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SFB-Report No. 2015-020   April 2015

This article is a preprint. It was published in Discrete and Continuous Dynamical Systems - Series A by American Institute of Mathematical Sciences. The version of record is available at
http://dx.doi.org/doi:10.3934/dcds.2015.35.4477.

Supported by the Austrian Science Fund (FWF)

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DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **35**, Number **9**, September **2015**

doi:10.3934/dcds.2015.35.xx

pp. **X–XX**

UNIFORM CONVERGENCE OF THE POD METHOD AND APPLICATIONS TO OPTIMAL CONTROL

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(Communicated by the associate editor name)

Abstract. We consider proper orthogonal decomposition (POD) based Galerkin approximations to parabolic systems and establish uniform convergence with respect to forcing functions. The result is used to prove convergence of POD approximations to optimal control problems that automatically update the POD basis in order to avoid problems due to unmodeled dynamics in the POD reduced order system. A numerical example illustrates the results.

1. **Introduction.** Proper orthogonal decomposition (POD) is a powerful Galerkin type technique for model reduction of evolution systems. A POD basis presents an optimal representation of "snapshots" of the dynamical system and it is used to derive reduced-order models (ROM) of the system. From the rich literature on POD, we can only select a few contributions that are in some way related to the present work: [1, 2, 5, 4, 15, 17].

In this work, we focus on the convergence of POD approximations as the number of basis elements tends to infinity. For the case of fixed forcing functions, such error estimates were first obtained in [6]. In the present paper, we derive error estimates for linear abstract parabolic evolution problems that establish the *uniform* convergence of the POD ROM solution with respect to forcing functions and also give a convergence rate.

Using the POD method in the context of optimal control, the problem of "unmodeled dynamics" may arise: The POD basis elements are computed from a reference trajectory which may contain features that are quite different from those of the optimally controlled trajectory. In Optimality Systems POD (OS-POD), this problem is avoided by augmenting the optimality system of the control problem with the POD basis generation criteria. Here, for a linear quadratic control problem, we apply the uniform POD convergence result in order to prove the (weak) convergence of the optimal "OS-POD control" to the full-order optimal control as the dimension of the POD-ROM system goes to infinity.

2010 *Mathematics Subject Classification.* 35K20, 49J20, 49K20, 65K10.

Key words and phrases. Proper orthogonal decomposition, reduced order systems, Galerkin approximations, parabolic equations, optimal control.

The authors were supported in part by the Fonds zur Förderung der wissenschaftlichen Forschung under SFB 32, "Mathematical Optimization and Applications in the Biomedical Sciences".

Finally, we give numerical examples which illustrate that even under bad starting conditions that cause the classical POD method to fail, the OS-POD basis update still leads to satisfactory optimal control results. Moreover, we show that convergence of the OS-POD controls as the dimension of the ROM tends to infinity is also obtained numerically.

The structure of the paper is the following: In Section 2, necessary prerequisites are summarized. Section 3 contains the uniform POD convergence result, which is applied to OS-POD in Section 4. The numerical tests are given in Section 5. The proof of a POD convergence result is deferred to an Appendix.

The paper is based on the second author's thesis [11], which is available under http://media.obvsg.at/p-AC08836134-2001.

2. **Problem Statement and Preliminaries.**

2.1. **Parabolic State Equation.**

Here, we gather some facts on parabolic equations that will be used in the remainder of the paper.

Let V and H be real separable Hilbert spaces and let (V, H, V^*) be a Gelfand triple. In particular, there exists a constant $C_V > 0$ such that

$$\|\cdot\|_H \leq C_V \|\cdot\|_V.$$

Denote the V^*-V-duality pairing by $(\cdot,\cdot)_{V^*,V}$ and let $a : V \times V \to \mathbb{R}$ be a symmetric bilinear form such that a is bounded and coercive, i.e., there exist constants $\beta, \kappa > 0$ such that for all $v, w \in V$, we have

$$|a(v,w)| \leq \beta \|v\|_V \|w\|_V \quad \text{and} \quad a(v,v) \geq \kappa \|v\|_V^2. \tag{1}$$

For $f \in L^2(0,T;V^*)$, $g \in H$ and $0 < T < \infty$, we consider

$$\frac{d}{dt}(y(t),\varphi)_H + a(y(t),\varphi) = (f(t),\varphi)_{V^*,V}, \qquad \varphi \in V, \text{ a.e. } t \in (0,T], \tag{2a}$$

$$(y(0),\varphi)_H = (g,\varphi)_H, \qquad \varphi \in V. \tag{2b}$$

Alternatively, we may introduce the isomorphism $\mathcal{A} : V \to V^*$ such that $a(u,v) = (\mathcal{A}u, v)_{V^*,V}$. (Note that $\mathcal{A}$ presents a linear unbounded self-adjoint operator in H with domain $D(\mathcal{A}) = \{\varphi \in V : \mathcal{A}\varphi \in H\}$.) Then, (2) is equivalent to

$$\frac{d}{dt} y(t) + \mathcal{A}y(t) = f(t) \quad \text{for a.e. } t \in (0,T], \tag{3a}$$

$$y(0) = g. \tag{3b}$$

We recall an existence and regularity result that will be used in Section 3.2.

Proposition 2.1 (Solvability/Regularity)**.** *Problem* (2) *admits a unique solution*

$$y \in W(0,T) := L^2(0,T;V) \cap H^1(0,T;V^*).$$

For an initial value $g \in H$, the solution operator $S_g : L^2(0,T;V^*) \to W(0,T)$, $y = S_g(f)$ is bounded.

If further $g \in V$ and $f \in H^k(0,T;V^*)$ for some $k \in \mathbb{N}$ and they are chosen such that the compatibility conditions $\partial_t^j y(0) \in V$, $j = 0,\dots,k-1$ and $\partial_t^k y(0) \in H$ hold, problem (2) *admits a solution $y \in H^k(0,T;V) \cap H^{k+1}(0,T;V^*)$ and there exists a constant $C > 0$ such that*

$$\|y\|_{H^k(0,T;V)}^2 + \|y\|_{H^{k+1}(0,T;V^*)}^2 \leq C\Big(\|f\|_{H^k(0,T;V^*)}^2 + \sum_{j=0}^{k} \left\|\partial_t^j y(0)\right\|_H^2\Big).$$

Proof. For $k = 0$, the solvability of the problem and the boundedness of S are established in [18, Satz 26.1]. In particular, assumption a) therein is naturally fulfilled since we have assumed a to be constant in time. Furthermore, b) and a special case of c) are given by (1). The additional regularity ($k > 0$) is stated in [18, Satz 27.2]. (Note that assumption d) therein is fulfilled since in our case, a is assumed to be time independent.) The final estimate is given in [9, Satz 8.7], for instance. □

As a special case of the evolution problem above, we consider a parabolic initial-value problem with homogeneous Dirichlet boundary conditions and a right-hand side (RHS) that is *space-time-separable*. Different types of boundary conditions as for instance inhomogeneities in (5b) or Neumann- or Robin-type conditions can be treated provided that regularity and a-priori estimates as in Proposition 2.3 can be established (see [18, Theorem 27.5], for instance).

Problem 2.2 (Initial Boundary Value Problem)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$, choose $q \in \mathbb{N}$ *even* and define the control operator

$$\mathcal{B}_q : \quad \mathbb{R}^M \to V, \quad \mathcal{B}_q(v) = \sum_{k=1}^{M} v_k \, b_k \quad \text{where} \quad b_k \in H_0^{q+1}(\Omega). \tag{4}$$

For $\hat{u} \in L^2(0,T;\mathbb{R}^M)$, $y_0 \in H^{q+1}(\Omega)$ and L a second-order strongly elliptic operator, define the *initial boundary value problem*

$$\frac{d}{dt}y + Ly = u := \mathcal{B}_q\hat{u}, \qquad \text{in } Q := (0,T] \times \Omega, \tag{5a}$$

$$y = 0, \qquad \text{on } \Sigma := (0,T] \times \partial\Omega, \tag{5b}$$

$$y(0) = y_0, \qquad \text{on } \{t = 0\} \times \Omega. \tag{5c}$$

Note that $u = \mathcal{B}_q\hat{u} \in L^2(0,T;H_0^{q+1}(\Omega))$ and $u(t)(x) = \sum_{k=1}^M \hat{u}_k(t)b_k(x)$. We will need additional regularity in space as provided by the following proposition, for which a proof can be found in [11, Proposition 1.1.6].

Proposition 2.3 (Regularity for Space-Time RHS)**.** *Problem* (5) *admits a unique solution*

$$y \in L^2(0,T;H^{q+2}(\Omega)) \cap H^1(0,T;H^q(\Omega)),$$

satisfying the a-priori estimate

$$\begin{aligned} &\|y\|_{L^2(0,T;H^{q+2}(\Omega))} + \left\|\frac{d}{dt}y\right\|_{L^2(0,T;H^q(\Omega))} \\ &\qquad \le C(T)\left(\|\hat{u}\|_{L^2(0,T;\mathbb{R}^M)} \max_{1\le k\le M}\|b_k\|_{H^{q+1}(\Omega)} + \|y_0\|_{H^{q+1}(\Omega)}\right) \end{aligned} \tag{6}$$

with a constant $C(T) > 0$. In other words, for each initial value $y_0 \in H^{q+1}(\Omega)$, the solution operator $S_{y_0} : L^2(0,T;\mathbb{R}^M) \to H^1(0,T;H^q(\Omega))$, $\hat{u} \mapsto y$ is bounded.

2.2. **Optimal Control Problem.** We turn to the optimal control problem with (2) as constraint. It admits a unique solution; cf. [8, III. (2.10)], for instance.

Problem 2.4 (Optimal Control Problem)**.** Choose a target state $z \in L^2(0,T;H)$ and a control operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^M, V^*)$. Set $\mathcal{U} := L^2(0,T;\mathbb{R}^M)$ and let $\mathcal{U}_{ad} \subset \mathcal{U}$ be closed, convex and non-empty. Find $\hat{u}^* \in \mathcal{U}_{ad}$ that solves

$$\min_{\hat{u}\in\mathcal{U}_{ad}} J(\hat{u}, y) := \frac{1}{2}\left\|y - z\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\hat{u}\right\|^2_{L^2(0,T;\mathbb{R}^M)}$$

such that $(\hat{u}, y)$ satisfies (3) with RHS $f = \mathcal{B}\hat{u}$ and initial condition $g \in V$ fixed, i.e., there holds

$$\frac{d}{dt}\, y(t) + \mathcal{A}y(t) = \mathcal{B}(\hat{u}(t)) \quad \text{for a.e. } t \in (0, T], \tag{7a}$$

$$y(0) = g. \tag{7b}$$

In order to characterize an optimal solution, let us quote first order necessary optimality conditions; cf. [8, Theorem III.2.1, Theorem III.2.2], for instance. Since the admissible set and the cost functional are convex, these conditions are also sufficient.

Proposition 2.5 (First Order Optimality Condition)**.** *The pair $x^* = (y^*, \hat{u}^*)$ is the (unique) solution of Problem 2.4 if and only if $\bar{x}$ fulfills the* state equation (7) *and, with the unique Lagrange-multiplier $p \in W(0,T)$, satisfies the following* adjoint equation

$$-\frac{d}{dt}p(t) + \mathcal{A}^*p(t) = y^*(t) - z(t), \quad \textit{for} \quad t \in [0, T] \textit{ a.e.}, \tag{8a}$$

$$p(T) = 0 \tag{8b}$$

as well as the optimality condition

$$\left(G(\hat{u}^*), u - \hat{u}^*\right)_{\mathcal{U}} \geq 0 \quad \textit{for all} \quad u \in \mathcal{U}_{ad} \tag{9}$$

where the operator $G : \mathcal{U} \to \mathcal{U}$ is defined by

$$G(u) = \alpha u + \mathcal{B}^*p.$$

Note that in the unconstrained case $\mathcal{U}_{ad} := \mathcal{U}$, the inequality in (9) *reduces to the equation $G(\hat{u}^*) = 0$, which can be rearranged to $\hat{u}^* = -\frac{1}{\alpha}\mathcal{B}^*p$.*

2.3. **Approximation Numbers of Compact Operators.** For convenience, we briefly recall the theory on approximation numbers of compact operators that shall be used later. Note that we consider real-valued Hilbert spaces. We start off with the definition of singular values and their existence.

