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ISOMETRIES OF THE SPACE OF COMPACT SUBSETS OF E^d

by

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Summary

The isometries of the space of compact subsets of E^d with respect to the Hausdorff-metric are the mappings generated by isometries of E^d.

1. Introduction

Let $\mathscr{K}$ denote the class of all (non-empty) compact subsets of d-dimensional Euclidean space E^d. For $A, B \subset E^d$ and $\lambda \in \mathbf{R}$ define $A+B := \{x+y | x \in A, y \in B\}$ and $\lambda A := \{\lambda x | x \in A\}$. Let U denote the unit ball and $\|\cdot\|$ the Euclidean norm of E^d. Then the *Hausdorff-metric* δ on $\mathscr{K}$ may be defined by

$$\delta(C, D) := \min \{\lambda \in \mathbf{R}^+ | C \subset D + \lambda U,\ D \subset C + \lambda U\}$$

for $C, D \in \mathscr{K}$ or, equivalently,

$$\delta(C, D) := \max \{\max_{x \in C} \min_{y \in D} \|x-y\|,\ \max_{y \in D} \min_{x \in C} \|x-y\|\}$$

for $C, D \in \mathscr{K}$ (see, e.g., HAUSDORFF [9], p. 145 or BLASCHKE [2], p. 60).

Let $\mathscr{K}$ be endowed with the topology induced by δ. Properties of $\mathscr{K}$ or, more generally, of spaces of closed or compact subsets of general metric and topological spaces have been investigated intensively during the last 50 years (see, e.g., HAUSDORFF [9], KURATOWSKI [11] and POPOV [12]). δ also plays an important role in convexity and approximation theory (see, e.g., BLASCHKE [2], HADWIGER [8], KELLY and WEISS [10] and SENDOV [15]). These investigations indicate that the particular metric space $\langle \mathscr{K}, \delta \rangle$ is of independent interest.

R. SCHNEIDER [14] conjectured that the isometries of $\langle \mathscr{K}, \delta \rangle$ into itself are precisely the mappings of the form $C \to i(C) (:= \{i(x) | x \in C\})$ for $C \in \mathscr{K}$ where i is a fixed isometry of E^d into itself, i.e., the isometries of $\langle \mathscr{K}, \delta \rangle$ are *generated* by isometries of the underlying space E^d. For the case of surjective isometries the conjecture has been confirmed by GRUBER [5] who has also proved some related results in [4]. In the present note we give a proof of the general case. This proof partly follows the proof of the corresponding result for the class $\mathscr{C}$ of compact convex subsets of E^d (see [7]).

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THEOREM. *A mapping $I\colon \mathcal{K}\to\mathcal{K}$ is an isometry of $\langle\mathcal{K},\delta\rangle$ into itself precisely when there is an isometry i of E^d into itself such that $I(C)=i(C)$ for all $C\in\mathcal{K}$.*

We conjecture that the Theorem holds also when the Euclidean norm is replaced by an arbitrary norm.

The above Theorem as well as the results of [6] and [7] belong to the following general problem: Let a class of functions $\mathcal{F}$ on, or a class of subsets $\mathcal{S}$ of some space X be given, such that one can metrisize $\mathcal{F}$ or $\mathcal{S}$ in a simple manner using a metric or a measure or some other property of X. Then one may expect that the isometries of $\mathcal{F}$ or $\mathcal{S}$ into itself are closely related to special mappings of X into itself. Well-known examples for this are the classical theorems of BANACH [1], p. 173, and STONE [17] (see [3], p. 115) and Banach [1], p. 174, and their modern descendants. (The paper of SOURMOUR [16] contains a review of some of them.)

Let $\langle\ ,\ \rangle$ denote the inner product in E^d. There is no danger of confusing $\langle x, y\rangle$ and $\langle\mathcal{K},\delta\rangle$. For $x, y\in E^d$ let $[x, y]$ be the line segment with endpoints x, y. o denotes the origin, S the $(d-1)$-dimensional unit sphere of E^d. We will not distinguish between $x\in E^d$ and $\{x\}\in\mathcal{K}$. bd, diam, conv, lin and pos stand, respectively, for boundary, diameter and convex, linear and positive hull. $o(\cdot)$ is the Bachmann—Landau symbol.

2. Preliminaries

This section contains a collection of simple results which will be needed in the proof of the theorem in Section 3.

(1) *Let $I_1, I_2, \dots$ be a sequence of isometries of $\langle\mathcal{K},\delta\rangle$ (into itself) such that $I_1(o), I_2(o), \dots$ are contained in a bounded subset of E^d. Then there exist a subsequence $I_{k_1}, I_{k_2}, \dots$ and an isometry I_0 such that $I_0(C)=\lim_{l\to+\infty} I_{k_l}(C)$ for all $C\in\mathcal{K}$.*

This has been proved for $\langle\mathcal{C},\delta\rangle$ in [7]. The same proof is valid in the present case if the more general version of the Blaschke selection theorem (as stated e.g. in [8], p. 154, or [13], p. 91) is used. The next proposition is due to GRUBER [4]:

(2) *Any isometry of $\langle\mathcal{K},\delta\rangle$ which maps some point onto a point is generated by an isometry of E^d.*

For $C, D\in\mathcal{K}$ the definition of $\delta(C, D)$ shows that there is a point $c\in C$ (or D) such that for each point $d\in D$ (or C, respectively) which is nearest to c we have $\|c-d\|=\delta(C, D)$. In general c and d will not be unique. $c-d$ will be called a *δ-vector* and $(c-d)/\|c-d\|$ a *δ-unit vector* from D to (the point c of) C. We show:

(3) *Let $C, C_0, C_l\in\mathcal{K}$ be given such that $\delta(C, C_0)+\delta(C_0, C_l)=\delta(C, C_l)$ and suppose, that there exists a δ-vector from C_l to $c\in C$. Then there exists a δ-vector from C_l to C_0.*

