 4$ exist.

Theorem 2.6. (A priori estimate for weak solutions) [19)] *Assume that the data within* $(\mathrm{B})_2$ *obey Assumptions 2.2., 1) – 6) with* $2 \leqslant p \leqslant 6$. *If a triple*

$$(\Phi_{tr}, \Phi_e, W) \in \Big(C^0\big[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \cap L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big] \cap L^p(\Omega_T) \Big) \times L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big] \times C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big] \tag{2.41}$$

forms a weak solution of the bidomain system $(\mathrm{B})_2$ *on* $[\,0\,,\;T\,]$ *then the following estimate holds:*

$$\begin{aligned} &\|\,\Phi_{tr}\,\|^2_{C^0[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\,]} + \|\,\Phi_{tr}\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,]} + \|\,\Phi_{tr}\,\|^p_{L^p(\Omega_T)} + \|\,\partial\Phi_{tr}/\partial t\,\|^q_{L^q[\,(\,0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]} \\ &\quad + \|\,\Phi_e\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,]} + \|\,W\,\|^2_{C^0[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\,]} + \|\,\partial W/\partial t\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]} \\ &\leqslant C \cdot \Big(1 + \|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,I_i\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]} + \|\,I_e\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]} \Big) \end{aligned} \tag{2.42}$$

where $1/p + 1/q = 1$. *The constant* $C > 0$ *does not depend on* Φ_0, W_0, I_i *and* I_e.

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16) See [Aliev/Panfilov 96] . The linearized model is taken from [Bourgault/Coudière/Pierre 09] , p. 480. The original model contains a Riccati equation for the gating variable instead.

17) [Bourgault/Coudière/Pierre 09] , pp. 479 – 481.

18) [Bourgault/Coudière/Pierre 09] , p. 473, Theorem 30.

19) [Nagaiah/Kunisch/Plank 09] , p. 10, Lemma 3.5.

d) Uniqueness of weak solutions.

Note that Theorem 2.5. does not guarantee the uniqueness of the weak solution of (2.7) – (2.10) in correspondence to a given pair of excitation variables I_i, I_e. The following error estimate, using assumptions, which are in accordance with the analytical framework of the optimal control problem (P) to be analyzed later, allows to prove a unique correspondence between (I_i, I_e) and the weak solution (Φ_{tr}, Φ_e, W).

Theorem 2.7. (Error estimates for weak solutions) *Assume that the data within* $(\mathrm{B})_2$ *obey Assumptions 2.2., 1) – 6) together with* $p = 4$*, and specify within* $(\mathrm{B})_2$ *one of the models from Subsection 2.b). If two weak solutions* $({\Phi_{tr}}', {\Phi_e}', W')$, $({\Phi_{tr}}'', {\Phi_e}'', W'')$ *of* $(\mathrm{B})_2$ *correspond to initial values* $\Phi_0' = \Phi_0'' = \Phi_0 \in L^2(\Omega)$, $W_0' = W_0'' = W_0 \in L^4(\Omega)$ *and inhomogeneities* ${I_i}'$, ${I_e}'$, ${I_i}''$ *and* ${I_e}'' \in L^\infty\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]$*, whose norms are bounded by* $R > 0$*, then the following estimates hold:*

$$\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2\big[\,(0\,,T\,)\,,\,W^{1,2}(\Omega)\big]} + \|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{C^0\big[\,[0\,,T\,]\,,\,L^2(\Omega)\big]} + \|\,{\Phi_e}' - {\Phi_e}''\,\|^2_{L^2\big[\,(0\,,T\,)\,,\,W^{1,2}(\Omega)\big]} \tag{2.43}$$
$$+ \|\,W' - W''\,\|^2_{L^2\big[\,(0\,,T\,)\,,\,L^2(\Omega)\big]} + \|\,W' - W''\,\|^2_{C^0\big[\,[0\,,T\,]\,,\,L^2(\Omega)\big]} + \|\,W' - W''\,\|^2_{W^{1,2}\big[\,(0\,,T\,)\,,\,L^2(\Omega)\big]}$$
$$\leqslant C\,\Big(\,\|\,{I_i}' - {I_i}''\,\|^2_{L^\infty\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]} + \|\,{I_e}' - {I_e}''\,\|^2_{L^\infty\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]}\,\Big)\,.$$

$$\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|_{W^{1,4/3}\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]} \leqslant C \cdot \mathrm{Max}\,\Big(\,\|\,{I_i}' - {I_i}''\,\|_{L^2\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]}\,, \tag{2.44}$$
$$\|\,{I_e}' - {I_e}''\,\|_{L^2\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]}\,,\,\|\,{I_i}' - {I_i}''\,\|^2_{L^2\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]}\,,\,\|\,{I_e}' - {I_e}''\,\|^2_{L^2\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]}\,\Big)\,.$$

The constant $C > 0$ *does not depend on* ${I_i}'$, ${I_e}'$, ${I_i}''$ *and* ${I_e}''$ *but possibly on* Ω, R, Φ_0, W_0 *and* $p = 4$.

This estimate yields the following uniqueness theorem:

Theorem 2.8. (Uniqueness of weak solutions) *Assume that the data within* $(\mathrm{B})_2$ *obey Assumptions 2.2., 1) – 6) together with* $p = 4$*, and specify within* $(\mathrm{B})_2$ *one of the models from Subsection 2.b). Then for initial values* $\Phi_0 \in L^2(\Omega)$, $W_0 \in L^4(\Omega)$ *and inhomogeneities* I_i, $I_e \in L^\infty\big[\,(0\,,T\,)\,,\,\big(W^{1,2}(\Omega)\big)^*\,\big]$*, the bidomain system* $(\mathrm{B})_2$ *admits a unique weak solution* (Φ_{tr}, Φ_e, W) *in the sense of Definition 2.1., 1) on* $[\,0\,,T\,]$.

e) Proofs.

Proof of Theorem 2.7. Throughout the proof, C denotes a generical positive constant, which may appropriately change from line to line. C will never depend on the data ${I_i}'$, ${I_e}'$, ${I_i}''$ and ${I_e}''$ but possibly on Ω, R, Φ_0, W_0 and $p = 4$. The proof will be divided into three parts according to the underlying ionic current model.

Part A. The Rogers-McCulloch model.

• **Step A1.** *The difference of the reduced parabolic equations.* By Theorem 2.3., 1), the pairs $({\Phi_{tr}}', W')$ and $({\Phi_{tr}}'', W'')$ satisfy for almost all $t \in [\,0\,,T\,]$ the equations

$$\langle\,\frac{d}{dt}\,{\Phi_{tr}}'(t)\,,\,\psi\,\rangle + A\big(\,{\Phi_{tr}}'(t)\,,\,\psi\,\big) + \int_\Omega I_{ion}({\Phi_{tr}}'(t), W'(t))\,\psi\,dx \;=\; \langle\,S'(t)\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega)\,; \tag{2.45}$$

$$\langle\,\frac{d}{dt}\,{\Phi_{tr}}''(t)\,,\,\psi\,\rangle + A\big(\,{\Phi_{tr}}''(t)\,,\,\psi\,\big) + \int_\Omega I_{ion}({\Phi_{tr}}''(t), W''(t))\,\psi\,dx \;=\; \langle\,S''(t)\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega)\,. \tag{2.46}$$

Consequently, we obtain the equation

$$\langle\,\frac{d}{dt}\,\big(\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\big)\,,\,\psi\,\rangle + A\big(\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,,\,\psi\,\big) \tag{2.47}$$
$$+ \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}'(t), W'(t)) - I_{ion}({\Phi_{tr}}''(t), W''(t))\,\Big)\,\psi\,dx \;=\; \langle\,S'(t) - S''(t)\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega)$$

where

$$\langle S'(t), \psi \rangle = \langle I_i'(t), \psi \rangle - \int_\Omega \nabla\psi^{\mathrm{T}} M_i \nabla\overline{\psi}_e' \, dx; \quad \langle S''(t), \psi \rangle = \langle I_i''(t), \psi \rangle - \int_\Omega \nabla\psi^{\mathrm{T}} M_i \nabla\overline{\psi}_e'' \, dx, \tag{2.48}$$

and $\overline{\psi}_e'$, $\overline{\psi}_e'' \in W^{1,2}(\Omega)$ are the uniquely determined solutions of the variational equations

$$\int_\Omega \nabla\psi^{\mathrm{T}}(M_i + M_e)\nabla\overline{\psi}_e' \, dx = \langle I_i' + I_e', \psi \rangle \quad \forall\, \psi \in W^{1,2}(\Omega) \text{ and} \tag{2.49}$$

$$\int_\Omega \nabla\psi^{\mathrm{T}}(M_i + M_e)\nabla\overline{\psi}_e'' \, dx = \langle I_i'' + I_e'', \psi \rangle \quad \forall\, \psi \in W^{1,2}(\Omega) \tag{2.50}$$

with $\int_\Omega \overline{\psi}_e'(x,t)\, dx = 0$, $\int_\Omega \overline{\psi}_e''(x,t)\, dx = 0 \quad (\forall)\, t \in (0, T)$.

Lemma 2.9. *Under the assumptions of Theorem 2.7., for arbitrary* $\varepsilon > 0$ *the following estimate holds:*

$$|\langle S'(t) - S''(t), \psi \rangle| \leqslant \frac{C}{2\varepsilon}\Big(\| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \| I_e'(t) - I_e''(t) \|^2_{(W^{1,2}(\Omega))^*} \Big) + \frac{3\varepsilon}{4} \| \psi \|^2_{W^{1,2}(\Omega)}.$$

The constant $C > 0$ *does not depend on* ε, Φ_0, W_0, I_i *and* I_e. (2.51)

Proof. Forming the difference of (2.49) and (2.50), we get the variational equation

$$\int_\Omega \nabla\psi^{\mathrm{T}}(M_i + M_e)\nabla(\overline{\psi}_e' - \overline{\psi}_e'')\, dx = \langle (I_i' - I_i'') + (I_e' - I_e''), \psi \rangle \quad \forall\, \psi \in W^{1,2}(\Omega), \tag{2.52}$$

and by insertion of $\psi = \overline{\psi}_e' - \overline{\psi}_e''$ as a feasible test function, we get from Assumption 2.2., 2) and the Poincaré inequality: [20]

$$\begin{aligned} C\, \| \overline{\psi}_e' - \overline{\psi}_e'' \|_{W^{1,2}(\Omega)} &\leqslant \int_\Omega \nabla(\overline{\psi}_e' - \overline{\psi}_e'')^{\mathrm{T}}(M_i + M_e)\nabla(\overline{\psi}_e' - \overline{\psi}_e'')\, dx \qquad (2.53)\\ &\leqslant |\langle I_i' - I_i'', \overline{\psi}_e' - \overline{\psi}_e'' \rangle| + |\langle I_e' - I_e'', \overline{\psi}_e' - \overline{\psi}_e'' \rangle|. \end{aligned}$$

For arbitrary $\varepsilon > 0$, the generalized Cauchy inequality [21] implies

$$\begin{aligned} C\, \| \overline{\psi}_e' - \overline{\psi}_e'' \|_{W^{1,2}(\Omega)} &\leqslant \frac{1}{2\varepsilon} \| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon}{2} \| \overline{\psi}_e' - \overline{\psi}_e'' \|^2_{W^{1,2}(\Omega)} \qquad (2.54)\\ &\quad + \frac{1}{2\varepsilon} \| I_e'(t) - I_e''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon}{2} \| \overline{\psi}_e' - \overline{\psi}_e'' \|^2_{W^{1,2}(\Omega)}, \end{aligned}$$

and with $\varepsilon = C/2$, we obtain

$$\frac{C}{2} \| \overline{\psi}_e' - \overline{\psi}_e'' \|_{W^{1,2}(\Omega)} \leqslant \frac{1}{C}\Big(\| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \| I_e'(t) - I_e''(t) \|^2_{(W^{1,2}(\Omega))^*} \Big). \tag{2.55}$$

Further, it holds that

$$|\langle S'(t) - S''(t), \psi \rangle| = |\langle I_i'(t) - I_i''(t), \psi \rangle - \int_\Omega \nabla\psi^{\mathrm{T}} M_i \nabla(\overline{\psi}_e' - \overline{\psi}_e'')\, dx| \tag{2.56}$$

$$\leqslant |\langle I_i'(t) - I_i''(t), \psi \rangle| + |\langle \nabla(\overline{\psi}_e' - \overline{\psi}_e''), M_i \nabla\psi \rangle| \tag{2.57}$$

$$\leqslant \frac{1}{2\varepsilon} \| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon}{2} \| \psi \|^2_{W^{1,2}(\Omega)} + \frac{1}{2\varepsilon'} \| \nabla(\overline{\psi}_e' - \overline{\psi}_e'') \|^2_{L^2(\Omega)} + \frac{\varepsilon'}{2} \| M_i \|^2 \cdot \| \psi \|^2_{L^2(\Omega)} \tag{2.58}$$

$$\leqslant \frac{1}{2\varepsilon} \| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon}{2} \| \psi \|^2_{W^{1,2}(\Omega)} + \frac{1}{2\varepsilon'} \| \overline{\psi}_e' - \overline{\psi}_e'' \|^2_{W^{1,2}(\Omega)} + \frac{\varepsilon'}{2} \| M_i \|^2 \cdot \| \psi \|^2_{W^{1,2}(\Omega)} \tag{2.59}$$

$$\begin{aligned} &\leqslant \frac{1}{2\varepsilon} \| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon}{2} \| \psi \|^2_{W^{1,2}(\Omega)}\\ &\qquad + \frac{C}{2\varepsilon'}\Big(\| I_i'(t) - I_i''(t) \|^2_{(W^{1,2}(\Omega))^*} + \| I_e'(t) - I_e''(t) \|^2_{(W^{1,2}(\Omega))^*} \Big) + \frac{C\,\varepsilon'}{2} \| \psi \|^2_{W^{1,2}(\Omega)} \qquad (2.60) \end{aligned}$$

[20] Together with the Rellich-Kondrachov theorem, [Evans 98], p. 275, Theorem 1 holds true even on a bounded strongly Lipschitz domain, cf. [Adams/Fournier 07], p. 168, Theorem 6.3. Note that $\overline{\psi}_e'$ and $\overline{\psi}_e''$ admit zero spatial means.

