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Stickelberger elements and cotangent numbers

Günter Lettl

Abstract. Using Leopoldt's theorem, which describes the structure of the integers of an abelian number field K as Galois module, we will show a connection between Stickelberger elements and cotangent numbers. Both of them have been used for index formulas containing h_K^-, the minus part of the class number of K.

1. Introduction

Let K be an abelian number field, i. e. a Galois extension of $\mathbf{Q}$ of finite degree with abelian Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. For any such K , Girstmair [1] defined a cotangent number $\eta_K \in K$ and calculated the index of the $\mathbf{Z}$- module generated by the conjugates of η_K in the imaginary part of o_K , the ring of integers of K . This index turns out to be h_K^- , the minus part of the class number of K , multiplied with a rational factor, which depends only on well-known arithmetical constants of K . This new algebraic description of h_K^- is in contrast to the known relation of h_K^- to the index of the Stickelberger ideal in the imaginary part of the group ring over G ([2], [6], [7]), where the computation of the additional factor in the index formula is rather difficult in general. A formal analogy between cotangent numbers and Stickelberger elements is obvious, but there was no explanation for that.

In this paper we develop a connection between cotangent numbers and Stickelberger elements, using Leopoldt's description of o_K as Galois module ([4], [5]). It turns out that the cotangent number of K is obtained by applying a sum of (modified) Stickelberger elements to a "Basiszahl" of K . Going back to the group ring, we find fairly easy index formulas (also for Stickelberger elements of higher degree), using an -in general fractional- principal Stickelberger ideal.

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2. Modified Stickelberger Elements

For $n \in \mathbf{N}$ let $\zeta_n = e^{\frac{2\pi i}{n}} \in \mathbf{C}$ be the analytically normed, primitive n- th root of unity and $\mathbf{Q}^{(n)} = \mathbf{Q}(\zeta_n) \subset \mathbf{C}$ the n- th cyclotomic field. Without loss of generality, every abelian number field shall be considered as a subfield of $\mathbf{C}$, thus by the theorem of Kronecker – Weber as subfield of some $\mathbf{Q}^{(n)}$. The natural operation of $\mathbf{Z}/(n)^\times$ on μ_n, the group of n- th roots of unity, yields

$$\begin{array}{ccccc} \{t \in \mathbf{Z} \mid (t,n)=1\} & \to & \mathbf{Z}/(n)^\times & \xrightarrow{\sim} & G^{(n)} = \mathrm{Gal}(\mathbf{Q}^{(n)}/\mathbf{Q}) \\ t & \mapsto & t+(n) & \mapsto & \sigma_t \end{array}$$

where σ_t is defined by $\sigma_t(\zeta_n) = \zeta_n^t$. Note that σ_{-1} is just complex conjugation for any n. For $m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ the m- th Bernoulli polynomial $B_m(X) \in \mathbf{Q}[X]$ is defined by

$$\frac{te^{Xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(X)\,\frac{t^m}{m!}\,.$$

For $x \in \mathbf{R}$ let $\langle x \rangle = x - [x] \in [0,1)$ denote the least non-negative element of $x + \mathbf{Z}$. To establish the connection with cotangent numbers in the next chapter, we need modified Stickelberger elements. In contrast to the usual definition (see e. g. [3], p.44) we take the inverse of each element of the Galois group.

Definition 1: Let $m, n \in \mathbf{N}$, $K/\mathbf{Q}$ be an abelian number field with $G = \mathrm{Gal}(K/\mathbf{Q})$ and $K_n = K \cap \mathbf{Q}^{(n)}$.

a) $\vartheta_m(\mathbf{Q}^{(n)}, n) = \frac{n^{m-1}}{m} \sum\limits_{\substack{t=1 \\ (t,n)=1}}^{n} B_m(\langle \frac{t}{n} \rangle)\, \sigma_t \in \mathbf{Q}[G^{(n)}]$ is called (modified) Stickelberger element of $\mathbf{Q}^{(n)}$ of degree m.

b) $\vartheta_m(K,n) = \mathrm{cor}_{K/K_n}\, \mathrm{res}_{\mathbf{Q}^{(n)}/K_n}\, \vartheta_m(\mathbf{Q}^{(n)}, n) =$
$\frac{n^{m-1}}{m} \sum\limits_{\substack{t=1 \\ (t,n)=1}}^{n} \Big(B_m(\langle \frac{t}{n} \rangle) \sum\limits_{\substack{\sigma \in G \\ \sigma|_{K_n} = \sigma_t|_{K_n}}} \sigma \Big) \in \mathbf{Q}[G]$ is called (modified) Stickelberger element of K of degree m and level n.

