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WELL-POSEDNESS OF THE WESTERVELT AND THE KUZNETSOV EQUATION WITH NONHOMOGENEOUS NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper we show wellposedness of two equations of nonlinear acoustics, as relevant e.g. in applications of high intensity ultrasound. After having studied the Dirichlet problem in previous papers, we here consider Neumann boundary conditions which are of particular practical interest in applications. The Westervelt and the Kuznetsov equation are quasilinear evolutionary wave equations with potential degeneration and strong damping. We prove local in time well-posedness as well as global existence and exponential decay for a slightly modified model, the proofs being based on energy estimates and Banach's fixed point theorem for local well-posedness, with small initial data, as well as barrier's method for global existence. A key step of the proof is an appropriate extension of the Neumann boundary data to the interior along with exploitation of singular estimates associated with the analytic semigroup generated by the strongly damped wave equation.

1. **Introduction.** Motivated by applications of high-intensity focused ultrasound (HIFU) such as lithotripsy, thermotherapy, ultrasound cleaning or welding and sonochemistry, see [1], [13], and the references therein, we study the two most frequently used models of nonlinear acoustics, namely the Westervelt and the Kuznetsov equation, see also [7] for some analytical results on Kuznetsov's equation.

The Kuznetsov equation reads as

$$\frac{1}{c^2}p_{\sim tt} - \Delta p_\sim - \frac{b}{c^2}\Delta p_{\sim t} = \frac{1}{\varrho_0 c^4}\frac{B}{2A}p^2_{\sim tt} + \frac{\varrho_0}{c^2}(\vec{v}\cdot\vec{v})_{tt} \tag{1}$$

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where $p_\sim$ denotes the acoustic pressure fluctuations, $\vec{v}$ the acoustic particle velocity, c is the speed of sound, b the diffusivity of the sound, ϱ_0 the mass density, B/A the parameter of nonlinearity, and the identity $\varrho_0 \vec{v}_t = -\nabla p_\sim$ holds. All parameters b, c, ϱ_0, B/A are positive real numbers. A slightly simplified model resulting from neglecting local nonlinear effects modelled by the quadratic velocity term, is the Westervelt equation

$$\frac{1}{c^2} p_{\sim tt} - \Delta p_\sim - \frac{b}{c^2} \Delta(p_{\sim t}) = \frac{\beta_a}{\varrho_0 c^4} p^2_{\sim tt} \tag{2}$$

with $\beta_a = 1 + B/(2A)$. For a detailed derivation of the PDE we refer to [6], [13], [14], [20]. These two equations can be equivalently rewritten as:

$$(1 - 2ku)u_{tt} - c^2 \Delta u - b\Delta(u_t) = 2k(u_t)^2 + \sigma \left(|\nabla u|^2 + \nabla \mathcal{I} u \nabla u_t \right) , \tag{3}$$

respectively, where $u = p$, $k = \beta_a/(\varrho c^2)$ and

$$\mathcal{I} : v \mapsto \int_0^{\cdot} v(\tau)\, d\tau + U_0 \tag{4}$$

denotes the antiderivative operator. Here the space dependent function $U_0 \in H^1_0(\Omega)$ is to be chosen appropriately, see Theorem 1.2 below. The parameter σ allows to formally switch between both equations: $\sigma = \frac{2}{\varrho_0}$ in case of the Kuznetsov equation, and $\sigma = 0$ in case of the (simpler) Westervelt equation. From formulation (3) it gets obvious that these are quasilinear strongly damped wave equations with potential degeneracy.

After having established well-posedness results for these equations with Dirichlet boundary conditions in [8], [9], [10], [11], our aim is to cover also the Neumann boundary case, which is of particular practical relevance in view of an appropriate modelling of boundary excitation, e.g., by electromagnetomechanical or piezoelectric devices, see, e.g. [13]. On the other hand, the mathematical treatment of Neumann boundary conditions presents us with new challenges resulting from a nontrivial nullspace associated with the linear generator as well as a different semigroup framework owning to the variational character of the Neumann problem.

Our particular emphasis is on problems that have *nonhomogenuous* boundary conditions. It is known that the analysis of non-homogenuous on the boundary problems can be quite subtle due to intrinsic incompatibility between fractional powers of generators and the spaces where solutions reside. This is particularly pronounced in the case of strongly damped dynamics (as considered in this paper) where the said incompatibility also involves time derivatives. Recent years have witnessed rapid development of PDE analysis enabling effective treatment of inhomogenity on the boundary (also with limited regularity) so that the generated results can be also applied in the context of nonlinear problems [2, 3, 16, 12]. The methods used rely on a combination of semigroup theory and analytic properties of the semigroup generated by homogenous on the boundary dynamics. These allow to build appropriate boundary kernels whose singularity can be controlled see [16, 3] and references therein. The approach taken in this paper, and also in [10, 11], is rooted in these methods.

1.1. **Main Results.**

The main goal of this paper is to provide results on existence of solutions for the Kuznetsov and the Westervelt equation on a bounded domain

$\Omega \subseteq R^n, n = 1, 2, 3$, with Lipschitz boundary and Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma = \partial\Omega \tag{5}$$

and initial data

$$(u(0), u_t(0)) = (u_0, u_1)\,. \tag{6}$$

For the model (3) with boundary conditions (5), due to the zero eigenvalue of the Laplace operator with Neumann boundary conditions, only local in time well-posedness can be shown. Global existence and exponential decay will be shown for the slightly modified model

$$(1-2ku)u_{tt} - c^2\Delta u + du - b\Delta(u_t) + eu_t = 2k(u_t)^2 + \sigma\left(|\nabla u|^2 + \nabla\mathcal{I}u\nabla u_t\right)\,, \tag{7}$$

$$\frac{\partial u}{\partial \nu} + hu = g \quad \text{on } \Gamma = \partial\Omega \tag{8}$$

with initial data $(u(0), u_t(0)) = (u_0, u_1)$. Throughout the remainder of this paper, all parameters are real and part of them satisfy sign conditions

$$b, c > 0\,,\ \ d, e, h \geq 0 \tag{9}$$

(the original model corresponding to $d = e = h = 0$) whereas the signs of k, σ may be arbitrary. For global existence and exponential decay we will additionally assume that

$$\min\{d, e\} + h > 0\,. \tag{10}$$

The parameter $b > 0$ means that we assume strong damping. The following compatibility conditions will be imposed throughout this paper:

$$g(t=0) \;=\; \left(\frac{\partial u_0}{\partial \nu} + hu_0\right)|_\Gamma\,, \quad g_t(t=0) \;=\; \left(\frac{\partial u_1}{\partial \nu} + hu_1\right)|_\Gamma\,. \tag{11}$$

In order to formulate our results we introduce the following energy functions:

$$\begin{aligned} E_{u,0}(t) &= \frac{1}{2}\left\{|u_t(t)|^2 + |\nabla u(t)|^2\right\}\,, \\ E_{u,1}(t) &= \frac{1}{2}\left\{|u_{tt}(t)|^2 + |\nabla u_t(t)|^2 + |\Delta u(t)|^2\right\} \end{aligned}$$

where $|u| \equiv |u|_{L_2(\Omega)}$. For $t = 0$, $E_{u,1}(0) \equiv \frac{1}{2}\{|(1-2ku_0)^{-1}[c^2\Delta u_0 - du_0 + b\Delta u_1 - eu_1 + 2ku_1^2 + \sigma|\nabla u_0|^2 + \nabla U_0\nabla u_1]|^2 + |\nabla u_1(t)|^2 + |\Delta u_0|^2\}\,.$