[21] Cf. [Evans 98], p. 622, b.

where (2.55) and Assumption 2.2., 2) have been used. Taking $\varepsilon' = \varepsilon/(2C)$ and equalizing the constants in the first and third term in an appropriate way, we arrive at the claimed estimate (2.51). ■

Turning back to equation (2.47), we may insert $\psi = {\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)$ as a feasible test function since ${\Phi_{tr}}'(t)$, ${\Phi_{tr}}''(t) \in W^{1,2}(\Omega)$. Using the constant $\beta > 0$ from Theorem 2.3., 2), (2.31), we obtain

$$\frac{1}{2}\,\frac{d}{dt}\,\|\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\|^2_{L^2(\Omega)} + A\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big) + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+ \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'')\,\Big)\,\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)\,dx$$
$$= \langle\, S'(t) - S''(t)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} \quad \Longrightarrow \tag{2.61}$$
$$\frac{1}{2}\,\frac{d}{dt}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \int_\Omega \Big(\,I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'')\,\Big)\,\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)\,dx$$
$$\leqslant\;\big|\,\langle\, S'(t) - S''(t)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle\,\big| + \beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}\,. \tag{2.62}$$

By Lemma 2.9., the first term on the right-hand side can be estimated through

$$\big|\,\langle\, S'(t) - S''(t)\,,\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\rangle\,\big| \;\leqslant\; \frac{C}{2\varepsilon_1}\,\Big(\,\|\,{I_i}'(t) - {I_i}''(t)\,\|^2_{(W^{1,2}(\Omega))^*} + \|\,{I_e}'(t) - {I_e}''(t)\,\|^2_{(W^{1,2}(\Omega))^*}\,\Big) \tag{2.63}$$
$$+ \frac{3\varepsilon_1}{4}\,\|\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\|^2_{W^{1,2}(\Omega)}$$

with a number $\varepsilon_1 > 0$ to be fixed later. The estimation of the second term can be performed literally as in [Kunisch/Wagner 11], pp. 7 ff., (2.33) – (2.43), and we end up with the inequality

$$\frac{d}{dt}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + 2\,\beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} \tag{2.64}$$
$$\leqslant C\,\frac{\varepsilon_2}{2}\,\Big(\,\|\,{\Phi_{tr}}'\,\|^4_{L^4(\Omega)} + \|\,{\Phi_{tr}}''\,\|^4_{L^4(\Omega)}\,\Big)^{1/2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{C}{2\,\varepsilon_2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+\; C\,\varepsilon_3\,\|\,W'\,\|^2_{L^4(\Omega)}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_3}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$
$$+\; C\,\varepsilon_4\,\|\,{\Phi_{tr}}''\,\|^2_{L^4(\Omega)}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{1}{\varepsilon_4}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}$$
$$+\; \frac{C}{\varepsilon_1}\,\Big(\,\|\,{I_i}'(t) - {I_i}''(t)\,\|^2_{(W^{1,2}(\Omega))^*} + \|\,{I_e}'(t) - {I_e}''(t)\,\|^2_{(W^{1,2}(\Omega))^*}\,\Big)$$
$$+\; \frac{3\varepsilon_1}{2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + 2\,\beta\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}\,,$$

which is identical with [Kunisch/Wagner 11], p. 9, (2.43). The positive numbers ε_2, ε_3 and ε_4, arising from the repeated application of the generalized Cauchy's inequality, will be fixed below. As a consequence of the assumed L^∞-regularity of the excitations ${I_i}'$, ${I_e}'$, ${I_i}''$ and ${I_e}''$, the inequalities

$$\int_\Omega {\Phi_{tr}}'(t)^4\,dx \;\leqslant\; C\,\Big(\,1 + \varrho\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} \tag{2.65}$$
$$+ \|\,{I_i}'\,\|^2_{L^\infty[\,(0,T)\,,\,(W^{1,2}(\Omega))^*\,]} + \|\,{I_e}'\,\|^2_{L^\infty[\,(0,T)\,,\,(W^{1,2}(\Omega))^*\,]}\,\Big)\,;$$
$$\int_\Omega {\Phi_{tr}}''(t)^4\,dx \;\leqslant\; C\,\Big(\,1 + \varrho\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} \tag{2.66}$$
$$+ \|\,{I_i}''\,\|^2_{L^\infty[\,(0,T)\,,\,(W^{1,2}(\Omega))^*\,]} + \|\,{I_e}''\,\|^2_{L^\infty[\,(0,T)\,,\,(W^{1,2}(\Omega))^*\,]}\,\Big)$$

may be taken over from loc. cit., p. 9, (2.46) and (2.48), as well. Further, W' is the unique (strong or weak) solution of the initial value problem (2.3), (2.6). Consequently, it belongs to the space $C^1\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \cap C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ and admits the representation [22)]

$$W'(x,t) \;=\; W_0(x)\,e^{-\varepsilon t} + \varepsilon\,\kappa\,e^{-\varepsilon t}\int_0^t \Phi_{tr}{}'(x,\tau)\,e^{\varepsilon\tau}\,d\tau\,, \tag{2.67}$$

which implies the estimate

$$\int_\Omega W'(t)^4\,dx \;\leqslant\; C\,\|\,W_0\,\|^4_{L^4(\Omega)} + C\,\Big(1 + \|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,I_i{}'\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_e{}'\,\|^2_{L^\infty\left[\,(\,0\,,\,T\,)\,,\,\left(W^{1,2}(\Omega)\right)^*\,\right]}\Big)\,, \tag{2.68}$$

which has been already obtained in [Kunisch/Wagner 11], p. 10, (2.51). Combining now (2.64) with (2.65), (2.66) and (2.68), we may fix the numbers ε_1, ε_2, ε_3, $\varepsilon_4 > 0$ in such a way that the terms with $\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{W^{1,2}(\Omega)}$ will be annihilated. We arrive at

$$\frac{d}{dt}\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} \leqslant \Big(\frac{C}{2\,\varepsilon_2} + \frac{1}{\varepsilon_3} + 2\,\beta\Big)\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon_4}\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \frac{C}{\varepsilon_1}\,\Big(\|\,I_i{}'(t) - I_i{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e{}'(t) - I_e{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)\,. \tag{2.69}$$

• **Step A2.** *The difference of the gating equations.* The weak solutions $(\Phi_{tr}{}', W')$ and $(\Phi_{tr}{}'', W'')$ satisfy for almost all $t \in [\,0\,,\,T\,]$ the equations

$$\langle\,\frac{d}{dt}\,W'(t)\,,\,\psi\,\rangle \;=\; -\int_\Omega \big(\,\varepsilon\,W'(t) - \varepsilon\,\kappa\,\Phi_{tr}{}'(t)\,\big)\,\psi\,dx \quad \forall\,\psi \in L^2(\Omega) \;\;\text{and} \tag{2.70}$$

$$\langle\,\frac{d}{dt}\,W''(t)\,,\,\psi\,\rangle \;=\; -\int_\Omega \big(\,\varepsilon\,W''(t) - \varepsilon\,\kappa\,\Phi_{tr}{}''(t)\,\big)\,\psi\,dx \quad \forall\,\psi \in L^2(\Omega)\,. \tag{2.71}$$

Inserting the feasible test function $\psi = W'(t) - W''(t)$ into the difference of (2.70) and (2.71), we get

$$\langle\,\frac{d}{dt}\,\big(\,W'(t) - W''(t)\,\big)\,,\,W'(t) - W''(t)\,\rangle = -\,\varepsilon\,\int_\Omega \big(\,W' - W''\,\big)^2\,dx + \varepsilon\,\kappa\,\int_\Omega \big(\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\big)\,\big(\,W' - W''\,\big)\,dx \quad\Longrightarrow \tag{2.71}$$

$$\frac{d}{dt}\,\Big(\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \;\leqslant\; \Big(\,2\,\varepsilon + \varepsilon\,\kappa\Big)\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)}\,. \tag{2.72}$$

• **Step A3.** *The estimates for the differences* $\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{C^0\left[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\right]}$, $\|\,W' - W''\,\|^2_{L^2\left[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\right]}$ *and* $\|\,W' - W''\,\|^2_{C^0\left[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\right]}$. After equalizing of the constants on the right-hand sides, the inequalities (2.69) and (2.72) give together

$$\frac{d}{dt}\,\Big(\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) \;\leqslant\; C\cdot\Big(\,\|\,\Phi_{tr}{}' - \Phi_{tr}{}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\Big) + \frac{C}{\varepsilon_1}\,\Big(\|\,I_i{}'(t) - I_i{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*} + \|\,I_e{}'(t) - I_e{}''(t)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\Big)\,, \tag{2.73}$$

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22) [Warga 72], p. 192, Theorem II.4.6.

After application of Gronwall's inequality,[23] (2.73) implies

$$\| \Phi_{tr}'(t) - \Phi_{tr}''(t) \|^2_{L^2(\Omega)} + \| W'(t) - W''(t) \|^2_{L^2(\Omega)} \leqslant e^{C\,t} \Big(\| \Phi_{tr}'(0) - \Phi_{tr}''(0) \|^2_{L^2(\Omega)} \tag{2.74}$$

$$+ \| W'(0) - W''(0) \|^2_{L^2(\Omega)} + \frac{C}{\varepsilon_1} \int_0^t \big(\| I_i'(\tau) - I_i''(\tau) \|^2_{(W^{1,2}(\Omega))^*} + \| I_e'(\tau) - I_e''(\tau) \|^2_{(W^{1,2}(\Omega))^*} \big)\, d\tau \Big)$$

$$\leqslant e^{C\,T} \frac{C}{\varepsilon_1} \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,. \tag{2.75}$$

From the last inequality, we get the desired estimates:

$$\| \Phi_{tr}' - \Phi_{tr}'' \|^2_{C^0[\,[0,T],\,L^2(\Omega)\,]} \leqslant e^{C\,T} \frac{C}{\varepsilon_1} \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,; \tag{2.76}$$

$$\| W' - W'' \|^2_{C^0[\,[0,T],\,L^2(\Omega)\,]} \leqslant e^{C\,T} \frac{C}{\varepsilon_1} \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,; \tag{2.77}$$

$$\| W' - W'' \|^2_{L^2[\,(0,T),\,L^2(\Omega)\,]} \leqslant T\, e^{C\,T} \frac{C}{\varepsilon_1} \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,. \tag{2.78}$$

• **Step A4.** *The estimate for the difference* $\| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2[\,(0,T),\,W^{1,2}(\Omega)\,]}$. The considerations from [Kunisch/Wagner 11] , p. 11, (2.65) – (2.68) may be taken over literally in order to confirm that

$$\| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2[\,(0,T),\,W^{1,2}(\Omega)\,]} \leqslant C \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,. \tag{2.79}$$

• **Step A5.** *The estimate for the difference* $\| W' - W'' \|^2_{W^{1,2}[\,(0,T),\,L^2(\Omega)\,]}$. Taking over the calculations from loc. cit., p. 11 f., (2.69) – (2.72), we get the estimate

$$\| W' - W'' \|^2_{W^{1,2}[\,(0,T),\,L^2(\Omega)\,]} \leqslant C \Big(\| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} + \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,. \tag{2.80}$$

• **Step A6.** *The estimate for the difference* $\| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,4/3}[\,(0,T),\,(W^{1,2}(\Omega))^*\,]}$. Even in this step, the calculations from [Kunisch/Wagner 11] , pp. 12 ff., (2.73) – (2.101), may be repeated with a single modification, namely the replacement of the monodomain form $M(\psi_1, \psi_2)$ by the bidomain form $A(\psi_1, \psi_2)$ from Theorem 2.3. We obtain the estimate

$$\| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,4/3}[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \leqslant C \cdot \mathrm{Max} \Big(\| I_i' - I_i'' \|_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]}\,, \tag{2.81}$$

$$\| I_e' - I_e'' \|_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]}\,,\ \| I_i' - I_i'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]}\,,\ \| I_e' - I_e'' \|^2_{L^2[\,(0,T),\,(W^{1,2}(\Omega))^*\,]} \Big)\,.$$

[23] [Evans 98] , p. 624, j.

• **Step A7.** *Derivation of the estimate for* $\| \Phi_e{}' - \Phi_e{}'' \|^2_{L^2[(0,T),W^{1,2}(\Omega)]}$ *from the difference of the elliptic equations.* Forming the difference of the equations

$$\int_\Omega \Big(\nabla\psi^{\mathrm{T}} M_i \nabla\Phi_{tr}{}' + \nabla\psi^{\mathrm{T}} (M_i + M_e)\, \nabla\Phi_e{}' \Big)\, dx \;=\; \int_\Omega \Big(I_i{}' + I_e{}' \Big)\, \psi\, dx \quad \text{and} \tag{2.82}$$

$$\int_\Omega \Big(\nabla\psi^{\mathrm{T}} M_i \nabla\Phi_{tr}{}'' + \nabla\psi^{\mathrm{T}} (M_i + M_e)\, \nabla\Phi_e{}'' \Big)\, dx \;=\; \int_\Omega \Big(I_i{}'' + I_e{}'' \Big)\, \psi\, dx \tag{2.83}$$

$$\forall\, \psi \in W^{1,2}(\Omega) \;\text{ with }\; \int_\Omega \psi(x)\, dx = 0\,,$$

which are valid for the pairs $(\Phi_{tr}{}', \Phi_e{}')$ and $(\Phi_{tr}{}'', \Phi_e{}'')$ for almost all $t \in [\,0\,,\,T\,]$, we get

$$\int_\Omega \Big(\nabla\psi^{\mathrm{T}} M_i \nabla\big(\Phi_{tr}{}' - \Phi_{tr}{}'' \big) + \nabla\psi^{\mathrm{T}} (M_i + M_e)\, \nabla\big(\Phi_e{}' - \Phi_e{}'' \big) \Big)\, dx \tag{2.84}$$

$$= \int_\Omega \Big((I_i{}' - I_i{}'') + (I_e{}' - I_e{}'' \Big)\, \psi\, dx \quad \forall\, \psi \in W^{1,2}(\Omega) \;\text{ with }\; \int_\Omega \psi(x)\, dx = 0\,.$$

Inserting $\psi = \Phi_e{}'(t) - \Phi_e{}''(t)$ as a feasible test function (cf. Definition 2.1., 2)), we obtain from Assumption 2.2., 2) and the Poincaré inequality:

$$C\, \| \Phi_e{}'(t) - \Phi_e{}''(t) \|^2_{W^{1,2}(\Omega)} \;\leqslant\; \int_\Omega \nabla\big(\Phi_e{}' - \Phi_e{}'' \big)^{\mathrm{T}}\, M_i\, \nabla\big(\Phi_{tr}{}' - \Phi_{tr}{}'' \big)\, dx \tag{2.85}$$

$$+ \int_\Omega (I_i{}' - I_i{}'')\, (\Phi_e{}' - \Phi_e{}'')\, dx + \int_\Omega (I_e{}' - I_e{}'')\, (\Phi_e{}' - \Phi_e{}'')\, dx$$

$$\leqslant\; \frac{1}{2\,\varepsilon_5}\, \| M_i \|^2 \cdot \| \nabla\Phi_{tr}{}' - \nabla\Phi_{tr}{}'' \|^2_{L^2(\Omega)} + \frac{\varepsilon_5}{2}\, \| \nabla\Phi_e{}' - \nabla\Phi_e{}'' \|^2_{L^2(\Omega)} + \frac{1}{2\,\varepsilon_6}\, \| I_i{}'(t) - I_i{}''(t) \|^2_{(W^{1,2}(\Omega))^*} \tag{2.86}$$

$$+\; \frac{\varepsilon_6}{2}\, \| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)} + \frac{1}{2\varepsilon_7}\, \| I_e{}'(t) - I_e{}''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon_7}{2}\, \| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)}$$

$$\leqslant\; \frac{C}{2\,\varepsilon_5}\, \| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{W^{1,2}(\Omega)} + \frac{\varepsilon_5}{2}\, \| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)} + \frac{1}{2\,\varepsilon_6}\, \| I_i{}'(t) - I_i{}''(t) \|^2_{(W^{1,2}(\Omega))^*} \tag{2.87}$$

$$+\; \frac{\varepsilon_6}{2}\, \| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)} + \frac{1}{2\varepsilon_7}\, \| I_e{}'(t) - I_e{}''(t) \|^2_{(W^{1,2}(\Omega))^*} + \frac{\varepsilon_7}{2}\, \| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)}\,.$$