Proposition 2.6 (Schmidt Representation of Compact Operators)**.** *Let H and G be Hilbert spaces. For each* compact *linear operator $A : H \to G$, there exist a decreasing zero-sequence $(\sigma_n)_n \subset [0, \infty)$ as well as orthonormal systems $(e_n)_n$ in H and $(f_n)_n$ in G such that*

$$A = \sum_{n=1}^{\infty} \sigma_n \left(\cdot, e_n\right) f_n,$$

where the series converges w.r.t. the operator norm. Furthermore, $(\sigma_n^2)_n$ is the monotonically decreasing sequence of eigenvalues of the operator A^*A.

The existence of the Schmidt representation is proved in [10, Proposition 16.3]. The second assertion follows since $A^*A = \sum_{n=1}^{\infty} \sigma_n^2 \left(\cdot, e_n\right) e_n$.

The sequence $(\sigma_n)_n$ of Proposition 2.6 is called the sequence of *singular values* of the respective compact operator A, which we shall denote by $(\sigma_n(A))_n$. Note that acccording to [10, Lemma 16.5], in our context, the n-th singular value $\sigma_n(A)$ coincides with the n-th approximation number $\zeta_n(A)$ for all $n \in \mathbb{N}$:

$$\zeta_n(A) := \inf\big\{\|A - B\|_{\mathcal{L}(H,G)} \,:\, B \in \mathcal{L}(H,G),\, \dim \operatorname{range}(B) < n\big\} \;=\; \sigma_n(A).$$

The following inequality for singular values will play a crucial role in the proofs of the theorems in Subsection 3.2. A proof can be found in [10, Lemma 16.6 (6)], for instance.

Lemma 2.7 (Courant Fischer Inequality)**.** *Let F, H and G be Hilbert spaces. Let $A : H \to G$ be a* compact *linear operator and $T \in \mathcal{L}(F,H)$. Then, there holds*

$$\sigma_n(AT) \le \sigma_n(A)\, \|T\|_{\mathcal{L}(F,H)} \quad \text{for all } n \in \mathbb{N}.$$

The uniform convergence theorems will also require the following result on Sobolev embeddings:

Lemma 2.8 (Approximation Numbers of Sobolev Embeddings)**.** *Let $\Omega \in \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let be $j, r \in \mathbb{N}$ with $r > 0$ and $j \ge 0$. Then, the embedding*

$$J : H^{j+r}(\Omega) \hookrightarrow H^j(\Omega)$$

is compact. Furthermore, there exists a constant $C_\Omega > 0$ such that we have for all $k \in \mathbb{N}$ that the k-th singular value σ_k of J satisfies

$$\sigma_k(J) \le C_\Omega \frac{1}{k^{r/d}}.$$

In particular, for all $s \in \mathbb{N}$ such that $sr > d$ and $C_{\Omega,r} := \frac{d}{sr-d} C_\Omega$, we have the following convergence result:

$$\sum_{k=\ell+1}^{\infty} (\sigma_k(J))^s \le C_{\Omega,r} \frac{1}{\ell^{(sr/d)-1}} \overset{\ell\to\infty}{\longrightarrow} 0.$$

Proof. The compactness of the embedding map as well as the first assertion on its singular values are given in [14, Theorem 1.107]. According to Remark 1.108 therein, this includes the case of Sobolev embeddings.

For the second assertion, let be $\ell \in \mathbb{N}$. For all $k \in \mathbb{N}$, there holds $\sigma_k(J) \ge 0$ and due to the first assertion, we have for all $m \in \mathbb{N}$ with $m > \ell$ that

$$\sum_{k=\ell+1}^{m} (\sigma_k(J))^s \le C_\Omega \sum_{k=\ell+1}^{m} \frac{1}{k^{sr/d}}.$$

Thus, both limits exist for $m \to \infty$ since $sr/d > 1$. Then, the integral test for convergence yields

$$\frac{1}{C_\Omega} \sum_{k=\ell+1}^{\infty} (\sigma_k(J))^s \le \sum_{k=\ell+1}^{\infty} \frac{1}{k^{sr/d}} \le \int_\ell^\infty \frac{1}{x^{sr/d}}\, \mathrm{d}x = \frac{1}{\frac{sr}{d} - 1} \frac{1}{\ell^{(sr/d)-1}}.$$

Note that the integrand $\frac{1}{x^{sr/d}}$ is monotonically decreasing and positive. ☐

3. Uniform Convergence for Continuous POD.

3.1. **Continuous POD Theory.** In the continuous POD setting that we follow, we consider the trajectories of y_u and $\partial_t y_u$ as our snapshot set. In order to ensure the existence of a POD basis, we construct a suitable POD operator, show the existence of eigenvalues and eigenvectors and then give a definition of a POD basis. Finally, we comment on optimality and convergence properties of the basis. Henceforth, $y_u := y(u)$ stands for the solution of (2) with $f := u \in L^2(0,T;V^*)$. We further assume that

$$y_u \in H^1(0,T;V), \tag{10}$$

which by Proposition 2.1, is satisfied for $u \in H^1(0,T;V^*)$, for instance. We next define the continuous POD problem for the snapshot set

$$\mathcal{V}_u = \{(y_u(t), \partial_t y_u(t)),\ t \in [0,T]\}. \tag{11}$$

Problem 3.1 (Continuous POD Problem)**.** Choose $X \in \{V, H\}$ and fix $\ell \in \mathbb{N}$. Find an orthonormal basis $\mathcal{B}^\ell = \{\psi_k\}_{k=1}^\ell$ that minimizes

$$\frac{1}{T}\int_0^T \Big\| y_u(t) - \sum_{k=1}^{\ell} (y_u(t), \psi_k)_X\, \psi_k \Big\|_X^2 \,\mathrm{d}t + \frac{1}{T}\int_0^T \Big\| \partial_t y_u(t) - \sum_{k=1}^{\ell} (\partial_t y_u(t), \psi_k)_X\, \psi_k \Big\|_X^2 \,\mathrm{d}t.$$

POD Operator. We define the operator $Y_u : (L^2(0,T))^2 \to X$,

$$(Y_u w)(x) = \frac{1}{\sqrt{T}}\int_0^T w_1(t) y_u(t)(x)\,\mathrm{d}t + \frac{1}{\sqrt{T}}\int_0^T w_2(t) \partial_t y_u(t)(x)\,\mathrm{d}t. \tag{12}$$

Let us show that the adjoint of Y_u is given by

$$Y_u^* : \quad X \to (L^2(0,T))^2, \quad (Y_u^* v)(t) = \frac{1}{\sqrt{T}}\left((y_u(t), v)_X\,, (\partial_t y_u(t), v)_X\right). \tag{13}$$

For this purpose, let be $W := L^2(0,T)$, $v \in X$ as well as $w \in W^2$ and observe

$$\begin{aligned}\sqrt{T}\,(Y_u w, v)_X &= ((w_1, y_u)_W + (w_2, \partial_t y_u)_W\,, v)_X \\ &= (w_1, (y_u, v)_X)_W + (w_2, (\partial_t y_u, v)_X)_W \\ &= ((w_1, w_2), ((y_u, v)_X\,, (\partial_t y_u, v)_X))_{W^2} = \sqrt{T}\,(w, Y^* v)_{W^2}\,.\end{aligned}$$

Finally, we set the *POD operator* to be $K_u := Y_u Y_u^* : X \to X$,

$$K_u v = \frac{1}{T}\int_0^T (y_u(t), v)_X\, y_u(t)\,\mathrm{d}t + \frac{1}{T}\int_0^T (\partial_t y_u(t), v)_X\, \partial_t y_u(t)\,\mathrm{d}t$$

and define an auxiliary operator

$$\hat{K}_u := Y_u^* Y_u : \quad (L^2(0,T))^2 \to (L^2(0,T))^2, \quad (\hat{K}_u v)(t) = \int_0^T k_u(s,t) v(s)\,\mathrm{d}s \tag{14}$$

with

$$k_u \in \left(L^2([0,T]^2)\right)^{2\times 2}, \quad k_u(s,t) = \frac{1}{T}\begin{pmatrix} (y_u(t), y_u(s))_X\,, & (y_u(t), \partial_t y_u(s))_X \\ (\partial_t y_u(t), y_u(s))_X\,, & (\partial_t y_u(t), \partial_t y_u(s))_X \end{pmatrix}.$$

Existence of a POD Basis. We now wish to define a POD basis by means of the operator introduced above. Beforehand, we need an auxiliary result on the spectrum of the POD operators such that a POD basis is well-defined.

Proposition 3.2 (Spectra of POD Operators)**.** *The operator K_u possesses at most countably many eigenvalues $\{\lambda_i^u\}_i$. All these eigenvalues are non-negative and can be ordered, taking into account their multiplicities. Furthermore, there exists an* orthonormal *system of eigenvectors*

$$\Psi^u = \{\psi_k^u\}_k \quad \textit{corresponding to} \quad \lambda_1^u \geq \lambda_2^u \geq \ldots \geq 0.$$

Additionally, ψ is an eigenvector of K_u if and only if $Y_u^*\psi$ is an eigenvector of $\hat{K}_u$.

Proof. • Spectrum of $\hat{K}_u$: Let us observe that $\hat{K}_u$ is *self-adjoint*

$$\hat{K}_u^* = (Y_u^* Y_u)^* = Y_u^* Y_u^{**} = Y_u^* Y_u = \hat{K}_u$$

and set $W := (L^2(0,T))^2$ to see that $\hat{K}_u$ is *non-negative*:

$$\big(\hat{K}_u v, v\big)_W = (Y_u^* Y_u v, v)_W = (Y_u v, Y_u v)_X = \|Y_u v\|_X^2 \geq 0 \quad \text{for all } v \in W.$$

Furthermore, k presents an L^2-kernel of $\hat{K}_u$. Thus, by [10, Proposition 16.12], we infer that $\hat{K}_u$ is Hilbert-Schmidt and in particular *compact* (due to [12, Théorème (10,2), 1]).

Hence, by [13, Subsection 4.2.6, Theorem 1], there exist at most countably many eigenvalues. These eigenvalues may be ordered, taking into account their multiplicities, to a non-increasing sequence $(\lambda_k^u)_k$. Since the operator is non-negative, all its eigenvalues are non-negative as well. Furthermore, there exists an orthonormal system of corresponding eigenvectors $V^u = (v_k^u)_k$.