Definition 1.b) employs the usual method to transport Stickelberger elements from $\mathbf{Q}^{(n)}$ to K. $\mathrm{res}_{\ldots/\ldots}$ and $\mathrm{cor}_{\ldots/\ldots}$ denote restriction and corestriction between the rational group rings of the corresponding Galois groups (see e. g. [7], p.185).

The following lemma shows how Stickelberger elements behave, when they are restricted to a cyclotomic subfield.

LEMMA 1. *Let $m, n, N \in \mathbf{N}$ with $n|N$ and put $\mathcal{P} = \{p \in \mathbf{P} |\ p|N$ and $p \nmid n\}$. Then*

$$\vartheta_m(\mathbf{Q}^{(n)}, N) = \prod_{p \in \mathcal{P}} (1 - p^{m-1}\sigma_p)\ \vartheta_m(\mathbf{Q}^{(n)}, n)\ .$$

Using the distribution relation for $B_m(X)$ (e. g. [3], pp.35f.), the usual straightforward calculation reducing the range of summation from $\mathbf{Z}/(N)^\times$ to $\mathbf{Z}/(n)^\times$ gives the proof of Lemma 1.

If K is an abelian number field with Galois group G, let the group ring $\mathbf{Q}[G]$ operate on the additive structure of K as usual, i. e. for $\rho = \sum_{\sigma \in G} r_\sigma \sigma \in \mathbf{Q}[G]$ and $\alpha \in K$ put $\rho\alpha = \sum_{\sigma \in G} r_\sigma\, \sigma(\alpha)$. Next we recall an obvious extension of Lemma 3 of [5].

LEMMA 2. *Let $p \in \mathbf{P}$, $n \in \mathbf{N}$, $\sigma \in G^{(n)}$ and $\zeta \in \mu_{np}$ an np-th root of unity. Then*

$$(\mathrm{cor}_{\mathbf{Q}^{(np)}/\mathbf{Q}^{(n)}}\sigma)\,\zeta = \begin{cases} [\mathbf{Q}^{(np)} : \mathbf{Q}^{(n)}]\,\sigma(\zeta) & \text{if } \zeta \in \mu_n \qquad \textbf{(i)} \\ 0 & \text{if } \zeta \notin \mu_n \text{ and } p|n \qquad \textbf{(ii)} \\ -\sigma_p^{-1}\sigma(\zeta^p) & \text{if } \zeta \notin \mu_n \text{ and } p \nmid n \qquad \textbf{(iii)} \end{cases}$$

For $n \in \mathbf{N}$ let $q(n)$ be the powerful part of n, i. e. $n = q(n) \prod_{\substack{p \in \mathbf{P} \\ p|n,\, p^2 \nmid n}} p$. If $n, N \in \mathbf{N}$ with $n|N$, then $\mathbf{Q}^{(N)}/\mathbf{Q}^{(n)}$ is wildly ramified if and only if $q(n) \neq q(N)$.

As usual, let φ denote Euler's totient function.

LEMMA 3. *Let $d, m, n', n \in \mathbf{N}$, put $n_0 = (d, n')$, $n_1 = \mathrm{lcm}(d, n')$ and suppose $n_1|n$. Further let ζ be a root of unity of order d. Then we have:*

a) $\vartheta_m(\mathbf{Q}^{(n)}, n')\,\zeta = 0$, *if $q(n_0) \neq q(d)$ (i. e. $\mathbf{Q}^{(d)}/\mathbf{Q}^{(n_0)}$ is wildly ramified).*

b) $\vartheta_m(\mathbf{Q}^{(n)}, n')\,\zeta = \frac{\varphi(n)}{\varphi(n_1)} \prod_{p|\frac{d}{n_0}} (-\sigma_p^{-1}) \prod_{p|n', p\nmid n_0} (1 - p^{m-1}\sigma_p) \cdot$
$\vartheta_m(\mathbf{Q}^{(n_0)}, n_0)\,\zeta^{\frac{d}{n_0}}$, *if* $q(n_0) = q(d)$.

