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FINDING ZEROES OF POLYNOMIALS USING POWER SERIES

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Short Communication[1]

Giving explicitly the formal power series, which solves some functional equation, we can use this result to calculate the roots of a polynomial. If the polynomial consists of $k+2$ monomials, the power series have k indeterminates. The case of trinomials (i.e. $k=1$) was already published in [1]. Now we report about the generalizations for $k>1$, and also confirm the conjectures, which were raised in [1]. The explicit formulas were found by Ing. Otto Cella, an engineer from Graz, who spent most of his spare time of the last past thirty years in this problem.

1. Formal solution of a functional equation.

Let K be a field of characteristic 0. For $k \in \mathbb{N}$ let $K[\![\mathbf{X}]\!] := K[\![X_1, \ldots, X_k]\!]$ denote the ring of formal power series over K in k indeterminates $X_1, \ldots, X_k$. For $a \in K$ and integers $j_1, \ldots, j_k \geq 0$ define the multinomial coefficient by

$$\binom{a}{j_1, \ldots, j_k} := \frac{\prod_{i=0}^{J-1} (a-i)}{j_1! \cdot \ldots \cdot j_k!} \qquad \text{with} \quad J := \sum_{i=1}^{k} j_i .$$

If $f \in K[\![\mathbf{X}]\!]$ is a power series with $f(0, \ldots, 0) = 1$, we define the power of f to an exponent $n \in K$ by using the binomial series, i.e.

$$f^n := \sum_{l=0}^{\infty} \binom{n}{l} (f-1)^l \in K[\![\mathbf{X}]\!] .$$

The following theorem generalizes Proposition 1 in [1].

[1]The full version (with proofs) will be published elsewhere.

Theorem 1. *Let* $n, m, n_1, \ldots, n_k \in K$ *with* $n \neq m$. *Then*

$$f = \sum_{j_1,\ldots,j_k \geq 0} \frac{1}{A} \binom{\frac{A}{n-m}}{j_1, \ldots, j_k} X_1^{j_1} \cdot \ldots \cdot X_k^{j_k} \tag{1}$$

$$\text{with} \quad A := 1 - \sum_{i=1}^{k} j_i (m - n_i)$$

is the unique solution in $K[\![\mathbf{X}]\!]$ *of the equation*

$$f^n - f^m = \sum_{i=1}^{k} X_i \, f^{n_i} \qquad \text{with} \quad f(0, \ldots, 0) = 1 .$$

If $A = 0$ for some term in (1), one must replace the corresponding coefficient by using some algebraic identity for multinomial coefficients, e.g.

$$\frac{1}{A} \binom{\frac{A}{n-m}}{j_1, \ldots, j_k} = \frac{1}{A - (n-m) \sum_{i=1}^{k} j_i} \binom{\frac{A}{n-m} - 1}{j_1, \ldots, j_k} .$$

The proof of theorem 1 uses the multi-dimensional form of the Bürmann-Lagrange formula, usually attributed to Good, (see e.g. [2], p. 21).

2. Finding zeroes of a complex polynomial.

Let $g = \sum_{i=0}^{N} a_i Z^i \in \mathbb{C}[Z]$ be a complex polynomial of degree N with $k+2$ nonvanishing coefficients a_i and $k \geq 1$. Choosing two of the nonvanishing coefficients, say a_n and a_m with $n > m$, we have

$$g = a_n Z^n + a_m Z^m + \sum_{i=1}^{k} a_{n_i} Z^{n_i} \qquad \text{with} \quad a_{n_i} \neq 0 .$$

Substituting $Z = \alpha Y$, where α is any of the $(n-m)$ different values of $\sqrt[n-m]{-\frac{a_m}{a_n}}$, the equation $g(Z) = 0$ is transformed to

$$Y^n - Y^m = \sum_{i=1}^{k} b_i \, Y^{n_i} \qquad \text{with} \quad b_i = \frac{a_{n_i}}{a_m} \alpha^{n_i - m} . \tag{2}$$

If the power series f, as given in (1), converges at the point $(b_1, \ldots, b_k)$, theorem 1 implies that $\alpha \cdot f(b_1, \ldots, b_k)$ is a root of the given polynomial g. In the case of convergence one can show that the $(n-m)$ different values of α indeed yield roots of g with appropriate algebraic multiplicities.