Choose $c_0\in C_0$ nearest to c, and $c_l\in C_l$ nearest to c_0. Then the definition of δ implies $\|c-c_0\|\leqq\delta(C, C_0)$, $\|c_0-c_l\|\leqq\delta(C_0, C_l)$ and thus $\|c-c_l\|\leqq\|c-c_0\|+\|c_0-c_l\|\leqq\delta(C, C_0)+\delta(C_0, C_l)=\delta(C, C_l)$. Our assumptions show that c has distance

$\geqq\delta(C, C_l)$ from each point of C_l. Therefore equality holds in each of the above inequalities. In particular $\|c_0-c_l\|=\delta(C_0, C_l)$. Since $c_l\in C_l$ is nearest to $c_0\in C_0$ we infer that c_0-c_l is a δ-vector from C_l to C_0. This proves (3). Essentially the same proof yields the following result:

(4) *Let $C_0, C_1, C_2, \dots, C_l\in\mathscr{K}$ be such that $\delta(C_0, C_1)+\delta(C_1, C_2)+\dots+\delta(C_{l-1}, C_l)=$ $=\delta(C_0, C_l)$ and suppose that there is a δ-vector from C_l to $c_0\in C_0$. Then there exists a vector u which is a δ-unit vector from each of $C_1, \dots, C_l$ to $c_0\in C_0$.*

Choose $c_1\in C_1$ nearest to c_0, $c_2\in C_2$ nearest to $c_1, \dots, c_l\in C_l$ nearest to c_{l-1}. As before

$$\|c_0-c_1\| \leqq \delta(C_0, C_1), \dots, \|c_{l-1}-c_l\| \leqq \delta(C_{l-1}, C_l).$$

Hence

$$\|c_0-c_l\| \leqq \|c_0-c_1\|+\dots+\|c_{l-1}-c_l\| \leqq \delta(C_0, C_1)+\dots+\delta(C_{l-1}, C_l) = \delta(C_0, C_l).$$

Since the assumptions in (4) imply that c_0 has distance $\geqq\delta(C_0, C_l)$ from each point of C_l, equality holds throughout. This implies that $\|c_0-c_i\|=\delta(C_0, C_i)$ for $i\in\{1, \dots, l\}$ and that $c_0-c_1, c_0-c_2, \dots, c_0-c_l$ are positive multiples of each other, thus confirming (4).

The next two propositions are obvious:

(5) *Let $C, D\in\mathscr{K}$, $c\in C$, $d\in D$ be given. Then* $\big|\|c-d\|-\delta(C, D)\big|\leqq \operatorname{diam} C+\operatorname{diam} D$.

(6) *Let $C, D\in\mathscr{K}$. Then* $\delta(C, D)\geqq\delta(\operatorname{conv} C, \operatorname{conv} D)$.

The smoothness of $\|\cdot\|$ yields the following results:

(7) *Let R be a ray in E^d and $\alpha\in\mathbf{R}^+$. Denote by $'$ the orthogonal projection onto R. Suppose that for $\lambda\in\mathbf{R}^+$ compact sets $C, C(\lambda)\in\mathscr{K}$ are given such that $C', C(\lambda)'\neq\emptyset$, $C, C(\lambda)\subset R+\alpha U$ and* $\min\{\|y\| \,|\, y\in C(\lambda)\}\to+\infty$ *as $\lambda\to+\infty$. Then, as $\lambda\to+\infty$,*

$$\delta(C, C(\lambda)) = \delta(\operatorname{conv} C', C(\lambda))+o(1)$$
$$= \delta(\operatorname{conv} C', \operatorname{conv} C(\lambda)')+o(1).$$

(8) *Let $C, D\in\mathscr{C}$. Then* $\bigcap_{x\in E^d}\{x+C+\delta(x, D)\,U\}=C+D$.

We will also need the following properties of convergent sequences in $\mathscr{K}$:

(9) *If $C_0, C_1, \ldots\in\mathscr{K}$ with $C_1, C_2, \ldots\to C_0$ then*

$$\operatorname{conv} C_1, \operatorname{conv} C_2, \ldots\to\operatorname{conv} C_0.$$

(10) *If $C_0, C_1, \dots, D_0, D_1, \ldots\in\mathscr{C}$ with $C_1, C_2, \ldots\to C_0$, $D_1, D_2, \ldots\to D_0$ then*

$$C_1+D_1, C_2+D_2, \ldots\to C_0+D_0.$$

Let $i_1, \dots, i_n$ be orthogonal transformations of E^d and $\lambda_1, \dots, \lambda_n\in[0, 1]$ with $\lambda_1+\dots+\lambda_n=1$. The mapping $m\colon \mathscr{C}\to\mathscr{C}$ defined by

$$m(C):=\{\lambda_1 i_1(x_1)+\dots+\lambda_n i_n(x_n)|x_1, \dots, x_n\in C\} \quad \text{for} \quad C\in\mathscr{C}$$

is called a *rotational mean* (see [8], p. 168). Then the sphericity theorem of Hadwiger may be formulated in the following way:

(11) *Let $D\in\mathscr{C}$ consist of more than one point. Then there exist a sequence $m_1, m_2, \dots$ of rotational means and a number $\varrho\in\mathbf{R}^+$ such that $m_1(D), m_2(D), \ldots\to\varrho U$.*

If D has non-empty interior this is the theorem as stated in [8], p. 170. If D has empty interior choose some rotational mean l such that $l(D)$ has non-empty interior. Then there are rotational means $l_1, l_2, \ldots$ such that $l_1(lD), l_2(lD), \ldots$ converges to a ball with center o. Now let $m_1 := l_1 \circ l, m_2 := l_2 \circ l, \ldots$. Taking into account that a composition of rotational means is again a rotational mean, we see that (11) holds also in the case when D has empty interior.