PROOF: $\vartheta_m(\mathbf{Q}^{(n)}, n')\,\zeta = [\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_1)}]\,\vartheta_m(\mathbf{Q}^{(n_1)}, n')\,\zeta = [\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_1)}]\,\vartheta_m(\mathbf{Q}^{(d)}, n')\,\zeta = [\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_1)}]\ \mathrm{cor}_{\mathbf{Q}^{(d)}/\mathbf{Q}^{(n_0)}}\ \vartheta_m(\mathbf{Q}^{(n_0)}, n')\,\zeta = [\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_1)}]\ \mathrm{cor}_{\mathbf{Q}^{(d)}/\mathbf{Q}^{(n_0)}}\Big(\prod_{p|n', p\nmid n_0} (1 - p^{m-1}\sigma_p)\ \vartheta_m(\mathbf{Q}^{(n_0)}, n_0)\Big)\zeta$, using Lemma 1. If $\mathbf{Q}^{(d)}/\mathbf{Q}^{(n_0)}$ is wildly ramified, this equals 0 by Lemma 2.(ii), which proves part a).

Now suppose that $q(n_0) = q(d)$ and use Lemma 2.(iii). Then the above expression equals $[\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_1)}] \prod_{p|n', p\nmid n_0} (1 - p^{m-1}\sigma_p)$
$\frac{n_0^{m-1}}{m} \sum_{\substack{t=1 \\ (t,n_0)=1}}^{n_0} B_m(\langle\frac{t}{n_0}\rangle)\ \mathrm{cor}_{\mathbf{Q}^{(d)}/\mathbf{Q}^{(n_0)}}\ \sigma_t\ \zeta =$
$\frac{\varphi(n)}{\varphi(n_1)} \prod_{p|n', p\nmid n_0} (1 - p^{m-1}\sigma_p) \prod_{p|\frac{d}{n_0}} (-\sigma_p^{-1})\ \vartheta_m(\mathbf{Q}^{(n_0)}, n_0)\ \zeta^{\frac{d}{n_0}}$. ∎

LEMMA 4. *Let* $m, n \in \mathbf{N}$ *with* n *odd. Then*

$$\vartheta_m(\mathbf{Q}^{(2n)}, 2n)\,\zeta_{2n} = (2^{m-1} - \sigma_2^{-1})\ \vartheta_m(\mathbf{Q}^{(n)}, n)\,\zeta_n\ .$$

PROOF: Using Lemma 2.(iii) with $\zeta = \zeta_{2n}$, $p = 2$ and $\sigma = \sigma_1$, we have $\zeta_{2n} = -\sigma_2^{-1}\zeta_n$. Therefore $\vartheta_m(\mathbf{Q}^{(2n)}, 2n)\,\zeta_{2n} =$
$-\sigma_2^{-1}\ \vartheta_m(\mathbf{Q}^{(n)}, 2n)\,\zeta_n = -\sigma_2^{-1}(1 - 2^{m-1}\sigma_2)\,\vartheta_m(\mathbf{Q}^{(n)}, n)\,\zeta_n =$
$(2^{m-1} - \sigma_2^{-1})\,\vartheta_m(\mathbf{Q}^{(n)}, n)\,\zeta_n$, where we made use of Lemma 1 again. ∎

3. Cotangent Numbers and Main Result

For the rest of this paper, K denotes an abelian number field with $G = \mathrm{Gal}(K/\mathbf{Q})$ and conductor n , i. e. n is the smallest positive integer with $K \subset \mathbf{Q}^{(n)}$.

DEFINITION 2: For $m \in \mathbf{N}$ and $K \neq \mathbf{Q}$

$$\eta_{m,K} = \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\Big(\frac{i}{2}\Big)^m \cot^{(m-1)}\Big(\frac{\pi}{n}\Big) - \frac{1}{2}\,\delta_{1,m}\Big) =$$
$$= \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\frac{n^{m-1}}{m}\sum_{j=1}^{n} B_m\Big(\Big\langle\frac{j}{n}\Big\rangle\Big)\,\zeta_n^{-j}\Big)$$

is called cotangent number of K of degree m.

$$\eta_K = (1-\sigma_{-1})\,\eta_{1,K} = \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\,(i\,\cot\frac{\pi}{n})$$

is called cotangent number of K.