• Spectrum of K_u: Finally, note that $\hat{K}_u$ possesses the same eigenvalues (with identical multiplicities) as K_u, except for possibly zero. This fact is shown in [7, Proposition 2.1], together with the last assertion of the proposition. □

Proposition 3.3 (POD Basis Criterion)**.** *A POD basis Ψ_ℓ^u of order ℓ that solves Problem 3.1 is given by the eigenvectors of K_u corresponding to the first ℓ eigenvalues (taking into account their multiplicities), i.e., we have $\Psi_\ell^u = (\psi_k^u)_{k=1}^\ell$. In other words, a* POD basis criterion *reads:*

$$K_u \psi_i^u = \lambda_i^u \psi_i^u \quad \textit{with} \quad \big(\psi_i^u, \psi_j^u\big)_X = \delta_{ij} \quad \textit{for } 1 \leq i, j \leq \ell. \tag{15}$$

For a proof, refer to [3, Proposition 2.2.3], for instance. Note that if among the first ℓ eigenvalues, an eigenvalue appears k times, any orthonormal basis of the corresponding k-dimensional subspace can be used. Furthermore, we define a projection onto $\mathcal{V}_\ell^u := span(\Psi_\ell^u)$ by

$$P_\ell^{\Psi^u}: \quad X \to \mathcal{V}_\ell^u, \quad P_\ell^{\Psi^u} v := \sum_{k=1}^{\ell} (v, \Psi_k^u)_X \, \Psi_k^u. \tag{16}$$

Remark 3.4 (Optimality and Convergence of POD)**.** Due to the construction, a POD basis approximates the trajectories that it is calculated from better than any other orthonormal basis. Also, this approximation is arbitrarily good for increasing dimension ℓ of the POD subspace $\mathcal{V}_\ell^u$. In particular, for $X \in \{V, H\}$ and a POD basis $\{\psi_k^u\}_{k=1}^\ell$ based on the trajectories of $(y(u), \partial_t y(u))$, for any integer $\ell \geq 1$, there holds for all orthonormal basis $(\varphi_k)_k$ of X that

$$\begin{aligned}\frac{1}{T}&\left\|y(u)-\sum_{k=1}^{\ell}(y(u),\varphi_k)_X\,\varphi_k\right\|^2_{H^1(0,T;X)}\\ &\geq\ \frac{1}{T}\left\|y(u)-\sum_{k=1}^{\ell}(y(u),\psi_k^u)_X\,\psi_k^u\right\|^2_{H^1(0,T;X)}=\sum_{k=\ell+1}^{\infty}\lambda_k^u\ \overset{\ell\to\infty}{\longrightarrow}\ 0.\end{aligned}\tag{17}$$

Continuous POD ROM Problem. For a RHS f, determine a POD basis Ψ^f based on the snapshots set (11). Using $\mathcal{V}_\ell^f := span(\Psi_\ell^f)$ as a test space in problem (2) and choosing a RHS $u \in L^2(0,T;V^*)$ for the reduced problem, we arrive at:

Problem 3.5 (POD ROM)**.** Find the (unique) solution $y_\ell^{\Psi^f}(u) \in C(0,T;\mathcal{V}_\ell^f)$ to the POD reduced-order model for problem (2), such that for a.e. $t \in (0,T]$:

$$\frac{d}{dt}\left(y_\ell^{\Psi^f}(u)(t),\varphi\right)_H + a\big(y_\ell^{\Psi^f}(u)(t),\varphi\big) = (u(t),\varphi)_{V',V}\,, \qquad \varphi\in\mathcal{V}_\ell^f, \tag{18a}$$

$$\left(y_\ell^{\Psi^f}(u)(0),\varphi\right)_H = (g,\varphi)_H\,, \qquad \varphi\in\mathcal{V}_\ell^f. \tag{18b}$$

In the case that the RHS f for the determination of the basis is equal to the RHS u of the low-order system, we abbreviate the notation of the low-order solution by

$$y_\ell(u) := y_\ell^{\Psi^u}(u).$$

As the eigenvectors of the POD operators are not uniquely determined necessarily, a POD basis Ψ_ℓ^f need not be unique. However, for $\ell \in \mathbb{N}$ with $\lambda_{\ell+1}^f \neq \lambda_\ell^f$, a "POD ROM problem" is uniquely setup by the snapshot trajectory since the span $\mathcal{V}_\ell^f$ of the POD basis is uniquely determined. Otherwise, uniqueness does not hold necessarily.

Convergence Estimate for POD ROM for Fixed Data. The main part of the following result (estimate (19)) was already derived in [4, Theorem 10]; see also [3, Proposition 3.3.2]. Since the assumptions here are weaker and to verify the remaining part, an independent proof is given in the Appendix. Note also that there is an implicit requirement on the regularity of the RHS u by assuming $y(u) \in H^1(0,T;V)$ such that a POD basis can be constructed. According to Proposition 2.1, it suffices to choose $u \in H^1(0,T;V^*)$, for instance.

Proposition 3.6 (Convergence of POD ROM for Fixed Data)**.** *Fix u and let $y(u) \in H^1(0,T;V)$ be the solution to problem (2) with RHS $f := u$. For $X := V$, construct a POD basis Ψ^u from the trajectories of $(y(u), \partial_t y(u))$ and let $y_\ell(u) := y_\ell^{\Psi^u}(u)$ be a solution to the ℓ-ROM approximation (18) of problem (2) with the same RHS u and ansatz space $\mathcal{V}_\ell^u = span(\Psi_\ell^u)$. Choose $\epsilon > 0$ such that $\gamma_\epsilon := \kappa - \frac{1}{2\epsilon}(C_V + \beta) > 0$.*

Then, for $c_\epsilon := 1 + \frac{\epsilon}{2\gamma_\epsilon}\max(C_V,\beta) > 0$, independent of u, the following estimates hold:

$$\|y(u)-y_\ell(u)\|^2_{L^2(0,T;V)} \leq c_\epsilon T\sum_{k=\ell+1}^{\infty}\lambda_k^u + \frac{1}{2\gamma_\epsilon}\left\|\sum_{k=\ell+1}^{\infty}(g,\psi_k^u)_V\,\psi_k^u\right\|^2_H \overset{\ell\to\infty}{\longrightarrow} 0 \tag{19}$$

as well as

$$\|y(u)-y_\ell(u)\|^2_{L^2(0,T;V)} \leq c_\epsilon T\left(1+\frac{C_V C_H}{2\gamma_\epsilon}\right)\sum_{k=\ell+1}^{\infty}\lambda_k^u \overset{\ell\to\infty}{\longrightarrow} 0 \tag{20}$$

where C_H is the constant of the continuous embedding of $H^1(0,T;V)$ in $C([0,T];H)$.

Remark 3.7 (Why to use $X = V$?)**.** For $X = H$, an error estimate as in Proposition 3.6 cannot be derived in a similar fashion. In fact, note that for $X = H$, a POD representation estimate as in Remark 3.4 does not hold in V, but only in H. Thus, looking at the proof of Proposition 3.6 in the appendix, we try to estimate all terms involving the V-norm in (43) and infer

$$\begin{aligned}\frac{1}{C_V}\left(\kappa - \frac{\beta}{2\epsilon}\right)\|\xi_\ell\|^2_{L^2(0,T;H)} &\le \left(\kappa - \frac{\beta}{2\epsilon}\right)\|\xi_\ell\|^2_{L^2(0,T;V)} \\ &\le \frac{\epsilon}{2}\|\partial_t\tilde{y}\|^2_{L^2(0,T;H)} + \frac{1}{2\epsilon}\|\xi_\ell\|^2_{L^2(0,T;H)} + \frac{\beta\epsilon}{2}\|\tilde{y}\|^2_{L^2(0,T;V)} + \frac{1}{2}\|\tilde{y}(0)\|^2_H,\end{aligned}$$

which we may rearrange to

$$\begin{aligned}&\frac{1}{C_V}\left(\kappa - \frac{\beta+1}{2\epsilon}\right)\|\xi_\ell\|^2_{L^2(0,T;H)} \\ &\qquad \le \frac{\epsilon}{2}\|\partial_t\tilde{y}\|^2_{L^2(0,T;H)} + \frac{\beta\epsilon}{2}\|\tilde{y}\|^2_{L^2(0,T;V)} + \frac{1}{2}\|\tilde{y}(0)\|^2_H.\end{aligned}$$

However, the term $\|\tilde{y}\|^2_{L^2(0,T;V)}$ still cannot be "POD estimated" in the case $X = H$. Yet, this term is unavoidable since $a(\cdot,\cdot)$ can only be bounded in the V-Norm. Note that an analogue phenomenon was observed in the discrete context as well (cf. the occurrence of the stiffness matrix in [6, Lemma 3, Theorem 7]).

Remark 3.8 (Why to use POD on $(y(u), \partial_t y(u))$?)**.** Using a POD basis based on solely the trajectory of $y(u)$, we may derive an analogue of estimation (19) – c.f. [4, Theorem 9]. Yet according to (43) in the proof of Proposition 3.6, the RHS of this estimate includes the term $\|\partial_t\tilde{y}(t)\|_H$, which in this case, cannot be estimated by $\sum_{k=\ell+1}^{\infty}\lambda_k^u$.

In [1, Corollary 1], an alternative error estimate is presented which avoids the addition of a time derivative term. However, the estimate includes a constant depending on the ROM dimension ℓ and is hence not suitable for our purpose of deriving a uniform convergence estimate.

Note that according to the previous remarks, the choice $X = V$ and the inclusion of the trajectory of y_t into the snapshot set are necessary for the present way of obtaining a suitable error estimate, yet in numerical applications, these issues do not appear to be essential in general.

3.2. **Uniform POD Convergence.** In standard Galerkin methods, the Galerkin basis is fixed and therefore, error estimates that are uniform w.r.t. varying right-hand sides are standard. This is not the case for the POD reduced-order problem since the POD basis depends on the snapshot set, which in turn depends on the RHS u. Assuming more regularity for the RHS, and hence obtaining more regularity for the solution y, we show that the convergence of the POD low-order solution is actually uniform w.r.t. the RHS u in bounded subsets of a suitable set of functions. Also, we now find a rate of the convergence w.r.t. the ROM dimension.

First, we will assume additional regularity in time and afterwards, we will require additional spatial regularity in an abstract sense. Finally, we consider the more specific Problem 2.2 with space-time-separable RHS.

The additional regularity allows to estimate the eigenvalues of the POD operator by the Courant Fischer inequality. In this way, the sum of eigenvalues in the RHS of

Proposition 3.6 can be estimated by the known decay of singular values of Sobolev embeddings and the a-priori estimate of the solution.

Additional Regularity in Time. The first way for attaining the desired uniform convergence uses additional regularity of the solution in time:

Theorem 3.9 (Uniform POD Convergence, Temporal Regularity). *Let $\mathcal{W}$ and U_g be such that for each $(u,g) \in \mathcal{W} \times U_g$, problem (2) admits a solution $y(u) \in H^2(0,T;V)$. Moreover, assume that there exists a constant $C_a > 0$ such that the a-priori estimate*

$$\|y(u)\|^2_{H^2(0,T;V)} \leq C_a \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \right) \tag{21}$$

holds for all $(u,g) \in \mathcal{W} \times U_g$.

For $X := V$, construct a POD basis Ψ^u from the trajectories of $(y(u), \partial_t y(u))$, depending on u. Let $y_\ell(u) := y_\ell^{\Psi^u}(u)$ be the solution to the ℓ-ROM approximation (18) of problem (2) with the same RHS u and ansatz space $\mathcal{V}_\ell^u = span(\Psi_\ell^u)$.