3. Proof of the Theorem

If i is an isometry of E^d, then the mapping $I\colon \mathcal{K} \to \mathcal{K}$ generated by i is obviously an isometry of $\langle \mathcal{K}, \delta \rangle$.

Conversely assume that $I\colon \mathcal{K} \to \mathcal{K}$ is an isometry of $\langle \mathcal{K}, \delta \rangle$ (into itself).

In the first part of the proof we construct an isometry J which is closely related to I and will permit us to construct an isometry i of E^d.

For each $k \in \mathbf{N}$ define an isometry I_k of $\langle \mathcal{K}, \delta \rangle$ by $I_k(C) := (1/k) I(kC)$ for $C \in \mathcal{K}$. Since $I_1(o), I_2(o), \ldots \to \{o\}$, (1) yields the existence of a subsequence $I_{k_1}, I_{k_2}, \ldots$ and of an isometry I_0 of $\langle \mathcal{K}, \delta \rangle$ such that

$$(12) \qquad I_0(C) = \lim_{l \to \infty} I_{k_l}(C) \quad \textit{for all} \quad C \in \mathcal{K}.$$

In particular $I_0(o) = \{o\}$. Hence by (2) there exists an isometry i_0 of E^d such that

$$(13) \qquad I_0(C) = i_0(C) \quad \textit{for all} \quad C \in \mathcal{K}.$$

Since $I_0(o) = \{o\}$, we have $i_0(o) = o$ and thus $i_0(U) = U$. Therefore $\lim_{l \to \infty} I_{k_l}(o) = I_0(o) = \{o\}$ and $\lim_{l \to \infty} I_{k_l}(U) = I_0(U) = i_0(U) = U$. Choose $\varepsilon \in]0, 1/2[$ and $m \in \mathbf{N}$ such that the mapping

$$(14) \qquad J(:= I_{k_m}) \;\; \textit{defined by} \;\; J(C) := \frac{1}{k_m} I(k_m C) \;\; \textit{for each} \;\; C \in \mathcal{K}$$

is an isometry of $\langle \mathcal{K}, \delta \rangle$ satisfying the inequalities

$$(15) \qquad \delta(o, J(o)) \left[= \delta(I_0(o), I_{k_m}(o)) \right] < \varepsilon,$$

$$(16) \qquad \delta(U, J(U)) \left[= \delta(I_0(U), I_{k_m}(U)) \right] < \varepsilon.$$

We shall establish the inclusion

$$(17) \quad J(x) \subset \{y \mid \|x\| - \varepsilon \leqq \|y\| \leqq \|x\| + \varepsilon\} \quad \text{for each} \quad x \in E^d \quad \text{with} \quad \|x\| \geqq 1.$$

Choose x. Then $\delta(J(o), J(x)) = \delta(o, x) = \|x\|$ yields $J(x) \subset J(o) + \|x\| U$ and (15) implies $J(o) \subset \varepsilon U$. Thus

$$(18) \qquad J(x) \subset [J(o) + \|x\| U \subset \varepsilon U + \|x\| U =] \{y \mid \|y\| \leqq \|x\| + \varepsilon\}.$$

This proves half of (17). Since by (16) $U \subset J(U) + \varepsilon U$, we deduce from (18), $\|x\| \geqq 1$ and $\varepsilon \in]0, 1/2[$ that

$$(19) \qquad J(x) \subset [(\varepsilon + \|x\|) U = U + (\varepsilon + \|x\| - 1) U \subset$$
$$\subset J(U) + (\|x\| + 2\varepsilon - 1) U \subset] J(U) + \|x\| U.$$

Because of (16) $J(U)\subset(1+\varepsilon)U$. Hence for each $y\in J(x)$,

(20) $$J(U)\subset[(1+\varepsilon)U\subset y+(\|y\|+1+\varepsilon)U\subset]\,J(x)+(\|y\|+1+\varepsilon)U.$$

The definition of δ together with (19) and (20) shows that

$$\max\{\|x\|,\|y\|+1+\varepsilon\}\geqq\delta(J(x),J(U))=\delta(x,U)=\|x\|+1.$$

Thus $\|y\|+1+\varepsilon\geqq\|x\|+1$ for each $y\in J(x)$, i.e., $J(x)\subset\{y\,|\,\|x\|-\varepsilon\leqq\|y\|\}$. This proves the second half of (17).

Now we take the first step towards a description of the images of points and prove

(21) $$\operatorname{diam} J(k_lx)=o(k_l)\quad as\quad l\to\infty\quad for\ each\quad x\in E^d\setminus\{o\}.$$

If x is given, then (14), (12) and (11) imply that

$$\frac{1}{k_l}J(k_lx)=\frac{1}{k_m}\cdot\frac{1}{k_l}I(k_l(k_mx))\to\frac{1}{k_m}I_0(k_mx)=\frac{1}{k_m}i_0(k_mx).$$

Consequently, diam $(1/k_l)J(k_lx)\to0$ as $l\to\infty$. This proves (21).

We show:

(22) *For each $x\in S$ and $l\in\mathbf{N}$ the set T_l of pairs $(u,c)\in S\times J(o)$ where u is a δ-unit vector from $J(k_lx)$ to $c\in J(o)$ is non-empty and compact.*

Choose x, l. $J(k_lx)\subset(k_l+\varepsilon)U$ by (17). Since (16) implies $U\subset J(U)+\varepsilon U$ we conclude that $J(k_lx)\subset J(U)+(k_l+2\varepsilon-1)U\subset J(U)+k_lU$. Therefore each point of $J(k_lx)$ has distance $\leqq k_l$ from some point of $J(U)$. Taking into account that $\delta(J(k_lx),J(U))=$ $=\delta(k_lx,U)=k_l+1>k_l$ the definition of δ-vectors implies that there is a δ-vector from $J(k_lx)$ to $J(U)$ and

$$\delta(J(U),J(o))+\delta(J(o),J(k_lx))=[\delta(U,o)+\delta(o,k_lx)=1+k_l]=\delta(J(U),J(k_lx)).$$