Our first result pertains to local existence and uniqueness of solution. For this purpose we define, for some time interval I,

$$X_I \equiv H^1(I; H^{1/2}(\partial\Omega)) \cap H^2(I; H^{-1/2}(\partial\Omega)) = \bigcap_{r=1,2} H^{r,3/2-r}(I \times \partial\Omega).$$

Theorem 1.1. *Let $T > 0$ be arbitrary and $I \equiv (0, T)$. There exist $\rho_T, \tilde\rho_T > 0$ such that if*

$$E_{u,0}(0) + E_{u,1}(0) \leq \rho_T,\ \text{and } g \in X_{(0,T)}, \|g\|^2_{X_{(0,T)}} \leq \tilde\rho_T$$

with the compatibility conditions (11) *then there exists a unique solution* (u, u_t) *solving the weak form*

$$\begin{aligned} &\textstyle\int_\Omega\{[(1-2ku(t))u_{tt}(t) + du(t) + eu_t(t) - 2k(u_t(t))^2 - \sigma(|\nabla u(t)|^2 + \nabla(\mathcal{I}u)(t)\nabla u_t(t))]\phi \\ &\quad +[c^2\nabla u(t) + b\nabla u_t(t)]\nabla\phi\}\,dx + \textstyle\int_{\partial\Omega}[c^2(hu(t) - g(t)) + b(hu_t(t) - g_t(t))]\phi\,ds = 0 \end{aligned}$$

for all $\phi \in H^1(\Omega)$ *and all* $t \in (0,T)$, *of* (7), (8) *with initial conditions* (6) *and such that*

$$u \in C([0,T],H^2(\Omega))\cap C^1([0,T],H^1(\Omega))\cap C^2([0,T];L_2(\Omega))\,,\ u_{tt} \in L_2((0,T);H^1(\Omega))$$

The said solution is unique and depends continuously (with respect to the topology generated by $E_{u,1}$*) on the initial data.*

Remark 1.1. *The boundary space* X_I *is optimal with respect to parabolic maximal regularity displayed by the energy function* $E_{u,1}(t)$ *for solutions with Neumann data.*

Our next theoren deals with global wellposedness.

Theorem 1.2. *Set* $I \equiv R^+$, *let* (10) *hold and let* $U_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ *in the definition of* $\mathcal{I} : v \mapsto \int_0^\cdot v(\tau)\,d\tau + U_0$ *be a solution to*

$$\begin{array}{lcl} -c^2\Delta U_0 + dU_0 - \sigma\nabla u_0 \nabla U_0 &=& -(1-2ku_0)u_1 + b\Delta u_0 - eu_0 \quad \text{in } \Omega \\ \frac{\partial U_0}{\partial\nu} + hU_0 &=& 0 \quad \text{on } \partial\Omega\,, \end{array}$$

For any $M > 0$ *there exist* $\rho > 0$, $\tilde\rho > 0$, *such that solutions corresponding to initial and boundary data* (u_0,u_1,g) *with* (11),

$$E_{u,0}(0) < \infty\,,\ E_{u,1}(0) < \infty\,,\ \|g\|_{X_{R^+}} < \infty\,,\ \|\mathcal{I}^\Gamma g\|^2_{L_2(R^+,H^{1/2}(\partial\Omega))} < \infty\,, \tag{12}$$

$$\begin{array}{l} E_{u,1}(0) \le \rho, \\ \sup_{t\ge0}[\|\mathcal{I}^\Gamma g(t)\|^2_{H^{1/2}(\partial\Omega)} + \sum_{l=0}^1 \|\frac{d^l}{dt^l}g(t)\|^2_{H^{1/2-l}(\partial\Omega)}] \\ +\sum_{l=0}^1 \|\frac{d^l}{dt^l}g\|^2_{L_2(R^+,H^{1/2-l}(\partial\Omega))} + \sum_{l=1}^2 \|\frac{d^l}{dt^l}g\|^2_{L_2(R^+,H^{3/2-l}(\partial\Omega))} \ \le \tilde\rho \end{array} \tag{13}$$

where $\mathcal{I}^\Gamma : g \mapsto \int_0^\cdot g(\tau)\,d\tau$, *exist for all* $t > 0$ *and satisfy* $E_{u,1}(t) \le M$ *for all* $t > 0$.

Note that for $\sigma\|\nabla u_0\|_{L_3}$ small enough, which follows from $E_{u,1}(0) \le \rho$ sufficiently small, and $d + h > 0$, a fixed point argument yields existence of a solution U_0 to (12). Moreover, $\frac{\partial U_0}{\partial\nu} + hU_0 = 0$, $\mathcal{I}^\Gamma g(0) = 0$ and (11) guarantee compatibility of initial and boundary data. Condition (12) is needed for showing enhanced regularity estimates of $U = \mathcal{I}u$ in the analysis of the Kuznetsov equation, see Lemma 3.6 in [10]. For the pure Westervelt case $\sigma = 0$, condition (12) is not required.

Finally, we shall present results on energy decays.

Theorem 1.3. *Let the assumptions of Theorem 1.2 be satisfied. Fix* $M > 0$ *and correspondingly* $\rho, \tilde\rho$ *according to Theorem 1.2. If the Cauchy data satisfy* (11), (12), (13) *and additionally*

$$|g(t)|_{H^{1/2}(\partial\Omega)} + |g_t(t)|_{H^{-1/2}(\partial\Omega)} + |g_{tt}(t)|_{H^{-1/2}(\partial\Omega)} \le C_g e^{-\omega_g t}\,, \tag{14}$$

for some $C_g > 0$, $\omega_g > 0$, *then* u *satisfies the exponential decay estimate*

$$E_{u,1}(t) + E_{u,0}(t) \le C_\rho e^{-\tilde\omega t}$$

for some $C_\rho > 0$, $\tilde\omega > 0$.

Like in [10] local well-posedness will be established by an application of the Banach Fixed Point Theorem together with appropriate energy estimates. Global well-posedness will be shown by means of barrier's method, i.e., by assuming that after finite time degeneration occurs and deriving a contradiction by means of energy estimates. In order to take into account the inhomogeneous boundary in an appropriate way, we will consider its extension to the interior using an abstract strongly damped linear wave equation as well as a related variation of parameters formula as developed in [11] (see also [2], [15] for the purely parabolic case).

The main technical difference with respect to the Dirichlet case lies in the following facts (i) that we have to take care of a nontrivial nullspace of the Laplace operator with Neumann boundary conditions (this is critical when dealing with quasilinear problems that depend crucially on total dissipation) , (ii) the treatment of boundary extensions is different due to the variational nature associated with Neumann problems, (iii) the characterization of fractional powers of the operators associated with strongly damped wave operators subject to Neumann boundary conditions provides an additional tool that needs to be skillfully used in resolving existence questions of quasilinear dynamics.

Being able to proceed along the lines of the Dirichlet case [10], [11], we keep the present paper short by only pointing out the major differences of the Neumann as compared to the Dirichlet case.