Here $\mathrm{Tr}_{\mathbf{Q}^{(n)}/K}$ denotes the trace from $\mathbf{Q}^{(n)}$ to K, $\cot^{(m-1)}$ the $(m-1)$-st derivative of the cotangent function, and $\delta_{1,m} = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{if } m\neq 1 \end{cases}$ Kronecker's delta. The equality of the two expressions defining $\eta_{m,K}$ can be found in [1], considering the following remark: since $B_m(1)=B_m(0)$ for even m and $B_m(1)=-B_m(0)=0$ for odd $m\geq 3$, the only difference between Girstmair's definition of $\eta_{m,K}$ and Definition 2 is for $m=1$, where Girstmair considers $\eta_{1,K}+[\mathbf{Q}^{(n)}:K] = \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\,(\frac{i}{2}\cot\frac{\pi}{n}+\frac{1}{2})$. If we use $\eta_{1,K}$ to obtain the cotangent number $\eta_K = (1-\sigma_{-1})\,\eta_{1,K}$, this difference becomes immaterial.

Splitting the sum in Definition 2 by summing over those j, for which (j,n) is the same, and using the modified Stickelberger elements, we obtain

$$\eta_{m,K} = \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\sigma_{-1}\sum_{\substack{t\\1\leq t|n}}\Big(\frac{n}{t}\Big)^{m-1}\,\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big)\,. \tag{1}$$

Let us define modified cotangent numbers of degree m by

$$\begin{aligned}\widetilde{\eta_{m,K}} &= \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\sigma_{-1}\sum_{\substack{t\\1\leq t|n}}\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big) = \\ &= \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\frac{n^{m-1}}{m}\sum_{j=1}^{n}\frac{1}{(j,n)^{m-1}}\,B_m(\Big\langle\frac{j}{n}\Big\rangle)\,\zeta_n^{-j}\Big)\,.\end{aligned} \tag{2}$$

Obviously $\widetilde{\eta_{1,K}} = \eta_{1,K}$.

Now we will recall Leopoldt's theorem. Put $n^* = \prod_{p\in\mathbf{P},\,p|n} p$, $\mathcal{D} = \{d\in\mathbf{N}|\ n^*|d \text{ and } d|n\}$, and define a "Basiszahl" of K by

$$T = \sum_{d\in\mathcal{D}}\mathrm{Tr}_{\mathbf{Q}^{(d)}/K_d}\,\zeta_d \tag{3}$$

where $K_d = K \cap \mathbf{Q}^{(d)}$. Note that in [5] a slightly different set $\mathcal{D}$ is used, namely the set of all possible conductors of Dirichlet characters of K. Since for odd $k \in \mathbf{N}$ $\mathbf{Q}^{(k)} = \mathbf{Q}^{(2k)}$ and $\zeta_{2k} = -\sigma_2^{-1}\zeta_k$, Leopoldt's theorem holds with T as defined in (3) as well:

Multiplication with T yields isomorphisms between $\mathbf{Q}[G]$ and K, as well as between $\mathcal{O}_K$ and $o_K = \bigoplus_{d\in\mathcal{D}} \mathbf{Z}[G]\operatorname{Tr}_{\mathbf{Q}^{(d)}/K_d}\zeta_d$, where $\mathcal{O}_K$ is Leopoldt's order (see [4], [5]) contained in the maximal order of $\mathbf{Q}[G]$, and o_K is the ring of integers of K.