Thus by (3) there is a δ-vector from $J(k_lx)$ to some $c\in J(o)$. Hence $T_l\neq\emptyset$. The compactness of T_l is obvious. This concludes the proof of (22). The following is a refinement of (22):

(23) *For each $x\in S$ there is a unique vector $u\in S$ and some point $c\in J(o)$ such that u is a δ-unit vector from $J(k_lx)$ to $c\in J(o)$ for each $l\in\mathbf{N}$.*

Let $x\in S$ be given. For each $l\in\mathbf{N}$

$$\delta(J(o),J(k_1x))+\delta(J(k_1x),J(k_2x))+\ldots+\delta(J(k_{l-1}x),J(k_lx))=$$
$$=k_1+(k_2-k_1)+\ldots+(k_l-k_{l-1})=\delta(J(o),J(k_lx)).$$

By (22) there exists a δ-unit vector from $J(k_lx)$ to some $c\in J(o)$. Now (4) implies existence of a vector $u\in S$ such that u is a δ-unit vector from each of $J(k_1x),\ldots,J(k_lx)$ to $c\in J(o)$. Using the notation of (22) we have $T_1\cap\ldots\cap T_l\neq\emptyset$ for each $l\in\mathbf{N}$. Since $T_1,T_2,\ldots$ are compact $T:=\bigcap\{T_l\,|\,l\in\mathbf{N}\}\neq\emptyset$. If $(u,c)\in T$ and $l\in\mathbf{N}$, the vector k_lu is a δ-vector from $J(k_lx)$ to $c\in J(o)$. To prove uniqueness of u choose

$(u, c), (v, d) \in T$. Then $c - k_l u, d - k_l v \in J(k_l x)$ for all $l \in \mathbf{N}$ by the definition of δ-vectors, and thus

$$\big| k_l \|u-v\| - \|c-d\| \big| [\leqq \|(c-k_l u)-(d-k_l v)\| \leqq \operatorname{diam} J(k_l x)] = o(k_l) \quad as \quad l \to \infty$$

by (21). Hence $u=v$, concluding the proof of (23).

Let $-i$ denote the map $x \to u$ of S into itself appearing in (23). Then

(24) *i: $S \to S$ is an isometry.*

Let $x, y \in S$ and let $u := -i(x)$, $v := -i(y)$. Then for each $l \in \mathbf{N}$ the vectors $k_l u$, $k_l v$ are δ-vectors from $J(k_l x)$ and $J(k_l y)$ to points $c, d \in J(o)$. Hence $c - k_l u \in J(k_l x)$, $d - k_l v \in J(k_l y)$. From this, together with (5) and (21) we infer

$$\begin{aligned} &\big| \|(c-d) - k_l(u-v)\| - k_l \|x-y\| \big| \\ &\big[= \big| \|(c-k_l u) - (d-k_l v)\| - \delta\big(J(k_l x), J(k_l y)\big) \big| \\ &\leqq \operatorname{diam} J(k_l x) + \operatorname{diam} J(k_l y) \big] \\ &= o(k_l) \quad \text{as} \quad l \to \infty, \end{aligned}$$

thus proving (24).

i can be extended uniquely to an isometry of E^d which will also be denoted by i. Let

(25) K: $\mathcal{K} \to \mathcal{K}$ *be defined by* $K(C) := i^{-1}\big(J(C)\big)$ *for* $C \in \mathcal{K}$.

Obviously, K is an isometry of $\langle \mathcal{K}, \delta \rangle$. The propositions (17), (21) and (23) now take the following form:

(26) $K(x) \subset \{y \,|\, \|x\| - \varepsilon \leqq \|y\| \leqq \|x\| + \varepsilon\}$ *for* $x \in E^d$ *with* $\|x\| \geqq 1$,

(27) $\operatorname{diam} K(k_l x) = o(k_l)$ *as* $l \to \infty$ *for* $x \in E^d \setminus \{o\}$,

(28) *for all* $x \in S$ *and* $l \in \mathbf{N}$ *the vector* $-k_l x$ *is a* δ*-vector from* $K(k_l x)$ *to a* $c \in K(o)$.

(26)—(28) will be used to prove that

(29) $K(k_l x) \subset \operatorname{lin}\{x\} + 2\varepsilon U$ *for* $x \in S$ *and* $l \in \mathbf{N}$.

Let x and l be given. Suppose there exists a $w \in K(k_l x)$ which is not contained in the cylinder on the right-hand side of the inclusion (29). Then one can choose a point

$$z \in S \quad \textit{with} \quad \langle x, z \rangle = 0, \quad w = \xi x + \zeta z, \quad \zeta < -2\varepsilon.$$

Let $\varphi \in]0, \pi/2[$ be so small that

(30) $$|\xi| \sin\varphi + \zeta \cos\varphi < -2\varepsilon.$$

Since z is orthogonal to x we have

(31) $\|k_l x - k_m z\| < k_m + \varepsilon$ *for all sufficiently large* $m \in \mathbf{N}$.