2. **Strongly damped abstract wave equation.** In what follows we recall results from [10], [11] on the following non-homogenuous and nonautonomus abstract strongly damped wave equation:

$$\alpha(t)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = f(t) \tag{15}$$

with initial conditions

$$u(0) = u_0 \in \mathcal{D}(\mathcal{A}^{1/2}),\ u_t(0) = u_1 \in \mathcal{H}\,.$$

Here

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

is a positive selfadjoint operator, $\mathcal{H}$ is a suitable Hilbert space. We shall introduce the following notation

$$|u| \equiv |u|_{\mathcal{H}}\,, \quad (u,v) \equiv (u,v)_{\mathcal{H}}\,.$$

We are interested in studying regularity properties of solutions u, u_t due to the forcing f and initial conditions u_0, u_1.

To take into account inhomogeneous boundary conditions, we will decompose $u = u^0 + \bar{g}$ with u^0 having to satisfy homogeneous Cauchy (boundary and initial) conditions and $\bar{g}$ will denote an appropriate extension of the Cauchy data, i.e.,

$$\begin{aligned} \frac{\partial \bar{g}}{\partial \nu} &= g \text{ on } \partial\Omega \\ \bar{g}(t=0) &= u_0\,, \quad \bar{g}_t(t=0) \ = \ u_1 \end{aligned} \tag{16}$$

For carrying out this extension, we will use results on the strongly damped abstract wave equation with $\alpha \equiv 1$.

2.1. **Extension of nonhomogeneous boundary data to the interior.** We consider

$$w_{tt} - c^2\Delta w + dw - b\Delta w_t + ew_t = f(t) \qquad \text{in } (0,T)\times\Omega \tag{17}$$

$$\frac{\partial w}{\partial \nu} + hw = g \qquad \text{on } \partial\Omega \tag{18}$$

$$w(0) = w_0\,, \quad w_t(0) = w_1 \qquad \text{in } \Omega\,.$$

which can be rewritten as

$$
\begin{aligned}
w_{tt}+\delta w_t+c^2(-\Delta+\gamma\mathrm{id})w+b(-\Delta+\gamma\mathrm{id})w_t &= f(t) && \text{in } (0,T)\times\Omega\\
\frac{\partial w}{\partial\nu}+\beta w &= g(t) && \text{on } \partial\Omega\\
w(0)=w_0\,,\quad w_t(0) &= w_1 && \text{in } \Omega
\end{aligned}
\tag{19}
$$

where we have $\beta=h$, $\gamma=\frac{d}{c^2}$, $\delta=e-b\gamma$.

Moroever, we define the harmonic extension operator $N^\Delta=N^\Delta_{\beta,\gamma}$ by

$$
v=N^\Delta z,\ \text{iff}\ \begin{cases} -\Delta v+\gamma v=0, & \text{in } \Omega\\ \frac{\partial v}{\partial\nu}+\beta v=z & \text{on } \partial\Omega\end{cases}
\tag{20}
$$

with nonnegative parameters β,γ, where the elliptic boundary value problem is to be satisfied in the weak sense. For all $s\in R$ the operator N^Δ is a bounded mapping

$$
N^\Delta : H^s_\diamondsuit(\partial\Omega)\to H^{s+3/2}(\Omega)\,,
\tag{21}
$$

where $H^s_\diamondsuit(\partial\Omega)=H^s(\partial\Omega)$ if $\beta+\gamma>0$ and $H^s_\diamondsuit(\partial\Omega)=\{z\in H^s(\partial\Omega)\,:\,\int_{\partial\Omega}z\,ds=0\}$ if $\beta=\gamma=0$, cf., e.g, Theorem 6.6 in [18]. [1] Using the definition of N^Δ, we can restate (17) as $w_{tt}+c^2(-\Delta+\gamma\mathrm{id})(w-N^\Delta g)+b(-\Delta+\gamma\mathrm{id})(w_t-N^\Delta g_t)+\delta w_t=f(t)$ and hence, setting $\mathcal{A}=\mathcal{A}_{\beta,\gamma}=-\Delta+\gamma\mathrm{id}$ with zero Robin boundary conditions $\frac{\partial}{\partial\nu}+\beta\mathrm{id}$, $\mathcal{H}=L_2(\Omega)$, we write (17), (18) as the abstract second order ODE:

$$
w_{tt}+c^2\mathcal{A}w+b\mathcal{A}w_t+\delta w_t=c^2\mathcal{A}N^\Delta g+b\mathcal{A}N^\Delta g_t+f(t)
\tag{22}
$$

where in splitting the brackets, we admit representation of the equation in the dual space to $\mathcal{D}(\mathcal{A})$. This procedure is standard by now [16], [3] and references therein.

As in [9], [10], [11] we make use of the semigroup generated by the nondegenerate operator

$$
A=\begin{pmatrix} 0 & I\\ -c^2\mathcal{A} & -b\mathcal{A}-\delta\mathrm{id}\end{pmatrix}
$$

with the domain $D(A)=\{(w,v)\in\mathcal{D}(\mathcal{A}^{1/2})\times\mathcal{D}(\mathcal{A}^{1/2}),c^2w+bv\in\mathcal{D}(\mathcal{A})\}$ which, up to the δ term, corresponds to $\alpha(t)=1$ in (15).

In what follows we shall use regularity properties of the generator $e^{At}\in\mathcal{L}(H)$ where

$$
H\equiv\mathcal{D}(\mathcal{A}^{1/2})\times\mathcal{H}
$$

It was shown in [5] that e^{At} is an analytic, strongly continuous semigroup defined on H and that

$$
|A^\theta e^{At}|_{\mathcal{L}(H)}\le Ce^{-\omega t}t^{-\theta},t>0
\tag{23}
$$

where

$$
\omega=\min\{b,\frac{c^2}{b}\}
\tag{24}
$$

[1] For $s<0$, one uses duality and replaces $\int_{\partial\Omega}z\,ds$ by the duality bracket $\langle 1,z\rangle_{H^s(\partial\Omega),H^{-s}(\partial\Omega)}$

In addition we have that [5, 16]

$$D(A^{1/2}) = \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}) = H^1(\Omega) \times H^1(\Omega)$$

$$\int_0^\infty |A^{1/2} e^{At} x|_H^2 dt \leq C|x|_H^2$$

$$L \in \mathcal{L}(L_2(\mathbb{R}^+; H), L_2(\mathbb{R}^+; H)) \text{ where } LF(\cdot) \equiv \int_0^\cdot Ae^{A(\cdot - s)} F(s) ds$$

$$|\int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds|_H^2 \leq C \int_0^t e^{-2\omega(t-s)} |\mathcal{A}^{-1/2} f(s)|^2 ds, \; t > 0 \quad (25)$$

Denoting $W(t) \equiv (w(t), w_t(t))$, we can rewrite (22) as

$$W_t(t) = AW(t) + \begin{pmatrix} 0 \\ c^2 \mathcal{A} N^\Delta g(t) + b\mathcal{A} N^\Delta g_t(t) + f(t) \end{pmatrix}$$

and obtain the following "variation of parameter formula" representing weak solutions to the nonhomogenuous boundary value problem driven by the non-degenerate ($\alpha = 1$) damped wave equation as introduced in [11]:

