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Feedback Solutions for a Class of Quantum Control Problems

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Abstract Control of quantum systems described by the linear Schrödinger equation are considered. Control inputs enter through coupling operators and results in a bilinear control system. Feedback control laws are developed for orbit tracking. The sysmptotic properties of the feedback laws are analyzed by the LaSalle-type invariance principle. Numerical integrations via time-splitting is also investigated and used to demonstrate the feasibility of the proposed feedback laws and to compare their performance.

1 Introduction

Consider a quantum system with internal Hamiltonian $\mathcal{H}_0$ prepared in the initial state $\Psi_0(x)$, where x denotes the relevant spatial coordinate. The state $\Psi(x,t)$ satisfies the time-dependent Schrödinger equation. In the presence of an external interaction taken as an electric field modeled by a coupling operator with amplitude $\epsilon(t) \in \mathbb{R}$ and a time independent dipole moment operator μ results in the controlled Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \epsilon(t)\mu$ and the following dynamical system:

$$i\,\frac{\partial}{\partial t}\Psi(x,t) = (\mathcal{H}_0 + \epsilon(t)\mu)\Psi(x,t), \quad \Psi(x,0) = \Psi_0(x). \tag{1.1}$$

where $\mathcal{H}_0$ is a positive, closed, self-adjoint operator in the Hilbert space H, $\mu \in \mathcal{L}(H)$ is self-adjoint, and $\epsilon \in L^1(0,\infty)$ is the control input. Let X be the complexified Hilbert space corresponding to H, so that the inner product of X is defined by

$$(\Phi,\Psi)_X = (\Phi_1,\Psi_1)_H + (\Phi_2,\Psi_2)_H + i\big((\Phi_2,\Psi_1)_H - (\Phi_1,\Psi_2)_H\big),$$

where $\Phi = (\Phi_1,\Phi_2)$, $\Psi = (\Psi_1,\Psi_2)$. Throughout we normalize the initial state by $|\Psi_0|_X = 1$.

We consider the control problem of driving the state $\Psi(t)$ of (1.1) to an orbit $\mathcal{O}(t)$ of the uncontrolled dynamics

$$i\,\frac{d}{dt}\mathcal{O}(t) = \mathcal{H}_0\mathcal{O}(t), \tag{1.2}$$

specifically to the one that corresponds to an eigen-state or the manifold spanned by finite many eigen-states. An element $\psi \in dom\,(\mathcal{H}_0)$ is an eigen-state of $\mathcal{H}_0$ if $\mathcal{H}_0\psi = \lambda\,\psi$ for $\lambda > 0$. Then, the corresponding orbit is given by

$$\mathcal{O}(t) = e^{-i(\lambda t-\theta)}\psi, \tag{1.3}$$

where $\theta \in [0, 2\pi)$ is the phase factor. We have $|\mathcal{O}(t)|_X = 1$ if ψ is normalized as $|\psi|_H = 1$. We consider the discrete spectrum case: i.e. assume $\mathcal{H}_0$ only has discrete eigenvalues $\{\lambda_k\}$, the family of eigenfunctions $\{\psi_k\}_{k=1}^\infty$ forms an orthonormal basis of X and that $\{\lambda_k\}$ are arranged in increasing order.

We employ a variational approach based on the Lyapunov functional

$$V(t) = V(\Psi(t), \mathcal{O}(t)) = \frac{1}{2}\,|\Psi(t) - \mathcal{O}(t)|_X^2. \tag{1.4}$$

The variational approaches were previously discussed in [BCMR, MRT, IK], for example. In general we shall consider the case

$$\mathcal{O}(t) = \sum_{k=1}^{N} A_k e^{-i(\lambda_k t-\theta_k)}\psi_k, \tag{1.5}$$

where $\{(\lambda_k, \psi_k)\}_{k=1}^N$ are the first N eigen-pairs of $\mathcal{H}_0$ and $\sum_{k=1}^N A_k^2 = 1$. But we refer [IK2] for analyzing the general case.

We shall see in Section 2 that $|\Psi(t)|_X = 1$ for all $t \geq 0$. Together with $|\mathcal{O}(t)|_X = 1$ this implies that the functional V can equivalently be expressed as

$$V(\Psi(t), \mathcal{O}(t)) = 1 - \mathrm{Re}\,(\mathcal{O}(t), \Psi(t))_X. \tag{1.6}$$

It will be shown that

$$\frac{d}{dt}V(\Psi(t), \mathcal{O}(t)) = \epsilon(t)\,\mathrm{Im}\,(\mathcal{O}(t), \mu\Psi(t))_X. \tag{1.7}$$

We propose the feedback law

$$\begin{aligned} \epsilon(t) &= -\frac{1}{\alpha}(u(t) + \beta\,\mathrm{sign}(u(t))V(t)^\gamma) = F(\Psi(t), \mathcal{O}(t)), \\ u(t) &= \mathrm{Im}\,(\mathcal{O}(t), \mu\Psi(t))_X, \quad V(t) = V(\Psi(t), \mathcal{O}(t)), \end{aligned} \tag{1.8}$$

for $\alpha > 0,\ \beta \geq 0,\ \gamma \in (0,1]$. The case $\beta = 0$ is analyzed in [IK]. From (1.7)

$$\frac{d}{dt} V(\Psi(t), \mathcal{O}(t)) = -\frac{1}{\alpha}\,(|u(t)|^2 + \beta\,|u(t)| V(t)^\gamma). \tag{1.9}$$

Note that $u(t)$ is a linear in $\Psi(t)$. It will be shown that the performance of feedback laws significantly increases by incorporating the switching control term with $\beta > 0$.