Since $\|z\|=1$, (26) yields

$$(32) \qquad K(k_m z) \subset \{y | k_m - \varepsilon \leqq \|y\| \leqq k_m + \varepsilon\} \quad \textit{for each} \quad m \in \mathbf{N}.$$

Applying (28) to z we see that there exists a point $d \in K(o)$ such that for all m the vector $-k_m z$ is a δ-vector from $K(k_m z)$ to $d \in K(o)$. (27) implies $\operatorname{diam} K(k_m z) = o(k_m)$ as $m \to \infty$. Hence $K(k_m z) \subset \{y | \|y - k_m z\| \leqq o(k_m)\}$ as $m \to \infty$ and thus

$$(33) \qquad K(k_m z) \subset \left\{ y \left| \left\langle \frac{y}{\|y\|}, z \right\rangle \geqq \cos \varphi \right. \right\} \quad \textit{for all sufficiently large} \quad m \in \mathbf{N}.$$

Fix an $m \in \mathbf{N}$ for which (31) and (33) hold. Then the definition of δ, (32), (33) and (31) imply

$$\begin{aligned}
(34) \qquad w \in K(k_l x) &\subset K(k_m z) + \delta(K(k_l x), K(k_m z)) U \\
&= K(k_m z) + \|k_l x - k_m z\| U \\
&\subset \left\{ y | k_m - \varepsilon \leqq \|y\| \leqq k_m + \varepsilon, \left\langle \frac{y}{\|y\|}, z \right\rangle \geqq \cos \varphi \right\} + (k_m + \varepsilon) U \\
&=: A + (k_m + \varepsilon) U.
\end{aligned}$$

lin $\{z\}$ is the axis of rotation of the compact set A. Thus there exists a point $y \in A \cap \cap \operatorname{lin}\{x, z\}$ which is nearest to $w = \xi x + \zeta z \in \operatorname{lin}\{x, z\}$. An elementary argument shows that $y = (k_m - \varepsilon)(\sin \varphi \cdot \operatorname{sign} \xi \cdot x + \cos \varphi \cdot z)$. (Here sign $0 = 0$.) From this together with (34) and (30) we deduce that

$$\begin{aligned}
(k_m + \varepsilon)^2 \geqq \|w - y\|^2 &= (\xi - (k_m - \varepsilon) \sin \varphi \operatorname{sign} \xi)^2 + (\zeta - (k_m - \varepsilon) \cos \varphi)^2 \\
&= (k_m - \varepsilon)^2 - 2|\xi|(k_m - \varepsilon) \sin \varphi - 2|\zeta|(k_m - \varepsilon) \cos \varphi + \xi^2 + \zeta^2 \\
&\geqq k_m^2 - 2k_m \varepsilon + \varepsilon^2 + 2(k_m - \varepsilon)(-|\xi| \sin \varphi - \zeta \cos \varphi) + \zeta^2 \\
&> k_m^2 - 2k_m \varepsilon + \varepsilon^2 + 2(k_m - \varepsilon) 2\varepsilon + 4\varepsilon^2 = (k_m + \varepsilon)^2.
\end{aligned}$$

This contradiction shows that our assumption was false, thus confirming (29).

The next part of our proof contains a rough description of the images of points. This will be used to obtain a rough description of the images of arbitrary compact sets. It will turn out that the convex hull of the image (under K) of a compact set can be obtained in a simple way from the convex hull of the set itself.

At first we show the following proposition:

(35) *Let $x \in S$ and denote by $H(x)$ the supporting half space of $K(o)$ with exterior normal vector $-x$. Then*

$$K(k_l x) \subset H(x) + (k_l + o(1)) x \quad \textit{as} \quad l \to \infty.$$

Because of (28) there is a point $c \in K(o)$ such that for each l the vector $-k_l x$ is a δ-vector from $K(k_l x)$ to $c \in K(o)$. Therefore the definition of δ-vectors shows that $K(k_l x) \subset \{y | \|y - c\| \geqq k_l (= \|-k_l x\|)\}$. Furthermore $c + k_l x \in K(k_l x)$, together with (27) and (29) shows that $K(k_l x) \subset \operatorname{pos}\{x\} + 2\varepsilon U$. It is obvious that $c \in \operatorname{bd} H(x)$. From these statements we conclude that $K(k_l x) \subset H(x) + (k_l + o(1))x$ as $l \to \infty$. This proves (35). Next we prove the following:

(36) *Let $z \in E^d$, $x \in S$ and let $H(x)$ denote the supporting half space of $K(o)$ with exterior normal vector $-x$. Then $K(z) \subset z + H(x)$.*

Let μx be the orthogonal projection of z onto lin $\{x\}$. Clearly,

(37) $$\|z - k_l x\| = k_l - \mu + o(1) \quad as \quad l \to \infty.$$

Since $\delta(\text{conv}\, K(z),\ \text{conv}\ K(k_l x)) \leqq \delta(K(z),\ K(k_l x)) = \|z - k_l x\|$ by (6) it follows from (35) and (37) that

$$\begin{aligned} K(z) &\subset \text{conv}\, K(z) \subset \text{conv}\, K(k_l x) + \|z - k_l x\|\, U \\ &\subset H(x) + (k_l + o(1))x + (k_l - \mu + o(1))U \\ &= H(x) + \mu x + o(1)x = H(x) + z + o(1)x \quad as \quad l \to \infty \end{aligned}$$

and thus $K(z) \subset H(x) + z$ concluding the proof of (36). An immediate consequence of (36) is that

(38) $$\text{conv}\, K(z) \subset z + \text{conv}\, K(o) \quad for\ each \quad z \in E^d.$$

The following refinement of (38) will be required later on

(39) $$\text{conv}\, K(z) = z + \text{conv}\, K(o) \quad for\ each \quad z \in E^d.$$

Considering (38), (25), (14) and the definition of I_k we see that so far we have proved: For each isometry of $\langle \mathscr{K}, \delta \rangle$ the convex hull of the image of an arbitrary point z is contained in a translate of the convex hull of the image of o. Applying this to the isometry K_z of $\langle \mathscr{K}, \delta \rangle$ defined by $K_z(C) := K(C+z)$ for $C \in \mathscr{K}$ one sees that for each $z \in E^d$ the set conv $K(o) = \text{conv}\ K_z(-z)$ is contained in a translate of conv $K(z) = = \text{conv}\ K_z(o)$. Together with (38) this proves (39).

Given $x \in S$ let $'$ denote the orthogonal projection onto lin $\{x\}$. Line segments of the form $[\alpha x, \beta x]$ will be written simply as $[\alpha, \beta]$. The following propositions (40) and (41) will be used in extending (39) from points to arbitrary compact sets C.