Lemma 2.1. *Let $\beta, \gamma \geq 0$ and either*

a) *$\beta + \gamma > 0$ and $I = \mathbb{R}^+$ or*

b) *$\beta = \gamma = 0$ and $I = (0, T)$ for some finite T (depending on $\|w_0\|_{H^2(\Omega)}$, $\|w_1\|_{H^2(\Omega)}$) and $\int_{\partial\Omega} g \, ds = 0$.*

Moreover, let $f \in L_2(I; H^1(\Omega)) \cap L_\infty(I; L_2(\Omega))$, $g \in L_\infty(I; H^{1/2}(\partial\Omega)) \cap L_2(I; H^{1/2}(\partial\Omega))$, $g_t \in L_2(I; H^{1/2}(\partial\Omega))$, $g_{tt} \in L_2(I; H^{-1/2}(\partial\Omega))$,

$$w_0, w_1 \in H^2(\Omega)\,, \quad \frac{\partial}{\partial\nu} w_0 + hw_0 = g(0)\,, \; \frac{\partial}{\partial\nu} w_1 + hw_1 = g_t(0) \;\; \text{on } \partial\Omega\,.$$

1. *The following representations hold:*

$$W(t) = e^{At} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ c^2 \mathcal{A} N^\Delta g + b\mathcal{A} N^\Delta g_t + f \end{pmatrix} ds \quad (26)$$

$$\begin{aligned} W_t &= e^{At} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ c^2 \mathcal{A} N^\Delta g_t + b\mathcal{A} N^\Delta g_{tt} + f_t \end{pmatrix} ds \\ &= e^{At} \begin{pmatrix} w_1 \\ w_2 - f(0) \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ c^2 \mathcal{A} N^\Delta g_t + b\mathcal{A} N^\Delta g_{tt} \end{pmatrix} ds \\ &\quad + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} - \int_0^t Ae^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \end{aligned} \quad (27)$$

where $w_2 = w_{tt}(0) = -c^2\mathcal{A}(w_0 - N^\Delta g(0)) - b\mathcal{A}(w_1 - N^\Delta g_t(0)) - \delta w_1 + f(0)$.

2. *$w_{tt} \in L_\infty(I; \mathcal{H})$, $\mathcal{A}^{1/2} w_{tt} \in L_2(I; \mathcal{H})$, $\mathcal{A}^{1/2} w_t \in L_2(I; \mathcal{H}) \cap L_\infty(I; \mathcal{H})$, $w_t \in L_2(I; H^2(\Omega))$, $w \in L_2(I; H^2(\Omega)) \cap L_\infty(I; H^2(\Omega))$,*
3. *In case a) with $f = 0$, and if additionally (14) for some C_g, $\omega_g > 0$ holds for all $t > 0$, then there exists $\hat{C} > 0$, $\hat{\omega} > 0$ such that for all $0 < s < T$*

$$\int_s^T \left(|\Delta w(t)|^2 + |\nabla w_t(t)|^2 + |w_{tt}(t)|^2 + |\nabla w_{tt}(t)|^2 \right) dt \leq \hat{C} e^{-\hat{\omega} s}\,. \quad (28)$$

Proof. The derivation of "variation of parameters formula" follows from the same arguments as used in [16] (see also [4] for related problems in the context of strongly damped wave equations). The Dirichlet case considered in [10] is somewhat different, however many steps are conceptually the same. One of the main differences is that in the Neumann case we have $\mathcal{A}^{1/2}N^{\Delta} \in \mathcal{L}(H^{-1/2}(\Gamma); L_2(\Omega))$ which fact along with the analyticity of the semigroup e^{At} allows to deal with formulas (26), (27) within the framework of the basic state space H (not valid in the Dirichlet case which requires a different handling [11]). In fact, in the formula (27) there is a cancellation of singularity in the first term on the right hand side which is due to assumed compatability conditions and the above mentioned regularity of the Neumann map.

The estimates on $\|w_{tt}\|_{L_\infty(I;\mathcal{H})}$, $\|\mathcal{A}^{1/2}w_{tt}\|_{L_2(I;H^1(\Omega))}$, $\|\mathcal{A}^{1/2}w_t\|_{L_2(I;\mathcal{H})}$, $\|\mathcal{A}^{1/2}w_t\|_{L_\infty(I;\mathcal{H})}$, directly follow from (26), (27): From (26) we get

$$\begin{aligned}
&|w_t(t)|^2 + |\mathcal{A}^{1/2}w(t)|^2 + 2\int_0^t |\mathcal{A}^{1/2}w_t|^2 ds \\
\leq\ & C[|w_t(0)|^2 + |\mathcal{A}^{1/2}w(0)|^2 + \int_0^t |g|^2_{-1/2} + |g_t|^2_{-1/2} + |\mathcal{A}^{-1/2}f|^2]\,,
\end{aligned}$$

and from (27)

$$\begin{aligned}
&|w_{tt}(t)|^2 + |\mathcal{A}^{1/2}w_t(t)|^2 + 2\int_0^t |\mathcal{A}^{1/2}w_{tt}|^2 ds \\
\leq\ & C\Big(\Big|\begin{pmatrix} w_1 \\ -c^2\mathcal{A}(w_0 - N^{\Delta}g(0)) - b\mathcal{A}(w_1 - N^{\Delta}g_t(0)) - \delta w_1 \end{pmatrix}\Big|_H \\
&+ \int_0^t |g_t|^2_{-1/2} + |g_{tt}|^2_{-1/2} + |\mathcal{A}^{-1/2}f_t|^2\Big)\,.
\end{aligned}$$

The improved estimates for $|w|_{H^2(\Omega)}$, $|w_t|_{H^2(\Omega)}$ can be derived by multiplication of

$$w_{tt} + c^2\mathcal{A}(w - N^{\Delta}g) + b\mathcal{A}(w_t - N^{\Delta}g_t) + \delta w_t = f \tag{29}$$

with $\mathcal{A}(w - N^{\Delta}g)$ and with $\mathcal{A}(w_t - N^{\Delta}g_t)$, respectively: Firstly we get

$$\begin{aligned}
&c^2\int_s^T |\mathcal{A}(w - N^{\Delta}g)|^2 dt + \frac{b}{2}(|\mathcal{A}(w - N^{\Delta}g)|^2(T) - |\mathcal{A}(w - N^{\Delta}g)|^2(s) \\
=\ & -\int_s^T (\mathcal{A}(w - N^{\Delta}g), w_{tt})dt - \delta\int_s^T (\mathcal{A}(w - N^{\Delta}g), w_t)dt + \int_s^T (\mathcal{A}(w - N^{\Delta}g), f)dt
\end{aligned}$$

hence

$$\begin{aligned}
&\Big(\frac{c^2}{2} - \epsilon\Big)\int_s^T |\mathcal{A}(w - N^{\Delta}g)|^2 dt + \frac{b}{2}\Big(|\mathcal{A}(w - N^{\Delta}g)(T)|^2 - |\mathcal{A}(w - N^{\Delta}g)(s)|^2\Big) \\
\leq\ & \frac{1}{c^2}\int_s^T |w_{tt}|^2 dt + \frac{\delta^2}{c^2}\int_s^T |w_t|^2 dt + \frac{4}{\epsilon}\int_s^T |f| dt\,,
\end{aligned} \tag{30}$$

secondly,

$$\begin{aligned}
&\frac{c^2}{2}(|\mathcal{A}(w - N^{\Delta}g)(T)|^2 - |\mathcal{A}(w - N^{\Delta}g)(s)|^2) + \Big(\frac{b}{2} - \epsilon\Big)\int_s^T |\mathcal{A}(w_t - N^{\Delta}g_t)|^2(t)\, dt \\
\leq\ & \frac{1}{b}\int_s^T |w_{tt}|^2 dt + \frac{\delta^2}{b}\int_s^T |w_t|^2 dt + \frac{4}{\epsilon}\int_s^T |f| dt\,.
\end{aligned}$$