Then, for each $g \in U_g$, the sequence $(y_\ell(u))_\ell$ converges strongly to $y(u)$ in $L^2(0,T;V)$, uniformly w.r.t. u in bounded subsets of $\mathcal{W}$ for $\ell \to \infty$. In particular, for a constant $C = C(T, \beta, \kappa) > 0$, the following estimate holds true for each $\ell \geq 1$:

$$\|y(u) - y_\ell(u)\|^2_{L^2(0,T;V)} \;\leq\; \frac{C}{\ell} \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \right).$$

Proof. • Introduction of Embedding: Due to the representation of Y_u^* in (13), there holds $Y_u^* v \in (L^2(0,T))^2$ for all $v \in X := V$. Since we assumed $y(u) \in H^2(0,T;V)$, we see that there even holds $Y_u^* v \in (H^1(0,T))^2$ for all $v \in V$. We can thus introduce the operator

$$\tilde{Y_u^*} \;:\; V \to (H^1(0,T))^2, \quad (\tilde{Y_u^*} v)(t) = \frac{1}{\sqrt{T}} \big((y(u)(t), v)_V \,, (\partial_t y(u)(t), v)_V \big)$$

such that there holds for $J : (H^1(0,T))^2 \to (L^2(0,T))^2$ being the compact embedding,

$$Y_u^* = J \tilde{Y_u^*}. \tag{22}$$

• Estimation of λ_k^u: Let $\sigma_k(Y_u^*)$ and $\sigma_k(J)$ denote the k-th singular values of Y_u^* and J, respectively. Setting $A := Y_u^*$ in Proposition 2.6, we have for the eigenvalues λ_k^u of $K_u = Y_u Y_u^* = A^* A$, that for all integers $k \geq 1$, there holds

$$\lambda_k^u = \sigma_k(Y_u^*)^2. \tag{23}$$

Since J is compact and $\tilde{Y_u^*}$ is bounded, we may use Lemma 2.7 and together with (22) and (23), we have for all integers $k \geq 1$:

$$\lambda_k^u = \sigma_k(Y_u^*)^2 = \sigma_k(J\tilde{Y_u^*})^2 \leq \sigma_k(J)^2 \left\| \tilde{Y_u^*} \right\|^2_{\mathcal{L}(V,(H^1(0,T))^2)}. \tag{24}$$

• Estimation of $\sum_{k=\ell+1}^{\infty} \lambda_k^u$: Let $(\sigma_k(I))_k$ be the sequence of singular values of the compact embedding $I : H^1(0,T) \hookrightarrow L^2(0,T)$. Choosing $\Omega = (0,T)$, $j = 0$, $s = 2$ and $d = r = 1$ in Lemma 2.8, we find

$$\sum_{k=\ell+1}^{\infty} \sigma_k(I)^2 \leq C_T \, \frac{1}{\ell}$$

since $sr/d = 2 > 1$. Clearly, the same result holds for $J : (H^1(0,T))^2 \hookrightarrow (L^2(0,T))^2$. Thus, we are allowed to sum over (24) and since the norm term is independent of the summation index k, we arrive at

$$\sum_{k=\ell+1}^{\infty} \lambda_k^u \leq \left\|\tilde{Y_u^*}\right\|^2_{\mathcal{L}(V,(H^1(0,T))^2)} \sum_{k=\ell+1}^{\infty} \sigma_k(J)^2 \leq C_T \frac{1}{\ell} \left\|\tilde{Y_u^*}\right\|^2_{\mathcal{L}(V,(H^1(0,T))^2)}. \tag{25}$$

• Estimation of $||\tilde{Y}_u^*||$: By the definition of $\|\cdot\|_{(H^1(0,T))^2}$, the Cauchy Schwarz inequality and finally by assumption (21), we have for a constant $C_a > 0$, independent of u:

$$\begin{aligned}
T &\left\|\tilde{Y_u^*}\right\|^2_{\mathcal{L}(V,(H^1(0,T))^2)} \\
&= \sup_{v\in V\setminus\{0\}} \frac{1}{\|v\|_V^2} \left\|\left((y(u)(t),v)_V, (\partial_t y(u)(t),v)_V\right)\right\|^2_{(H^1(0,T))^2} \\
&= \sup_{v\in V\setminus\{0\}} \left(\frac{1}{\|v\|_V^2} \int_0^T \left((y(u)(t),v)_V^2 + (\partial_t y(u)(t),v)_V^2\right) \mathrm{d}t \right. \\
&\qquad \left. + \frac{1}{\|v\|_V^2} \int_0^T \left((\partial_t y(u)(t),v)_V^2 + (\partial_{tt} y(u)(t),v)_V^2\right) \mathrm{d}t \right) \\
&\leq \int_0^T \left(\|y(u)(t)\|_V^2 + 2\|\partial_t y(u)(t)\|_V^2 + \|\partial_{tt} y(u)(t)\|_V^2\right) \mathrm{d}t \\
&\leq 2\|y(u)\|^2_{H^2(0,T;V)} \leq 2C_a \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g}\right).
\end{aligned} \tag{26}$$

• Finalization: Note that our assumptions are stronger than those of Proposition 3.6. Thus, we may in particular use (20), estimate the sum term therein by (25) and use (26) to arrive at

$$\begin{aligned}
\|y(u) - y_\ell(u)\|^2_{L^2(0,T;V)} &\leq c_\epsilon T \left(1 + \frac{C_V C_H}{2\gamma_\epsilon}\right) \sum_{k=\ell+1}^{\infty} \lambda_k^u \\
&\leq \underbrace{2C_a C_T c_\epsilon \left(1 + \frac{C_V C_H}{2\gamma_\epsilon}\right)}_{:=C(T,\beta,\kappa)} \frac{1}{\ell} \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g}\right),
\end{aligned}$$

which clearly converges to zero for $\ell \to \infty$, completing the proof. □

Note that from a practical point of view, the estimate of the POD eigenvalues in (25) is very pessimistic; in fact, if it was sharp, the size of low-order models would be too large to be feasible in most cases. However, estimating the u-dependent POD eigenvalues by the u-independent singular values of J implies the uniform convergence of the POD method.

Furthermore, we remark that in the context of optimal control, the estimate of Theorem 3.9 is restrictive due to the regularity requirements for the controls: For the compatibility condition for the initial value g in Proposition 2.1 with $k = 2$, it is sufficient to require $\mathcal{W} = H^1(0,T;V)$ in order to obtain $\mathcal{A}g + u(0) \in V$ and for the second temporal derivative, $g \in D(\mathcal{A}^2)$, $u(0) = u_t(0) = 0$ suffices, for instance. Together with the requirement for the RHS that $u \in H^2(0,T;V^*)$, a sufficient choice for the control space would be $\mathcal{W} = \{u \in H^2(0,T;V^*) \cap H^1(0,T;V) : u(0) = u_t(0) = 0\}$.

Additional Regularity in Space. In a second approach, additional regularity of the solution in space is required. We can then use the same idea of estimation, but make use of a *spatial* Sobolev embedding. (The space $\tilde{U}$ is introduced in order to take care of boundary conditions imposed on y_u, for instance $\tilde{U} := H^2(\Omega) \cap H_0^1(\Omega)$.)

Theorem 3.10 (Uniform POD Convergence, Spatial Regularity)**.** *For a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, set*

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad U := H^{1+r}(\Omega), \quad r \in \mathbb{N},\ r > \frac{d}{2}$$

and let $\tilde{U}$ be a function space continuously embedded into U. Let $\mathcal{W}$ and U_g be such that for each $(u, g) \in \mathcal{W} \times U_g$, problem (2) admits a solution $y(u) \in H^1(0, T; \tilde{U})$. Moreover, assume that there exists a constant $C_a > 0$ such that the a-priori estimate

$$\|y(u)\|^2_{H^1(0,T;\tilde{U})} \leq C_a \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \right) \tag{27}$$

holds for all $(u, g) \in \mathcal{W} \times U_g$.

For $X := V$, construct a POD basis Ψ^u from the trajectories of $(y(u), \partial_t y(u))$, depending on u. Let $y_\ell(u) := y_\ell^{\Psi^u}(u)$ be a solution to the ℓ-ROM approximation (18) of problem (2) with the same RHS u and ansatz space $\mathcal{V}_\ell^u = span(\Psi_\ell^u)$.

Then, for each $g \in U_g$, the sequence $(y_\ell(u))_\ell$ converges strongly to $y(u)$ in $L^2(0, T; V)$, uniformly w.r.t. u in bounded subsets of $\mathcal{W}$. In particular, there exists a constant $C > 0$, independent of ℓ, such that for each $\ell \geq 1$, we have

$$\|y(u) - y_\ell(u)\|^2_{L^2(0,T;V)} \leq \frac{C}{\ell^{(2r/d)-1}} \Big(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \Big).$$

Proof. • Introduction of Embedding: Due to the definition of Y_u in (12), there holds $Y_u v \in X = V$ for all $v \in (L^2(0, T))^2$. Since we assumed $y(u) \in H^1(0, T; \tilde{U})$, there even holds $Y_u v \in \tilde{U}$ for all $v \in (L^2(0, T))^2$. We can thus introduce the operator $\overline{Y_u} : (L^2(0, T))^2 \to \tilde{U}$,

$$(\overline{Y_u} v)(x) = \frac{1}{\sqrt{T}} \left(v_1, y_u(\cdot)(x)\right)_{L^2(0,T)} + \frac{1}{\sqrt{T}} \left(v_2, \partial_t y_u(\cdot)(x)\right)_{L^2(0,T)}$$

and set for the continuous embedding $E_U : \tilde{U} \hookrightarrow U$,

$$\tilde{Y_u} := E_U \overline{Y_u} \ :\ (L^2(0, T))^2 \to U.$$

By the definition of U and V, there exists a compact embedding $E : U \hookrightarrow V$ for which we have

$$Y_u = E\tilde{Y_u}. \tag{28}$$

• Estimation of λ_k^u: Let $\sigma_k(Y_u)$ and $\sigma_k(E)$ denote the k-th singular values of Y_u and E, respectively. According to Proposition 3.2, we have $(\lambda_k^u)_k = (\hat{\lambda}_k^u)_k$, where $(\lambda_k^u)_k$ and $(\hat{\lambda}_k^u)_k$ denote all *non-zero* eigenvalues of K_u and $\hat{K}_u = Y_u^* Y_u$ in (14), respectively. Setting $A := Y_u$ in Proposition 2.6, we have $\hat{K}_u = A^* A$ and hence for all integers $k \geq 1$, there holds

$$\lambda_k^u = \hat{\lambda}_k^u = \sigma_k(Y_u)^2. \tag{29}$$

Since E is compact and $\tilde{Y_u}$ is bounded, we may use Lemma 2.7 and together with (28) and (29), we have for all integers $k \geq 1$:

$$\lambda_k^u = \sigma_k(Y_u)^2 = \sigma_k(E\tilde{Y_u})^2 \leq \sigma_k(E)^2 \big\|\tilde{Y_u}\big\|^2_{\mathcal{L}((L^2(0,T))^2,U)}. \tag{30}$$

Note that estimate (30) is trivially true for all $\lambda_k^u = 0$ since the RHS is non-negative. Hence, the estimate is true for all eigenvalues $(\lambda_k^u)_k$ of K_u.

• Estimation of $\sum_{k=\ell+1}^{\infty} \lambda_k^u$: Recall that $(\sigma_k(E))_k$ denotes the singular values of the compact embedding $E: U = H^{1+r}(\Omega) \hookrightarrow V = H^1(\Omega)$. Choosing $j = 1$ and $s = 2$ in Lemma 2.8, there exists a constant $C_{\Omega,r}$ such that

$$\sum_{k=\ell+1}^{\infty} \sigma_k(E)^2 \leq C_{\Omega,r} \frac{1}{\ell^{(2r/d)-1}}$$

since by the assumption on r, there holds $2r > d$. Thus, we may sum over (30), in which the norm term is independent of the summation index k. Hence, we arrive at

$$\begin{aligned} \sum_{k=\ell+1}^{\infty} \lambda_k^u &\leq \left\| \tilde{Y}_u \right\|^2_{\mathcal{L}((L^2(0,T))^2,U)} \sum_{k=\ell+1}^{\infty} \sigma_k(E)^2 \\ &\leq C_{\Omega,r} \frac{1}{\ell^{(2r/d)-1}} \left\| \tilde{Y}_u \right\|^2_{\mathcal{L}((L^2(0,T))^2,U)} . \end{aligned} \tag{31}$$

• Estimation of $||\tilde{Y}_u||$: Set $W := L^2(0,T)$. Due to the continuous embedding E_U with constant C_U, by definition of $\|\cdot\|_{\mathcal{L}(W^2,U)}$ and $\|\cdot\|_{W^2}$, the Cauchy Schwarz inequality and the fact that for all $v = (v_1, v_2) \in W^2$, there holds $\|v_i\|_W \leq \|v\|_{W^2}, i = 1, 2$, we infer

$$\begin{aligned} &\frac{\sqrt{T}}{C_U} \left\| \tilde{Y}_u \right\|_{\mathcal{L}((L^2(0,T))^2,U)} \leq \sqrt{T} \left\| \overline{Y_u} \right\|_{\mathcal{L}((L^2(0,T))^2,\tilde{U})} \\ &= \sup_{v \in W^2 \setminus \{0\}} \frac{1}{\|v\|_{W^2}} \left\| (v_1, y_u)_W + (v_2, \partial_t y_u)_W \right\|_{\tilde{U}} \\ &\leq \sup_{v \in W^2 \setminus \{0\}} \left\| \frac{\|v_1\|_W}{\|v\|_{W^2}} \|y_u\|_W + \frac{\|v_2\|_W}{\|v\|_{W^2}} \|\partial_t y_u\|_W \right\|_{\tilde{U}} \\ &\leq \left\| \|y_u\|_W + \|\partial_t y_u\|_W \right\|_{\tilde{U}} \leq \|y(u)\|_{L^2(0,T;\tilde{U})} + \|\partial_t y(u)\|_{L^2(0,T;\tilde{U})} , \end{aligned}$$

where the last step is due to the triangular inequality for integrals and changing the order of integration. Hence, by assumption (27), we have for a constant $C_a > 0$, independent of u,

$$\begin{aligned} \left\| \tilde{Y}_u \right\|^2_{\mathcal{L}((L^2(0,T))^2,U)} &\leq 2 \frac{C_U^2}{T} \left(\|y(u)\|^2_{L^2(0,T;\tilde{U})} + \|\partial_t y(u)\|^2_{L^2(0,T;\tilde{U})} \right) \\ &= 2 \frac{C_U^2}{T} \|y(u)\|^2_{H^1(0,T;\tilde{U})} \leq 2 \frac{C_U^2 C_a}{T} \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \right) . \end{aligned} \tag{32}$$