Fixing ε_5, ε_6, $\varepsilon_7 > 0$ in an appropriate way and normalizing the constant on the left-hand side, we arrive at

$$\| \Phi_e{}'(t) - \Phi_e{}''(t) \|^2_{W^{1,2}(\Omega)} \;\leqslant\; C\, \Big(\| \Phi_{tr}{}'(t) - \Phi_{tr}{}''(t) \|^2_{W^{1,2}(\Omega)} \tag{2.88}$$

$$+ \| I_i{}'(t) - I_i{}''(t) \|^2_{(W^{1,2}(\Omega))^*} + \| I_e{}'(t) - I_e{}''(t) \|^2_{(W^{1,2}(\Omega))^*} \Big) \quad \Longrightarrow$$

$$\| \Phi_e{}' - \Phi_e{}'' \|^2_{L^2[(0,T),W^{1,2}(\Omega)]} \;\leqslant\; C\, \Big(\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|^2_{L^2[(0,T),W^{1,2}(\Omega)]} \tag{2.89}$$

$$+ \| I_i{}' - I_i{}'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} + \| I_e{}' - I_e{}'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big)\,,$$

from which we get with (2.79) the estimate

$$\| \Phi_e{}' - \Phi_e{}'' \|^2_{W^{1,2}(\Omega)} \tag{2.90}$$

$$\leqslant\; C\, \Big(\| I_i{}' - I_i{}'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} + \| I_e{}' - I_e{}'' \|^2_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \Big)\,.$$

• **Step A8.** *Conclusion of the proof of Part A.* Since $L^\infty[\,(0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]$ is continuously embedded into $L^2[\,(0\,,\,T\,)\,,\,(W^{1,2}(\Omega))^*\,]$, we have

$$\| I_i{}' - I_i{}'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \;\leqslant\; C\, \| I_i{}' - I_i{}'' \|_{L^\infty[(0,T),(W^{1,2}(\Omega))^*]}\,; \tag{2.91}$$

$$\| I_e{}' - I_e{}'' \|_{L^2[(0,T),(W^{1,2}(\Omega))^*]} \;\leqslant\; C\, \| I_e{}' - I_e{}'' \|_{L^\infty[(0,T),(W^{1,2}(\Omega))^*]}\,, \tag{2.92}$$

and the proof of Part A is complete.

Part B. The FitzHugh-Nagumo model.

• **Step B1.** *The difference of the reduced parabolic equations.* In comparison with the Rogers-McCulloch model, the FitzHugh-Nagumo model differs in the replacement of the nonlinear coupling term $\varphi\, w$ by w and the setting $b = 1$ within the ionic current. When proceeding as in Step A1, the first change applies to the estimation of the term

$$\int_\Omega \Big(I_{ion}({\Phi_{tr}}', W') - I_{ion}({\Phi_{tr}}'', W'') \Big) \, \big({\Phi_{tr}}' - {\Phi_{tr}}'' \big) \, dx \tag{2.93}$$

within (2.62). The modification can be traced in complete analogy to [Kunisch/Wagner 11], p. 14 f., (2.103) – (2.106), and we get (2.69) again (up to a change in the constants).

• **Step B2.** *The difference of the gating equations.* Since the gating equation is the same as in the Rogers-McCulloch model, Step A2 can be carried over without alterations.

• **Step B3.** *The estimates for the differences* $\| {\Phi_{tr}}' - {\Phi_{tr}}'' \|^2_{C^0 [\, [\,0\,,T\,]\,,\,L^2(\Omega)\,]}$, $\| W' - W'' \|^2_{L^2 [\,(\,0\,,T\,)\,,\,L^2(\Omega)\,]}$ *and* $\| W' - W'' \|^2_{C^0 [\,[\,0\,,T\,]\,,\,L^2(\Omega)\,]}$. Again, Step A3 can be carried over without changes, and we obtain the estimates (2.76 – (2.78).

• **Step B4.** *The estimate for the difference* $\| {\Phi_{tr}}' - {\Phi_{tr}}'' \|^2_{L^2 [\,(\,0\,,T\,)\,,\,W^{1,2}(\Omega)\,]}$. Proceeding as in [Kunisch/ Wagner 11], p. 15, (2.106) – (2.107), we get (2.79) again.

• **Step B5.** *The estimate for the difference* $\| W' - W'' \|^2_{W^{1,2} [\,(\,0\,,T\,)\,,\,L^2(\Omega)\,]}$. Step A5 can be carried over without alterations, and we obtain (2.80).

• **Step B6.** *The estimate for the difference* $\| {\Phi_{tr}}' - {\Phi_{tr}}'' \|_{W^{1,4/3} [\,(\,0\,,T\,)\,,\,(\,W^{1,2}(\Omega)\,)^*\,]}$. The considerations from [Kunisch/Wagner 11], p. 15 f., (2.108) – (2.112), may be repeated with the same modification as in Step A6, namely the replacement of the monodomain form $M(\psi_1, \psi_2)$ by the bidomain form $A(\psi_1, \psi_2)$ from Theorem 2.3. In the result, we arrive at (2.81) anew.

• **Step B7.** *The estimate for the difference* $\| {\Phi_e}' - {\Phi_e}'' \|^2_{L^2 [\,(\,0\,,T\,)\,,\,W^{1,2}(\Omega)\,]}$. Step A7 can be taken over without changes, and we get (2.90) again.

• **Step B8.** *Conclusion of the proof of Part B.* By application of (2.91) and (2.92), the proof of Part B will be completed.

Part C. The linearized Aliev-Panfilov model.

• **Step C1.** *The difference of the reduced parabolic equations.* Since the ionic currents in the Rogers-McCulloch and the linearized Aliev-Panfilov model are identical, Step A1 can be carried over literally until (2.69).

• **Step C2.** *The difference of the gating equations.* The weak solutions $({\Phi_{tr}}', W')$ and $({\Phi_{tr}}'', W'')$ satisfy for almost all $t \in [\,0\,,T\,]$

$$\langle \frac{d}{dt} W'(t)\,,\,\psi \,\rangle \;=\; -\int_\Omega \big(\varepsilon\, W'(t) - \varepsilon\,\kappa\,(a+1)\,{\Phi_{tr}}'(t) + \varepsilon\,\kappa\,{\Phi_{tr}}'(t)^2 \,\big)\, \psi\, dx \quad \forall\, \psi \in L^2(\Omega)\,; \tag{2.94}$$

$$\langle \frac{d}{dt} W''(t)\,,\,\psi \,\rangle \;=\; -\int_\Omega \big(\varepsilon\, W''(t) - \varepsilon\,\kappa\,(a+1)\,{\Phi_{tr}}''(t) + \varepsilon\,\kappa\,{\Phi_{tr}}''(t)^2 \,\big)\, \psi\, dx \quad \forall\, \psi \in L^2(\Omega) \quad \Longrightarrow \tag{2.95}$$

$$\begin{aligned} \langle \frac{d}{dt} \big(W'(t) - W''(t) \big)\,,\,\psi \,\rangle \;&=\; -\,\varepsilon \int_\Omega \big(W'(t) - W''(t) \big)\, \psi\, dx \\ &+ \varepsilon\,\kappa\,(a+1) \int_\Omega \big({\Phi_{tr}}'(t) - {\Phi_{tr}}''(t) \big)\, \psi\, dx - \varepsilon\,\kappa \int_\Omega \big({\Phi_{tr}}'(t)^2 - {\Phi_{tr}}''(t)^2 \big)\, \psi\, dx \quad \forall\, \psi \in L^2(\Omega)\,. \end{aligned} \tag{2.96}$$

Inserting now the feasible test function $\psi = W'(t) - W''(t)$, we obtain

$$\langle \frac{d}{dt}\big(W'(t) - W''(t)\big)\,,\, W'(t) - W''(t)\,\rangle \;=\; -\,\varepsilon\int_\Omega \big(W' - W''\big)^2\,dx \tag{2.97}$$

$$+\,\varepsilon\,\kappa\,(a+1)\int_\Omega \big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)\,\big(\,W' - W''\,\big)\,dx - \varepsilon\,\kappa\int_\Omega \big(\,({\Phi_{tr}}')^2 - ({\Phi_{tr}}'')^2\,\big)\,\big(\,W' - W''\,\big)dx \quad\Longrightarrow$$

$$\frac{d}{dt}\Big(\,\frac{1}{2}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\,\Big) \;\leqslant\; \varepsilon\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\,(a+1)\int_\Omega \big|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big|\cdot\big|\,W' - W''\,\big|\,dx \tag{2.98}$$

$$+\,\varepsilon\,\kappa\int_\Omega \big|\,({\Phi_{tr}}')^2 - ({\Phi_{tr}}'')^2\,\big|\cdot\big|\,W' - W''\,\big|\,dx \quad\Longrightarrow$$

$$\frac{d}{dt}\Big(\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\,\Big) \;\leqslant\; \Big(\,2\,\varepsilon + \varepsilon\,\kappa\,(a+1)\,\Big)\,\|\,W' - W''\,\|^2_{L^2(\Omega)} + \varepsilon\,\kappa\,(a+1)\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} \tag{2.99}$$

$$+\,2\,\varepsilon_8\int_\Omega \big(\,{\Phi_{tr}}' + {\Phi_{tr}}''\,\big)^2\,\big(\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\big)^2\,dx + \frac{1}{2\,\varepsilon_8}\int_\Omega \big(\,W' - W''\,\big)^2\,dx$$

$$\leqslant\; C\,\Big(\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\,\Big) \tag{2.100}$$

$$+\,C\,\varepsilon_8\,\Big(\int_\Omega \big(\,|\,{\Phi_{tr}}'\,|^4 + |\,{\Phi_{tr}}''\,|^4\,\big)\,dx\Big)^{1/2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^4(\Omega)} + \frac{1}{2\,\varepsilon_8}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}$$

where the generalized Cauchy's inequality with $\varepsilon_8 > 0$ has been applied. Together with (2.65), (2.66) and the imbedding inequality $\|\,\cdot\,\|_{L^4(\Omega)} \leqslant C\,\|\,\cdot\,\|_{W^{1,2}(\Omega)}$, we find

$$\frac{d}{dt}\Big(\,\|\,W' - W''\,\|^2_{L^2(\Omega)}\,\Big) \;\leqslant\; \big(\,C + \frac{1}{2\,\varepsilon_8}\,\big)\,\Big(\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)} + \|\,W' - W''\,\|^2_{L^2(\Omega)}\,\Big) \tag{2.101}$$

$$+\,C\,\varepsilon_8\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)}\,.$$

• **Step C3.** *The estimates for the differences* $\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{C^0[\,[\,0\,,T\,]\,,\,L^2(\Omega)\,]}$, $\|\,W' - W''\,\|^2_{L^2[\,(\,0\,,T\,)\,,\,L^2(\Omega)\,]}$ *and* $\|\,W' - W''\,\|^2_{C^0[\,[\,0\,,T\,]\,,\,L^2(\Omega)\,]}$. After addition of the inequalities (2.69) and (2.101), we may fix the numbers $\varepsilon_1, \ldots, \varepsilon_4, \varepsilon_8 > 0$ in such a way that the terms with $\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)}$ will be annihilated. Equalizing the constants on the right-hand side of the resulting inequality, Step A3 can be repeated literally, and we obtain (2.76), (2.77) and (2.78).

• **Step C4.** *The estimate for the difference* $\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2[\,(\,0\,,T\,)\,,\,W^{1,2}(\Omega)\,]}$. With an appropriate choice of $\varepsilon_8 > 0$, the calculations from Step A4 can be repeated without change, and we arrive at (2.79).

• **Step C5.** *The estimate for the difference* $\|\,W' - W''\,\|^2_{W^{1,2}[\,(\,0\,,T\,)\,,\,L^2(\Omega)\,]}$. Into equation (2.96), we insert the feasible test function $\psi = (\partial W'(t)/\partial t) - (\partial W''(t)/\partial t)$. With the generalized Cauchy's inequality, we get

$$\langle\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,,\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,\rangle \;=\; \|\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,\|^2_{L^2(\Omega)} \tag{2.102}$$

$$\leqslant\; \varepsilon\,\frac{\varepsilon_9}{2}\,\|\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\,\varepsilon_9}\,\|\,W' - W''\,\|^2_{L^2(\Omega)}$$

$$+\;\varepsilon\,\kappa\,(a+1)\,\frac{\varepsilon_{10}}{2}\,\|\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,\|^2_{L^2(\Omega)} + \frac{\varepsilon\,\kappa}{2\,\varepsilon_{10}}\,(a+1)\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{L^2(\Omega)}$$

$$+\;C\,\varepsilon_{11}\,\Big(\,\|\,{\Phi_{tr}}'\,\|^4_{L^4(\Omega)} + \|\,{\Phi_{tr}}''\,\|^4_{L^4(\Omega)}\,\Big)^{1/2}\,\|\,{\Phi_{tr}}' - {\Phi_{tr}}''\,\|^2_{W^{1,2}(\Omega)} + \frac{1}{2\,\varepsilon_8}\,\|\,\frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}\,\|^2_{L^2(\Omega)}$$

for arbitrary ε_9, ε_{10}, $\varepsilon_{11} > 0$. Considering (2.65) and (2.66), the arguments from Step A5 can be repeated after an appropriate choice of these constants, and we arrive at (2.80) again.

• **Step C6.** *The estimate for the difference* $\| \Phi_{tr}{}' - \Phi_{tr}{}'' \|_{W^{1,4/3}\left[\,(0,T),\,H^{-1}(\Omega)\right]}$. The calculations from Step A6 may be literally taken over, and we obtain (2.81).

• **Step C7.** *The estimate for the difference* $\| \Phi_e{}' - \Phi_e{}'' \|^2_{L^2\left[\,(0,T),\,W^{1,2}(\Omega)\right]}$. Step A7 can be taken over without changes, and we get (2.90) again.