In this paper we establish the wellposed-ness of the feedback law (1.8) and analyze its asymptotic tracking properties. Sufficient conditions will be obtained which guarantee orbit tracking.

In order to obtain improved tracking capability we shall also analyze multiple control potentials of the form

$$\mu(t) = \sum_{j=1}^{m} \epsilon_j(t)\,\mu_j \tag{1.10}$$

and the corresponding feedback law

$$\epsilon_j(t) = -\frac{1}{\alpha}(u_j(t) + \beta\,\mathrm{sign}(u_j(t)) V(t)^\gamma), \quad u_j(t) = \mathrm{Im}\,(\mathcal{O}(t), \mu_j \Psi(t))_X.$$

Section 2 is devoted to wellposedness of the dynamical system in open and closed loop form. In Section 3 it is shown that the feedback law F is optimal in the sense that $\epsilon(t) = F(\Psi(t), \mathcal{O}(t))$ minimizes

$$\int_0^T \frac{\alpha}{2}(|\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))|^2 + \frac{1}{\alpha}(\frac{1}{2}|u(t)|^2 + \beta\,|u(t)| V(t)^\gamma)\,dt + V(\Psi(T), \mathcal{O}(T))$$

$$u(t) = \mathrm{Im}\,(\mathcal{O}(t), \mu \Psi(t))_X.$$

An operator splitting method for solving (1.1) is discussed in Section 4. Section 5 is devoted to analyzing the asymptotic tracking properties of the feedback control laws. Section 6 contains numerical experiments that demonstrates the feasibility of the proposed feedback laws. The nonlinear feedback law ($\beta > 0$) significantly improves the tracking performance compared to the linear one ($\beta = 0$).

2 Wellposedness

Associated to the closed, positive, self-adjoint operator $\mathcal{H}_0$ densely defined in the Hilbert space H, we define the closed linear operator A_0 in $H \times H$ by

$$A_0 = \begin{pmatrix} 0 & \mathcal{H}_0 \\ \\ -\mathcal{H}_0 & 0 \end{pmatrix}$$

with $dom\,(A_0) = dom\,(\mathcal{H}_0) \times dom\,(\mathcal{H}_0)$. Here $\Psi = (\Psi_1, \Psi_2) \in H \times H$ is identified with $\Psi = \Psi_1 + i\,\Psi_2 \in X$. We note that

$$|(\Psi_1, \Psi_2)|_{H \times H} = |\Psi|_X, \text{ and } (\Phi, \Psi)_{H\times H} = Re(\Phi, \Psi)_X,$$

and that A_0 is skew-adjoint, i.e.,

$$(A_0\Psi, \hat{\Psi})_{H\times H} = -(A_0\hat{\Psi}, \Psi)_{H\times H} \text{ for all } \Psi, \hat{\Psi} \in dom\,(A_0).$$

Thus by Stone's theorem [P], A_0 generates C_0-group on X and $|S(t)\Psi_0|_X = |\Psi_0|_X$. Let $V = dom\,(\mathcal{H}_0^{\frac{1}{2}})$ and $X_2 = V \times V$. Then $\mathcal{H}_0 \in \mathcal{L}(V, V^*)$ with $V^* = dom\,(\mathcal{H}_0^{-\frac{1}{2}})$ and V is equipped with

$$|\phi|_V^2 = \langle \mathcal{H}_0\phi, \phi\rangle_{V^*\times V}$$

as norm. The restriction of $S(t)$ to X_2 defines a C_0 group.

Associated to the self-adjoint operator $\mu \in \mathcal{L}(H)$ we define the skew-adjoint operator

$$B = \begin{pmatrix} 0 & \mu \\ \\ -\mu & 0 \end{pmatrix}$$

Then for $\epsilon \in L^2(0,T)$ there exists a unique mild solution $\Psi(t) \in C(0,T;X)$ to

(2.1) $$\Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)\epsilon(s)\,B\Psi(s)\,ds, \quad t \in [0,T],$$

and

(2.2) $$\frac{d}{dt}\Psi = A_0\Psi(t) + \epsilon(t)B\Psi(t) \quad \text{in } (dom\,(A_0))^*,$$

where

$$(dom\,(A_0))^* = dom\,(\mathcal{H}_0^{-1}) \times dom\,(\mathcal{H}_0^{-1}),$$

[IK], Chapter2, [P], Chapter 4. Equivalently

$$\frac{d}{dt}\Psi(t) = -i\,(\mathcal{H}_0\Psi(t) + \epsilon(t)\mu\Psi(t)) \text{ in } dom(\mathcal{H}_0^{-1})$$

Since $\mathcal{O}(t) \in C(0,T;dom\,(A_0)) \cap C^1(0,T;X)$, we have

$$\frac{d}{dt}\mathcal{O}(t) = -i\,\mathcal{H}_0\mathcal{O}(t) \text{ in } H. \tag{2.3}$$

Thus,

$$\frac{d}{dt}\mathrm{Re}\,(\mathcal{O}(t),\Psi(t))_X = \mathrm{Re}\,\big((-i\,\mathcal{H}_0\mathcal{O}(t),\Psi(t))_X + (\mathcal{O}(t),-i(\mathcal{H}_0\Psi(t)+\epsilon(t)\mu\Psi(t))_X\big)$$

$$= \mathrm{Re}\big(\,i\,\epsilon(t)\,(\mathcal{O}(t),\mu\Psi(t))\big)_X = -\epsilon(t)\,\mathrm{Im}\,(\mathcal{O}(t),\mu\Psi(t))_X,$$

which proves (1.7). Thus, we obtain the closed loop system of the form

$$\Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)F(\Psi(s),\mathcal{O}(s))B\Psi(s)\,ds \tag{2.4}$$