• Finalization: Note that the assumptions of Proposition 3.6 are fulfilled (in particular, we have $y \in H^1(0,T;\tilde{U}) \hookrightarrow H^1(0,T;V)$). Thus, we can use (20), estimate the sum term therein by (31) and use (32) to arrive at

$$\begin{aligned} \|y(u) - y_\ell(u)\|^2_{L^2(0,T;V)} &\leq c_\epsilon T \left(1 + \frac{C_V C_H}{2\gamma_\epsilon} \right) \sum_{k=\ell+1}^{\infty} \lambda_k^u \\ &\leq \underbrace{2 C_U^2 C_a C_{\Omega,r} c_\epsilon \left(1 + \frac{C_V C_H}{2\gamma_\epsilon} \right)}_{=:C} \frac{1}{\ell^{(2r/d)-1}} \left(\|u\|^2_{\mathcal{W}} + \|g\|^2_{U_g} \right) , \end{aligned}$$

which clearly approaches zero for $\ell \to \infty$, completing the proof. □

Space-time-separable RHS. The previous results have given sufficient conditions for the POD convergence to be uniform. Now, we give a concrete example in which this uniformity holds.

Corollary 3.11. *In Problem 2.2, choose* q even *such that* $q > d/2 + 1$ *(where* d *is the dimension of* Ω*), let be* $\hat{u} \in L^2(0,T;\mathbb{R}^M)$ *and set* $u := \mathcal{B}_q \hat{u} \in L^2(0,T;H_0^{q+1}(\Omega))$.

For $X := V$*, construct a POD basis* Ψ^u *from the trajectories of* $(y(u), \partial_t y(u))$*, depending on* u*. Let* $y_\ell(u) := y_\ell^{\Psi^u}(u)$ *be a solution to the* ℓ*-ROM approximation of Problem 2.2 with the same RHS* u *and ansatz space* $\mathcal{V}_\ell^u = span(\Psi_\ell^u)$.

Then, for $\ell \to \infty$*, the sequence* $(y_\ell(u))_\ell$ *converges strongly to* $y(u)$ *in* $L^2(0,T;V)$ *and uniformly w.r.t.* u *in bounded subsets of* $L^2(0,T;H_0^{q+1}(\Omega))$*. In particular, there exists a constant* $C > 0$*, independent of* $\hat{u}$*,* b_k *and* ℓ*, such that*

$$\|y(u) - y_\ell(u)\|^2_{L^2(0,T;V)} \leq \frac{C}{\ell^s}\Big(\|\hat{u}\|^2_{L^2(0,T;\mathbb{R}^M)} \max_{1\leq k\leq M} \|b_k\|^2_{H^{q+1}(\Omega)} + \|y_0\|^2_{H^{q+1}(\Omega)} \Big)$$

where $s := \frac{2q-2}{d} - 1 > 0$.

Proof. The assertion is a direct consequence of Theorem 3.10 in which we choose: $g = y_0$, $U_g = H^{q+1}(\Omega)$, $r = q - 1$, $\tilde{U} = H^q(\Omega) \cap H_0^1(\Omega) \hookrightarrow U = H^q(\Omega)$ and $\mathcal{W} = L^2(0,T;H_0^{q+1}(\Omega))$. Then, the assumption $q > d/2 + 1$ implies $r > d/2$.

Furthermore, Proposition 2.3 implies that for $u = \mathcal{B}_q \hat{u} \in \mathcal{W}$, we have $y \in H^1(0,T;\tilde{U})$. By standard estimates, compare [11, (1.16)], we obtain for a suitable constant $C_R > 0$, that

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2(0,T;H^{q+1}(\Omega))} \leq M^2 C_R^2 \|\hat{u}\|^2_{L^2(0,T;\mathbb{R}^M)} \max_{1\leq k\leq M} \|b_k\|^2_{H^{q+1}(\Omega)}.$$

Therefore, an a-priori estimate of the form of (27) is given by (6) and all requirements of Theorem 3.10 are met. □

4. Convergence of Optimality Systems POD.

In this section, we use the uniform convergence result of Corollary 3.11 to establish convergence of Optimality Systems POD (OS-POD), which was first proposed in [7] and subsequently extended in [16].

4.1. OS-POD.

The starting point is the suboptimal control based on POD reduction w.r.t. a reference trajectory with input f:

Problem 4.1 (POD Suboptimal Control)**.** *For* $f \in L^2(0,T;V)$ *and* $z_\ell^f := P_\ell^{\Psi^f} z$*, find* $\big(u_f^*, (y_\ell^{\Psi^f})^*\big)$ *that solve*

$$\min J^\ell(u, y_\ell^{\Psi^f}) := \frac{1}{2}\left\| y_\ell^{\Psi^f}(u) - z_\ell^f \right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T;V)} \tag{33}$$

such that $(u, y_\ell^{\Psi^f})$ *satisfies the corresponding POD-reduced order model* (18).

Here, one is confronted with the problem that the POD basis is computed from a reference trajectory which is not the optimal one. Moreover, during the course of iterative strategies to solve (33), the trajectory changes and the controlled dynamics may not be sufficiently well represented in the POD modes computed from the state corresponding to f. OS-POD was introduced to overcome this problem of "unmodeled dynamics". In the OS-POD procedure, the basis, with fixed dimension

ℓ, is updated with the goal that in the asymptotic limit of the iteration, the POD-basis corresponds to the optimal trajectory at discretization level ℓ. Here, we shall further justify the OS-POD procedure by proving that as the dimension $\ell \to \infty$, the optimal controls of the finite dimensional OS-POD problems converge weakly to the solution u^* of the infinite dimensional optimal control problem.

In the OS-POD algorithm, this issue is taken care of by augmenting the optimization problem with the POD basis construction criteria. For that matter, we need to include the *full* system (7) as well as the POD basis condition (15). Altogether, this then reads:

For J^ℓ in (33), solve $\min J^\ell(\hat{u}, y_\ell^{\Psi^u})$ such that

- $(\hat{u},\, y(u))$ fulfills the full problem (7):

$$\frac{d}{dt}\, y(u)(t) + \mathcal{A}y(u)(t) = u = \mathcal{B}(\hat{u}(t)) \quad \text{for a.e. } t \in (0,T], \tag{34a}$$

$$y(u)(0) = g, \tag{34b}$$

- Ψ^u is a POD basis constructed according to:

$$K_u \psi_i^u = \lambda_i^u \psi_i^u \text{ for } 1 \le i \le \ell \quad \text{with} \quad \left(\psi_j^u, \psi_i^u\right)_X = \delta_{ij} \text{ for } 1 \le i,j \le \ell,$$

- $\left(\hat{u},\, y_\ell^{\Psi^u}\right)$ fulfills the reduced-order model (18) with $f = u$ and $\Psi^f = \Psi^u$.

Compact Statement. Let us gather the conditions into a more compact form and recall the following notation:

- $u^* = \mathcal{B}(\hat{u}^*)$ denotes the optimal solution of the full-order optimization problem and $u_\ell^* = \mathcal{B}(\hat{u}_\ell^*)$ is the optimal solution of the OS-POD problem of order ℓ.
- $y_\ell^{\Psi^{u_\ell}}(u_\ell^*)$ denotes the solution of the ℓ-th-order ROM with RHS u_ℓ^* and ansatz space $span(\Psi^{u_\ell})$ where Ψ^{u_ℓ} is determined based on the trajectory of $y(u_\ell)$.

Then, the OS-POD problem reads

Problem 4.2 (OS-POD$_\ell$ Problem)**.** *Find $\hat{u}_\ell^* \in L^2(0,T;\mathbb{R}^M)$ as the solution to*

$$\min J^\ell(\hat{u}_\ell, y_\ell^{\Psi^{u_\ell}}) := \frac{1}{2}\left\| y_\ell^{\Psi^{u_\ell}}(u_\ell) - z_\ell^{u_\ell} \right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\hat{u}_\ell\right\|^2_{L^2(0,T;\mathbb{R}^M)}$$

where $z_\ell^{u_\ell} := P_\ell^{\Psi^{u_\ell}} z$, and $y_\ell^{\Psi^{u_\ell}}$ satisfies (18) *with RHS $= u_\ell$, and the basis Ψ^{u_ℓ} is determined from the trajectories $\big(y(u_\ell), \partial_t y(u_\ell)\big)$ of the full solution $y(u_\ell)$ of* (34) *with RHS u_ℓ as well, i.e., $f = u_\ell$.*

To this problem, Theorem 2.2 in [7] guarantees the existence of a solution u_ℓ^*, provided that for each $\hat{u} \in L^2(0,T;\mathbb{R}^M)$, we have $\min \lambda(K_u(y)) > 0$, where y solves (34):

Proposition 4.3 (Existence of Solution)**.** *Let be $X \in \{H,V\}$ and set $X^\ell := \otimes_{i=1}^\ell X$. Assume that $D(\mathcal{A})$ embeds compactly into V and that for every $\hat{u} \in L^2(0,T,\mathbb{R}^M)$, there exists a unique solution $y = y(\mathcal{B}\hat{u}) \in L^2(0,T;D(\mathcal{A})) \cap H^1(0,T;V)$ of problem* (34)*, and moreover there exists a continuous function $c_1 : \mathbb{R} \to \mathbb{R}$ such that for all $\hat{u} \in L^2(0,T;\mathbb{R}^M)$, there holds*

$$\|y(u)\|_{L^2(0,T;D(\mathcal{A}))\cap H^1(0,T;V)} \le c_1\big(\,\|\hat{u}\|_{L^2(0,T;\mathbb{R}^M)} + \|g\|_V\,\big).$$

Further, assume that for the eigenvalues λ of the POD operator K_u, there holds

$$\min\Big\{\lambda(K_u(y))\,\big|\, y \text{ solves (34) with } \hat{u} \in L^2(0,T;\mathbb{R}^M)\Big\} > 0.$$

Then, for each $\ell \in \mathbb{N}$, the OS-POD$_\ell$ Problem 4.2 admits a (global) solution

$$(x_\ell^*, \Psi^{u_\ell^*}, \hat{u}_\ell^*) \in H^1(0,T;\mathbb{R}^\ell) \times X^\ell \times L^2(0,T;\mathbb{R}^M)$$

with $(\lambda_\ell^*, y_\ell^*) \in \mathbb{R}^\ell \times \big(L^2(0,T;D(\mathcal{A})) \cap H^1(0,T;V)\big)$, where y_ℓ^* denotes the full-order optimal solution. (Note that y_ℓ^* depends on ℓ as the RHS u_ℓ^* does.)

The OS-POD problem is certainly computationally demanding. For an efficient numerical treatment, see [11, Chapter 6], [7] and [16], for instance.

4.2. **Convergence of OS-POD.** In this subsection, we consider the optimal control Problem 2.4 associated to the initial boundary value Problem 2.2 and show that the solutions u_ℓ^* of the corresponding OS-POD$_\ell$ problems converge weakly to the full-order optimal control u^* as ℓ goes to infinity. First, we consider the case $z = 0$. Then, we allow a state $z \neq 0$.