To decide which of the $\binom{k+2}{2}$ choices of pairs of nonvanishing coefficients of the polynomial g leads to a converging solution, one has to know the domain of convergence $D(f)$ of f. $D(f)$ depends on the discriminant of the polynomial given in (2), and we can exhibit only some polycylinder containing $D(f)$.

Although many numerical examples behave well, one can show that for $k > 1$ there are cases, in which the above method does not lead to all zeroes of the polynomial. Nevertheless taking only a few terms of the power series, one obtains – even if the series diverges – astonishingly good approximations of the zeroes. So our method might be helpful in locating the roots of a polynomial.

3. Finding zeroes of a complex trinomial.

In the case of a complex trinomial, i.e.

$$g = a_1 Z^{n_1} + a_2 Z^{n_2} + a_3 \in \mathbb{C}[Z],$$

the above described method always works. We were able to prove the conjectures raised in [1], about the correct multiplicity of the roots and the convergence of the series on the boundary of the circle of convergence. For the latter question one can use an idea of Riemann, (see [3], p. 16).

Therefore proposition 4 of [1] can be strengthened as follows:

Theorem 2. *Let*

$$g(Z) = a_1 Z^n + a_2 Z^m + a_3 \in \mathbb{C}[Z]$$

be a complex polynomial with $n > m > 0$ *und* $a_i \in \mathbb{C} \setminus \{0\}$.

(1) *If*

$$\frac{|a_2|^n}{|a_1|^m \, |a_3|^{n-m}} \leq \frac{n^n}{m^m \, (n-m)^{n-m}}$$

then the numbers

$$\alpha \, f_{n-m,-m}(\beta)$$

are the n *zeroes of* g , *where* α *runs through the* n *roots of* $\alpha^n = -\frac{a_3}{a_1}$ *and* $\beta = \frac{a_2}{a_3} \cdot \alpha^m$.

(2) *If*

$$\frac{|a_2|^n}{|a_1|^m \, |a_3|^{n-m}} \geq \frac{n^n}{m^m \, (n-m)^{n-m}}$$

then the numbers

$$\alpha' \, f_{n,m}(\beta') \quad \text{and} \quad \alpha'' \, f_{m-n,-n}(\beta'')$$

are the n *zeroes of* g , *where* $\alpha', \alpha'' \in \mathbb{C}$ *run through the roots of* $(\alpha')^{n-m} = -\frac{a_2}{a_1}$ *and* $(\alpha'')^m = -\frac{a_3}{a_2}$, *respectively, and* $\beta' = \frac{a_3}{a_2}(\alpha')^{-m}$, $\beta'' = \frac{a_1}{a_3}(\alpha'')^n$, *respectively.*

References

[1] O. Cella & G. Lettl, *Power series and zeroes of trinomial equations*, Aequa. Math. **43** (1992), 94 – 102.

[2] I. P. Goulden & D. M. Jackson, *Combinatorial enumeration*, Wiley & Sons, New York, 1983.

[3] F. Klein, *Vorlesungen über die hypergeometrische Funktion*, Grundlehren Bd. 39, Springer Berlin, 1933.

Note added in proof.

Although the use of analytical methods for solving equations is a very old idea, it was just recently that we succeeded in our search through the old literature. A. CAPELLI, Hj. MELLIN, R. BIRKELAND, G. BELARDINELLI and many others worked on this subject at the beginning of this century. Our work rediscovers results of Hj. MELLIN, which were published in

[4] *Zur Theorie der trinomischen Gleichungen*, Ann. Ac. Sc. Fenn. **7** no. 7 (1915).

[5] *Ein allgemeiner Satz über algebraische Gleichungen*, Ann. Ac. Sc. Fenn. **7** no. 8 (1915).

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