We show that (2.4) has a solution. Let sign(u) be the maximal monotone function

$$\mathrm{sign}(u) = \begin{cases} [-1\ \ 1] & u = 0 \\ \dfrac{u}{|u|} & |u| > 0 \end{cases}$$

and sign$_\delta(u)$ be the Yosida approximation of sign(u) for $\delta > 0$:

$$\mathrm{sign}_\delta(u) = \begin{cases} \dfrac{u}{\delta} & |u| \le \delta \\ \dfrac{u}{|u|} & |u| \ge \delta. \end{cases}$$

Define the F_δ by

$$F_\delta(\Psi,\mathcal{O}) = -\frac{1}{\alpha}(u + \beta\,\mathrm{sign}_\delta(u)V^\gamma), \quad u = \mathrm{Im}\,(\mathcal{O}(t),\mu\Psi). \tag{2.5}$$

Then assuming $V \geq c > 0$ F_δ is Lipschitz continuous and $|F_\delta| \leq M$ for all $|\Psi|_X = |\mathcal{O}|_X = 1$. Thus, it can be proved [IK],[IK2] that

$$\Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)F_\delta(\Psi(s), \mathcal{O}(s))\,ds$$

has the unique solution $\Psi_\delta \in C(0,T;X)$ and

$$V_\delta(t) = V(0) - \int_0^t \frac{1}{\alpha}(|u_\delta(s)|^2 + \beta\,\mathrm{sign}_\delta(u_\delta)(s)V_\delta(s)^\gamma)u_\delta(s)\,ds$$

where

$$V_\delta(s) = 1 - \mathrm{Re}\,(\Psi_\delta(s), \mathcal{O}(s)), \quad u_\delta(s) = \mathrm{Im}\,(\mathcal{O}(s), \mu\Psi_\delta(s))$$

It follows from [BMS], Theorem 3.6 that there exists a subsequence δ for which Ψ_δ converges Ψ in $C(0,T;X)$. Thus $u_\delta(t) \to u(t) = \mathrm{Im}\,(\mathcal{O}(t))(t), \mu\Psi(t))$ strongly in $L^2(0,T;R)$. Since $\mathrm{sign}_\delta(u_\delta(t)) \in L^2(0,T;R)$ there exists a subsequence of δ such that $\mathrm{sign}_\delta(t) \to z(t)$ weakly in $L^2(0,T;R)$. Since sign is maximal monotone $z(t) = \mathrm{sign}(u(t))$ and since $V_\delta(t) \to V(t)$ in $C(0,T;R)$, $\epsilon_\delta(t) \to \epsilon(t) = u(t) + \mathrm{sign}(u(t))V(t)^\gamma$ weakly in $L^2(0,T;R)$. For all $\phi \in X$

$$(\Psi_\delta(t), \phi) = (S(t)\Psi_0, \phi) + \int_0^t \epsilon_\delta(t)(S(t-s)B\Psi_\delta(s), \phi)\,ds$$

and letting $\delta \to 0^+$ we have

$$(\Psi(t), \phi) = (S(t)\Psi_0, \phi) + \int_0^t \epsilon(t)(S(t-s)B\Psi(s), \phi)\,ds$$

which implies Ψ is the mild solution to (2.1) that corresponds to ϵ. Moreover, we have

$$V(t) = V(0) - \int_0^t \frac{1}{\alpha}\,(|u(s)|^2 + \beta\,|u(s)|V(s)^\gamma)\,ds. \tag{2.6}$$

3 Optimality

We argue that

$$V(t,\Psi) = 1 - (\mathcal{O}(t),\Psi)_{H\times H}$$

satisfies the Hamilton Jacobi equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \min_{\epsilon}[V_\Psi(A_0\Psi + \epsilon B\Psi)] \\ &\qquad + \frac{\alpha}{2}|\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + \frac{1}{\alpha}(\frac{1}{2}|u(t)|^2 + |u(t)|V^\gamma) = 0 \qquad (3.1) \\ &u(t) = \mathrm{Im}\,(\mathcal{O}(t),\mu\Psi) = -(\mathcal{O}(t),B\Psi)_{H\times H} \end{aligned}$$

where

$$V_\Psi(\Phi) = -(\mathcal{O}(t),\Phi)_{H\times H}.$$

In fact,

$$\begin{aligned} &\frac{\alpha}{2}|\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + u(t)(\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma) + \frac{1}{2\alpha}|u(t)|^2 \\ &= \frac{\alpha}{2}|\epsilon + \frac{1}{\alpha}(u(t) + \beta\,\mathrm{sign}(u(t))V^\gamma)|^2, \qquad (3.2) \end{aligned}$$

and thus $\epsilon^*(t)$ minimizes

$$\frac{\alpha}{2}|\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + u(t)(\epsilon + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma) + \frac{1}{2\alpha}|u(t)|^2.$$

This implies

$$\begin{aligned} &\frac{\partial V}{\partial t} + V_\Psi(A_0\Psi + \epsilon^* B\Psi) \\ &\quad + \frac{\alpha}{2}|\epsilon^*(t) + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + +u(t)(\frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + \frac{1}{2\alpha}|u(t)|^2 \\ &= -(A_0\mathcal{O}(t),\Psi)_{H\times H} - (\mathcal{O}(t), A_0\Psi + \epsilon^*(t)B\Psi)_{H\times H} \\ &\quad + \frac{\alpha}{2}|\epsilon^*(t) + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma|^2 + u(t)(\epsilon^*(t) + \frac{\beta}{\alpha}\mathrm{sign}(u(t))V^\gamma)|^2 + \frac{1}{2\alpha}|u(t)|^2 = 0 \end{aligned}$$

as desired.