Here we note that $w - N^{\Delta}g \in \mathcal{D}(A)$ is equivalent to the fact that both w and g have the right spatial regularity ($w \in H^2(\Omega), g \in H^{-1/2}((\Gamma))$. Thus, the resulting energy estimates along with the apriori regularity of g give the needed spatial regularity of w.

For exponential decay, we use (25) to obtain from (27) (with $f=0$)

$$
\begin{aligned}
&|\mathcal{A}^{1/2}w_t(t)|+|w_{tt}(t)|\\
&\leq\; e^{-\omega t}\left|\begin{pmatrix} w_1\\ -c^2\mathcal{A}(w_0-N^\Delta g(0))-b\mathcal{A}(w_1-N^\Delta g_t(0))-\delta w_1\end{pmatrix}\right|_H\\
&\quad+C\sqrt{\int_0^t e^{-2\omega(t-s)}\left|\begin{pmatrix}0\\ c^2\mathcal{A}^{1/2}N^\Delta g_t+b\mathcal{A}^{1/2}N^\Delta g_{tt}\end{pmatrix}\right|_H^2 ds}\\
&\leq\; e^{-\omega t}\Big(|\mathcal{A}^{1/2}w_1|+|-c^2\mathcal{A}(w_0-N^\Delta g(0))-b\mathcal{A}(w_1-N^\Delta g_t(0))-\delta w_1|\Big)\\
&\quad+\tilde{C}e^{-\omega t}\sqrt{\int_0^t e^{2(\omega-\omega_g)s}ds}\\
&\leq\; \check{C}e^{-\breve{\omega}t}
\end{aligned}
\tag{31}
$$

with $0<\breve{\omega}<\min\{\omega,\omega_g\}$.

Considering the sum of $\Big(1+\frac{2}{c^4}\max\{1,\delta^2C_0^2\}\Big)$ times (31) and the integral from s to T of the square of (30), we get for the energy

$$E(t)=\max\{|\mathcal{A}^{1/2}w_t(t)|+|w_{tt}(t)|\,,\,\check{C}e^{-\breve{\omega}t}\}^2+|\mathcal{A}(w-N^\Delta g)(t)|^2$$

the estimate

$$\int_s^T E(t)\,dt\leq 2\breve{\omega}E(s)\quad\text{with } 2\breve{\omega}=\max\{\frac{\check{C}^2}{2\breve{\omega}}\Big(1+\frac{2}{c^4}\max\{1,\delta^2C_0^2\}\Big)\,,\,\frac{b}{c^2}\}\,. \tag{32}$$

Here C_0 is the constant in the continuous embedding estimate $|z|\leq C_0|\mathcal{A}^{1/2}z|$, (see also Assumption (3.1) below). Using a standard argument from [19] and $\|N^\Delta g(t)\|_{H^2(\Omega)}\leq C_g e^{-\omega_g t}$ we obtain from (32)

$$|\mathcal{A}^{1/2}w_t(t)|^2+|w_{tt}(t)|^2+|w(t)|^2_{H^2(\Omega)}\leq\check{C}e^{-2\min\{\breve{\omega},\omega_g\}t}\,.$$

The decay estimate of $\mathcal{A}^{1/2}w_{tt}$ is obtained by differentiating (22) (with $f=0$) wrt. time and multiplying with w_{tt}:

$$
\begin{aligned}
&\frac{1}{2}\Big(|w_{tt}(T)|^2-|w_{tt}(s)|^2\Big)+\frac{c^2}{2}\Big(|\mathcal{A}^{1/2}w_t(T)|^2-|\mathcal{A}^{1/2}w_t(s)|^2\Big)\\
&+\int_s^T(b|\mathcal{A}^{1/2}w_{tt}|^2+\delta|\mathcal{A}^{1/2}w_{tt}|^2)dt\\
&\quad=\; c^2\int_s^T(\mathcal{A}^{1/2}N^\Delta g_t,\mathcal{A}^{1/2}w_{tt})dt+b\int_s^T(\mathcal{A}^{1/2}N^\Delta g_{tt},\mathcal{A}^{1/2}w_{tt})dt
\end{aligned}
$$

hence

$$
\begin{aligned}
&\frac{b}{2}\int_s^T|\mathcal{A}^{1/2}w_{tt}|^2dt\\
&\quad\leq\; \frac{1}{2}\Big(|w_{tt}(s)|^2-|w_{tt}(T)|^2\Big)+\frac{c^2}{2}\Big(|\mathcal{A}^{1/2}w_t(s)|^2-|\mathcal{A}^{1/2}w_t(T)|^2\Big)\\
&\qquad+\frac{c^4}{b}\int_s^T|\mathcal{A}^{1/2}N^\Delta g_t(t)|^2dt+b\int_s^T|\mathcal{A}^{1/2}N^\Delta g_{tt}(t)|^2dt\\
&\quad\leq\; \bar{C}e^{-2\min\{\breve{\omega},\omega_g\}t}
\end{aligned}
$$

□

Considering (17) we thus define an extension (16)

$$\begin{aligned} &\bar{g} = w \text{ according to (17) with} \\ &f = 0\,,\ w_0 = u_0\,,\ w_1 = u_1\,. \end{aligned} \tag{33}$$

2.2. **Energy estimates for the variable coefficient model.** In this section we will consider the strongly damped abstract wave equation (15) with $\alpha(t) \in L_\infty(\Omega)$ being a positive multiplier on $\mathcal{H} = L_2(\Omega)$ i.e.,

Assumption 2.1.

$$(\alpha(t)u, u) \geq \underline{\alpha}(t)|u|^2 \geq \underline{\alpha}_0|u|^2\,, \quad u \in \mathcal{H} \tag{34}$$

$$(\alpha(t)u, v)_{L_2(\Omega)} \leq \overline{\alpha}(t)|u||v| \leq \overline{\alpha}_0|u||v|\,, \quad u, v \in \mathcal{H} \tag{35}$$

Our assumptions on the initial and boundary data are the folllowing.

Assumption 2.2. .