• **Step C8.** *Conclusion of the proof of Part C.* By application of (2.91) and (2.92), the proof of Part C is complete. ∎

Proof of Theorem 2.8. The existence of a weak solution (Φ_{tr}, Φ_e, W) of $(\mathrm{B})_2$ is ensured by Theorem 2.5. In order to prove its uniqueness, Theorem 2.7. has to be applied to $I_i{}' = I_i{}'' = I_i$ and $I_e{}' = I_e{}'' = I_e$. ∎

3. Higher regularity of weak solutions for the bidomain system.

a) Veneroni's iterative procedure.

In order to prove a further regularity theorem for the weak solutions of $(\mathrm{B})_2$, we strengthen the assumptions for the data. In particular, we will assume throughout this section that the sum of the excitations I_i and I_e possesses a weak time derivative.

Assumptions 3.1. (Stronger assumptions on the data in $(\mathrm{B})_1$ **and** $(\mathrm{B})_2$**)**

1) $\Omega \subset \mathbb{R}^3$ is a bounded strongly Lipschitz domain with $C^{1,1}$-boundary.

2) $M_i,\ M_e :\ \mathrm{cl}\,(\Omega) \to \mathbb{R}^{3\times 3}$ are symmetric, positive definite matrix functions with $W^{1,\infty}(\Omega)$-coefficients, obeying uniform ellipticity conditions:

$$0 \leqslant \mu_1 \| \xi \|^2 \leqslant \xi^{\mathrm{T}} M_i(x)\, \xi \leqslant \mu_2 \| \xi \|^2 \quad \text{and} \quad 0 \leqslant \mu_1 \| \xi \|^2 \leqslant \xi^{\mathrm{T}} M_e(x)\, \xi \leqslant \mu_2 \| \xi \|^2 \quad \forall \xi \in \mathbb{R}^3 \ \ \forall x \in \Omega \tag{3.1}$$

with $\mu_1,\ \mu_2 > 0$.

3) I_{ion} and G are affine-linear with respect to W with

$$I_{ion}(\varphi, w) = F_1(\varphi) + F_2(\varphi)\, w \quad \text{and} \quad G(\varphi, w) = G_1(\varphi) + g_2\, w \tag{3.2}$$

with continuous functions $F_1,\ F_2,\ G_1 :\ \mathbb{R} \to \mathbb{R}$ and $g_2 \in \mathbb{R}$.

4) The functions F_1, F_2 and G_1 obey the following growth conditions: For all $\varphi \in \mathbb{R}$, it holds that

$$\big| F_1(\varphi) \big| \ \leqslant\ c_1 + c_2 \,|\, \varphi \,|^{p-1}\,; \tag{3.3}$$

$$\big| F_2(\varphi) \big| \ \leqslant\ c_3 + c_4 \,|\, \varphi \,|^{p/2-1}\,; \tag{3.4}$$

$$\big| G_1(\varphi) \big| \ \leqslant\ c_5 + c_6 \,|\, \varphi \,|^{p/2} \tag{3.5}$$

with nonnegative constants $c_1, \ldots, c_6 \geqslant 0$ and some $2 \leqslant p \leqslant 6$. Further, for all $\varphi,\ w \in \mathbb{R}$, it holds that

$$a_0 \,|\, \varphi \,|^p - b_0 \left(\varrho \,|\, \varphi \,|^2 + |\, w \,|^2 \right) - c_0 \ \leqslant\ \varrho \left(F_1(\varphi) + F_2(\varphi)\, w \right) \cdot \varphi + \left(G_1(\varphi) + g_2\, w \right) \cdot w \tag{3.6}$$

with constants $a_0 > 0$, $\varrho > 0$, $b_0,\ c_0 \geqslant 0$ and $2 \leqslant p \leqslant 6$ as above.

5) The initial values belong to the following spaces: $\Phi_0 \in W^{2,2}(\Omega)$, $W_0 \in L^\infty(\Omega)$. Moreover, Φ_0 satisfies the compatibility condition described in [Veneroni 09], p. 853, and p. 854, Remark 3. [24)]

[24)] In contrast to the condition in [Bourgault/Coudière/Pierre 09], p. 469, Theorem 20, this condition can be verified in a constructive way.

6) I_i, I_e belong to the space $L^r\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$ with $4 < r \leqslant 6$; the sum $I_i + I_e$ belongs even to the Sobolev space $W^{1,2}\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$, and it holds that

$$\int_\Omega \Big(I_i(x,t) + I_e(x,t) \Big)\, dx \,=\, 0 \quad (\forall)\, t \in (\,0\,,\,\infty\,)\,. \tag{3.7}$$

Note that Assumption 3.1., k) implies Assumption 2.2., k) from Subsection 2.a), $1 \leqslant k \leqslant 6$.

The following theorem is part of an iterative solution procedure for the bidomain system, which has been presented in [Veneroni 09]. In the present investigation, Veneroni's result will be used in order to improve the regularity of a given weak solution of $(\mathrm{B})_2$.

Theorem 3.2. (Iterative solution of the bidomain system) [25] *Let the Assumptions 3.1., 1) – 6) hold. Further, let a function* $\widetilde{\Phi}_{tr} \in L^2\big[\,(0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big]$ *be given, which is regular enough to ensure that* $I_{ion}\big(\,\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr})\big)$ *belongs to* $L^q\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$, $1 < q \leqslant r$, *where* $W(\widetilde{\Phi}_{tr}) \in C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ *is defined as the (weak or strong) solution of the initial value problem*

$$(\widetilde{\mathrm{B}})_{0,gat} \quad \frac{\partial W}{\partial t} + G(\widetilde{\Phi}_{tr}, W) \,=\, 0 \;\; (\forall)\,(x,t) \in \Omega \times (\,0\,,\,T\,)\,, \quad W(x,0) = W_0(x) \;\; (\forall)\, x \in \Omega\,. \tag{3.8}$$

1) Then the initial-boundary value problem

$$(\widetilde{\mathrm{B}})_{0,par} \quad \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr})) - \mathrm{div}\,\big(M_i \,\nabla \Phi_i \big) \,=\, \quad I_i \quad (\forall)\,(x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{3.9}$$

$$\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr})) + \mathrm{div}\,\big(M_e \,\nabla \Phi_e \big) \,=\, -I_e \quad (\forall)\,(x,t) \in \Omega \times [\,0\,,\,T\,]\,; \tag{3.10}$$

$$\mathfrak{n}^{\mathrm{T}} M_i \,\nabla \Phi_i \,=\, 0 \quad \forall\,(x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{3.11}$$

$$\mathfrak{n}^{\mathrm{T}} M_e \,\nabla \Phi_e \,=\, 0 \quad \forall\,(x,t) \in \partial\Omega \times [\,0\,,\,T\,]\,; \tag{3.12}$$

$$\Phi_{tr}(x,0) \,=\, \Phi_i(x,0) - \Phi_e(x,0) \,=\, \Phi_0(x) \quad (\forall)\, x \in \Omega\,; \quad \int_\Omega \Phi_e(x,t)\, dx \,=\, 0 \quad (\forall)\, t \in [\,0\,,\,T\,] \tag{3.13}$$

possesses a uniquely determined solution $(\Phi_i, \Phi_e) \in L^q\big[\,(0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big] \,\times\, L^q\big[\,(0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]$ *with*

$$\Phi_{tr} \,=\, \Phi_i - \Phi_e \,\in\, W^{1,q}\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, L^q\big[\,(0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]\,. \tag{3.14}$$

2) The solutions Φ_i, Φ_e *satisfy the estimates*

$$\begin{aligned} \|\,\Phi_i\,\|_{L^q[\,(0,T),\,W^{2,2}(\Omega)]} + \|\,\Phi_{tr}\,\|_{W^{1,q}[\,(0,T),\,L^2(\Omega)]} \,&\leqslant\, C\,\Big(\|\,\Phi_0\,\|_{W^{2,2}(\Omega)} \\ + \|\,I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr}))\,\|_{L^q[\,(0,T),\,L^2(\Omega)]} &+ \|\,I_i\,\|_{L^q[\,(0,T),\,L^2(\Omega)]} + \|\,I_i + I_e\,\|_{W^{1,2}[\,(0,T),\,L^2(\Omega)]} \Big)\,; \end{aligned} \tag{3.15}$$

$$\begin{aligned} \|\,\Phi_e\,\|_{L^q[\,(0,T),\,W^{2,2}(\Omega)]} + \|\,\Phi_{tr}\,\|_{W^{1,q}[\,(0,T),\,L^2(\Omega)]} \,&\leqslant\, C\,\Big(\|\,\Phi_0\,\|_{W^{2,2}(\Omega)} \\ + \|\,I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr}))\,\|_{L^q[\,(0,T),\,L^2(\Omega)]} &+ \|\,I_e\,\|_{L^q[\,(0,T),\,L^2(\Omega)]} + \|\,I_i + I_e\,\|_{W^{1,2}[\,(0,T),\,L^2(\Omega)]} \Big)\,. \end{aligned} \tag{3.16}$$

3) If $q \geqslant 2$ *then for a pair of solutions* $({\Phi_{tr}}', {\Phi_e}', W')$, $({\Phi_{tr}}'', {\Phi_e}'', W'')$ *of* $(\widetilde{\mathrm{B}})_{0,gat}$ *and* $(\widetilde{\mathrm{B}})_{0,par}$ *corresponding to the entries* $(\widetilde{\Phi}'_{tr}, \widetilde{\Phi}'_e, \widetilde{W}')$ *and* $(\widetilde{\Phi}''_{tr}, \widetilde{\Phi}''_e, \widetilde{W}'')$, *we have the estimate*

$$\|\,{\Phi_{tr}}'(t) - {\Phi_{tr}}''(t)\,\|^2_{L^2(\Omega)} \,\leqslant\, C \int_0^t \|\,I_{ion}(\widetilde{\Phi}'_{tr}(\vartheta), \widetilde{W}'(\vartheta)) - I_{ion}(\widetilde{\Phi}''_{tr}(\vartheta), \widetilde{W}''(\vartheta))\,\|^2_{L^2(\Omega)}\, d\vartheta \;\; \forall\, t \in [\,0\,,\,T\,]\,. \tag{3.17}$$

[25] [Veneroni 09], p. 856 f., Proposition 3.1., together with p. 861.

Proof. Parts 1) and 2) of Theorem 3.1. are identical with [VENERONI 09], p. 856 f., Proposition 3.1., except for the fact that $I_{ion}\big(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr})\big)$ is assumed to belong to $L^q\big[\,(0\,,T\,)\,,\,L^2(\Omega)\,\big]$, $1 < q \leqslant r$, instead of $L^r\big[\,(0\,,T\,)\,,\,L^2(\Omega)\,\big]$. However, a closer inspection of the proof in [VENERONI 09], pp. 858 − 861, reveals that all arguments remain in virtue for the above assumed weaker regularity since the $W^{1,q}$-regularity of the solution of $(\widetilde{\mathrm{B}})_{0,par}$ depends on the regularity of

$$\frac{I_i(t)+I_e(t)}{2} - I_{ion}\big(\,\widetilde{\Phi}_{tr}(t), W(\widetilde{\Phi}_{tr}(t))\,\big) \in L^q(\Omega) \tag{3.18}$$

only. Then the estimates in Part 2) and 3) follow from [VENERONI 09], p. 861, Theorem 3.1., together with p. 860, (3.27), (3.28) and (3.30), in a completely analogous way. ∎

b) Higher regularity of weak solutions.

Imposing the Assumptions 3.1., the regularity of a given weak solution of $(\mathrm{B})_2$ can be considerably improved.

Theorem 3.3. (Higher regularity of weak solutions) *Let the Assumptions 3.1., 1) − 6) with* $p = 4$ *and* $4 < r \leqslant 6$ *hold, and specify within* $(\mathrm{B})_2$ *one of the models from Subsection 2.b). Assume that a triple* (Φ_{tr}, Φ_e, W) *forms a weak solution of the bidomain system* $(\mathrm{B})_2$ *on* $[\,0\,,\,T\,]$ *in the sense of Definition 2.1., 2) in correspondence to initial values* $\Phi_0 \in W^{2,2}(\Omega)$ *and* $W_0 \in L^\infty(\Omega)$. *Then the functions possess the higher regularity*

$$\Phi_{tr} \in W^{1,r}\big[\,(0\,,T\,)\,,\,L^2(\Omega)\,\big] \cap L^r\big[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\,\big] \cap C^0(\Omega_T)\,; \tag{3.19}$$

$$\Phi_e \in L^2\big[\,(0\,,T\,)\,,\,L^2(\Omega)\,\big] \cap L^r\big[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\,\big]\,; \tag{3.20}$$

$$W \in C^1\big[\,(0\,,T\,)\,,\,L^\infty(\Omega)\,\big] \cap C^0\big[\,[\,0\,,T\,]\,,\,L^\infty(\Omega)\,\big]\,. \tag{3.21}$$

Consequently, (Φ_{tr}, Φ_e, W) *forms even a strong solution of* $(\mathrm{B})_0$ *and* $(\mathrm{B})_1$.