We next show that ϵ^* minimizes

$$J(\epsilon) = \int_0^T (\frac{\alpha}{2}|\epsilon(t)+\frac{\beta}{\alpha}\text{sign}(u(t))V(t)^\gamma|^2+\frac{1}{\alpha}(\frac{1}{2}|u(t)|^2+\beta\,|u(t)|)\,dt+V(\Psi(t),\mathcal{O}(T),$$

over $\epsilon \in L^2(0,T)$. To this end choose any $\epsilon \in L^2(0,T)$ and let $\Psi(t) \in C(0,T;X)$ be the solution to (2.1)-(2.2). Since $\mathcal{O}(t) \in C^1(0,T;X)\cap C(0,T;dom(A_0)$ we have

$$\frac{d}{dt}V(\mathcal{O}(t),\Psi(t)) = -(A_0\mathcal{O}(t),\Psi(t))_{H\times H} - (\mathcal{O}(t),A_0\Psi(t)+\epsilon(t)B\Psi(t))_{H\times H}.$$

Integrating this over $(0,T)$ and using (3.2) we find

$$V(\Psi(T),\mathcal{O}(T)) + \int_0^T (\frac{\alpha}{2}|\epsilon(t)+\frac{\beta}{\alpha}\,\text{sign}(u(t))V(t)^\gamma|^2+\frac{1}{\alpha}(\frac{1}{2}|u(t)|^2+\beta\,|u(t)|V(t)^\gamma)\,dt$$

$$= V(\Psi(0),\mathcal{O}(0)) + \int_0^T \frac{\alpha}{2}\,|\epsilon(t)+\frac{1}{\alpha}(u(t)+\beta\,\text{sign}(u(t))V(t)^\gamma|^2\,dt$$

$$u(t) = -(\mathcal{O}(t),B\Psi(t))_{H\times H}$$

Hence

$$\epsilon^*(t) = F(\Psi^*(t),\mathcal{O}(t)).$$

where $\Psi^*(t)$ is the trajectory corresponding to $\epsilon^*(t)$ minimizes $J(\epsilon)$ over $L^2(0,T)$.

4 Operator Splitting and Numerical Methods

Since the Hamiltonian is the sum of $\mathcal{H}_0$ and $\epsilon(t)\mu$ it is very natural to consider time integration based on the operator splitting method. For the stepsize $h>0$ consider the Strang splitting method:

$$\frac{\hat{\Psi}_{k+1}-\hat{\Psi}_k}{h} = \epsilon_k\,B\frac{\hat{\Psi}_{k+1}+\hat{\Psi}_k}{2}, \quad \hat{\Psi}_k = S(\frac{h}{2})\Psi_k, \tag{4.1}$$

$$\Psi_{k+1} = S(\tfrac{h}{2})\hat{\Psi}_{k+1},$$

where

$$\epsilon_k = \frac{1}{h}\int_{kh}^{(k+1)h} \epsilon(s)\,ds.$$

For time integration of the controlled Hamiltonian we employ the Crank-Nicolson scheme since it is a norm preserving scheme. In fact, since B is skew adjoint

$$(\frac{\Psi_{k+1}-\hat{\Psi}_k}{h}, \Psi_{k+1}+\hat{\Psi}_k)_X = 0,$$

and thus $|\Psi_{k+1}|_X^2 = |\hat{\Psi}_k|_X^2$. The Strang splitting is of second order as time-integration. We have the convergence of (4.1);

Theorem 4.1 If we define $\Psi_h(t) = \Psi_k$ on $[kh,(k+1)h)$, then

$$|\Psi_h(t)-\Psi(t)|_X \to 0 \text{ uniformly in } t\in[0,T]$$

where $\Psi(t)$, $t\geq 0$, satisfies

$$\Psi(t) = S(t)\Psi_0 + \int_0^t S(t-s)\epsilon(s)B\Psi(s)\,ds.$$

Proof. Define the one step transition operator

$$\Psi_{k+1} = T_h(t)\Psi_k$$

by

$$T_h(t) = S(\frac{h}{2})(I-\frac{\epsilon_k h}{2}B)^{-1}(I+\frac{\epsilon_k h}{2}B)S(\frac{h}{2})\Psi.$$

Then, $|T_h(t)\Psi|_X = |\Psi|_X$ and

$$A_h(t)\Psi = \frac{T_h(t)\Psi-\Psi}{h} = S(\frac{h}{2})\frac{J_{h/2}(\epsilon_k B)-I}{h/2}S(\frac{h}{2})\Psi + \frac{S(h)\Psi-\Psi}{h}$$

where

$$J_{h/2}(\epsilon_k B) = (I-\frac{\epsilon_k h}{2}B)^{-1}.$$

Since for $\Psi\in X$

$$\lim_{h\to 0^+}\frac{J_{h/2}(\epsilon_k B)-I}{h/2}\Psi = \epsilon(t)B\Psi$$

and for $\Psi \in dom\,(A)$

$$\lim_{h\to 0^+} \frac{S(h)\Psi - \Psi}{h} = A_0\Psi,$$

we have for $\Psi \in dom\,(A)$ and $\epsilon \in C(0,T)$

$$|A_h(t)\Psi - (A_0\Psi + \epsilon(t)B)\Psi)|_X \to 0 \text{ as } h \to 0^+.$$

It thus follows from the Chernoff theorem [IK] that $|\Psi_h(t) - \Psi(t)|_X \to 0$ uniformly in $t \in [0,T]$.