Now one can guess our main result: an appropriate sum of Stickelberger elements corresponds to the cotangent number $\eta_{1,K}$ under the above isomorphism. For $m \geq 2$ the analogous sum of higher Stickelberger elements will correspond to the modified cotangent numbers $\widetilde{\eta_{m,K}}$. First we need another lemma:

LEMMA 5. *Let $d \in \mathcal{D}$, $m, n' \in \mathbf{N}$ with $n'|n$, and put $n_0 = (d, n')$ and $n_1 = \operatorname{lcm}(d, n')$.*

a) *If $q(d) \nmid n'$ then $\vartheta_m(K, n')\operatorname{Tr}_{\mathbf{Q}^{(d)}/K_d}\zeta_d = 0$.*

b) *If $q(d)|n'$ then* $\vartheta_m(K, n')\operatorname{Tr}_{\mathbf{Q}^{(d)}/K_d}\zeta_d = \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n')}][\mathbf{Q}^{(n)}:K\mathbf{Q}^{(d)}]}$

$$\operatorname{Tr}_{\mathbf{Q}^{(n)}/K}\left(\frac{\varphi(n)}{\varphi(n_1)}\prod_{p|\frac{d}{n_0}}(-\sigma_p^{-1})\prod_{\substack{p|n'\\ p\nmid n_0}}(1-p^{m-1}\sigma_p)\,\vartheta_m(\mathbf{Q}^{(n_0)}, n_0)\,\zeta_{n_0}\right).$$

c) $$[\mathbf{Q}^{(n)} : K\mathbf{Q}^{(d)}] = \begin{cases} 2 & \text{if } d \equiv 2\ (4) \text{ and } [\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n^*)}] = 2 \\ 1 & \text{else} \end{cases}$$

PROOF: $\vartheta_m(K, n')\ \operatorname{Tr}_{\mathbf{Q}^{(d)}/K_d}\zeta_d = \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n')}]}\vartheta_m(\mathbf{Q}^{(n)}, n')\cdot$ $\operatorname{Tr}_{\mathbf{Q}^{(d)}/K_d}\zeta_d = \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n')}]}\operatorname{Tr}_{K\mathbf{Q}^{(d)}/K}\left(\vartheta_m(\mathbf{Q}^{(n)}, n')\,\zeta_d\right) =$ $\frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n')}]\,[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(d)}]}\operatorname{Tr}_{\mathbf{Q}^{(n)}/K}\left(\vartheta_m(\mathbf{Q}^{(n)}, n')\,\zeta_d\right)$.
Now using Lemma 3, gives assertions a) and b).

To prove c) we will use the notations of Chapter 2 of [5]. Let X (resp. Y) be the group of primitive Dirichlet characters of K (resp. of $K\mathbf{Q}^{(d)}$). Then Y is generated by X and $X^{(d)}$, the character group of $\mathbf{Q}^{(d)}$. Let ω_2 denote the Dirichlet character with conductor 4. Since the projection of X onto the characters of second kind modulo n is surjective ([5], Lemma 1) and $X^{(d)}$ contains all characters of first kind except ω_2 if $d \equiv 2\ (4)$, one deduces

$$(X^{(n)} : Y) = \begin{cases} 2 & \text{if } n \equiv 0\ (8),\ d \equiv 2\ (4) \text{ and } f_\chi \not\equiv 4\ (8) \text{ for all } \chi \in X \\ 1 & \text{else} \end{cases}$$

The proof of assertion c) now follows easily. ∎

We introduce the following notations: let $n = \prod_{i=1}^{r} p_i^{e_i}$ with pairwise different primes $p_i \in \mathbf{P}$ be the prime factorization of the conductor n of K , putting $p_1 = 2$ if n is even. For any $I \subset \{1, \dots, r\}$ let

$$n_I = \prod_{i \in I} p_i^{e_i} \quad \text{and} \quad c_I = \frac{[\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n_I)}]}{\prod_{i \notin I} p_i^{e_i - 1}} \prod_{i \notin I} (-\mathrm{Frob}(p_i)) \in \mathbf{Q}[G] \ ,$$

where $\mathrm{Frob}(p_i)$ is any extension of $\mathrm{res}_{\mathbf{Q}^{(l_i)}/K_{l_i}} \sigma_{p_i}$ with $l_i = \frac{n}{p_i^{e_i}}$.