Corollary 3.4. (Higher regularity and uniqueness of weak solutions) *Let the assumptions of Theorem 3.3. hold. Besides of Assumption 3.1., 6), let the inhomogeneities* I_i *and* I_e *belong even to the space* $L^\infty\big[\,(0\,,T\,)\,,\,L^2(\Omega)\,\big]$. *Then the bidomain system* $(\mathrm{B})_2$ *possesses on* $[\,0\,,T\,]$ *a uniquely determined weak solution* (Φ_{tr}, Φ_e, W), *which admits the regularity described by* (3.19) − (3.21) *and, consequently, forms a uniquely determined strong solution of* $(\mathrm{B})_0$ *and* $(\mathrm{B})_1$ *as well.*

As pointed out e. g. in [COLLI FRANZONE/GUERRI/TENTONI 90], p. 159 f., the eigenvector bases of the conductivity tensors M_i, $M_e\colon \mathrm{cl}\,(\Omega) \to \mathbb{R}^{3\times 3}$, reflecting the direction of the muscle fibers within the heart tissue, should be identical. Consequently, let us assume that the representations

$$M_i(x) \;=\; \begin{pmatrix} \mathfrak{e}_1(x)^{\mathrm{T}} \\ \mathfrak{e}_2(x)^{\mathrm{T}} \\ \mathfrak{e}_3(x)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \lambda_{1,i}(x) & 0 & 0 \\ 0 & \lambda_{2,i}(x) & 0 \\ 0 & 0 & \lambda_{3,i}(x) \end{pmatrix} \big(\,\mathfrak{e}_1(x) \quad \mathfrak{e}_2(x) \quad \mathfrak{e}_3(x)\,\big) \quad \forall\, x \in \mathrm{cl}\,(\Omega)\,; \tag{3.22}$$

$$M_e(x) \;=\; \begin{pmatrix} \mathfrak{e}_1(x)^{\mathrm{T}} \\ \mathfrak{e}_2(x)^{\mathrm{T}} \\ \mathfrak{e}_3(x)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \lambda_{1,e}(x) & 0 & 0 \\ 0 & \lambda_{2,e}(x) & 0 \\ 0 & 0 & \lambda_{3,e}(x) \end{pmatrix} \big(\,\mathfrak{e}_1(x) \quad \mathfrak{e}_2(x) \quad \mathfrak{e}_3(x)\,\big) \quad \forall\, x \in \mathrm{cl}\,(\Omega) \tag{3.23}$$

hold. As a consequence, the boundary conditions as well as the compatibility condition for the initial datum Φ_0 may be simplified in the following way:

Corollary 3.5. (Geometrical consequences of higher regularity) *Let the assumptions of Theorem 3.3. hold. Assume further that the descriptions* (3.22), (3.23) *and*

$$M_i(x)\,\mathfrak{n}(x) = \begin{pmatrix} \lambda_{1,i}(x) & 0 & 0 \\ 0 & \lambda_{2,i}(x) & 0 \\ 0 & 0 & \lambda_{3,i}(x) \end{pmatrix} \mathfrak{n}(x)\,,\;\; M_e(x)\,\mathfrak{n}(x) = \begin{pmatrix} \lambda_{1,e}(x) & 0 & 0 \\ 0 & \lambda_{2,e}(x) & 0 \\ 0 & 0 & \lambda_{3,e}(x) \end{pmatrix} \mathfrak{n}(x) \tag{3.24}$$

$$\forall\, x \in \partial\Omega$$

are valid while $\mathfrak{e}_1,\ \mathfrak{e}_2,\ \mathfrak{e}_3:\ \mathrm{cl}\,(\Omega) \to \mathbb{R}^3$ *are continuous functions.*

1) Then any weak solution of $(\mathrm{B})_2$ *on* $[\,0\,,\,T\,]$ *is a strong solution of* $(\mathrm{B})_0$ *and* $(\mathrm{B})_1$ *where the boundary conditions* (1.4) − (1.5) *and* (2.4) − (2.5) *are replaced by*

$$\mathfrak{n}^{\mathrm{T}}\,\nabla\Phi_i \;=\; 0\,,\;\; \mathfrak{n}^{\mathrm{T}}\,\nabla\Phi_e \;=\; 0 \quad \forall\,(x,t)\in\partial\Omega\times[\,0\,,\,T\,] \quad \textit{and} \tag{3.25}$$

$$\mathfrak{n}^{\mathrm{T}}\,\nabla\Phi_{tr} \;=\; 0\,,\;\; \mathfrak{n}^{\mathrm{T}}\,\nabla\Phi_e \;=\; 0 \quad \forall\,(x,t)\in\partial\Omega\times[\,0\,,\,T\,]\,, \tag{3.26}$$

respectively.

2) The compatibility condition for the function $\Phi_0\in W^{2,2}(\Omega)$ *from Assumption 3.1., 5) reduces to*

$$\mathfrak{n}^{\mathrm{T}}\,\nabla\Phi_0 \;=\; 0 \quad \forall\,x\in\partial\Omega\,. \tag{3.27}$$

The proofs of Theorem 3.3. and the Corollaries 3.4. and 3.5. will be given in the following subsection.

Remark. By a slight modification of the full iteration procedure from [Veneroni 09] , the uniqueness of the weak solutions of $(\mathrm{B})_2$ can be confirmed exclusively under the assumptions of Theorem 3.3. without reference to Theorem 2.8. In the optimal control problems for the bidomain system analyzed in Part III, however, the intracellular excitation I_i is set zero for physiological reasons; consequently, Assumption 3.1., 6) implies $I_e\in W^{1,2}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$, which is continuously imbedded into $C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]\subset L^\infty\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$ by [Kunisch/Wagner 11] , p. 30, Theorem 4.5, and we arrive at the analytical situation of Corollary 3.4. from the outset.

c) Proofs.

Before starting with the proof of Theorem 3.3., we establish the following two propositions.

Proposition 3.6. (Iterative definition of the entries for the bidomain system I) *Let the assumptions of Theorem 3.3. hold. If a triple* $(\widetilde{\Phi}_{tr},\widetilde{\Phi}_e,\widetilde{W})\in\Big(C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]\cap L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big]\cap L^4(\Omega_T)\Big)\times L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big]\times C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ *forms a weak solution of the bidomain system* $(\mathrm{B})_2$ *on* $[\,0\,,\,T\,]$ *then the (weak or strong) solution* W *of the initial value problem* $(\widetilde{\mathrm{B}})_{0,gat}$ *from Theorem 3.2. belongs to the space* $C^1\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]\cap C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$ *and coincides with* $\widetilde{W}$.

Proof. In the case of the Rogers-McCulloch model and the FitzHugh-Nagumo model, the (weak or strong) solution of $(\widetilde{\mathrm{B}})_{0,gat}$ is given through

$$W(x,t) \;=\; W_0(x)\,e^{-\varepsilon t}+\varepsilon\,\kappa\,e^{-\varepsilon t}\int_0^t\widetilde{\Phi}_{tr}(x,\tau)\,e^{\varepsilon\tau}\,d\tau\,, \tag{3.28}$$

cf. [Warga 72] , p. 192, Theorem II.4.6., and (2.67) above. Obviously, this function admits the claimed regularity. Since (3.28) is the uniquely determined weak solution of $(\widetilde{\mathrm{B}})_{0,gat}$ as well, the solution W of $(\widetilde{\mathrm{B}})_{0,gat}$ must agree with $\widetilde{W}$, which is already known as a weak solution of $(\widetilde{\mathrm{B}})_{0,gat}$. Consequently, W and $\widetilde{W}$ agree even within the space $C^1\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]\cap C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]$. In the case of the linearized Aliev-Panfilov model, the (weak or strong) solution of $(\widetilde{\mathrm{B}})_{0,gat}$ reads as

$$W(x,t) \;=\; W_0(x)\,e^{-\varepsilon t}+\varepsilon\,\kappa\,e^{-\varepsilon t}\int_0^t\Big(\,(a+1)\,\widetilde{\Phi}_{tr}(x,\tau)-\widetilde{\Phi}_{tr}^2(x,\tau)\Big)\,e^{\varepsilon\tau}\,d\tau\,, \tag{3.29}$$

and the same implications are true. ∎

Proposition 3.7. (Iterative definition of the entries for the bidomain system II) *Let the assumptions of Theorem 3.3. hold. Consider a triple* $(\widetilde{\Phi}_{tr},\widetilde{\Phi}_e,\widetilde{W})\in\Big(C^0\big[\,[\,0\,,\,T\,]\,,\,L^2(\Omega)\,\big]\cap L^2\big[\,(\,0\,,\,T\,)\,,\,W^{1,2}(\Omega)\,\big]\cap$

$L^4(\Omega_T)\Big) \times L^2\big[\,(0\,,\,T\,)\,,\,W^{1,2}(\Omega)\big] \times C^0\big[\,[0\,,\,T\,)\,,\,L^2(\Omega)\big]$, which forms on $[\,0\,,\;T\,]$ a weak solution of the bidomain system $(\mathrm{B})_2$ *with initial values $\Phi_0 \in W^{2,2}(\Omega)$ and $W_0 \in L^\infty(\Omega)$. Assume further that $\widetilde{\Phi}_{tr}$ is regular enough to ensure that $I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr}))$ belongs to $L^q\big[\,(0\,,\,T\,)\,,\,L^2(\Omega)\big]$, $1 < q \leqslant r$. Then the solutions Φ_{tr}, Φ_e of $(\widetilde{\mathrm{B}})_{0,par}$ coincide with $\widetilde{\Phi}_{tr}$ and $\widetilde{\Phi}_e$ within the space $L^2\big[\,(0\,,\,T\,)\,,\,W^{1,2}(\Omega)\big]$.*

Proof. The assumptions about $\widetilde{\Phi}_{tr}$ guarantee the applicability of Theorem 3.2., 1). Consequently, the system $(\widetilde{\mathrm{B}})_{0,par}$ admits the uniquely determined solutions

$$\Phi_i\,,\;\Phi_e \,\in\, L^q\big[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\big] \quad \text{with} \tag{3.30}$$

$$\Phi_{tr} \,=\, \Phi_i - \Phi_e \,\in\, W^{1,q}\big[\,(0\,,T\,)\,,\,L^2(\Omega)\big] \cap L^q\big[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\big]\,. \tag{3.31}$$

Since $(\widetilde{\mathrm{B}})_{0,par}$ may be equivalently stated in an elliptic-parabolic form analogous to $(\mathrm{B})_1$, the triple (Φ_{tr}, Φ_e, W) forms a weak solution of a system, which is identical with $(\mathrm{B})_2$ except for the replacement of $I_{ion}(\Phi_{tr}, W)$ by $I_{ion}(\widetilde{\Phi}_{tr}, W(\widetilde{\Phi}_{tr}))$. Due to Proposition 3.6., we have $W = W(\widetilde{\Phi}_{tr}) = \widetilde{W}$, and we obtain from $(\mathrm{B})_2$ and $(\widetilde{\mathrm{B}})_{0,par}$:

$$\langle\, \frac{d}{dt}\,\big(\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big)\,,\,\psi\,\rangle + \int_\Omega \nabla\psi^{\mathrm{T}} M_i\,\Big(\nabla\big(\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big) + \nabla\big(\,\widetilde{\Phi}_e(t) - \Phi_e(t)\,\big)\Big)\,dx \tag{3.32}$$

$$+ \int_\Omega \big(\,I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) - I_{ion}(\widetilde{\Phi}_{tr}(t), W(t))\,\big)\,\psi\,dx \,=\, 0 \quad \forall\,\psi \in W^{1,2}(\Omega) \quad \Longrightarrow$$

$$\langle\, \frac{d}{dt}\,\big(\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big)\,,\,\psi\,\rangle + \int_\Omega \nabla\psi^{\mathrm{T}} M_i\,\Big(\nabla\big(\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big) + \nabla\big(\,\widetilde{\Phi}_e(t) - \Phi_e(t)\,\big)\Big)\,dx \,=\, 0 \tag{3.33}$$

$$\forall\,\psi \in W^{1,2}(\Omega)\,;$$

$$\int_\Omega \nabla\psi^{\mathrm{T}} M_i \nabla\big(\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big)\,dx + \int_\Omega \nabla\psi^{\mathrm{T}}(M_i + M_e)\,\nabla\big(\,\widetilde{\Phi}_e(t) - \Phi_e(t)\,\big)\,dx \,=\, 0 \tag{3.34}$$

$$\forall\,\psi \in W^{1,2}(\Omega) \;\text{ with }\; \int_\Omega \psi(x)\,dx = 0\,;$$

$$\widetilde{\Phi}_{tr}(x,0) - \Phi_{tr}(x,0) \,=\, 0 \quad \text{ and } \quad \widetilde{W}(x,0) - W(x,0) \,=\, 0 (\forall)\,x \in \Omega\,. \tag{3.35}$$

Together with the difference of the gating equations, we arrive at a "homogeneous" bidomain system in the variables $(\widetilde{\Phi}_{tr} - \Phi_{tr})$, $(\widetilde{\Phi}_e - \Phi_e)$ and $(\widetilde{W} - W)$ with the inhomogenities $I_i = I_e = \mathfrak{o}$. Observe that Assumptions 2.2., 1), 2), 5) and 6) hold for the system (3.32) – (3.35) from the outset. With F_1, F_2, $G_1 = \mathfrak{o}$ and $g_2 = 0$, Assumptions 2.2., 3) and 4) can be satisfied with $p = 2$, $c_1 = c_3 = c_5 = 0$, $c_2 = c_4 = c_6 = 1$, $a_0 = \varrho = 1$, $b_0 = 1$ and $c_0 = 0$. Consequently, we may apply Theorem 2.3., 1) in order to get

$$\frac{d}{dt}\langle\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\,\psi\,\rangle + A(\,\widetilde{\Phi}_{tr} - \Phi_{tr}\,,\,\psi\,) \,=\, \langle\, S_0\,,\,\psi\,\rangle \quad \forall\,\psi \in W^{1,2}(\Omega) \quad (\forall)\,t \in [\,0\,,\,T\,]\,; \tag{3.36}$$

$$\widetilde{\Phi}_{tr}(x,0) - \Phi_{tr}(x,0) \,=\, 0 \quad (\forall)\,x \in \Omega \tag{3.37}$$

where

$$\langle\, S_0(t)\,,\,\psi\,\rangle \,=\, -\int_\Omega \nabla\overline{\psi}_e^{\mathrm{T}}(t) M_i\,\nabla\psi\,dx\,, \tag{3.38}$$

and $\overline{\psi}_e^{\mathrm{T}}(t) \in W^{1,2}(\Omega)$ is the uniquely determined solution of the variational equation

$$\int_\Omega \nabla\overline{\psi}_e^{\mathrm{T}}(M_i + M_e)\,\nabla\psi\,dx \,=\, 0 \quad \forall\,\psi \in W^{1,2}(\Omega) \text{ with } \int_\Omega \overline{\psi}_e^{\mathrm{T}}(x,t)\,dx = 0 \tag{3.39}$$

for almost all $t \in [\,0\,,T\,]$. Consequently, $\overline{\psi}_e^{\,\mathrm{T}} = \mathfrak{o}$, and $\langle\, S_0(t)\,,\,\psi\,\rangle = 0$ for almost all $t \in [\,0\,,T\,]$ as well. Inserting now into (3.36) the admissible test function $\psi = \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)$, we find with (2.31):

$$\frac{d}{dt}\langle\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\rangle + A\big(\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big) \;=\; 0 \quad \Longrightarrow \tag{3.40}$$

$$\frac{d}{dt}\langle\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\rangle + A\big(\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\big) \tag{3.41}$$

$$+\,\beta\,\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)} \;=\; \beta\,\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)} \quad \Longrightarrow$$

$$\frac{d}{dt}\langle\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,,\, \widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\rangle + \|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{W^{1,2}(\Omega)} \;\leqslant\; \beta\,\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)} \quad \Longrightarrow \tag{3.42}$$

$$\frac{1}{2}\,\frac{d}{dt}\,\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)} \;\leqslant\; \beta\,\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)}\,. \tag{3.43}$$

Application of the Gronwall inequality yields

$$\|\,\widetilde{\Phi}_{tr}(t) - \Phi_{tr}(t)\,\|^2_{L^2(\Omega)} \;\leqslant\; e^{2\beta t}\,\|\,\widetilde{\Phi}_{tr}(0) - \Phi_{tr}(0)\,\|^2_{L^2(\Omega)} = 0\,, \tag{3.44}$$

consequently, $\widetilde{\Phi}_{tr}(t)$ and $\Phi_{tr}(t)$ agree for almost all $t \in [\,0\,,T\,]$ as $W^{1,2}(\Omega)$-functions. Then (3.34) implies together with the uniform ellipticity of $(M_i + M_e)$ and the Poincaré inequality:

$$0 \;=\; \int_\Omega \nabla(\widetilde{\Phi}_e(t) - \Phi_e(t))^{\mathrm{T}}\,(M_i + M_e)\,\nabla(\widetilde{\Phi}_e(t) - \Phi_e(t))\,dx \;\geqslant\; C\,\|\,\widetilde{\Phi}_e(t) - \Phi_e(t)\,\|^2_{W^{1,2}(\Omega)}\,, \tag{3.45}$$

and $\widetilde{\Phi}_e(t)$, $\Phi_e(t)$ agree as $W^{1,2}(\Omega)$-functions almost everywhere as well. ∎

Proof of Theorem 3.3. Throughout the proof, C denotes a generical positive constant, which may appropriately change from line to line. C will never depend on the data Φ_0, W_0, I_i and I_e but possibly on Ω and $p = 4$. The proof will be divided into three parts according to the underlying ionic current model. In order to apply Theorem 3.2. and Propositions 3.6. and 3.7., we rename the given weak solution by $(\widetilde{\Phi}_{tr}, \widetilde{\Phi}_e, \widetilde{W})$ and study the outcomes $W = W(\widetilde{\Phi}_{tr})$ and (Φ_{tr}, Φ_e) of Veneronis' procedure.