Note that

$$\Psi_{k+1} = S(h)\Psi_k + hS(\frac{h}{2})\epsilon_k J_{h/2}(\epsilon_k B)S(\frac{h}{2})\Psi_k$$

and thus

$$\Psi^m = S(mh)\Psi_0 + \sum_{k=1}^{m} h\,S((m-k)h)S(\frac{h}{2})\epsilon_k B J_{h/2}(\epsilon_k B)S(\frac{h}{2})\Psi_{k-1}.$$

Thus, letting $h \to 0$ in this expression, $\Psi(t) \in C(0,T;X)$ satisfies (2.1). □

Suppose for (4.1) there exists an ϵ_k on $[kh,(k+1)h)$ such that for $\mathcal{O}_{k+1/2} = S(\frac{h}{2})\mathcal{O}_k$

(4.2)
$$\epsilon_k = F(\Psi_{k+1/2}, \mathcal{O}_{k+1/2}) = \frac{1}{\alpha}(u_{k+1/2} + \beta\,\mathrm{sign}(u_{k+1/2})V_k^\gamma),$$
$$u_{k+1/2} = (\mathcal{O}_{k+1/2}, B\Psi_{k+1/2}), \quad \Psi_{k+1/2} = \frac{\hat{\Psi}_{k+1} + \hat{\Psi}_k}{2}.$$

Then Ψ_k satisfies closed loop system

(4.3)
$$\frac{\hat{\Psi}_{k+1} - \hat{\Psi}_k}{h} = \epsilon_k\,B\frac{\hat{\Psi}_{k+1} + \hat{\Psi}_k}{2}, \quad \hat{\Psi}_k = S(\frac{h}{2})\Psi_k,$$
$$\epsilon_k = F(\Psi_{k+1/2}, \mathcal{O}_{k+1/2}), \quad \Psi_{k+1} = S(\frac{h}{2})\hat{\Psi}_{k+1}.$$

Since

$$V(S(\frac{h}{2})\hat{\Psi}_{k+1}, S(\frac{h}{2})\mathcal{O}_{k+1/2}) = V(\hat{\Psi}_{k+1}, \mathcal{O}_{k+1/2}),$$

the discrete analog of (2.6)

$$V(\Psi_{k+1},\mathcal{O}_{k+1})=V(\Psi_k,\mathcal{O}_k)+\frac{1}{\alpha}(|u_k|^2+\beta\,|u_k|V(Psi_k,\mathcal{O}_k))^\gamma.$$

holds for the closed loop (4.3).

Now, we show that there exists a unique ϵ_k that satisfies (4.2). Let

$$\chi(u)=u+\beta\,\mathrm{sign}(u)$$

Then, it is equivalent to find $\epsilon\in R$ that satisfies

$$\chi^{-1}(\alpha\,\epsilon)=(B\hat{\Psi}(\epsilon),\mathcal{O}_{k+1/2}), \tag{4.4}$$

where $\hat{\Psi}(\epsilon)$ is the solution to

$$\frac{\hat{\Psi}-\hat{\Psi}_k}{h}=\epsilon B\frac{\hat{\Psi}-\hat{\Psi}_k}{2}.$$

Note that

$$\hat{\Psi}-\hat{\Psi}_k=h\epsilon\,B(I-\frac{h\epsilon}{2}B)^{-1}\hat{\Psi}_k$$

and thus

$$(B\hat{\Psi}(\epsilon),\mathcal{O}_{k+1/2})=(B\hat{\Psi}_k,\mathcal{O}_{k+1/2})+h\epsilon(B^2(I-\frac{h\epsilon}{2}B)^{-1}\hat{\Psi}_k,\mathcal{O}_{k+1/2})$$

Since

$$B^2=\begin{pmatrix}-\mu^2 & 0\\ 0 & -\mu^2\end{pmatrix}$$

one can assume that there exists $c>0$ for all k

$$(B^2\hat{\Psi}_k,\mathcal{O}_{k+1/2})\le -c$$

Thus,

$$(B^2(I-\frac{h\epsilon}{2}B)^{-1}\hat{\Psi}_k,\mathcal{O}_{k+1/2})\le -\frac{c}{2},$$

for $h>0$ is sufficiently small and hence (4.4) has a unique solution.

5 Asymptotic Tracking

The objective of this section is to analyze the asymptotic properties of the controlled system (1.1). Let is of the form

$$\mathcal{O}(t) = e^{-i(\lambda_{k_0} t - \hat{\theta})} \psi_{k_0}$$

for some eigenpair $(\lambda_{k_0}, \psi_{k_0})$ of $\mathcal{H}_0$ and phase $\hat{\theta}$. We assume that

$$\mu_{k_0}^k = (\psi_{k_0}, \mu\psi_k)_X \neq 0 \text{ for all } k = 1, 2, \ldots, \tag{5.1}$$

and that

$$\{S(t)\Psi_0, \; \geq 0\} \text{ is compact in } H \times H. \tag{5.2}$$

Assumption (5.2) holds, for example if $dom(\mathcal{H}_0)$ is compact in H and $\psi_0 \in V \times V$. In case Ω is unbounded we may assume that $W = V \cap L^p(\Omega), p > 2$, is compactly embedded in $H = L^2(\Omega)$. Then, if $\Psi_0 \in W \times W$ and $S(t)$ leaves $W \times W$ invariant [IK1], we have (5.2). $V(t) \geq 0$, it follows from (2.6) that either $V(t) \to \infty$ or $\int_0^\infty |u(t)|\, de < \infty$. We also assume that

$$\int^\infty |\epsilon(t)|\, dt < \infty \tag{5.3}$$

This assumption holds if either we use the regularized feedback law (2.5) for arbitrary $\delta > 0$ or $\beta = 0$. Thus,

$$\lim_{t\to\infty} \int_0^t S(t-s)\epsilon(s)B\Psi(s)\, ds \text{exists.}$$

It follows that $\{\int_0^t S(t-s)\epsilon(s)B\psi(s) : t \geq 0\}$ is compact in $H \times H$. Together with (5.2) we conclude that $\{\Psi(t) : t \geq 0\}$ is compact. We shall proceed with the asymptotic analysis utilizing assumptions (5.1)–(5.3) and summarize the results in a theorem at the end.