For $m \in \mathbf{N}$ we define

$$\kappa_m = \begin{cases} 0 & \text{if } [\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n^*)}] = 1 \\ 1 + 2^{m-1} - \mathrm{Frob}(2)^{-1} & \text{if } [\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n^*)}] = 2 \end{cases} , \tag{4}$$

$$\Theta_m = \sum_{I \subset \{1, \dots, r\}} c_I \, \vartheta_m(K, n_I) \tag{5}$$

and

$$\Theta'_m = \sigma_{-1} \Big(\Theta_m + \kappa_m \sum_{I \subset \{2, \dots, r\}} c_I \, \vartheta_m(K, n_I) \Big) \ . \tag{6}$$

With these notations we have the following result:

THEOREM 1.

$$\Theta'_m \ T = \widetilde{\eta_{m,K}}$$

Specializing the theorem for $m = 1$, yields the relations

$$\Theta'_1 \, T = \eta_{1,K} \quad \text{and} \quad (1 - \sigma_{-1}) \, \Theta'_1 \, T = \eta_K \ . \tag{7}$$

Note that for $K = \mathbf{Q}^{(n)}$ the definition of Θ_m uses the Stickelberger elements of the same subfields of $\mathbf{Q}^{(n)}$ as were used by Ramachandra to obtain circular units, which generate a subgroup of finite index in the full group of units (see [8], Theorem 8.3).

PROOF: Let $I \subset \{1,\ldots,r\}$ and $d \in \mathcal{D}$. Since Lemma 5.a) yields $\vartheta_m(K,n_I)\,\mathrm{Tr}_{\mathbf{Q}^{(d)}/K_d}\,\zeta_d = 0$ whenever $q(d) \nmid n_I$, we will always assume that $q(d)|n_I$. Then we have $n_0 = (d,n_I) = q(d) \prod_{i\in I,\, p_i \nmid q(d)} p_i$ and $\frac{d}{n_0} = \prod_{i\notin I} p_i$. Furthermore $n_1 = \mathrm{lcm}(d,n_I) = n_I \prod_{i\notin I} p_i$ and $\frac{\varphi(n)}{\varphi(n_1)} = \prod_{i\notin I} p_i^{e_i-1}$. Using Lemma 5.b) a short calculation shows $c_I\,\vartheta_m(K,n_I)\,\mathrm{Tr}_{\mathbf{Q}^{(d)}/K_d}\,\zeta_d = \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(d)}]}\,\mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\big(\vartheta_m(\mathbf{Q}^{(n_0)},n_0)\,\zeta_{n_0}\big)$.

If I is fixed and d runs through $\mathcal{D}$ with $q(d)|n_I$, then $n_0 = (d,n_I)$ runs through $N_I = \{t \mid \prod_{i\in I} p_i \mid t \mid n_I\}$. This correspondence between d and n_0 is one-to-one, so we have $c_I\,\vartheta_m(K,n_I)\,T = \sum_{t\in N_I} \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(d_t)}]}\,\mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big)$, where $d_t = q(t) \prod_{\substack{1\le i\le r \\ p_i \nmid q(t)}} p_i$.

Since $\{t \in \mathbf{N} \mid t \mid n\} = \dot{\bigcup}_{I\subset\{1,\ldots,r\}} N_I$, we obtain

$$\begin{aligned}\Theta_m\,T = &\sum_{\substack{1\le t|n \\ t\equiv 0\ (4)}} \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big) + \\ &+ \frac{1}{[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n^*)}]} \sum_{\substack{1\le t|n \\ t\not\equiv 0\ (4)}} \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big)\,.\end{aligned} \tag{8}$$

If $[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n^*)}] = 1$, just multiply both sides with σ_{-1}, and remember $\kappa_m = 0$ and (2) to finish the proof of Theorem 1.

If $[\mathbf{Q}^{(n)}:K\mathbf{Q}^{(n^*)}] = 2$, n must be even and $p_1 = 2$. In this case we calculate as above $\kappa_m \sum_{I\subset\{2,\ldots,r\}} c_I\,\vartheta_m(K,n_I)\,T = \kappa_m\,\frac{1}{2} \sum_{\substack{1\le t|n \\ t\equiv 1\ (2)}} \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big)$.

Using Lemma 4, this equals $\frac{1}{2} \sum_{\substack{1\le t|n \\ t\equiv 1\ (2)}} \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t + \vartheta_m(\mathbf{Q}^{(2t)},2t)\,\zeta_{2t}\Big) = \frac{1}{2} \sum_{\substack{1\le t|n \\ t\not\equiv 0\ (4)}} \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}\Big(\vartheta_m(\mathbf{Q}^{(t)},t)\,\zeta_t\Big)$.