Part A. The Rogers-McCulloch model.

• **Step A1.** *The function $\widetilde{\Phi}_{tr}$ belongs even to the space $L^4\big[\,(\,0\,,T\,)\,,L^6(\Omega)\,\big]$.* We start with the following estimate, which is taken from the proof of Theorem 2.5., cf. [Nagaiah/Kunisch/Plank 09], p. 10, (33):

$$\frac{\varrho\,\beta}{4}\,\|\,\widetilde{\Phi}_{tr}(t)\,\|^2_{W^{1,2}(\Omega)} \;\leqslant\; 2\,(\beta + b_0)\,e^{2\,(\beta+b_0)\,t}\,\Big(\,\varrho\,\|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \int_0^t c_0\,|\,\Omega\,|\,d\tau \tag{3.46}$$

$$+\;C\int_0^t \|\,I_i(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,d\tau + C\int_0^t \|\,I_e(\tau)\,\|^2_{\left(W^{1,2}(\Omega)\right)^*}\,d\tau\,\Big) + c_0\,|\,\Omega\,| + \frac{C\varrho}{2\beta}\,\Big(\,\|\,I_i(t)\,\|^2_{L^2(\Omega)} + \|\,I_e(t)\,\|^2_{L^2(\Omega)}\,\Big)\,.$$

Due to Assumption 3.1., 6), the following may be derived from (3.46):

$$\|\,\widetilde{\Phi}_{tr}(t)\,\|^4_{W^{1,2}(\Omega)} \;\leqslant\; C\,\Big(\,|\,\Omega\,| + \|\,\Phi_0\,\|^2_{L^2(\Omega)} + \|\,W_0\,\|^2_{L^2(\Omega)} + \|\,I_i\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} \tag{3.47}$$

$$+\,\|\,I_e\,\|^2_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_i(t)\,\|^2_{L^2(\Omega)} + \|\,I_e(t)\,\|^2_{L^2(\Omega)}\,\Big)^2$$

$$\leqslant\; C\,\Big(\,|\,\Omega\,|^2 + \|\,\Phi_0\,\|^4_{L^2(\Omega)} + \|\,W_0\,\|^4_{L^2(\Omega)} + \|\,I_i\,\|^4_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} \tag{3.48}$$

$$+\,\|\,I_e\,\|^4_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_i(t)\,\|^4_{L^2(\Omega)} + \|\,I_e(t)\,\|^4_{L^2(\Omega)}\,\Big) \quad \Longrightarrow$$

$$\|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4\left[\,(\,0\,,T\,)\,,W^{1,2}(\Omega)\,\right]} \;\leqslant\; C\,T\,\Big(\,|\,\Omega\,|^2 + \|\,\Phi_0\,\|^4_{L^2(\Omega)} + \|\,W_0\,\|^4_{L^2(\Omega)} + \|\,I_i\,\|^4_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} \tag{3.49}$$

$$+\,\|\,I_e\,\|^4_{L^2\left[\,(\,0\,,T\,)\,,\left(W^{1,2}(\Omega)\right)^*\,\right]} + \|\,I_i\,\|^4_{L^4\left[\,(\,0\,,T\,)\,,L^2(\Omega)\,\right]} + \|\,I_e\,\|^4_{L^4\left[\,(\,0\,,T\,)\,,L^2(\Omega)\,\right]}\,\Big)\,.$$

Since the imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ is continuous, $\widetilde{\Phi}_{tr}$ is contained in the space $L^4\big[\,(\,0\,,\,T\,)\,,\,L^6(\Omega)\,\big]$.

• **Step A2.** *The function* $I_{ion}(\widetilde{\Phi}_{tr}, W) = I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W})$ *belongs to the space* $L^{4/3}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big]$.[26] We consider

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^2_{L^2(\Omega)} \;=\; \int_\Omega \Big(\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\Big)^2\, dx$$

$$= \int_\Omega \Big(\, b\,\widetilde{\Phi}_{tr}(\widetilde{\Phi}_{tr} - a)\,(\widetilde{\Phi}_{tr} - 1) + e^{-\varepsilon t}\, W_0\,\widetilde{\Phi}_{tr} + \varepsilon\,\kappa\, e^{-\varepsilon t}\,\widetilde{\Phi}_{tr} \int_0^t \widetilde{\Phi}_{tr}(x, \vartheta)\, e^{\varepsilon\,\vartheta}\, d\vartheta\,\Big)^2\, dx \tag{3.50}$$

$$\leqslant C \int_\Omega \Big(\, \widetilde{\Phi}^2_{tr}\,(\widetilde{\Phi}^2_{tr} + a^2)\,(\widetilde{\Phi}^2_{tr} + 1) + |\, W_0\,|^2\, \widetilde{\Phi}^2_{tr} + \varepsilon^2\,\kappa^2\,\widetilde{\Phi}^2_{tr}\,\Big(\int_0^t |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,| \cdot |\, e^{\varepsilon\vartheta}\,|\, d\vartheta\,\Big)^2\,\Big)\, dx \tag{3.51}$$

$$\leqslant C\,\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\, dx \tag{3.52}$$

$$+ \int_\Omega |\, W_0 \cdot \widetilde{\Phi}_{tr}\,|^2\, dx + e^{2\varepsilon T} \int_\Omega \widetilde{\Phi}^2_{tr} \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx\,\Big) \quad \Longrightarrow$$

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^{4/3}_{L^2(\Omega)} \;\leqslant\; C\,\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\, dx \tag{3.53}$$

$$+ \int_\Omega |\, W_0 \cdot \widetilde{\Phi}_{tr}\,|^2\, dx + \int_\Omega \widetilde{\Phi}^2_{tr} \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx\,\Big)^{2/3}$$

$$\leqslant C\,\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\, dx\,\Big)^{2/3} + C\,\Big(\,1 + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\, dx \tag{3.54}$$

$$+ \int_\Omega |\, W_0 \cdot \widetilde{\Phi}_{tr}\,|^2\, dx + \int_\Omega \widetilde{\Phi}^2_{tr} \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx\,\Big)^{2/3}$$

$$\leqslant C\,\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\, dx\,\Big)^{2/3} + C\,\Big(\,1 + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\, dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\, dx \tag{3.55}$$

$$+ \int_\Omega |\, W_0 \cdot \widetilde{\Phi}_{tr}\,|^2\, dx + \int_\Omega \widetilde{\Phi}^2_{tr} \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx\,\Big) \quad \Longrightarrow$$

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W})\,\|^{4/3}_{L^{4/3}[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,]} \;=\; \int_0^T \|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^{4/3}_{L^2(\Omega)}\, dt \tag{3.56}$$

$$\leqslant C\,\Big(\,1 + \|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4[\,(\,0\,,\,T\,)\,,\,L^6(\Omega)\,]} + \|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4[\,(\,0\,,\,T\,)\,,\,L^4(\Omega)\,]} + \|\,\widetilde{\Phi}_{tr}\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,]} \tag{3.57}$$

$$+ \|\,\mathrm{W}_0\,\|^2_{L^\infty(\Omega)} \cdot \|\,\widetilde{\Phi}_{tr}\,\|^2_{L^2[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,]}\,\Big) + C \cdot \int_0^T \int_\Omega \widetilde{\Phi}^2_{tr}(x, t) \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx\, dt$$

$$= C\,\Big(\,\ldots\,\Big) + C \int_\Omega \int_0^T \widetilde{\Phi}^2_{tr}(x, t)\, dt \cdot \Big(\int_0^T |\,\widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta\,\Big)^2\, dx \tag{3.58}$$

$$= C\,\Big(\,\ldots\,\Big) + C \int_\Omega \|\,\widetilde{\Phi}_{tr}(x)\,\|^2_{L^2(\,0\,,\,T\,)} \cdot \|\,\widetilde{\Phi}_{tr}(x)\,\|^2_{L^1(\,0\,,\,T\,)}\, dx \tag{3.59}$$

$$\leqslant C\,\Big(\,\ldots\,\Big) + C \int_\Omega \|\,\widetilde{\Phi}_{tr}(x)\,\|^2_{L^2(\,0\,,\,T\,)} \cdot \|\,\widetilde{\Phi}_{tr}(x)\,\|^2_{L^2(\,0\,,\,T\,)}\, dx \leqslant C\,\Big(\,\ldots\,\Big) + C\,\|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4[\,\Omega\,,\,L^2(\,0\,,\,T\,)\,]}\,. \tag{3.60}$$

Due to the continuous imbeddings $L^4\big[\,(\,0\,,\,T\,)\,,\,L^6(\Omega)\,\big] \hookrightarrow L^4(\Omega_T) = L^4\big[\,\Omega\,,\,L^4(\,0\,,\,T\,)\,\big] \hookrightarrow L^4\big[\,\Omega\,,\, L^2(\,0\,,\,T\,)\,\big]$, all norms of the right-hand side are finite, and the claim has been proved.

• **Step A3.** *The functions* $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ *and* $\widetilde{W} = W$ *belong to the spaces*

$$\Phi_{tr} \in W^{1,4/3}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, L^{4/3}\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big] \,\cap\, L^s(\Omega_T)\,,\;\; 4 \leqslant s < \infty\,; \tag{3.61}$$

$$\Phi_e \in L^2\big[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, L^{4/3}\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]\,; \tag{3.62}$$

$$W \in C^1\big[\,(\,0\,,\,T\,)\,,\,L^\infty(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^\infty(\Omega)\,\big]\,. \tag{3.63}$$

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[26] Note that, from [Bourgault/Coudière/Pierre 09], p. 471, Lemma 25, we only know that $I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))$ belongs to $L^{4/3}(\Omega)$ for almost all $t \in [\,0\,,\,T\,]$.

In view of Step A2, we may apply Proposition 3.6., Theorem 3.2. and Proposition 3.7. to the triple $(\widetilde{\Phi}_{tr}, \widetilde{\Phi}_e, \widetilde{W})$. Consequently, the corresponding solution (Φ_{tr}, Φ_e, W) of $(\widetilde{\mathrm{B}})_{0,gat}$ and $(\widetilde{\mathrm{B}})_{0,par}$ coincides with $(\widetilde{\Phi}_{tr}, \widetilde{\Phi}_e, \widetilde{W})$ for almost all $t \in (0, T)$ in the first and second component, and for all $t \in [0, T]$ for the third component. This proves the claimed regularity except the relation $\Phi_{tr} \in L^s(\Omega_T)$, which will be derived from the Aubin-Dubinskij lemma:[27]) Choosing $\mathrm{X}_0 = W^{2,2}(\Omega)$, $\mathrm{X} = C^0(\Omega)$ and $\mathrm{X}_1 = L^2(\Omega)$, we find that

$$\Phi_{tr} \in \Big\{ f \in L^{4/3}\big[(0, T), W^{2,2}(\Omega)\big] \;\Big|\; \frac{df}{dt} \in L^{4/3}\big[(0, T), L^2(\Omega)\big] \Big\} \tag{3.64}$$

belongs to $L^s\big[(0, T), C^0(\Omega)\big] \subset L^s(\Omega_T)$ for all $1 < s < \infty$. Hereby the claimed regularity of W with respect to the spatial variables is confirmed as well.

• **Step A4.** *The function $I_{ion}(\widetilde{\Phi}_{tr}, W) = I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W})$ belongs even to the space $L^r\big[(0, T), L^2(\Omega)\big]$.* We estimate

$$\begin{aligned}
\| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|^6_{L^6(\Omega)} &= \int_\Omega \Big(I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \Big)^6 dx \\
&= \int_\Omega \Big(b\, \widetilde{\Phi}_{tr}(\widetilde{\Phi}_{tr} - a)\,(\widetilde{\Phi}_{tr} - 1) + e^{-\varepsilon t}\, W_0\, \widetilde{\Phi}_{tr} + \varepsilon\, \kappa\, e^{-\varepsilon t}\, \widetilde{\Phi}_{tr} \int_0^t \widetilde{\Phi}_{tr}(x, \vartheta)\, e^{\varepsilon\,\vartheta}\, d\vartheta \Big)^6 dx && (3.65) \\
&\leqslant C \int_\Omega \Big(\widetilde{\Phi}^6_{tr}\, (\widetilde{\Phi}^6_{tr} + a^6)\, (\widetilde{\Phi}^6_{tr} + 1) + |\, W_0\,|^6\, \widetilde{\Phi}^6_{tr} + \widetilde{\Phi}^6_{tr} \Big(\int_0^t |\, \widetilde{\Phi}_{tr}(x, \vartheta)\,| \cdot |\, e^{\varepsilon\vartheta}\,|\, d\vartheta \Big)^6 \Big) dx && (3.66) \\
&\leqslant C \Big(\int_\Omega |\, \widetilde{\Phi}_{tr}\,|^{18} dx + \int_\Omega |\, \widetilde{\Phi}_{tr}\,|^{12} dx + \int_\Omega |\, \widetilde{\Phi}_{tr}\,|^6 dx + \int_\Omega |\, W_0\,|^6 \cdot |\, \widetilde{\Phi}_{tr}\,|^6 dx && (3.67) \\
&\qquad + \int_\Omega |\, \widetilde{\Phi}_{tr}\,|^6 \cdot \Big(\int_0^t |\, \widetilde{\Phi}_{tr}(x, \vartheta)\,| \cdot |\, e^{\varepsilon\vartheta}\,|\, d\vartheta \Big)^6 dx \Big) \quad \Longrightarrow
\end{aligned}$$

$$\begin{aligned}
\| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|^6_{L^6[(0,T), L^6(\Omega)]} &= \int_0^T \| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|^6_{L^6(\Omega)}\, dt && (3.68) \\
&\leqslant C \Big(1 + \| \widetilde{\Phi}_{tr} \|^{18}_{L^{18}(\Omega_T)} + \| \widetilde{\Phi}_{tr} \|^{12}_{L^{12}(\Omega_T)} + \| \widetilde{\Phi}_{tr} \|^6_{L^6(\Omega_T)} && (3.69) \\
&\qquad + \| \mathrm{W}_0 \|^6_{L^\infty(\Omega)} \cdot \| \widetilde{\Phi}_{tr} \|^6_{L^6(\Omega_T)} \Big) + C \cdot \int_0^T \int_\Omega |\, \widetilde{\Phi}_{tr}(x, t)|^6 \cdot \Big(\int_0^T |\, \widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta \Big)^6 dx\, dt \\
&= C\Big(\dots \Big) + C \int_\Omega \int_0^T |\, \widetilde{\Phi}_{tr}(x, t)\,|^6\, dt \cdot \Big(\int_0^T |\, \widetilde{\Phi}_{tr}(x, \vartheta)\,|\, d\vartheta \Big)^6 dx && (3.70) \\
&= C\Big(\dots \Big) + C \int_\Omega \| \widetilde{\Phi}_{tr}(x) \|^6_{L^6(0,T)} \cdot \| \widetilde{\Phi}_{tr}(x) \|^6_{L^1(0,T)}\, dx && (3.71) \\
&\leqslant C\Big(\dots \Big) + C \int_\Omega \| \widetilde{\Phi}_{tr}(x) \|^6_{L^6(0,T)} \cdot \| \widetilde{\Phi}_{tr}(x) \|^6_{L^6(0,T)}\, dx \leqslant C\Big(\dots \Big) + C\, \| \widetilde{\Phi}_{tr} \|^{12}_{L^{12}[\Omega, L^6(0,T)]} && (3.72)
\end{aligned}$$

Since $4 < r \leqslant 6$ and

$$\begin{aligned}
\| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^r[(0,T), L^2(\Omega)]} &\leqslant C\, \| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^6[(0,T), L^2(\Omega)]} && (3.73) \\
&\leqslant C\, \| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^6[(0,T), L^6(\Omega)]},
\end{aligned}$$

the claim has been proved.