Since $\{\Psi(t) : t \geq 0\}$ and $\{\mathcal{O}(t) : t \geq 0\}$ are compact in X there exists a sequence $\{t_n\} \to \infty$ and elements $\Psi_\infty \in X$, $\mathcal{O}_\infty \in X$ such that

$$\lim_{n\to\infty} \Psi(t_n) = \Psi_\infty \text{ and } \lim_{n\to\infty} \mathcal{O}(t_n) = \mathcal{O}_\infty, \tag{5.4}$$

in particular, $\Psi_\infty, \mathcal{O}_\infty$ are in the $\omega-$ limit sets of (2.2) and (2.3), respectively. Since $\epsilon \in L^2(0,\infty)$ it follows from (2.1) that $\Psi(t_n+\tau) \to S(\tau)\Psi_\infty$ and analogously $\mathcal{O}(t_n+\tau) \to S(\tau)\mathcal{O}_\infty$ uniformly with respect to $\tau \in (0,\infty)$. Here $S(\tau)\Psi_\infty$ and $S(\tau)\mathcal{O}_\infty$ are the mild solutions to

$$\frac{d}{dt}\Psi_\infty(t) = A_0\Psi_\infty(t), \quad \Psi_\infty(0) = \Psi_\infty,$$

$$\frac{d}{dt}\mathcal{O}_\infty(t) = A_0\mathcal{O}_\infty(t), \quad \mathcal{O}_\infty(0) = \mathcal{O}_\infty.$$

Moreover

$$\Psi_\infty(\tau) = \sum_{k=1}^{\infty} A_k e^{-i(\lambda_k\tau-\theta_k)}\psi_k,$$

$$\mathcal{O}_\infty(\tau) = e^{-i(\lambda_{k_0}\tau-\tilde{\theta}_{k_o})}\psi_{k_0},$$

with $0 \le \theta_k,\ \tilde{\theta}_{k_0} < \pi$ and $\sum |A_k|^2 = 1$. Since

$$u(t_n+\cdot) = \mathrm{Im}\,(\mathcal{O}(t_n+\cdot), \mu\Psi(t_n+\cdot) \to 0 \text{ in } L^2(0,\infty), \text{ as } t_n \to \infty,$$

we have

$$u(\tau) = \mathrm{Im}\,(\mathcal{O}_\infty(\tau), \mu\Psi_\infty(\tau)) = 0, \text{ for } \tau \ge 0. \tag{5.5}$$

It follows now that
(5.6)

$$u(\tau) = \mathrm{Im}\,(\sum_{k=1}^{\infty} A_k e^{i((\lambda_k-\lambda_{k_0})\tau-\theta_k+\tilde{\theta}_{k_0}))}\mu_{k_0}^k)$$

$$= \sum_{k=1}^{\infty} \mu_{k_0}^k A_k \Big(\cos(\theta_k-\tilde{\theta}_{k_0})\sin((\lambda_k-\lambda_{k_0})\tau) - \sin(\theta_k-\tilde{\theta}_{k_0})\cos((\lambda_k-\lambda_{k_0})\tau)\Big) = 0.$$

Suppose the family

$$\{\cos((\lambda_k-\lambda_{k_0})\tau), \sin((\lambda_k-\lambda_{k_0})\tau)\} \text{ is } \omega- \text{ independent in } L^2(0,T), \tag{5.7}$$

i.e., a family $\{\varphi_k\}_{k=-\infty}^{\infty}$ is called ω-independent if $\sum_{k=-\infty}^{\infty} c_k\varphi_k = 0$ implies that $c_k = 0$ for all k. Then, $\mu_{k_0}^k A_k = 0$ for $k \ne k_0$ and $\mu_{k_0}^{k_0} A_{k_0} - \sin(\theta_k - \tilde{\theta}_{k_0}) =$

0, Thus, by (5.1) $A_k = 0$ for $k \neq k_0$. Moreover, since $|\Psi_\infty| = 1$, we have $\theta_{k_0} = \tilde{\theta}_{k_0}$ and $A_{k_0} = 1$. Here the case $A_{k_0} = -1$ can be excluded since it implies that

$$V(\Psi_\infty(\tau), \mathcal{O}_\infty(\tau)) = 1 + \mathrm{Re}\,(e^{-i(\lambda_{k_0}\tau-\theta_{k_0})}\psi_{k_0}, e^{-i(\lambda_{k_0}\tau-\theta_{k_0})}\psi_{k_0})_X = 2,$$

and

$$V(\Psi_0, \mathcal{O}(0)) = 1 - \mathrm{Re}\big(e^{i\tilde{\theta}_{k_0}}\psi_{k_0}, \Psi_0\big)_X = 1 - Re\big(e^{i\tilde{\theta}_{k_0}}(\psi_{k_0}, \Psi_0)_X\big) < 2,$$

since $\tilde{\theta}_{k_0} \in [0, \pi)$. Hence, $A_{k_0} = -1$ is impossible since $\frac{d}{dt}V(\Psi(t), \mathcal{O}(t)) \le 0$.