- *$f \in L_2(0,T;\mathcal{H}) \cap H^1(0,T;[\mathcal{D}(\mathcal{A}^{1/2})]')$*
- *$u_0 \in \mathcal{D}(\mathcal{A})$, $u_1 \in \mathcal{D}(\mathcal{A}^{1/2})$, and $\frac{1}{\alpha(0)}(f(0) - c^2\mathcal{A}u_0 - b\mathcal{A}u_1) \in \mathcal{H}$.*

Assumptions 2.1, 2.2 will be imposed throughout the rest of this section.
In order to derive energy estimates for the problem

$$\begin{aligned} \alpha u_{tt} - c^2\Delta u + du - b\Delta u_t + eu_t = f(t) \qquad &\text{in } (0,T)\times\Omega \\ \frac{\partial u}{\partial \nu} + hu = 0 \qquad &\text{on } \partial\Omega \\ u(0) = u_0\,, \quad u_t(0) = u_1 \qquad &\text{in } \Omega \end{aligned} \tag{36}$$

in case $d = e = h = 0$, we use a transformation $u = e^{\lambda t}\tilde{u}$, so that (36) becomes

$$\alpha\tilde{u}_{tt} + (\alpha\lambda^2 - \tilde{c}^2\gamma)\tilde{u} + (2\alpha\lambda - b\gamma)\tilde{u}_t + \tilde{c}^2\mathcal{A}\tilde{u} + b\mathcal{A}\tilde{u}_t = e^{-\lambda t}f(t) \tag{37}$$

with initial conditions $\tilde{u}(0) = u_0$, $\tilde{u}_t(0) = u_1$, where for fixed $\gamma > 0$ we choose λ sufficiently large so that

$$\underline{\alpha}_0\lambda^2 - (c^2 + b\lambda)\gamma \geq 0 \text{ and } 2\underline{\alpha}_0\lambda - b\gamma \geq 0 \tag{38}$$

and set $\tilde{c}^2 = c^2 + b\lambda$, as well as

$$\mathcal{A} = \mathcal{A}_{0,\gamma} = -\Delta + \gamma\text{id}, \text{ with zero Neumann boundary data}\,, \quad \mathcal{H} = L_2(\Omega)\,.$$

By inspection of the proofs of Theorem 2.1 in [9], and Theorem 2.1 in [10] we get

Lemma 2.2. *Consider* (36) *with $d = e = h = 0$,* (37) *with α satisfying (additionally to Assumption* (34)*) $\alpha \in C^1(0,T;\mathcal{H})$, $\nabla\alpha \in C(0,T;\mathcal{H})$ and $\tilde{f}(t) = e^{-\lambda t}f(t)$. Then the following energy estimates hold:*
By multiplication of (37) *with $\tilde{u}_t$:*

$$\begin{aligned} &E_{\tilde{u}}(t) + \frac{1}{2}(\tilde{d}(t)\tilde{u}(t), \tilde{u}(t)) + (b - \hat{\epsilon})\int_0^t |\mathcal{A}^{1/2}\tilde{u}_t(s)|^2 ds \\ &\leq \quad E_{\tilde{u}}(0) + \frac{1}{2}(\tilde{d}(0)u_0, u_0) + \frac{1}{4\hat{\epsilon}}\int_0^t |\mathcal{A}^{-1/2}\tilde{f}(s)|^2 ds + \frac{1}{2}\int_0^t \left\{(\alpha_t\tilde{u}_t, \tilde{u}_t) + \lambda^2(\alpha_t\tilde{u}, \tilde{u})\right\} ds\,; \end{aligned} \tag{39}$$

By time differentiation of (37) and multiplication with $\tilde{u}_{tt}$:

$$
\begin{aligned}
&E_{\tilde{u}_t}(t) + \frac{1}{2}(\tilde{d}(t)\tilde{u}_t(t), \tilde{u}_t(t)) + (b-\hat{\epsilon})\int_0^t |\mathcal{A}^{1/2}\tilde{u}_{tt}(s)|^2 ds \\
\leq\ & E_{\tilde{u}_t}(0) + \frac{1}{2}(\tilde{d}(0)u_1, u_1(t)) + \frac{1}{4\hat{\epsilon}}\int_0^t |\mathcal{A}^{-1/2}\tilde{f}_t(s)|^2 ds \qquad (40)\\
&+\frac{1}{2}\int_0^t \left\{-(\alpha_t\tilde{u}_t, \tilde{u}_t) + \lambda^2(\alpha_t\tilde{u}_t, \tilde{u}_t) - 2\lambda^2(\alpha_t\tilde{u}, \tilde{u}_{tt}) - 4\lambda(\alpha_t\tilde{u}_t, \tilde{u}_{tt})\right\} ds\,;
\end{aligned}
$$

By multiplication of (37) with $\mathcal{A}\tilde{u}_t$:

$$
\begin{aligned}
&\tilde{c}^2|\mathcal{A}\tilde{u}(t)|^2 + b\int_0^t |\mathcal{A}\tilde{u}_t(s)|^2 ds + \gamma(\tilde{d}(t)\tilde{u}(t), \tilde{u}(t)) + (\tilde{d}(t)\nabla\tilde{u}(t), \nabla\tilde{u}(t)) \qquad (41)\\
\leq\ & \tilde{c}^2|\mathcal{A}u_0|^2 + \frac{2}{b}\int_0^t |\tilde{f}(s)|^2 ds + \frac{2\overline{\alpha}_0}{b}\int_0^t (\alpha\tilde{u}_{tt}, \tilde{u}_{tt})\, ds + \gamma(\tilde{d}(0)u_0, u_0) + (\tilde{d}(0)\nabla u_0, \nabla u_0)\\
&+\int_0^t \left\{2\lambda^2(\tilde{u}\nabla\alpha, \nabla\tilde{u}_t) - \gamma\lambda^2(\alpha_t\tilde{u}, \tilde{u}) - \lambda^2(\alpha_t\nabla\tilde{u}, \nabla\tilde{u}) - 4\lambda(\tilde{u}_t\nabla\alpha, \nabla\tilde{u}_t)\right\} ds\,;
\end{aligned}
$$

By multiplication of (37) with $\mathcal{A}\tilde{u}$:

$$
\begin{aligned}
&b|\mathcal{A}\tilde{u}(t)|^2 + \tilde{c}^2\int_0^t |\mathcal{A}\tilde{u}(s)|^2 ds + \gamma(\tilde{e}(t)\tilde{u}(t), \tilde{u}(t)) + (\tilde{e}(t)\nabla\tilde{u}(t), \nabla\tilde{u}(t)) \qquad (42)\\
\leq\ & b|\mathcal{A}u_0|^2 + \frac{2}{\tilde{c}^2}\int_0^t |\tilde{f}(s)|^2 ds + \frac{2\overline{\alpha}_0}{\tilde{c}^2}\int_0^t (\alpha\tilde{u}_{tt}, \tilde{u}_{tt})\, ds + \gamma(\tilde{e}(0)u_0, u_0) + (\tilde{e}(0)\nabla u_0, \nabla u_0)\\
&+\int_0^t \left\{4\lambda(\tilde{u}_t\nabla\alpha, \nabla\tilde{u}) - 2\gamma\lambda(\alpha_t\tilde{u}, \tilde{u}) - 2\lambda(\alpha_t\nabla\tilde{u}, \nabla\tilde{u}) - 2\lambda^2(\tilde{u}\nabla\alpha, \nabla\tilde{u})\right\} ds\,;
\end{aligned}
$$