Theorem 4.4 (OS-POD Convergence for $z = 0$)**.** *Set $X = V$, choose the target state $z = 0$ and let $\mathcal{B}_q$ be as in Problem 2.2 with q* even *such that $q > d/2+1$, where d is the dimension of Ω. Let $\hat{u}^* \in L^2(0,T;\mathbb{R}^M)$ be the optimal control of Problem 2.4 with differential equation constraint* (5a)–(5c) *and set $u^* := \mathcal{B}_q\hat{u}^*$. Further, let $\hat{u}_\ell^* \in L^2(0,T;\mathbb{R}^M)$ be a solution to the corresponding OS-POD$_\ell$ Problem 4.2 and set $u_\ell^* := \mathcal{B}_q\hat{u}_\ell^*$.*

Then, $(\hat{u}_\ell^*)_\ell$ converges weakly to $\hat{u}^*$ in $L^2(0,T;\mathbb{R}^M)$. Furthermore, the sequence of corresponding OS-POD states $(y_\ell^{\Psi^{u_\ell^*}}(u_\ell^*))_\ell$ converges to $y(u^*)$ strongly in $L^2(0,T;H)$.

Proof. • Convergence of OS-POD Control: In the OS-POD$_\ell$ Problem 4.2, fix $\hat{u}_\ell = 0$, i.e., $u_\ell = \mathcal{B}_q\hat{u}_\ell = 0$. By Corollary 3.11, the sequence of low-order solutions $(y_\ell^{\Psi^0}(0))_\ell$ converges in $L^2(0,T;V)$ and hence, this sequence is bounded in $L^2(0,T;V)$.

Furthermore, the *optimal* $\hat{u}_\ell^*$ cannot lead to a larger value of J^ℓ than $\hat{u}_\ell = 0$. We thus have for all integers $\ell \geq 1$ and a constant $\overline{C} > 0$, independent of ℓ,

$$\frac{\alpha}{2}\,\|\hat{u}_\ell^*\|^2_{L^2(0,T;\mathbb{R}^M)} \leq J^\ell\big(\hat{u}_\ell^*,\, y_\ell^{\Psi^{u_\ell^*}}(u_\ell^*)\big) \leq J^\ell\big(0,\, y_\ell^{\Psi^0}(0)\big) \leq \overline{C}.$$

Hence, we infer that $(\hat{u}_\ell^*)_\ell$ is bounded in $L^2(0,T;\mathbb{R}^M)$. Thus, there exists a subsequence $(\hat{u}_{\ell_k}^*)_k$ that converges weakly to some $\tilde{u} \in L^2(0,T;\mathbb{R}^M)$.

• Convergence of Full State: For S_{y_0} of Proposition 2.3, define the sequence

$$(y_k)_k := (y(u_{\ell_k}^*))_k = (S_{y_0}(\hat{u}_{\ell_k}^*))_k,$$

i.e., for each $k \in \mathbb{N}$, $y(u_{\ell_k}^*)$ solves problem (5) for the RHS $u_{\ell_k}^* = \mathcal{B}_q\hat{u}_{\ell_k}^*$. Since $(\hat{u}_{\ell_k}^*)_k$ is a *bounded* sequence in $L^2(0,T;\mathbb{R}^M)$, by Proposition 2.3, we have that $(y_k)_k$ is bounded in $H^1(0,T;H^q(\Omega))$ which is continuously embedded into $H^1(0,T;V)$ as $q \geq 2$. Hence, $(y_k)_k$ has got a subsequence $(y_{k_p})_p = (y(u_{\ell_{k_p}}^*))_p$ converging weakly to some $\overline{y} \in H^1(0,T;V)$. Since $H^1(0,T;V)$ embeds compactly into $L^2(0,T;H)$, the convergence is strong in $L^2(0,T;H)$.

In order to simplify the notation, let us re-index the sub-subsequence $(\hat{u}_{\ell_{k_p}}^*)_p$ to $(\hat{u}_{\ell_k}^*)_k$ since the former also converges to $\tilde{u} \in L^2(0,T;\mathbb{R}^M)$ due to the uniqueness of weak limits. We can then say that $(y_k)_k = (y(u_{\ell_k}^*))_k$ converges strongly to $\overline{y} \in H^1(0,T;V)$ in the sense of $L^2(0,T;H)$.

• Taking the Limit in the Equation: Let us now show that $(y_k)_k$ actually converges strongly to $y(\tilde{u})$ in the sense of $L^2(0,T;H)$, i.e., that in $L^2(0,T;H)$, we have $\overline{y} = y(\tilde{u})$.

As stated above, $(\hat{u}^*_{\ell_k})_k$ converges weakly to some $\tilde{u} \in L^2(0,T;\mathbb{R}^M)$. By Proposition 2.3, we have that the solution operator S_{y_0} is bounded. Furthermore, the embedding $E_2 : H^1(0,T,H^q(\Omega)) \hookrightarrow H^1(0,T,V) \hookrightarrow L^2(0,T;H)$ is compact. Hence, we infer for w-lim denoting the weak limit,

$$\begin{aligned} E_2 y(\tilde{u}) = E_2 S_{y_0}(\tilde{u}) &= E_2 S_{y_0}\big(\operatorname*{w-lim}_{k\to\infty} \hat{u}^*_{\ell_k}\big) = E_2 \operatorname*{w-lim}_{k\to\infty} S_{y_0}(\hat{u}^*_{\ell_k}) \\ &= \lim_{k\to\infty} E_2 S_{y_0}(\hat{u}^*_{\ell_k}) = \lim_{k\to\infty} E_2 y_k = E_2 \overline{y}. \end{aligned}$$

By uniqueness of the limit, we then have $\overline{y} = y(\tilde{u})$ in the sense of $L^2(0,T;H)$. Thus, $(y_k)_k = (y(u^*_{\ell_k}))_k$ converges strongly to $y(\tilde{u})$ in the sense of $L^2(0,T;H)$.

• Convergence of OS-POD State: We have that the sequence $(\hat{u}^*_{\ell_k})_k$ is bounded in $L^2(0,T;\mathbb{R}^M)$. Making use of Corollary 3.11 and zero-adding the term $y(u^*_{\ell_k})$, we infer, for a constant $C > 0$, using Young's inequality and $V \hookrightarrow H$,

$$\begin{aligned} &\left\| y(\tilde{u}) - y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k}) \right\|^2_{L^2(0,T;H)} \\ &\leq 2 \left\| y(\tilde{u}) - y(u^*_{\ell_k}) \right\|^2_{L^2(0,T;H)} + 2 \left\| y(u^*_{\ell_k}) - y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k}) \right\|^2_{L^2(0,T;H)} \\ &\leq 2 \left\| y(\tilde{u}) - y(u^*_{\ell_k}) \right\|^2_{L^2(0,T;H)} \\ &\qquad + 2C \, \frac{1}{\ell_k^s} \Big(\left\| \hat{u}^*_{\ell_k} \right\|^2_{L^2(0,T;\mathbb{R}^M)} \max_{1\leq k\leq M} \|b_k\|^2_{H^{q+1}(\Omega)} + \|y_0\|^2_{H^{q+1}(\Omega)} \Big) \end{aligned} \tag{35}$$

where $s := \frac{2q-2}{d} - 1 > 0$ due to the assumption on q. The previous step of the proof implies that the first summand of (35) approaches zero for $k \to \infty$ since $(y(u^*_{\ell_k}))_k$ converges strongly to $y(\tilde{u})$ in $L^2(0,T;H)$. The second summand approaches zero for $k \to \infty$ since $\left\| \hat{u}^*_{\ell_k} \right\|_{L^2(0,T;\mathbb{R}^M)}$ is bounded and $s > 0$.

Thus, we have that $y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k})$ approaches $y(\tilde{u})$ strongly in $L^2(0,T;H)$ for $k \to \infty$, which gives the second assertion of the theorem as soon as we proof $\tilde{u} = \hat{u}^*$.

• Optimality of $\tilde{u}$: Note that since $z = 0$, the cost functional J does not explicitly depend on ℓ. By weakly lower semi continuity of J and the previous step, we have for all $\hat{u} \in L^2(0,T;\mathbb{R}^M)$ and $u := \mathcal{B}_q\hat{u}$, that

$$\begin{aligned}
J(\tilde{u}, y(\mathcal{B}_q\tilde{u})) &= \frac{1}{2}\left\|y(\mathcal{B}_q\tilde{u})\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\tilde{u}\right\|^2_{L^2(0,T;\mathbb{R}^M)} \\
&\leq \lim_{k\to\infty}\frac{1}{2}\left\|y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k})\right\|^2_{L^2(0,T;H)} + \liminf_{k\to\infty}\frac{\alpha}{2}\left\|\hat{u}^*_{\ell_k}\right\|^2_{L^2(0,T;\mathbb{R}^M)} \\
&= \liminf_{k\to\infty}\left(\frac{1}{2}\left\|y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k})\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\hat{u}^*_{\ell_k}\right\|^2_{L^2(0,T;\mathbb{R}^M)}\right) \\
&= \liminf_{k\to\infty} J\left(\hat{u}^*_{\ell_k}, y_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k})\right) \leq \liminf_{k\to\infty} J\left(\hat{u}, y_{\ell_k}^{\Psi^u}(u)\right) \qquad (36) \\
&= \liminf_{k\to\infty}\left(\frac{1}{2}\left\|y_{\ell_k}^{\Psi^u}(u)\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\hat{u}\right\|^2_{L^2(0,T;\mathbb{R}^M)}\right) \\
&= \frac{1}{2}\left\|y(u)\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|\hat{u}\right\|^2_{L^2(0,T;\mathbb{R}^M)} = J(\hat{u}, y(u)) \qquad (37)
\end{aligned}$$

where estimation (36) holds for all $\hat{u} \in L^2(0,T;\mathbb{R}^M)$ since $(\hat{u}^*_{\ell_k}, y_{\ell_k}^{\Psi^{u^*_{\ell_k}}})$ is minimal for J for each integer $k \geq 1$. In (37), we used that the subsequence $(y_{\ell_k}^{\Psi^u}(u))_k$ converges strongly to $y(u)$ in $L^2(0,T;V) \hookrightarrow L^2(0,T;H)$ for $k \to \infty$, which is justified by Corollary 3.11.

Altogether, we infer that $\tilde{u}$ is a solution of the optimal control Problem 2.4 with contraint (5). Hence, by the uniqueness of the solution to this problem, we have $\tilde{u} = \hat{u}^*$.