Since the ω-limit pair $(\Psi_\infty, \mathcal{O}_\infty)$ was arbitrary it follows from (1.4) that $\lim_{t\to\infty} V(\Psi(t), \mathcal{O}(t)) = 0$, i.e. $\Psi(t)$ asymptotically approaches the orbit $\mathcal{O}(t)$. We summarize the above discussion as;

Theorem 5.1. *Assume that* (5.1)*,* (5.2) *and* (5.7) *hold. Then* $\lim_{t\to\infty} V(\Psi(t), \mathcal{O}(t)) = 0$*, for the feedback law* F*.*

The following lemma addresses condition (5.7).

Lemma 5.1. *If there exits a constant* $\delta > 0$ *such that* $|\lambda_k + \lambda_\ell - 2\lambda_{k_0}| \ge \delta$ *for all* $k, \ell \ge 1$ *with* $\ell \neq k_0$*, and* $|\lambda_k - \lambda_\ell| \ge \delta$ *for all* $k \neq \ell$*, then* $\{e^{i(\lambda_k-\lambda_{k_0})\tau}\} \cup \{e^{-i(\lambda_k-\lambda_{k_0})\tau}\}_{k\neq k_0}$ *is* $\omega-$ *independent for sufficiently large* $T > 0$*.*

Proof. Let $\{\mu_\ell\}_{\ell\in I}$ be a real number sequence defined by

$$\mu_k = \lambda_k - \lambda_{k_0}, \quad k \ge 1, \quad \mu_{-k} = -(\lambda_k - \lambda_{k_0}) \quad k \neq k_0.$$

It follows from the assumption that

$$|\mu_m - \mu_\ell| \ge \delta, \quad m \neq \ell$$

From the Ingham's theorem [I], if $T > \frac{2\pi}{\delta}$, there exits a constant c, depending on T and $\delta > 0$ such that

$$c \sum_{m\in I} |a_m|^2 \le \int_0^T f(\tau)|^2\, d\tau$$

for

$$f(\tau) = \sum_{m \in I} a_m e^{i\mu_m \tau}.$$

□

Remark 5.1. For the harmonic oscillator case we have

$$\mathcal{H}_0 \psi = -\frac{d^2}{dx^2}\psi + x^2 \psi, \quad x \in \mathbb{R} = \Omega.$$

Then the eigen-pairs $\{(\lambda_k, \psi_k)\}_{k=1}^{\infty}$ are given by

$$\lambda_k = 2k - 1, \quad \psi_k(x) = \hat{c}\, H_{k-1}(x) e^{-\frac{x^2}{2}}$$

where H_k is the Hermite polynomial of degree k and $\hat{c}$ is a normalizing factor. In this case we have

$$\lambda_{k_0-\ell} - \lambda_{k_0} = -(\lambda_{k_0+\ell} - \lambda_{k_0}), \quad 1 \le \ell \le k_0 - 1,$$

and the gap condition $|\lambda_k + \lambda_\ell - 2\lambda_{k_0}| > \delta$ is not satisfied. Thus, $\displaystyle\int_0^T |u(\tau))|^2 \, d\tau = 0$ implies

$$\operatorname{Im}\left(A_{k_0+\ell} e^{i(\lambda_\ell \tau - \theta_{k_0+\ell} + \tilde{\theta}_{k_0})} \mu_{k_0}^{k_0+\ell} + A_{k_0-\ell} e^{-i(\lambda_\ell \tau - \theta_{k_0-\ell} + \tilde{\theta}_{k_0})} \mu_{k_0}^{k_0-\ell}\right) = 0$$

for $1 \le \ell < k_0$. That is, $A_{k_0-\ell}$ and $A_{k_0-\ell}$ are not necessary zero and thus $\Psi_\infty(\tau)$ is distributed over energy levels $1 \le \ell \le 2k_0 - 1$.

5.1 Degenerated Case

We now turn to the case when the gap condition $|\lambda_k + \lambda_\ell - 2\lambda_{k_0}| > \delta$ is violated. Then more than one control operator μ is required and we consider (1.10). Then for $V(\Psi, \mathcal{O}) = 1 - \operatorname{Re}(\mathcal{O}, \Psi)_X$ we find

$$\frac{d}{dt} V(\Psi(t), \mathcal{O}(t)) = \sum_{j=1}^{m} \epsilon_j Im(\mathcal{O}(t), \mu_j \Psi(t))_X,$$

which suggests feedback laws of the form

$$\epsilon_j(t) = -\frac{1}{\alpha}(u_j(t) + \beta\,\text{sign}(u_j(t))V(t)^\gamma), \quad u_j(t) = \text{Im}\,(\mathcal{O}(t), \mu_j\Psi(t)). \tag{5.8}$$

As shown above, we have

$$V(t) - V(0) = -\frac{1}{\alpha}\int_0^t \sum_{j=1}^m (|u_j(s)|^2 + \beta\,|u_j(s)|V(s)^\gamma)\,ds.$$

In the following discussion we assume (5.2) i.e. that $\{S(t)\Psi_0 : t \geq 0\}$ is compact. Then using exactly the same arguments as above for all $\omega-$ limit $\Psi_\infty(\tau) : tau \geq 0\}$

$$u_j(\tau)) = \text{Im}\big(\mathcal{O}_\infty(\tau), \mu_j\Psi_\infty(\tau)\big) = 0, \text{ for } \tau \geq 0, j = 1, \ldots, m.$$