• **Step A5.** *The functions $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ and $\widetilde{W} = W$ belong to the spaces*

$$\Phi_{tr} \in W^{1,r}\big[(0, T), L^2(\Omega)\big] \cap L^r\big[(0, T), W^{2,2}(\Omega)\big] \cap C^0(\Omega_T)\,; \tag{3.74}$$

$$\Phi_e \in L^2\big[[0, T), L^2(\Omega)\big] \cap L^r\big[(0, T), W^{2,2}(\Omega)\big]\,; \tag{3.75}$$

$$W \in C^1\big[(0, T), L^\infty(\Omega)\big] \cap C^0\big[[0, T], L^\infty(\Omega)\big]\,. \tag{3.76}$$

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27) [Kunisch/Wagner 11], p. 30, Theorem 4.6.

In view of Step A4, Proposition 3.6., Theorem 3.2. and Proposition 3.7. may be applied again to the triple $(\widetilde{\Phi}_{tr}, \widetilde{\Phi}_e, \widetilde{W})$, and the new solution (Φ_{tr}, Φ_e, W) of $(\widetilde{\mathrm{B}})_{0,gat}$ and $(\widetilde{\mathrm{B}})_{0,par}$ coincides again with the former one. Using interpolation space arguments, the claim $\Phi_{tr} \in C^0(\Omega_T)$ follows from

$$\Phi_{tr} \in \Big\{ f \in L^r\big[\,(0\,,T\,)\,, W^{2,2}(\Omega)\big] \;\Big|\; \frac{df}{dt} \in L^r\big[\,(0\,,T\,)\,, L^2(\Omega)\big] \Big\}\,, \quad r > 4 \tag{3.77}$$

by repeating the arguments from [Veneroni 09] , p. 864 f.

Part B. The FitzHugh-Nagumo model.

• **Step B1.** *The function* $\widetilde{\Phi}_{tr}$ *belongs to the space* $L^4\big[\,(0\,,T\,)\,,L^6(\Omega)\big]$. This step is identical with Step A1.

• **Step B2.** *The function* $I_{ion}(\widetilde{\Phi}_{tr}, W) = I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W})$ *belongs to the space* $L^{4/3}\big[\,(0\,,T\,)\,,L^2(\Omega)\big]$. Let us consider

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^2_{L^2(\Omega)} \;=\; \int_\Omega \Big(I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\Big)^2\,dx$$

$$= \int_\Omega \Big(\widetilde{\Phi}_{tr}(\widetilde{\Phi}_{tr} - a)\,(\widetilde{\Phi}_{tr} - 1) + e^{-\varepsilon t}\,W_0 + \varepsilon\,\kappa\,e^{-\varepsilon t} \int_0^t \widetilde{\Phi}_{tr}(x,\vartheta)\,e^{\varepsilon\,\vartheta}\,d\vartheta \Big)^2\,dx \tag{3.78}$$

$$\leqslant C \int_\Omega \Big(\widetilde{\Phi}^2_{tr}\,(\widetilde{\Phi}^2_{tr} + a^2)\,(\widetilde{\Phi}^2_{tr} + 1) + |\,W_0\,|^2 + e^2\,\kappa^2 \Big(\int_0^t |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,| \cdot |\,e^{\varepsilon\vartheta}\,|\,d\vartheta \Big)^2 \Big)\,dx \tag{3.79}$$

$$\leqslant C \Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx \tag{3.80}$$

$$+ \int_\Omega |\,W_0\,|^2\,dx + e^{2\varepsilon T}\,T^2 \int_\Omega \Big(\frac{1}{T} \int_0^T |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|\,d\vartheta \Big)^2\,dx \Big)$$

$$\leqslant C \Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx + \int_\Omega |\,W_0\,|^2\,dx + \int_\Omega \frac{1}{T} \int_0^T |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\,d\vartheta\,dx \Big) \tag{3.81}$$

by application of Jensen's integral inequality [28] to the last member. It follows that

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^{4/3}_{L^2(\Omega)} \;\leqslant\; C \Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx \tag{3.82}$$

$$+ \int_\Omega |\,W_0\,|^2\,dx + \int_0^T \int_\Omega |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\,dx\,d\vartheta \Big)^{2/3}$$

$$\leqslant C \Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx \Big)^{2/3} + C \Big(1 + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx \tag{3.83}$$

$$+ \int_\Omega |\,W_0\,|^2\,dx + \int_0^T \int_\Omega |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\,dx\,d\vartheta \Big)^{2/3}$$

$$\leqslant C \Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx \Big)^{2/3} + C \Big(1 + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx \tag{3.84}$$

$$+ \int_\Omega |\,W_0\,|^2\,dx + \int_0^T \int_\Omega |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\,dx\,d\vartheta \Big) \quad \Longrightarrow$$

$$\|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^{4/3}_{L^{4/3}[\,(0\,,T\,)\,,L^2(\Omega)]} \;=\; \int_0^T \|\, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t))\,\|^{4/3}_{L^2(\Omega)}\,dt \tag{3.85}$$

$$\leqslant C \Big(1 + \|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4[\,(0\,,T\,)\,,L^6(\Omega)]} + \|\,\widetilde{\Phi}_{tr}\,\|^4_{L^4[\,(0\,,T\,)\,,L^4(\Omega)]} + \|\,\widetilde{\Phi}_{tr}\,\|^2_{L^2[\,(0\,,T\,)\,,L^2(\Omega)]}$$

$$+ \|\,\mathrm{W}_0\,\|^2_{L^\infty(\Omega)} + \|\,\widetilde{\Phi}_{tr}\,\|^2_{L^2[\,(0\,,T\,)\,,L^2(\Omega)]} \Big)\,.$$

[28] [Ioffe/Tichomirow 79] , p. 310.

Due to the continuous imbeddings $L^4\big[(0,T),L^6(\Omega)\big] \hookrightarrow L^4(\Omega_T) = L^4\big[\Omega, L^4(0,T)\big] \hookrightarrow L^4\big[\Omega, L^2(0,T)\big]$, all norms of the right-hand side are finite, and the claim has been proved.

• **Step B3.** *The functions* $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ *and* $\widetilde{W} = W$ *belong to the spaces*

$$\Phi_{tr} \in W^{1,4/3}\big[(0,T),L^2(\Omega)\big] \cap L^{4/3}\big[(0,T),W^{2,2}(\Omega)\big] \cap L^s(\Omega_T),\ \ 4 \leqslant s < \infty\,; \tag{3.86}$$

$$\Phi_e \in L^2\big[[0,T),L^2(\Omega)\big] \cap L^{4/3}\big[(0,T),W^{2,2}(\Omega)\big]\,; \tag{3.87}$$

$$W \in C^1\big[(0,T),L^\infty(\Omega)\big] \cap C^0\big[[0,T],L^\infty(\Omega)\big]\,. \tag{3.88}$$

This step is identical with Step A3.

• **Step B4.** *The function* $I_{ion}(\widetilde{\Phi}_{tr},W) = I_{ion}(\widetilde{\Phi}_{tr},\widetilde{W})$ *belongs even to the space* $L^r\big[(0,T),L^2(\Omega)\big]$. In analogy to Steps A4 and B2, we get

$$\|\,I_{ion}(\widetilde{\Phi}_{tr}(t),\widetilde{W}(t))\,\|^6_{L^6(\Omega)} = \int_\Omega \Big(I_{ion}(\widetilde{\Phi}_{tr}(t),\widetilde{W}(t))\Big)^6\,dx \tag{3.89}$$

$$\leqslant C\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^{18}\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^{12}\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^{6}\,dx + \int_\Omega |\,W_0\,|^6\,dx + T^6\int_\Omega \frac{1}{T}\int_0^T |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^6\,d\vartheta\,dx\Big) \quad\Longrightarrow$$

$$\|\,I_{ion}(\widetilde{\Phi}_{tr},\widetilde{W})\,\|^6_{L^6[(0,T),L^6(\Omega)]} = \int_0^T \|\,I_{ion}(\widetilde{\Phi}_{tr}(t),\widetilde{W}(t))\,\|^6_{L^6(\Omega)}\,dt$$

$$\leqslant C\Big(1 + \|\,\widetilde{\Phi}_{tr}\,\|^{18}_{L^{18}(\Omega_T)} + \|\,\widetilde{\Phi}_{tr}\,\|^{12}_{L^{12}(\Omega_T)} + \|\,\widetilde{\Phi}_{tr}\,\|^{6}_{L^{6}(\Omega_T)} + \|\,\mathrm{W}_0\,\|^6_{L^\infty(\Omega)} + \|\,\widetilde{\Phi}_{tr}\,\|^{6}_{L^{6}(\Omega_T)}\Big)\,. \tag{3.90}$$

Since $4 < r \leqslant 6$, (3.73) may be applied, and the claim has been proved.

• **Step B5.** *The functions* $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ *and* $\widetilde{W} = W$ *belong to the spaces*

$$\Phi_{tr} \in W^{1,r}\big[(0,T),L^2(\Omega)\big] \cap L^{r}\big[(0,T),W^{2,2}(\Omega)\big] \cap C^0(\Omega_T)\,; \tag{3.91}$$

$$\Phi_e \in L^2\big[[0,T),L^2(\Omega)\big] \cap L^{r}\big[(0,T),W^{2,2}(\Omega)\big]\,; \tag{3.92}$$

$$W \in C^1\big[(0,T),L^\infty(\Omega)\big] \cap C^0\big[[0,T],L^\infty(\Omega)\big]\,. \tag{3.93}$$

This step is identical with Step A5.

Part C. The linearized Aliev-Panfilov model.

• **Step C1.** *The function* $\widetilde{\Phi}_{tr}$ *belongs to the space* $L^4\big[(0,T),L^6(\Omega)\big]$. The proof of this claim is the same as in Step A1.

• **Step C2.** *The function* $I_{ion}(\widetilde{\Phi}_{tr},W) = I_{ion}(\widetilde{\Phi}_{tr},\widetilde{W})$ *belongs to the space* $L^{4/3}\big[(0,T),L^2(\Omega)\big]$. We will consider

$$\|\,I_{ion}(\widetilde{\Phi}_{tr}(t),\widetilde{W}(t))\,\|^2_{L^2(\Omega)} = \int_\Omega \Big(I_{ion}(\widetilde{\Phi}_{tr}(t),\widetilde{W}(t))\Big)^2\,dx \tag{3.94}$$

$$= \int_\Omega \Big(b\,\widetilde{\Phi}_{tr}(\widetilde{\Phi}_{tr}-a)\,(\widetilde{\Phi}_{tr}-1) + e^{-\varepsilon t}\,W_0\,\widetilde{\Phi}_{tr} + \varepsilon\,\kappa\,e^{-\varepsilon t}\,\widetilde{\Phi}_{tr}\int_0^t \Big((a+1)\,\widetilde{\Phi}_{tr}(x,\vartheta) - \widetilde{\Phi}^2_{tr}(x,\vartheta)\Big)\,e^{\varepsilon\,\vartheta}\,d\vartheta\Big)^2\,dx$$

$$\leqslant C\int_\Omega \Big(\widetilde{\Phi}^2_{tr}\,(\widetilde{\Phi}^2_{tr}+a^2)\,(\widetilde{\Phi}^2_{tr}+1) + |\,W_0\,|^2\,\widetilde{\Phi}^2_{tr} \tag{3.95}$$

$$+\,\varepsilon^2\,\kappa^2\,\widetilde{\Phi}^2_{tr}\Big(\int_0^t \Big(|\,(a+1)\,\widetilde{\Phi}_{tr}(x,\vartheta)\,| + |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\Big)\cdot|\,e^{\varepsilon\vartheta}\,|\,d\vartheta\Big)^2\Big)\,dx$$

$$\leqslant C\Big(\int_\Omega |\,\widetilde{\Phi}_{tr}\,|^6\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^4\,dx + \int_\Omega |\,\widetilde{\Phi}_{tr}\,|^2\,dx + \int_\Omega |\,W_0\cdot\widetilde{\Phi}_{tr}\,|^2\,dx \tag{3.96}$$

$$+\,e^{2\varepsilon T}\int_\Omega \widetilde{\Phi}^2_{tr}\cdot\Big(\int_0^T \Big(|\,(a+1)\,\widetilde{\Phi}_{tr}(x,\vartheta)\,| + |\,\widetilde{\Phi}_{tr}(x,\vartheta)\,|^2\Big)\,d\vartheta\Big)^2\,dx\Big) \quad\Longrightarrow$$

$$\| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^2(\Omega)}^{4/3} \leqslant C \Big(\int_\Omega | \widetilde{\Phi}_{tr} |^6 \, dx + \int_\Omega | \widetilde{\Phi}_{tr} |^4 \, dx + \int_\Omega | \widetilde{\Phi}_{tr} |^2 \, dx \tag{3.97}$$

$$+ \int_\Omega | W_0 \cdot \widetilde{\Phi}_{tr} |^2 \, dx + \int_\Omega \widetilde{\Phi}_{tr}^2 \cdot \Big(\int_0^T \Big(| (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) | + | \widetilde{\Phi}_{tr}(x,\vartheta) |^2 \Big) \, d\vartheta \Big)^2 dx \Big)^{2/3}$$