(39) together with (7) implies:

(40) *Let $x \in S$ be given. Then*

$$\delta(C, K(\lambda x)) = \delta(\text{conv}\, C',\ \text{conv}\, K(\lambda x)') + o(1)$$

as $\lambda \to \pm\infty$ for each $C \in \mathscr{K}$.

We show:

(41) *Let $x \in S$ and* conv $K(o)' = [\alpha, \beta]$ $(\alpha \leqq \beta)$. *Then* conv $K([\mu, \nu])' = [\alpha + \mu, \beta + \nu]$ *for all $\mu, \nu \in \mathbf{R}$ with $\nu - \mu > \beta - \alpha$ $(\geqq 0)$.*

Choose μ, ν and let conv $K([\mu, \nu])' =: [\sigma, \tau]\ (\sigma \leqq \tau)$. (39) yields

$$\begin{aligned} \text{conv}\, K(\lambda x)' &[= (\text{conv}\, K(o) + \lambda x)' = \text{conv}\, K(o)' + \lambda x] \\ &= [\alpha + \lambda, \beta + \lambda] \quad for\ each\ \lambda \in \mathbf{R}. \end{aligned}$$

From this together with (40) we conclude that

(42) $$\max\{\alpha+\lambda-\sigma, \beta+\lambda-\tau\} = \delta([\sigma,\tau], [\alpha+\lambda, \beta+\lambda])$$
$$= \delta(\operatorname{conv} K([\mu, \nu])', \operatorname{conv} K(\lambda x)')$$
$$= \delta(K([\mu,\nu]), K(\lambda x))+o(1) = \delta([\mu,\nu], \lambda x)+o(1)$$
$$= \lambda-\mu+o(1) \quad as \quad \lambda \to +\infty,$$

(43) $$\max\{\sigma-\alpha+\lambda, \tau-\beta+\lambda\} = \delta([\sigma,\tau], [\alpha-\lambda, \beta-\lambda])$$
$$= \delta(\operatorname{conv} K([\mu,\nu])', \operatorname{conv} K(-\lambda x)')$$
$$= \delta(K([\mu,\nu]), K(-\lambda x))+o(1) = \delta([\mu,\nu], -\lambda x)+o(1)$$
$$= \nu+\lambda+o(1) \quad as \quad \lambda \to +\infty.$$

In particular, $\alpha+\lambda-\sigma \leqq \lambda-\mu+o(1)$, $\tau-\beta+\lambda \leqq \nu+\lambda+o(1)$ (as $\lambda \to +\infty$) and therefore $\alpha+\mu \leqq \sigma$, $\tau \leqq \beta+\nu$. Suppose $\alpha+\mu<\sigma$. Then $\alpha+\lambda-\sigma<\lambda-\mu+o(1)$ and thus $\beta+\lambda-\tau=\lambda-\mu+o(1)$ by (42). Hence $\beta=\tau-\mu$. Together with $\mu<\nu$ this implies $\tau-\beta+\lambda=\mu+\lambda<\nu+\lambda+o(1)$. Now (43) shows that $\sigma-\alpha+\lambda=\nu+\lambda+o(1)$, i.e., $\alpha=\sigma-\nu$. It follows from $\beta=\tau-\mu$ and $\alpha=\sigma-\nu$ that $\beta-\alpha=(\nu-\mu)+(\tau-\sigma)\geqq\nu-\mu$, a contradiction. Therefore $\alpha+\mu=\sigma$. Similarly, one can show that $\tau=\beta+\nu$. Thus $[\sigma,\tau]=[\alpha+\mu, \beta+\nu]$, confirming (41).

(44) *For all $x\in S$ and $\mu, \nu\in \mathbf{R}^+$ $(\mu<\nu)$ the inclusion* $K([\mu,\nu])\subset[\mu,\nu]+\operatorname{conv} K(o)$ *holds.*

Choose x, μ, ν. Then (44) is a consequence of the definition of δ, (6), (39) and (8), namely

$$K([\mu,\nu]) \subset \operatorname{conv} K([\mu,\nu])$$
$$\subset \bigcap_{z\in E^d} \{\operatorname{conv} K(z)+\delta(\operatorname{conv} K(z), \operatorname{conv} K([\mu,\nu]))U\}$$
$$\subset \bigcap_{z\in E^d} \{z+\operatorname{conv} K(o)+\delta(z, [\mu,\nu])U\}$$
$$= [\mu,\nu]+\operatorname{conv} K(o).$$

It follows from (44) and (7) that

(45) *for all $x\in S$ and $C\in\mathscr{K}$ we have*

$$\delta(\operatorname{conv} C', \pm[\lambda, 2\lambda]) = \delta(C, \pm[\lambda, 2\lambda])+o(1)$$
$$= \delta(K(C), K(\pm[\lambda, 2\lambda]))+o(1)$$
$$= \delta(\operatorname{conv} K(C)', \operatorname{conv} K(\pm[\lambda, 2\lambda])')+o(1) \quad as \quad \lambda\to+\infty.$$

We are now in a position to prove that

(46) *for all $C\in\mathscr{K}$ and $x\in S$ the equality* $\operatorname{conv} K(C)'=\operatorname{conv} C'+\operatorname{conv} K(o)'$ *holds.*