Thus,

$$\text{Im}\Big(\sum_{k=1}^\infty A_k e^{i((\lambda_k - \lambda_{k_0})\tau - \theta_k + \tilde{\theta}_{k_0})}(\mu_j)^k_{k_0}\Big) = 0, \text{ for } j = 1, \ldots, m,$$

where

$$(\mu_j)^k_{k_0} = (\psi_{k_0}, \mu_j\psi_k)_X.$$

We henceforth consider the case $m = 2$. Suppose that $\lambda_{\bar{k}} + \lambda_{\bar{\ell}} - 2\lambda_{k_0} = 0$ for single pair $(\bar{k}, \bar{\ell})$, $\bar{\ell} \neq k_0$, and that otherwise (5.7) holds. Then $\lambda_{\bar{k}} - \lambda_{k_0} = -(\lambda_{\bar{\ell}} - \lambda_{k_0})$ and we have

$$\text{Im}\,\big(A_{\bar{k}}e^{i((\lambda_{\bar{k}} - \lambda_{k_0})\tau - \theta_{\bar{k}} + \tilde{\theta}_{k_0})}(\mu_j)^{\bar{k}}_{k_0} + A_{\bar{\ell}}e^{i(-(\lambda_{\bar{k}} - \lambda_{k_0})\tau - \theta_{\bar{\ell}} + \tilde{\theta}_{k_0})}(\mu_j)^{\bar{\ell}}_{k_0}\big) = 0, \tag{5.9}$$

for $j = 1,\ 2$. If

$$\text{rank}\begin{pmatrix} (\mu_1)^{\bar{k}}_{k_0} & (\mu_1)^{\bar{\ell}}_{k_0} \\ (\mu_2)^{\bar{k}}_{k_0} & (\mu_2)^{\bar{\ell}}_{k_0} \end{pmatrix} = 2, \tag{5.10}$$

then from (5.9), it follows that $A_{\bar{k}} = A_{\bar{\ell}} = 0$. If moreover

$$\text{for each } k \text{ there exists } j \in \{1, 2\} \text{ such that } (\mu_j)^k_{k_0} \neq 0, \tag{5.11}$$

then $A_k = 0$ for all $k \neq k_0$, $A_{k_0} = 1$ and $\theta_{k_0} = \tilde{\theta}_{k_0}$. As a consequence we have $\lim_{t\to\infty} V(\Psi(t), \mathcal{O}(t)) = 0$.

In general let

$$\lambda_{k_i} + \lambda_{\ell_i} - 2\lambda_{k_0} = 0$$

for multiple pairs (k_i, ℓ_i) with $\ell_i \neq k_0$. If we assume that

$$\text{rank} \begin{pmatrix} (\mu_1)_{k_0}^{k_i} & (\mu_1)_{k_0}^{\ell_i} \\ (\mu_2)_{k_0}^{k_i} & (\mu_2)_{k_0}^{\ell_i} \end{pmatrix} = 2 \tag{5.12}$$

for each i, then $A_{k_i} = A_{\ell_i} = 0$, and in particular $A_k = 0$ for all k. If in addition (5.11) holds then, again $\lim_{t\to\infty} V(\Psi(t), \mathcal{O}(t)) = 0$.

6 Numerical Tests

In this section we demonstrate the feasibility of our proposed feedback laws using a test example. We set $H = L^2(0,1)$ and

$$\mathcal{H}_0\psi = \sum_{k=1}^{\infty} \lambda_k (\psi, \psi_k)_H \, \psi_k,$$

where

$$\psi_k(x) = \sqrt{2}\sin(k\pi x) \quad \text{and} \quad \lambda_k = k\pi.$$

The control Hamiltonians are given by

$$(\mu_i \Psi)(x) = b_i(x)\Psi(x), \quad x \in (0,1),$$

with $i = 1, 2$. For computations we truncated the expansion of $\mathcal{H}_0$ at $N = 99$, so that

$$S_N(h)\Psi_0 = \sum_{k=1}^{N} e^{-i\lambda_k h} (\Psi_0, \psi_k) \, \psi_k.$$

To integrate the control Hamiltonian term the collocation method was used in the form

$$(B_i^N \psi)(x_n^N) = b_i(x_n^N)\psi(x_n^N), \quad i = 1,\ 2,$$

where $x_n^N = \frac{n}{N}$, $1 \le n \le N-1$. Thus, we implemented the feedback law based on the Strang splitting method in the form

$$\Psi^{k+1} = S_N(\tfrac{h}{2})\mathcal{F}_N(I - \tfrac{\epsilon_1^k h}{2}B_1^N - \tfrac{\epsilon_2^k h}{2}B_2^N)^{-1}(I + \tfrac{\epsilon_1^k h}{2}B_1^N + \tfrac{\epsilon_2^k h}{2}B_2^N)S_N(\tfrac{h}{2})$$

$$\epsilon_i^k = F_i(\Psi^{k+1/2}, \mathcal{O}^{k+1/2}), \quad i = 1,\ 2,$$

where $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$ are the discrete Fourier sine transform and its inverse transform, respectively and B_i^N is the diagonal matrix with diagonal

$$(b_i(x_1^N), \cdots, b_i(x_{N-1}^N) \quad \text{for each } i = 1,\ 2.$$

This is an implicit method and its wellposed-ness is discussed in Section for given $\beta > 0$ and $\gamma \in [0,1]$. The numerical tests that we report on are computed with $h = 0.01$, $\alpha = 1/500$ and

$$b_1(x) = (x - .5) + 1.75(x - .5)^2, \quad b_2(x) = 2.5(x - .5)^3 - 2.5(x - .5)^4.$$

These control potentials satisfy the rank condition in Section 5 and are selected by minimizing the tracking time by trial and error tests. Figure 1 shows the orbit tracking performance $V = \frac{1}{2}|\Psi(t) - \mathcal{O}(t)|_X^2$ comparison between different β and different power γ of V. As β increases, the performance V is significantly improved and the 10 % performance level is achieved in much shorter horizon. By decreasing the power of V, the performance V improves also and more rapidly in the beginning of the time horizon.

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Figure 1: Performance comparison

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