• Convergence of $(\hat{u}^*_\ell)_\ell$: We have shown that $(\hat{u}^*_\ell)_\ell$ has a subsequence $(\hat{u}^*_{\ell_k})_k$ that converges weakly to $\tilde{u} = \hat{u}^*$ in $L^2(0,T;\mathbb{R}^M)$. Since $\hat{u}^*$ is the unique solution to Problem 2.4, a standard argument implies weak convergence of the whole sequence; cf. [11, Theorem 4.2.1] for details. □

Corollary 4.5 (OS-POD Convergence for $z \neq 0$)**.** *Let be $\vec{z} \in H^1(0,T;\mathbb{R}^N)$ and $b^z_k \in H_0^{q+3}(\Omega)$, $k = 1,\dots,N$. Then, Theorem 4.4 holds true for*

$$z = \sum_{k=1}^{N} \vec{z}_k b^z_k \in H^1(0,T;H_0^{q+3}).$$

Proof. Throughout this proof, let H^q be a shorthand for $H^q(\Omega)$. In Theorem 4.4, we consider controls $\hat{u} \in L^2(0,T;\mathbb{R}^M)$. Thus, for the state y, solving Problem 2.2, we due to Proposition 2.3 have that $y \in H^1(0,T;H^q)$. As $z \in H^1(0,T;H_0^{q+3})$, we can define

$$w := y - z \in H^1(0,T;H^q).$$

Since $y = w + z$ satisfies Problem 2.2, we infer

$$w_t + z_t + L(w+z) = y_t + Ly = \mathcal{B}_q\hat{u}$$

and hence, w fulfills

$$w_t + Lw = \mathcal{B}_q\hat{u} - z_t - Lz \quad \text{in } Q. \qquad (38)$$

Since we assumed $y_0 \in H^{q+1}(\Omega)$ and there holds $z(0) \in H_0^{q+3}(\Omega)$, we have $w_0 := w(0) = y_0 - z(0) \in H^{q+1}(\Omega)$. Together with $w = y - z = 0$ on Γ, we infer that w satisfies Problem 2.2 with initial value w_0 and RHS

$$u := \mathcal{B}_q\hat{u} - z_t - Lz.$$

Let us now define $\tilde{u} \in L^2(0,T;\mathbb{R}^{M+2N})$ by

$$\tilde{u}_k = \begin{cases} \hat{u}_k, & k = 1,\dots,M, \\ \partial_t \vec{z}_{k-M}, & k = M+1,\dots,M+N, \\ \vec{z}_{k-M-N}, & k = M+N+1,\dots,M+2N \end{cases}$$

as well as $\tilde{b}_k \in H_0^{q+1}(\Omega)$, $k = 1,\dots,M+2N$, by

$$\tilde{b}_k = \begin{cases} b_k, & k = 1,\dots,M, \\ -b^z_{k-M}, & k = M+1,\dots,M+N, \\ -Lb^z_{k-M-N}, & k = M+N+1,\dots,M+2N. \end{cases}$$

Then, there holds $u = \sum_{k=1}^{M+2N} \tilde{u}_k \tilde{b}_k$ and we can apply Corollary 3.11 with $\tilde{u}$ in place of $\hat{u}$ and $\tilde{b}_k$ in place of b_k: For the low-order approximation w_ℓ of w and $s = \frac{2q-2}{d} - 1 > 0$, we find for a constant $C > 0$,

$$\begin{aligned} &\|w(u) - w_\ell(u)\|^2_{L^2(0,T;V)} \\ &\quad \le C\,\frac{1}{\ell^s}\Big(\|\tilde{u}\|^2_{L^2(0,T;\mathbb{R}^{M+2N})} \max_{1\le k\le M+2N} \Big\|\tilde{b}_k\Big\|^2_{H^{q+1}(\Omega)} + \|w_0\|^2_{H^{q+1}(\Omega)} \Big), \end{aligned}$$

which ensures the convergence of the corresponding OS-POD state as in (35):

$$\begin{aligned} &\Big\| w(u^*_{\ell_k}) - w_{\ell_k}^{\Psi^{u^*_{\ell_k}}}(u^*_{\ell_k}) \Big\|^2_{L^2(0,T;H)} \\ &\quad \le C\,\frac{1}{\ell^s}\Big(\Big\|\tilde{u}^*_{\ell_k}\Big\|^2_{L^2(0,T;\mathbb{R}^{M+2N})} \max_{1\le k\le M+2N} \Big\|\tilde{b}_k\Big\|^2_{H^{q+1}(\Omega)} + \|w_0\|^2_{H^{q+1}(\Omega)} \Big). \end{aligned}$$

We can thus repeat the proof of Theorem 4.4 with w in place of y. Since we have $J_{z=0}(u,w) = J_z(u,y)$, this completes the proof. □

Remark 4.6 (Non-separable Target State)**.** The OS-POD convergence can also be obtained for a target state $z \neq 0$ that is not space-time-separable (in contrast to the assumption in Corollary 4.5). In this case, it is sufficient to require

$$z_t + Lz \in H^2(0,T;V^*)$$

as well as the compatibility conditions (implicitly) assumed for the RHS f in Proposition 2.1:

As seen in (38) in the proof of Corollary 4.5, $w := y - z$ fulfills the equation of Problem 2.2 with RHS $\mathcal{B}_q\hat{u} - z_t - Lz$. Due to linearity, we can make the ansatz $w = w_1 + w_2$, where w_1 and w_2 solve Problem 2.2 with RHS $u_1 := \mathcal{B}_q\hat{u}$ and $u_2 := -z_t - Lz$, respectively. For u_1, we can apply Corollary 3.11 in order to obtain the uniform convergence of the low-order solution corresponding to w_1. For u_2, we can apply Theorem 3.9 due to the additional regularity assumed for z and obtain the uniform convergence of the low-order solution corresponding to w_2.

5. **Numerical Investigation.**

The purpose of this section is to illustrate the convergence result of Theorem 4.4 and to show that the OS-POD basis update leads to satisfactory control results when classical POD suboptimal control does not perform well.

In order to efficiently solve the OS-POD problem, we use a splitting algorithm that alternatingly considers the constraints given by the partial differential equation and the eigenvalue problems (cf. [7]). This approach involves the *OS-POD optimality condition* derived in [7, Theorem 2.3].

	FE	Iter 0	Iter 1	Iter 2	Iter 3	Iter 4	Iter 5
System Size	132098	3084	3084	3084	3084	3084	3084
Time (s)	15.2366	4.0504	3.5012	3.5120	3.5544	3.4981	3.5728
Cost Value	0.3759	0.2676	0.2671	0.3205	0.3900	0.3962	0.3956
Control Error		2.4839	2.5291	2.5673	0.0669	0.1192	0.1165
State Error		0.5748	0.5544	0.5954	0.1338	0.0267	0.0262

TABLE 1. Performance of OS-POD basis update for $\ell = 6$, where "Iter k" indicates that k OS-POD basis updates were performed.

Here, we consider an optimal boundary control problem for the heat equation with convection in the form

$$\min_{(u,y)} J(u,y) = \frac{1}{2}\left\|y - z\right\|^2_{L^2(0,T;H)} + \frac{\alpha}{2}\left\|u\right\|^2_{L^2(0,T;\mathbb{R}^2)}, \tag{39a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y - \Delta y + c y_x = f \qquad \text{in } (0,T] \times \Omega, \tag{39b}$$

$$-y_x + \sigma y = u_1 \qquad \text{on } (0,T] \times \{0\}, \tag{39c}$$

$$y_x + \sigma y = u_2 \qquad \text{on } (0,T] \times \{1\}, \tag{39d}$$

$$y(0) = y_0 \qquad \text{in } \Omega \tag{39e}$$

where $\Omega = (0,1)$, $H = L^2(\Omega)$, $u = (u_1, u_2)$,

$$z(t,x) = \begin{cases} \sin(2\pi x), & t \le T/2, \\ x, & t > T/2 \end{cases} \quad \text{and} \quad f(t,x) = \begin{cases} \exp(2xt), & t \le T/2, \\ 6\cos(4\pi t)\sin(4\pi x t), & t > T/2. \end{cases}$$

For the sake of comparison, Problem (39) is solved by an FE approach. The FE solution is obtained by discretizing the optimality system for the unconstrained case in Proposition 2.5 by piece-wise linear finite elements on an equidistant grid and using backward Euler for the time discretization. The discrete system is solved by a direct solver.

The POD reduction is based on (33) and is utilized within OS-POD. The OS-POD optimality systems were also solved by an exact solver.

In all cases, the initial snapshots are taken at every 8th time step, based on the trajectory of the solution corresponding to $u_1 = u_2 = 0$. All POD bases are used for the state as well as the adjoint state.

All computations were carried out using a Matlab 2008a implementation on an Intel Quad Core CPU with 2.83 GHz and 3.25 GB RAM.

The following parameter settings were used. For the model problem, we set: $T = 1$, $\sigma = 10^6$, $\alpha = 10^{-3}$, $c = 10$ and $y_0(x) = \sin(2\pi x)$. For the discretization, we used a spatial as well as a temporal grid size of $h = \tau = 1/256$. (Note that the solutions are plotted on an eight times coarser mesh than the computational grid.)

For this example, POD suboptimal control fails for $\ell = 6$; see column "Iter 0" of Table 1. However, Table 1 also shows that OS-POD yields satisfactory results after three basis updates.

Figure 1 depicts the FE optimal control, state and adjoint state. In the first/second row of Figure 2, we see the OS-POD optimal states/controls for zero to five basis updates. Figure 3 finally depicts the first two corresponding POD basis functions and we observe that during the first and the fourth OS-POD iterations, these two basis elements change their shapes significantly.

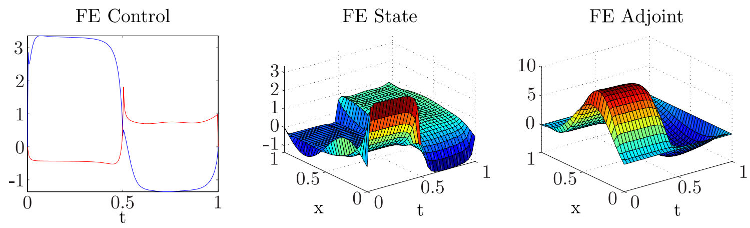

FIGURE 1. FE optimal solution.

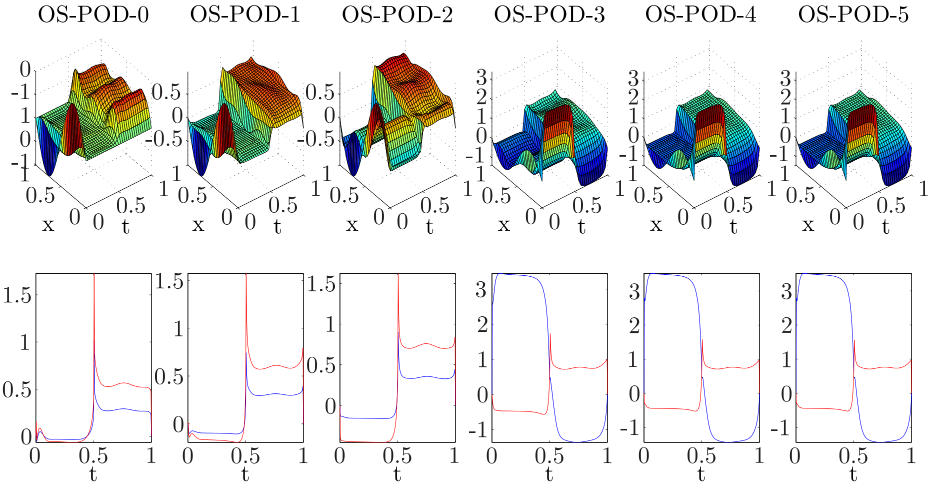

FIGURE 2. OS-POD optimal states (upper row) and controls (lower row) for $\ell = 6$.

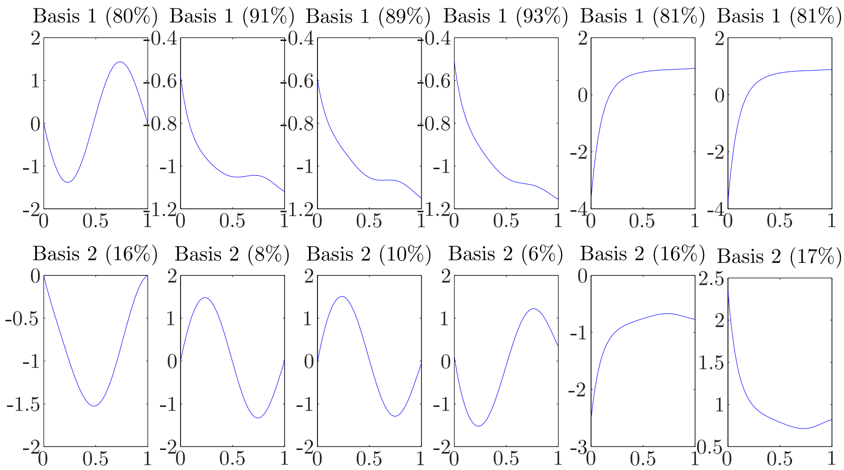

FIGURE 3. OS-POD optimal basis for $\ell = 6$. (The k-th column shows the first two basis elements after $k - 1$ basis updates.)