For given C, x let $\operatorname{conv} K(o)'=[\alpha, \beta]$ $(\alpha \leqq \beta)$ be as before and suppose $\operatorname{conv} C'=[\xi, \eta]$ $(\xi \leqq \eta)$, $\operatorname{conv} K(C)'=[\varphi, \psi]$ $(\varphi \leqq \psi)$. Then (45) and (41) yield

$$\begin{aligned} 2\lambda-\eta &= \delta([\xi, \eta],\ [\lambda, 2\lambda]) = \delta(\operatorname{conv} C',\ [\lambda, 2\lambda]) \\ &= \delta\big(\operatorname{conv} K(C)',\ \operatorname{conv} K([\lambda, 2\lambda])'\big)+o(1) \\ &= \delta([\varphi, \psi],\ [\alpha+\lambda, \beta+2\lambda])+o(1) = 2\lambda+\beta-\psi+o(1) \quad \text{as} \quad \lambda \to +\infty, \\ 2\lambda+\xi &= \delta([\xi, \eta],\ -[\lambda, 2\lambda]) = \delta(\operatorname{conv} C',\ -[\lambda, 2\lambda]) \\ &= \delta\big(\operatorname{conv} K(C)',\ \operatorname{conv} K(-[\lambda, 2\lambda])'\big)+o(1) \\ &= \delta([\varphi, \psi],\ [\alpha-2\lambda, \beta-\lambda])+o(1) = \varphi+2\lambda-\alpha+o(1) \quad \text{as} \quad \lambda \to +\infty. \end{aligned}$$

Consequently, $2\lambda-\eta=2\lambda+\beta-\psi+o(1)$, $2\lambda+\xi=\varphi+2\lambda-\alpha+o(1)$ as $\lambda \to +\infty$ and thus $\varphi=\alpha+\xi$, $\psi=\beta+\eta$, i.e., $[\varphi, \psi]=[\xi, \eta]+[\alpha, \beta]$. This proves (46). An immediate consequence of (46) is:

(47) *For each* $C \in \mathscr{K}$ *the equality* $\operatorname{conv} K(C)=\operatorname{conv} C+\operatorname{conv} K(o)$ *holds.*

We now come to the final part of the proof. First we suppose that $K(o)$ consists of more than one point. Then it is possible to construct an isometry L for which $L(o)=U$. Now an investigation of images of sets consisting of two points only shows that L cannot be an isometry. This shows that we need to consider only the case when $K(o)$ consists of a single point. Then apply (2) to prove that K and thus I is generated by an isometry of E^d.

Suppose $K(o)$ consists of more than one point and let $D:=\operatorname{conv} K(o)$. By (11) there exists a sequence of rotational means $m_1, m_2, \ldots$ such that

$$(48) \qquad m_1(D), m_2(D), \ldots \to \varrho U \quad \textit{for suitable} \quad \varrho \in \mathbf{R}^+.$$

Given $k \in \mathbf{N}$ suppose $m_k(C)=\lambda_1 i_1(C)+\ldots+\lambda_n i_n(C)$ for $C \in \mathscr{C}$ and let $K_k\colon \mathscr{K} \to \mathscr{K}$ be defined for $C \in \mathscr{K}$ by

$$K_k(C):= \varrho^{-1}\lambda_1 i_1\big(K(\lambda_1^{-1} i_1^{-1}(\lambda_2 i_2(K(\lambda_2^{-1} i_2^{-1} \ldots \big(\lambda_n i_n(K(\lambda_n^{-1} i_n^{-1}(\varrho C)) \ldots\big).$$

It is easy to see that K_k is an isometry of $\langle \mathscr{K}, \delta \rangle$ for each k. Furthermore (47) yields

$$(49) \qquad \begin{aligned} \operatorname{conv} K_k(C)\big[&= \varrho^{-1}\lambda_1 i_1\big(\operatorname{conv}(K(\lambda_1^{-1} i_1^{-1}(\lambda_2 i_2(K(\lambda_2^{-1} i_2^{-1} \ldots))))))\big) \\ &= \varrho^{-1}\lambda_1 i_1\big(\operatorname{conv}(\lambda_1^{-1} i_1^{-1}(\lambda_2 i_2(K(\lambda_2^{-1} i_2^{-1} \ldots)))) + D\big) \\ &= \operatorname{conv}\big(\varrho^{-1}\lambda_2 i_2(K(\lambda_2^{-1} i_2^{-1} \ldots))\big)+\varrho^{-1}\lambda_1 i_1(D) \\ &= \varrho^{-1}\lambda_2 i_2\big(\operatorname{conv}(K(\lambda_2^{-1} i_2^{-1} \ldots))\big)+\varrho^{-1}\lambda_1 i_1(D) \\ &= \varrho^{-1}\lambda_2 i_2\big(\operatorname{conv}(\lambda_2^{-1} i_2^{-1} \ldots)+D\big)+\varrho^{-1}\lambda_1 i_1(D) \\ &= \operatorname{conv}(\ldots)+\varrho^{-1}\big(\lambda_1 i_1(D)+\lambda_2 i_2(D)\big) = \ldots\big] \\ &= \operatorname{conv} C+\varrho^{-1} m_k(D) \quad \textit{for each} \quad C \in \mathscr{K}. \end{aligned}$$

This together with (48) implies that $\operatorname{conv} K_k(o)=\varrho^{-1} m_k(D) \to U$. Therefore the sequence $K_1(o), K_2(o), \ldots$ is contained in a bounded subset of E^d. Hence (1) shows that there are a subsequence $K_{k_1}, K_{k_2}, \ldots$ and an isometry L of $\langle \mathcal{K}, \delta \rangle$ such that

$$L(C) = \lim_{l \to \infty} K_{k_l}(C) \quad \text{for each} \quad C \in \mathcal{K}.$$

From this together with (9), (49), (48) and (10) we infer that

$$(50) \qquad L(C) \subset \operatorname{conv} L(C) [= \operatorname{conv} \lim_{l \to \infty} K_{k_l}(C) = \lim_{l \to \infty} \operatorname{conv} K_{k_l}(C)$$

$$= \lim_{l \to \infty} (\operatorname{conv} C + \varrho^{-1} m_k(D))] = \operatorname{conv} C + U.$$