By multiplication of (37) with $\tilde{u}_{tt}$:

$$
\begin{aligned}
&\frac{1}{4}\int_0^t (\alpha(s)\tilde{u}_{tt}(s), \tilde{u}_{tt}(s)) ds + \frac{b}{4}|\mathcal{A}^{1/2}\tilde{u}_t(t)|^2 + (\tilde{e}(t)\tilde{u}_t(t), \tilde{u}_t(t)) \qquad (43)\\
\leq\ & b|\mathcal{A}^{1/2}u_1|^2 + \frac{\tilde{c}^4}{2b}|\mathcal{A}^{1/2}u_0|^2 + \frac{1}{2\underline{\alpha}_0}\int_0^t |\tilde{f}(s)|^2 ds + \frac{\tilde{c}^4}{b}|\mathcal{A}^{1/2}\tilde{u}(t)|^2\\
&+\tilde{c}^2\int_0^t |\mathcal{A}^{1/2}\tilde{u}_t(s)|^2\, ds + (\tilde{e}(0)u_1, u_1) + \int_0^t \left\{(\frac{1}{\alpha}\tilde{d}\tilde{u}, \tilde{u}) + \lambda(\alpha_t\tilde{u}_t, \tilde{u}_t)\right\} ds
\end{aligned}
$$

with

$$
E_{\tilde{u}}(t) \equiv \frac{1}{2}\left\{(\alpha(t)\tilde{u}_t(t), \tilde{u}_t(t)) + \tilde{c}^2|\mathcal{A}^{1/2}\tilde{u}(t)|^2\right\}.
$$

where $\tilde{c}^2 = c^2 + b\lambda$, $\tilde{d}(t) = (\alpha(t)\lambda^2 - \tilde{c}^2\gamma)$ and $\tilde{e}(t) = (2\alpha(t)\lambda - b\gamma)$, so $\tilde{c}^2, \tilde{d}, \tilde{e} \geq 0$ for sufficiently large λ.

Proof. For (41) we use the fact that

$$\begin{aligned}
&\int_0^t ((\alpha\lambda^2-\tilde{c}^2\gamma)\tilde{u},\mathcal{A}\tilde{u}_t)ds\\
&= \int_0^t \left\{\gamma((\alpha\lambda^2-\tilde{c}^2\gamma)\tilde{u},\tilde{u}_t)+((\alpha\lambda^2-\tilde{c}^2\gamma)\nabla\tilde{u},\nabla\tilde{u}_t)+\lambda^2(\tilde{u}\nabla\alpha,\nabla\tilde{u}_t)\right\}ds\\
&= \frac{1}{2}\left\{\gamma((\alpha\lambda^2-\tilde{c}^2\gamma)\tilde{u},\tilde{u})+((\alpha\lambda^2-\tilde{c}^2\gamma)\nabla\tilde{u},\nabla\tilde{u})\right\}_{s=0}^{s=t}\\
&\quad+\lambda^2\int_0^t\left\{(\tilde{u}\nabla\alpha,\nabla\tilde{u}_t)-\gamma\tfrac{1}{2}(\alpha_t\tilde{u},\tilde{u})-\tfrac{1}{2}(\alpha_t\nabla\tilde{u},\nabla\tilde{u})\right\}ds
\end{aligned}$$

$$\begin{aligned}
&\int_0^t ((2\alpha\lambda-b\gamma)\tilde{u}_t,\mathcal{A}\tilde{u}_t)ds\\
&= \int_0^t \left\{\gamma((2\alpha\lambda-b\gamma)\tilde{u}_t,\tilde{u}_t)+((2\alpha\lambda-b\gamma)\nabla\tilde{u}_t,\nabla\tilde{u}_t)+2\lambda(\tilde{u}_t\nabla\alpha,\nabla\tilde{u}_t)\right\}ds
\end{aligned}$$

The proof of (42) is analogous to the one of (41). For (43) we used the estimate

$$\int_0^t ((\alpha\lambda^2-\tilde{c}^2\gamma)\tilde{u},\tilde{u}_{tt})ds \le \frac{1}{4}\int_0^t(\alpha\tilde{u}_{tt},\tilde{u}_{tt})ds+\int_0^t(\frac{1}{\alpha}(\alpha\lambda^2-\tilde{c}^2\gamma)\tilde{u},\tilde{u})ds$$

□

Remark 2.1. *If* (10) *holds in* (36)*, we can simply set* $\beta = h$*,* $\gamma = \min\{\frac{d}{c^2},\frac{e}{b}\}$*,* $\lambda = 0$*,* $\tilde{u} = u$*, and replace the multipliers* $\tilde{d} = (\alpha\lambda^2 - \tilde{c}^2\gamma)$ *and* $\tilde{e} = (2\alpha\lambda - b\gamma)$ *appearing in the estimates above by* $\tilde{d} = d - c^2\gamma \ge 0$ *and* $\tilde{e} = e - b\gamma \ge 0$*, respectively.*

3. **Back to the proof of Theorems 1.1-1.3.** For applying barrier's method (i.e., for creating a set of initial data that is invariant under the dynamics for all times) to prove global existence for (7), (8) with (10), we need an estimate of the higher level energy

$$\mathcal{E}(t) \equiv E_{\tilde{u}_t}(t) + |\mathcal{A}\tilde{u}(t)|^2\,,$$

(see Proposition 2.3 in [10]). For this purpose we impose, in addition to Assumptions 2.1, 2.2, also

Assumption 3.1. *The following continuous embeddings hold*

$$\begin{aligned}
\mathcal{D}(\mathcal{A}^{1/2}) &\subset L_2(\Omega)\,, && \textit{with } |w| \le C_0|\mathcal{A}^{1/2}w|\,,\\
\mathcal{D}(\mathcal{A}^{1/2}) &\subset L_6(\Omega)\,, && \textit{with } |w|_{L_6(\Omega)} \le C_1|\mathcal{A}^{1/2}w|\,,\ |\nabla w|_{L_6(\Omega)} \le C_1|\mathcal{A}w|\,,\\
\mathcal{D}(\mathcal{A}) &\subset C(\Omega)\,, && \textit{with } |w|_{L_\infty(\Omega)} \le C_2|\mathcal{A}w|\,,
\end{aligned}$$

which is obviously satisfied for $\mathcal{A} = -\Delta + \gamma\mathrm{id}$ with homogeneous Robin boundary conditions $\frac{\partial}{\partial\nu} + \beta\mathrm{id}$ if $\beta,\gamma \ge 0$ and $\beta+\gamma > 0$. Note that the constants C_0, C_1, C_2 will depend on β,γ and especially will be large for small $\beta+\gamma$.

Proposition 3.1. *Let* u *be a solution to* (36) *with* (10)*. Then there exist constants* $\hat{b} > 0$*,* $\tilde{C}^1$ *-* $\tilde{C}^4$ *such that*

$$\begin{aligned}
&\mathcal{E}(T) + \hat{b}\int_0^T \left[|\mathcal{A}u(t)|^2 + |\mathcal{A}u_t(t)|^2 + |\mathcal{A}^{1/2}u_{tt}(t)|^2\right]dt \qquad (44)\\
&\le \tilde{C}^1\mathcal{E}(0) + \int_0^T \left[\tilde{C}^2|\mathcal{A}^{-1/2}f_t(t)|^2 + \tilde{C}^3|f(t)|^2 + \tilde{C}^4|\alpha_t(t)|^4|u_{tt}(t)|^2\right]dt
\end{aligned}$$

where

$$\hat{b} = \min\{\frac{c^2}{2b}, \frac{b}{2c^2}, (b - \hat{\epsilon} - \epsilon)\}\,, \quad \tilde{C}^1 = 1 + \frac{2\overline{\alpha}_0 C_0^2}{bc^2(b - \hat{\epsilon} - \epsilon)} \tag{45}$$

We return to the (possibly modified) Kuznetsov/Westervelt equation (7) with inhomogeneous Neumann (or Robin) boundary conditions (8) and initial conditions (6).