ℓ	Iter 0	Iter 1	Iter 2	Iter 3	Iter 4	Iter 5	Iter 6
4	2.3936	0.9658	0.6213	0.7727	0.7662	0.7664	0.7663
5	2.5224	2.2875	0.1358	0.6164	0.7169	0.7328	0.7362
6	2.4839	2.5291	2.5673	0.0669	0.1192	0.1165	0.1152
7	2.5482	1.5947	0.0345	0.0341	0.0333	0.0363	0.0361
8	2.5416	0.1831	0.0324	0.0315	0.0306	0.0312	0.0328
9	2.5458	0.0342	0.0297	0.0283	0.0292	0.0284	0.0298
10	2.5529	0.0305	0.0300	0.0305	0.0323	0.0308	0.0294

TABLE 2. Control error $\left\|u_{\mathrm{FE}} - u^{\ell}_{\mathrm{OSPOD}}\right\|_{L^2(\mathbb{R}^2)}$ for OS-POD basis updates for different choices of ℓ.

In Table 2, we show the dependence of the error in the OS-POD control $u^{\ell}_{\mathrm{OSPOD}}$ on the size of the low-order system ℓ. Note that this presents a numerical justification for the convergence of the OS-POD algorithm that we established in Theorem 4.4: For sufficiently many basis updates, the error in the control approaches zero for ℓ being increased. On the other hand, we see in the first column that increasing ℓ without updating the basis does not significantly reduce the error, i.e., the classical POD method fails for our choices of ℓ. Also, we observe that for $\ell < 6$, the error in the control settles at a rather high value. Furthermore, the higher the choice of ℓ, the less basis updates are necessary to achieve a satisfactory result.

We report that for this example, updating the POD basis on the basis of the current suboptimal state without invoking OS-POD does not improve the POD suboptimal control result.

The OS-POD update also performs well with higher convection $c = 20$. However, we have to set $\ell \geq 7$ and carry out a minimum of two OS-POD basis updates to obtain satisfactory results.

Appendix. *Proof of Proposition 3.6.*

Proof. Since u is fixed, we shall simplify the notation by omitting the u-dependencies. Also, we sometimes denote $\frac{\mathrm{d}}{\mathrm{d}t}$ by ∂_t.

• Splitting the Error: Let $P^X_\ell : X \to \mathcal{V}_\ell$ denote the POD projection defined in (16). Note that for the low-order solution $y_\ell \in \mathcal{V}_\ell = span(\Psi_\ell)$ of problem (18), there holds for all $t \in [0, T]$,

$$y_\ell(t) = P^X_\ell y_\ell(t) = \sum_{k=1}^{\ell} (y_\ell(t), \psi_k)_X\, \psi_k.$$

For the full trajectory, let us write for all $t \in [0, T]$,

$$y(t) = P^X_\ell y(t) + \tilde{y}(t) \quad \text{with} \quad \tilde{y}(t) := y(t) - P^X_\ell y(t) = \sum_{k=\ell+1}^{\infty} (y(t), \psi_k)_X\, \psi_k.$$

By means of that, we split the approximation error into two components:

$$\begin{aligned} \|y - y_\ell\|^2_{L^2(0,T,X)} &= \left\|P^X_\ell y + \tilde{y} - y_\ell\right\|^2_{L^2(0,T,X)} \\ &= \left\|P^X_\ell y - y_\ell\right\|^2_{L^2(0,T,X)} + \|\tilde{y}\|^2_{L^2(0,T,X)}, \end{aligned} \tag{40}$$

where the second equation is true since $P^X_\ell y - y_\ell$ and $\tilde{y}$ are orthogonal in X.

• Estimation of $\|P_\ell^X y - y_\ell\|$: Set $\xi_\ell := y_\ell - P_\ell^X y \in \mathcal{V}_\ell$. Since $y = P_\ell^X y + \tilde{y}$ solves problem (2), we have for a.e. $t \in (0,T]$ and for all $\varphi \in V$

$$\frac{d}{dt}\left(P_\ell^X y(t),\varphi\right)_H + a(P_\ell^X y(t),\varphi) = (f(t),\varphi)_{V^*,V} - \frac{d}{dt}\left(\tilde{y}(t),\varphi\right)_H - a(\tilde{y}(t),\varphi),$$
$$\left(P_\ell^X y(t)(0),\varphi\right)_H = (g,\varphi)_H - (\tilde{y}(0),\varphi)_H\,.$$

As y_ℓ solves problem (18) with $u = f$ and test functions in $\mathcal{V}_\ell$, we due to the previous equation and the linearity have for a.e. $t \in (0,T]$ that

$$\begin{aligned}
&\frac{d}{dt}\left(\xi_\ell(t),\varphi\right)_H + a(\xi_\ell(t),\varphi)\\
&\quad= \frac{d}{dt}\left(y_\ell(t),\varphi\right)_H + a(y_\ell(t),\varphi) - \frac{d}{dt}\left(P_\ell^X y(t),\varphi\right)_H - a(P_\ell^X y(t),\varphi)\\
&\quad= (f(t),\varphi)_{V^*,V} - (f(t),\varphi)_{V^*,V} + \frac{d}{dt}\left(\tilde{y}(t),\varphi\right)_H + a(\tilde{y}(t),\varphi), \qquad \varphi \in \mathcal{V}_\ell.
\end{aligned}$$

Proceeding analogously for the initial condition, ξ_ℓ solves for a.e. $t \in (0,T]$

$$\frac{d}{dt}\left(\xi_\ell(t),\varphi\right)_H + a(\xi_\ell(t),\varphi) = \frac{d}{dt}\left(\tilde{y}(t),\varphi\right)_H + a(\tilde{y}(t),\varphi), \qquad \varphi \in \mathcal{V}_\ell, \tag{41a}$$
$$(\xi_\ell(0),\varphi)_H = (\tilde{y}(0),\varphi)_H\,, \qquad \varphi \in \mathcal{V}_\ell. \tag{41b}$$

Choosing $\varphi = \xi_\ell(0)$ in (41b), we obtain by the Cauchy Schwarz inequality

$$\|\xi_\ell(0)\|_H \le \|\tilde{y}(0)\|_H\,. \tag{42}$$

Choosing $\varphi = \xi_\ell(t) \in \mathcal{V}_\ell$ in (41a), we obtain after integration in $[0,T]$

$$\begin{aligned}
&\frac{1}{2}\|\xi_\ell(T)\|_H^2 + \int_0^T a(\xi_\ell(t),\xi_\ell(t))\,\mathrm{d}t\\
&\qquad = \int_0^T (\partial_t\tilde{y}(t),\xi_\ell(t))_H\,\mathrm{d}t + \int_0^T a(\tilde{y}(t),\xi_\ell(t))\,\mathrm{d}t + \frac{1}{2}\|\xi_\ell(0)\|_H^2\,.
\end{aligned}$$

Omitting the first term on the LHS, using coercivity and boundedness of a, the Cauchy Schwarz inequality as well as (42), we infer that

$$\begin{aligned}
&\kappa\int_0^T (\xi_\ell(t),\xi_\ell(t))_V\,\mathrm{d}t\\
&\quad \le \int_0^T \|\partial_t\tilde{y}(t)\|_H\|\xi_\ell(t)\|_H\,\mathrm{d}t + \beta\int_0^T \|\tilde{y}(t)\|_V\|\xi_\ell(t)\|_V\,\mathrm{d}t + \frac{1}{2}\|\tilde{y}(0)\|_H^2\,.
\end{aligned}$$

Then, for $\epsilon > 0$, by Young's inequality $ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ and using $V \hookrightarrow H$ with constant C_V for the second estimation, we obtain

$$\begin{aligned}
\kappa\|\xi_\ell\|_{L^2(0,T;V)}^2 &\le \int_0^T\left(\frac{\epsilon}{2}\|\partial_t\tilde{y}(t)\|_H^2 + \frac{1}{2\epsilon}\|\xi_\ell(t)\|_H^2\right)\mathrm{d}t\\
&\qquad + \int_0^T\left(\frac{\beta\epsilon}{2}\|\tilde{y}(t)\|_V^2 + \frac{\beta}{2\epsilon}\|\xi_\ell(t)\|_V^2\right)\mathrm{d}t + \frac{1}{2}\|\tilde{y}(0)\|_H^2\\
&\le \frac{C_V\epsilon}{2}\|\partial_t\tilde{y}\|_{L^2(0,T;V)}^2 + \frac{\beta\epsilon}{2}\|\tilde{y}\|_{L^2(0,T;V)}^2\\
&\qquad + \left(\frac{C_V}{2\epsilon} + \frac{\beta}{2\epsilon}\right)\|\xi_\ell\|_{L^2(0,T;V)}^2 + \frac{1}{2}\|\tilde{y}(0)\|_H^2\,,
\end{aligned} \tag{43}$$

which we may rearrange to

$$\underbrace{\Big(\kappa - \frac{1}{2\epsilon}(C_V + \beta)\Big)}_{=:\gamma_\epsilon} \|\xi_\ell\|^2_{L^2(0,T;V)} \leq \frac{\epsilon}{2} \max\big(C_V, \beta\big) \|\tilde{y}\|^2_{H^1(0,T;V)} + \frac{1}{2} \|\tilde{y}(0)\|^2_H$$
$$= T \frac{\epsilon}{2} \max\big(C_V, \beta\big) \sum_{k=\ell+1}^{\infty} \lambda_k^u + \frac{1}{2} \|\tilde{y}(0)\|^2_H, \tag{44}$$

where the last step is due to the definition of $\tilde{y}$ and (17) since the POD basis is constructed from the trajectories of $(y, \partial_t y)$.

• First Assertion: We can estimate the second error component in (40) by Remark 3.4, yielding

$$\|\tilde{y}\|^2_{L^2(0,T,V)} \leq \|\tilde{y}\|^2_{H^1(0,T,V)} = T \sum_{k=\ell+1}^{\infty} \lambda_k^u.$$

Since ϵ is chosen such that $\gamma_\epsilon > 0$, we can divide (44) by γ_ϵ in order to estimate the first term in (40). Altogether, we arrive at the assertion

$$\|y - y_\ell\|^2_{L^2(0,T,X)} \leq \underbrace{\Big(1 + \frac{\epsilon}{2\gamma_\epsilon} \max\big(C_V, \beta\big)\Big)}_{=:c_\epsilon} T \sum_{k=\ell+1}^{\infty} \lambda_k^u + \frac{1}{2\gamma_\epsilon} \|\tilde{y}(0)\|^2_H.$$

• Estimation of Initial Value: Since we constructed the POD basis based on $(y(u), \partial_t y(u))$, we due to (17) have

$$T \sum_{k=\ell+1}^{\infty} \lambda_k^u = \left\| y(u) - \sum_{k=1}^{\ell} (y(u), \psi_k^u)_V \, \psi_k^u \right\|^2_{H^1(0,T;V)}$$
$$= \left\| \sum_{k=\ell+1}^{\infty} (y(u), \psi_k^u)_V \, \psi_k^u \right\|^2_{H^1(0,T;V)}$$
$$\geq \frac{1}{C_H} \left\| \sum_{k=\ell+1}^{\infty} (y(u), \psi_k^u)_V \, \psi_k^u \right\|^2_{C([0,T];V)}$$

where the last step is due to the continuous embedding of $H^1(0,T;V)$ in $C([0,T];H)$ with constant C_H. We can thus choose $t = 0$ and obtain (using $V \hookrightarrow H$ with constant C_V):

$$\left\| \sum_{k=\ell+1}^{\infty} (g, \psi_k^u)_V \, \psi_k^u \right\|^2_H \leq C_V \left\| \sum_{k=\ell+1}^{\infty} (y(u), \psi_k^u)_V \, \psi_k^u \right\|^2_{C([0,T];V)}$$
$$\leq T C_V C_H \sum_{k=\ell+1}^{\infty} \lambda_k^u.$$

Together with (19), we arrive at (20). Clearly, the remainder of the sum converges to zero. □

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Received xxxx 20xx; revised xxxx 20xx.

Acknowledgements. We thank two anonymous referees for several useful remarks.

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