$$\leqslant C \Big(\int_\Omega | \widetilde{\Phi}_{tr} |^6 \, dx \Big)^{2/3} + C \Big(1 + \int_\Omega | \widetilde{\Phi}_{tr} |^4 \, dx + \int_\Omega | \widetilde{\Phi}_{tr} |^2 \, dx \tag{3.98}$$

$$+ \int_\Omega | W_0 \cdot \widetilde{\Phi}_{tr} |^2 \, dx + \int_\Omega \widetilde{\Phi}_{tr}^2 \cdot \Big(\int_0^T \Big(| (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) | + | \widetilde{\Phi}_{tr}(x,\vartheta) |^2 \Big) \, d\vartheta \Big)^2 dx \Big)^{2/3}$$

$$\leqslant C \Big(\int_\Omega | \widetilde{\Phi}_{tr} |^6 \, dx \Big)^{2/3} + C \Big(1 + \int_\Omega | \widetilde{\Phi}_{tr} |^4 \, dx + \int_\Omega | \widetilde{\Phi}_{tr} |^2 \, dx \tag{3.99}$$

$$+ \int_\Omega | W_0 \cdot \widetilde{\Phi}_{tr} |^2 \, dx + \int_\Omega \widetilde{\Phi}_{tr}^2 \cdot \Big(\int_0^T \Big(| (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) | + | \widetilde{\Phi}_{tr}(x,\vartheta) |^2 \Big) \, d\vartheta \Big)^2 dx \Big) \quad \Longrightarrow$$

$$\| I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W}) \|_{L^{4/3}[\,(0\,,T\,)\,,\,L^2(\Omega)\,]}^{4/3} = \int_0^T \| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^2(\Omega)}^{4/3} \, dt$$

$$\leqslant C \Big(1 + \| \widetilde{\Phi}_{tr} \|_{L^4[\,(0\,,T\,)\,,\,L^6(\Omega)\,]}^4 + \| \widetilde{\Phi}_{tr} \|_{L^4[\,(0\,,T\,)\,,\,L^4(\Omega)\,]}^4 + \| \widetilde{\Phi}_{tr} \|_{L^2[\,(0\,,T\,)\,,\,L^2(\Omega)\,]}^2 \tag{3.100}$$

$$+ \| \mathrm{W}_0 \|_{L^\infty(\Omega)}^2 \cdot \| \widetilde{\Phi}_{tr} \|_{L^2[\,(0\,,T\,)\,,\,L^2(\Omega)\,]}^2 \Big)$$

$$+ \, C \cdot \int_0^T \int_\Omega \widetilde{\Phi}_{tr}^2(x,t) \cdot \Big(\int_0^T \Big(| (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) | + | \widetilde{\Phi}_{tr}(x,\vartheta) |^2 \Big) \, d\vartheta \Big)^2 dx \, dt$$

$$= C \Big(\, ... \, \Big) + C \int_\Omega \int_0^T \widetilde{\Phi}_{tr}^2(x,t) \, dt \cdot \Big(\int_0^T | (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) | \, d\vartheta + \int_0^T \widetilde{\Phi}_{tr}^2(x,\vartheta) \, d\vartheta \Big)^2 dx \tag{3.101}$$

$$\leqslant C \Big(\, ... \, \Big) + C \int_\Omega \| \widetilde{\Phi}_{tr}(x) \|_{L^2(\,0\,,T\,)}^2 \cdot \Big(\| \widetilde{\Phi}_{tr}(x) \|_{L^1(\,0\,,T\,)}^2 + \| \widetilde{\Phi}_{tr}(x) \|_{L^2(\,0\,,T\,)}^4 \Big) \, dx \tag{3.102}$$

$$\leqslant C \Big(\, ... \, \Big) + C \int_\Omega \| \widetilde{\Phi}_{tr}(x) \|_{L^2(\,0\,,T\,)}^2 \cdot \Big(\| \widetilde{\Phi}_{tr}(x) \|_{L^2(\,0\,,T\,)}^2 + \| \widetilde{\Phi}_{tr}(x) \|_{L^2(\,0\,,T\,)}^4 \Big) \, dx \tag{3.103}$$

$$\leqslant C \Big(\, ... \, \Big) + C \Big(\| \widetilde{\Phi}_{tr} \|_{L^4[\,\Omega\,,\,L^2(\,0\,,T\,)\,]}^4 + \| \widetilde{\Phi}_{tr} \|_{L^6[\,\Omega\,,\,L^2(\,0\,,T\,)\,]}^6 \Big) \, . \tag{3.104}$$

From Step A2, we know that the first and second member on the right-hand side are finite. Due to [Kunisch/Wagner 11], p. 30, Proposition 4.3., the imbeddings $L^4[\,(0\,,T\,)\,,\,L^6(\Omega)\,] \hookrightarrow L^6[\,\Omega\,,\,L^4(\,0\,,T\,)\,] \hookrightarrow L^6[\,\Omega\,,\,L^2(\,0\,,T\,)\,]$ are continuous, and by Step C1, the third member is finite as well.

• **Step C3.** *The functions* $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ *and* $\widetilde{W} = W$ *belong to the spaces*

$$\Phi_{tr} \in W^{1,4/3}[\,(0\,,T\,)\,,\,L^2(\Omega)\,] \,\cap\, L^{4/3}[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\,] \,\cap\, L^s(\Omega_T)\,, \ \ 4 \leqslant s < \infty\,; \tag{3.105}$$

$$\Phi_e \in L^2[\,[\,0\,,T\,)\,,\,L^2(\Omega)\,] \,\cap\, L^{4/3}[\,(0\,,T\,)\,,\,W^{2,2}(\Omega)\,]\,; \tag{3.106}$$

$$W \in C^1[\,(0\,,T\,)\,,\,L^\infty(\Omega)\,] \,\cap\, C^0[\,[\,0\,,T\,]\,,\,L^\infty(\Omega)\,]\,. \tag{3.107}$$

This step is identical with Step A3.

• **Step C4.** *The function* $I_{ion}(\widetilde{\Phi}_{tr}, W) = I_{ion}(\widetilde{\Phi}_{tr}, \widetilde{W})$ *belongs even to the space* $L^r[\,(0\,,T\,)\,,\,L^2(\Omega)\,]$. We estimate

$$\| I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \|_{L^6(\Omega)}^6 = \int_\Omega \Big(I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \Big)^6 dx \tag{3.108}$$

$$= \int_\Omega \Big(b \, \widetilde{\Phi}_{tr} (\widetilde{\Phi}_{tr} - a) \, (\widetilde{\Phi}_{tr} - 1) + e^{-\varepsilon t} \, W_0 \, \widetilde{\Phi}_{tr} + \varepsilon \, \kappa \, e^{-\varepsilon t} \, \widetilde{\Phi}_{tr} \int_0^t \Big((a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) - \widetilde{\Phi}_{tr}^2(x,\vartheta) \Big) \, e^{\varepsilon \, \vartheta} \, d\vartheta \Big)^6 dx$$

$$\leqslant C \int_\Omega \Big(\widetilde{\Phi}_{tr}^6 \, (\widetilde{\Phi}_{tr}^6 + a^6) \, (\widetilde{\Phi}_{tr}^6 + 1) + |\, W_0 \,|^6 \, \widetilde{\Phi}_{tr}^6 \tag{3.109}$$

$$+ \widetilde{\Phi}_{tr}^6 \Big(\int_0^t \Big(\big|\, (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big| + \big|\, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big|^2 \Big) \cdot \big|\, e^{\varepsilon\vartheta} \,\big| \, d\vartheta \Big)^6 \Big) \, dx$$

$$\leqslant C \Big(\int_\Omega \big|\, \widetilde{\Phi}_{tr} \,\big|^{18} \, dx + \int_\Omega \big|\, \widetilde{\Phi}_{tr} \,\big|^{12} \, dx + \int_\Omega \big|\, \widetilde{\Phi}_{tr} \,\big|^6 \, dx + \int_\Omega \big|\, W_0 \,\big|^6 \cdot \big|\, \widetilde{\Phi}_{tr} \,\big|^6 \, dx \tag{3.110}$$

$$+ \int_\Omega \big|\, \widetilde{\Phi}_{tr} \,\big|^6 \cdot \Big(\int_0^T \Big(\big|\, (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big| + \big|\, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big|^2 \Big) \cdot \big|\, e^{\varepsilon T} \,\big| \, d\vartheta \Big)^6 dx \Big) \quad \Longrightarrow$$

$$\| \, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \, \|^6_{L^6[\,(0\,,T\,)\,,\,L^6(\Omega)\,]} \; = \; \int_0^T \| \, I_{ion}(\widetilde{\Phi}_{tr}(t), \widetilde{W}(t)) \, \|^6_{L^6(\Omega)} \, dt$$

$$\leqslant C \Big(1 + \| \, \widetilde{\Phi}_{tr} \, \|^{18}_{L^{18}(\Omega_T)} + \| \, \widetilde{\Phi}_{tr} \, \|^{12}_{L^{12}(\Omega_T)} + \| \, \widetilde{\Phi}_{tr} \, \|^{6}_{L^{6}(\Omega_T)} + \| \, \mathrm{W}_0 \, \|^6_{L^\infty(\Omega)} \cdot \| \, \widetilde{\Phi}_{tr} \, \|^{6}_{L^{6}(\Omega_T)} \Big) \tag{3.111}$$

$$+ C \cdot \int_0^T \int_\Omega \big|\, \widetilde{\Phi}_{tr}(x,t) \big|^6 \cdot \Big(\int_0^T \big|\, (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big| \, d\vartheta + \int_0^T \widetilde{\Phi}_{tr}^2(x,\vartheta) \, d\vartheta \Big)^6 dx \, dt$$

$$= C \Big(\; ... \; \Big) + C \int_\Omega \int_0^T \big|\, \widetilde{\Phi}_{tr}(x,t) \,\big|^6 \, dt \cdot \Big(\int_0^T \big|\, (a+1) \, \widetilde{\Phi}_{tr}(x,\vartheta) \,\big| \, d\vartheta + \int_0^T \widetilde{\Phi}_{tr}^2(x,\vartheta) \, d\vartheta \Big)^6 dx \tag{3.112}$$

$$\leqslant C \Big(\; ... \; \Big) + C \int_\Omega \| \, \widetilde{\Phi}_{tr}(x) \, \|^6_{L^6(\,0\,,\,T\,)} \cdot \Big(\| \, \widetilde{\Phi}_{tr}(x) \, \|^6_{L^1(\,0\,,\,T\,)} + \| \, \widetilde{\Phi}_{tr}(x) \, \|^{12}_{L^2(\,0\,,\,T\,)} \Big) \, dx \tag{3.113}$$

$$\leqslant C \Big(\; ... \; \Big) + C \int_\Omega \| \, \widetilde{\Phi}_{tr}(x) \, \|^6_{L^6(\,0\,,\,T\,)} \cdot \Big(\| \, \widetilde{\Phi}_{tr}(x) \, \|^6_{L^6(\,0\,,\,T\,)} + \| \, \widetilde{\Phi}_{tr}(x) \, \|^{12}_{L^6(\,0\,,\,T\,)} \Big) \, dx \tag{3.114}$$

$$\leqslant C \Big(\; ... \; \Big) + C \Big(\| \, \widetilde{\Phi}_{tr} \, \|^{12}_{L^{12}[\,\Omega\,,\,L^6(\,0\,,\,T\,)\,]} + \| \, \widetilde{\Phi}_{tr} \, \|^{18}_{L^{18}[\,\Omega\,,\,L^6(\,0\,,\,T\,)\,]} \Big) \, . \tag{3.115}$$

Since $4 < r \leqslant 6$ and (3.73), the claim has been proved.

• **Step C5.** *The functions* $\widetilde{\Phi}_{tr} = \Phi_{tr}$, $\widetilde{\Phi}_e = \Phi_e$ *and* $\widetilde{W} = W$ *belong to the spaces*

$$\Phi_{tr} \in W^{1,r}\big[\,(\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, L^r\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big] \,\cap\, C^0(\Omega_T)\,; \tag{3.116}$$

$$\Phi_e \in L^2\big[\,[\,0\,,\,T\,)\,,\,L^2(\Omega)\,\big] \,\cap\, L^r\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]\,; \tag{3.117}$$

$$W \in C^1\big[\,(\,0\,,\,T\,)\,,\,L^\infty(\Omega)\,\big] \,\cap\, C^0\big[\,[\,0\,,\,T\,]\,,\,L^\infty(\Omega)\,\big]\,. \tag{3.118}$$

Again this step is identical with Step A5.

Let us finally note that the regularity (3.19) – (3.21) of a weak solution (Φ_{tr}, Φ_e, W) of $(\mathrm{B})_2$ allows for partial integration within all terms where $\nabla\psi$ is occuring and is, at the same time, consistent with Definition 2.1., 1). Consequently, (Φ_{tr}, Φ_e, W) forms a strong solution of $(\mathrm{B})_1$ and $(\mathrm{B})_0$ as well, and the proof of Theorem 3.3. is complete. ∎

Proof of Corollary 3.4. Under the assumptions of the Corollary, $(\mathrm{B})_2$ possesses a uniquely determined weak solution (Φ_{tr}, Φ_e, W) on $[\,0\,,\,T\,]$ by Theorem 2.8. By Theorem 3.3., this solution admits the regularity described in (3.19) – (3.21) and is, in fact, a strong solution of $(\mathrm{B})_0$ and $(\mathrm{B})_1$. ∎

Proof of Corollary 3.5. 1) By Theorem 3.3., any weak solution of $(\mathrm{B})_2$ is in fact a strong solution of $(\mathrm{B})_0$ and $(\mathrm{B})_1$ with Φ_{tr}, $\Phi_e \in L^r\big[\,(\,0\,,\,T\,)\,,\,W^{2,2}(\Omega)\,\big]$ at least. By Assumption 3.1., 1), the boundary of Ω is $C^{1,1}$, and by Assumption 3.1., 2), the entries of M_i and M_e are Lipschitz continuous. Since the eigenvalues $\lambda_{k,i}(\,\cdot\,)$ and $\lambda_{k,e}(\,\cdot\,)$, $1 \leqslant k \leqslant 3$, depend Lipschitz-continuously on the matrix elements (cf. [Schaback/Wendland 05] , p. 270, Corollary 15.4.), the functions are continuous on $\mathrm{cl}\,(\Omega)$. Now the assertion follows from [Bourgault/Coudiére/Pierre 09] , p. 462, Lemma 1.

2) Under the assumptions above the compatibility condition for Φ_0 reduces to $\mathfrak{n}^{\mathrm{T}}\, M_e\, \nabla\Phi_0 = 0$ on $\partial\Omega$ after [Veneroni 09] , p. 854, Remark 3. By Part 1, this condition may be further reduced to the claimed form. ■

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