Proof of local well-posedness Theorem 1.1 is based on a suitable fixed point formulation $\mathcal{T}u = u$ of (3), (5), (6), where $\mathcal{T} : \mathcal{W} \subset \mathcal{X} \to \mathcal{W}$ with $\mathcal{T}(v) \equiv u = u^0 + \bar{g}$ with $\bar{g}$ according to (33), and u^0 given by

$$\alpha(t)u^0_{tt} - c^2\Delta u^0 - b\Delta u^0_t = f(t) \tag{46}$$

with homogeneous initial and boundary conditions on u^0, and

$$\alpha \equiv 1 - 2kv,\;\; f \equiv 2k(v_t u_t + v\bar{g}_{tt}) + \sigma\left(\nabla v \nabla u + \nabla \mathcal{I} v \nabla u_t\right)\,. \tag{47}$$

The space $\mathcal{X}$ and the domain of $\mathcal{T}$ are defined by

$$\mathcal{X} \equiv C(H^2(\Omega)) \cap C^1(H^1(\Omega)) \cap H^1(H^2(\Omega)) \cap H^2(L_2(\Omega))\,,$$

$$\begin{aligned} \mathcal{W} \;\; = \;\; & \{v \in L^\infty((0,T)\times\Omega) \mid v(0) = v_0, v_t(0) = u_1\,,\; ||v||_{L^\infty((0,T)\times\Omega)} \le m\,, \\ & ||\mathcal{A}v||_{C(L_2(\Omega))} \le \bar{a}\,,\; ||\mathcal{A}^{1/2}v_t||_{C(L_2(\Omega))} \le \bar{a}\,,\; ||\mathcal{A}^{1/2}v_{tt}||_{L_2(L_2(\Omega))} \le \bar{a}\}\,, \end{aligned} \tag{48}$$

where $m < \frac{1}{2k}$, and $\bar{a}$ is sufficiently small.

Using the energy estimates from Lemma 2.2 and Remark 2.1, the local well-posedness results can be done exactly along the lines of the proof of Theorem 3.1 in [10] both for the original model (3), (5) and for (7), (5) with $\min\{d, e\} + h > 0$. In this case the contractivity is achieved by calibrating the size of initial data with respect to the running time.

One remark should be made regarding the term f as given above. It contains $\bar{g}_{tt}$. Since the estimates in Theorem 2.2 call for f_t, one must be careful how to estimate $\bar{g}_{ttt}$ without involving three time derivatives of g. However, this can be done by using the method used in Proposition 3.5 in [10], which in turn relies on special properties of convolutions generated by analytic semigroups.

Proof of global existence and exponential decay - results stated in Theorems 1.2, 1.3 - is more involved. It requires the construction of a set of smooth initial data that is invariant under the dynamics. This set is defined by inequalities (12), (13) in Theorem 1.2. The crux of the problem lies in proving a certain observability inequality (as in controllability problems) for smooth solutions. This can only be shown for the modified problem (7), (5) with $\min\{d, e\}+h > 0$ by following the line of attack designed for the proofs of Theorems 3.2 and 3.3 in [10]. The procedure amounts to: (i) proving observability estimates which depend on dissipation present in the equation, (ii) construct an appropriate set of smooth initial data that is invariant under dynamics, (ii) for the latter use of the estimates in Lemma 2.2 which incorporate time dependent coefficients. We shall briefly outline the procedure. The operator $\mathcal{A}$ in [10] is replaced by $\mathcal{A} = \mathcal{A}_{\beta,\gamma}$, $\beta = h$, $\gamma = \min\{\frac{d}{c^2}, \frac{e}{b}\}$. The observability estimate is derived for solution u^0 of (46) with the time-space variable coefficients given in (47). The estimates of Lemma 2.2 along with estimates for the coefficients $\alpha(t), f(t)$ (see Proposition 3.5 , Lemma 3.6 in [10]) lead to the expression

$$\begin{aligned} &\sup_{t\in[s,T]} \mathcal{E}^0(t) + (\tilde{b} - \Psi(T) - H_1(\rho,\tilde{\rho})) \int_s^T (1-\Phi(t))\mathcal{E}^0(t)dt \\ &\leq \quad C^1\mathcal{E}^0(s) + \Theta(T)H_2(\rho,\tilde{\rho}) + H_3(\rho,\tilde{\rho}) \,, \end{aligned}$$

where $\mathcal{E}^0(t) = \frac{1}{2}\{(\alpha(t)u^0_{tt}(t), u^0_{tt}(t)) + |\nabla u^0_t(t)|^2 + |\Delta u^0_t(t)|^2\}$, $\tilde{b} > 0$,

$$\begin{aligned} \Psi(T) &\leq C[\sup_{t\in[s,T]} \mathcal{E}^0(t) + \int_s^T (\mathcal{E}^0(t) + \mathcal{E}^0(t)^2)\, dt] \\ \Phi(t) &\leq C(\mathcal{E}^0(t) + \mathcal{E}^0(t)^2) \\ \Theta(t) &\leq C[1 + \sup_{t\in[s,T]} \mathcal{E}^0(t) + \int_s^T (\mathcal{E}^0(t) + \mathcal{E}^0(t)^2)\, dt] \\ H_i(\rho,\tilde{\rho}) &\to 0 \text{ as } \rho,\tilde{\rho}\to 0 \quad i = 1,2,3\,. \end{aligned}$$

The above inequality with $s = 0$ allows to tune the parameters $\rho, \tilde{\rho}$ and the size of $\mathcal{E}^0(0)$ in order to obtain the desired invariance. In particular, using the energy estimates from Lemma 2.2 and Proposition 3.1, we can bound the factors $\alpha(t)$, $(\tilde{b}-\Psi(T)-H_1(\rho,\tilde{\rho}))$ and $(1-\Phi(t))$ away from zero and dominate the right hand side term $\Theta(T)H_2(\rho,\tilde{\rho})$ by a fraction of the left hand side. Under the decay assumption (14) and using (28), we can replace the remaining term $H_3(\rho,\tilde{\rho})$ on the right hand side by a refined exponentially decaying bound. Thus we end up with

$$\bar{b}\int_s^T \mathcal{E}^0(t)\, dt \leq \bar{C}^1\mathcal{E}^0(s) + Ce^{-\bar{\omega}t}$$

for some $\bar{b}, \bar{C}^1, C, \bar{\omega} > 0$, which by a standard semigroup argument [19] yields exponential decay of $\mathcal{E}^0(t)$ and therewith of $\mathcal{E}(t)$.

Note that, as expected, the required order of spatial differentiability of g is one order less for the Neumann as compared to the Dirichlet case and that there is no requirement for triple differentiation in time.

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