Let $x, y \in S$ be chosen so that $\delta(x, y) (= \|x-y\|) = 2 \sin (\pi/8)$. Then

$$(51) \qquad \delta(\{-\lambda x, \lambda x\}, \{-\lambda y, \lambda y\}) = 2\lambda \sin \frac{\pi}{8} \quad \textit{for each} \quad \lambda \in \mathbf{R}^+.$$

Let $'$ and $''$ be the orthogonal projections of E^d onto $\operatorname{lin}\{x\}$ and $\operatorname{lin}\{y\}$, respectively. For line segments $[\alpha x, \beta x] \subset \operatorname{lin}\{x\}$ we simply write $[\alpha, \beta]$. Let $S(x)$ denote the hemisphere $\{z \in S | \langle x, z \rangle \geqq 0\}$. Similar notations will be used for y. We have

$$(52) \quad (-S(x) - \lambda x) \cup (S(x) + \lambda x) \subset L(\{-\lambda x, \lambda x\}) \subset [-\lambda, \lambda] + U \quad \textit{for each} \quad \lambda \in \mathbf{R}^+,$$

$$L(\{-\lambda x, \lambda x\})' \subset [-\lambda - 1, -\lambda + o(1)] \cup [\lambda - o(1), \lambda + 1] \quad \textit{as} \quad \lambda \to +\infty.$$

To prove this choose $\lambda \in \mathbf{R}^+$. By (50) $\operatorname{conv}(L(\{-\lambda x, \lambda x\})) = [-\lambda, \lambda] + U$. Since each extreme point of the convex hull of a compact set belongs to the set, $(-S(x) - \lambda x) \cup (S(x) + \lambda x) \subset L(\{-\lambda x, \lambda x\})$. This proves half of (52). Now assume $\lambda > 2$. As before

$$\left(-S(x) - \frac{\lambda}{2} x\right) \cup \left(S(x) + \frac{\lambda}{2} x\right) \subset L\left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]\right).$$

This together with (48) and the first part of (52) implies

$$L\left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]\right) + \left(\frac{\lambda}{2} + 1\right) U \supset \left[\left(\left(-S(x) - \frac{\lambda}{2} x\right) \cup \left(S(x) + \frac{\lambda}{2} x\right)\right) + \left(\frac{\lambda}{2} + 1\right) U\right.$$

$$\left.\supset [-\lambda, \lambda] + U \supset \operatorname{conv} L(\{-\lambda x, \lambda x\})\right] \supset L(\{-\lambda x, \lambda x\}).$$

Thus $\delta([-\lambda/2, \lambda/2], \{-\lambda x, \lambda x\}) = \lambda > (\lambda/2) + 1$ and the definition of δ imply that

$$(53) \qquad L(\{-\lambda x, \lambda x\}) + \mu U \supset L\left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]\right)$$

holds for $\mu = \lambda$ *but not for* $\mu \in [0, \lambda[$.

In order to prove

$$L(\{-\lambda x, \lambda x\})' \cap]-\lambda+\varepsilon(\lambda), \lambda-\varepsilon(\lambda)[= \emptyset \tag{54}$$

$$\textit{where} \quad \varepsilon(\lambda) = \max\left\{\lambda-\sqrt{\lambda^2-2},\ 2\lambda-2\sqrt{\lambda^2-1}\right\}$$

assume that there is a point $z\in L(\{-\lambda x, \lambda x\})$ with $z'\in]-\lambda+\varepsilon(\lambda), \lambda-\varepsilon(\lambda)[$. Then by the first half of (52) and (47)

$$L(\{-\lambda x, \lambda x\})+\mu U \supset [((-S(x)-\lambda x)\cup(S(x)+\lambda x)\cup\{z\})+\mu U$$

$$\supset \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]+U = \operatorname{conv}\left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]\right)+U \supset \left] L\left(\left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]\right)\right.$$

for some $\mu\in[0, \lambda[$, which contradicts (53). Thus (54) holds. The first half of (52) together with (54) proves the second half of (52). The same reasoning yields:

$$(-S(y)-\lambda y)\cup(S(y)+\lambda y) \subset L(\{-\lambda y, \lambda y\}) \subset [-\lambda y, \lambda y]+U \quad \textit{for} \quad \lambda\in\mathbf{R}^+, \tag{55}$$

$$L(\{-\lambda y, \lambda y\})'' \subset [(-\lambda-1)y, (-\lambda+o(1))y]\cup[(\lambda-o(1))y, (\lambda+1)y] \quad \textit{as} \quad \lambda\to+\infty.$$

Choose a coordinate system in lin $\{x, y\}$ such that $x=(1, 0)$, $y=(1/\sqrt{2}, 1/\sqrt{2})$. Then

$$z := \lambda y+\left(-\sin\frac{\pi}{8}, \cos\frac{\pi}{8}\right)\in S(y)+\lambda y \subset L(\{-\lambda y, \lambda y\})$$

by (55). The point of $L(\{-\lambda x, \lambda x\})$ which is nearest to z is of the form $\lambda x+(0, 1)+w$ with $\|w\|=o(1)$ as $\lambda\to+\infty$ by (52). Hence

$$\left\|z-(\lambda x+(0, 1)+w)\right\| \geqq 2\lambda\sin\frac{\pi}{8}+\left(1-\cos\frac{\pi}{8}\right)+o(1) \quad \textit{as} \quad \lambda\to+\infty. \tag{56}$$

On the other hand the definition of δ and (51) imply that

$$\left\|z-(\lambda x+(0, 1)+w)\right\| \leqq \delta(L(\{-\lambda x, \lambda x\}),\ L(\{-\lambda y, \lambda y\})) = 2\lambda\sin\frac{\pi}{8}$$

which contradicts (56). Hence $K(o)$ cannot consist of more than one point.

Now assume that $K(o)$ consists of one point only. Then (2) implies that the isometry K of $\langle\mathscr{K}, \delta\rangle$ is generated by an isometry of E^d. Because of (25) and (14) this shows that I is generated by an isometry of E^d, concluding the proof of the theorem.

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