Adding this result to (8), one can finish the proof as above. ∎

4. Index Formulas

In this chapter we use the generalized module index: if L and L' are lattices spanning the same finite-dimensional $\mathbf{Q}$- vector space V (i. e. L and L' are $\mathbf{Z}$- modules with $\dim_{\mathbf{Q}} V = \mathrm{rg}_{\mathbf{Z}} L = \mathrm{rg}_{\mathbf{Z}} L' < \infty$), define

$$(L : L') = |\det \ \phi| \ ,$$

where ϕ is any automorphism of V with $\phi(L) = L'$ (see e. g. [7], p.187). If $L' \subset L$, $(L : L')$ equals the usual index of groups.

Let X be the group of primitive Dirichlet characters belonging to the abelian number field K , and let X^+ (resp. X^-) denote the group of even (resp. the set of odd) characters in X (see e. g. [8], chap.3). Using the cotangent number $\eta_K = (1 - \sigma_{-1})\, \eta_{1,K} = \mathrm{Tr}_{\mathbf{Q}^{(n)}/K}(i \cot\frac{\pi}{n})$, Girstmair [1] computed $\Big((1 - \sigma_{-1})\, o_K : \mathbf{Z}[G]\, \eta_K\Big)$ and found that h_K^- is a factor of this index. He also computed an analogous index for $\eta_{m,K}$, where $\prod_{\chi \in X^\pm} B_{m,\chi}$ appears as a factor. In this product χ runs through X^+ for even m , and through X^- for odd m . $B_{m,\chi} = f_\chi^{m-1} \sum_{t=1}^{f_\chi} \chi(t)\, B_m(\frac{t}{f_\chi})$ are Leopoldt's generalized Bernoulli numbers. If the conductor of K is a prime power, $\eta_{m,K}$ does not belong to o_K in general. Therefore in [1] the cotangent numbers are multiplied with appropriate rational integers to obtain $\eta^*_{m,K} \in o_K$, and the results are formulated for these numbers, using the usual index of groups.

If we assume that $[\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n^*)}] = 1$ and use Theorem 1, we can split the above index and transfer it with the help of Leopoldt's theorem into the group ring, abbreviating $R^- = (1 - \sigma_{-1})\, \mathbf{Z}[G]$:
$\big((1 - \sigma_{-1})\, o_K : \mathbf{Z}[G]\, \eta_K\big) = \big((1 - \sigma_{-1})\, \mathcal{O}_K\, T : R^-\, T\big) \cdot$
$\big(R^-\, T : \mathbf{Z}[G]\, (1 - \sigma_{-1})\sigma_{-1}\Theta_1\, T\big) = \big((1 - \sigma_{-1})\, \mathcal{O}_K : R^-\big)\big(R^- : \Theta_1\, R^-\big)$.
We will investigate the second index and show that it is h_K^- times a rational factor, similarly easy as in [1]. More generally we will derive an index formula for Θ_m as defined in (5).

For this purpose let K^+ be the maximal real subfield of K . For any rational prime $p \in \mathbf{P}$ let g_p be the number of prime ideals of K lying above p and p^{f_p} be their norm, and define g_p^+, f_p^+ analogously for K^+. If $K \neq K^+$, put

$$\psi(p) = \begin{cases} 0 \\ 1 \\ -1 \end{cases} \text{if the primes of } K^+ \text{ lying above } p \text{ are } \begin{cases} \text{ramified} \\ \text{split} \\ \text{inert} \end{cases}$$

in the relative extension K/K^+. This is just the Hecke character of K/K^+, which depends only on the rational primes, since K is abelian.

LEMMA 6. *Let z be algebraic independent over $\mathbf{C}$. Then*

$$\prod_{\chi \in X^+} (1 - \chi(p) z) = (1 - z^{f_p^+})^{g_p^+}$$

and if $K \neq K^+$

$$\prod_{\chi \in X^-} (1 - \chi(p) z) = (1 - \psi(p)\, z^{f_p^+})^{g_p^+} .$$

PROOF: If the product runs over a group of Dirichlet characters, the proof works with the usual idea of interpreting the ramification properties of primes via Dirichlet characters (see the proof of Theorem 4.3 in [8]). For the second product we have

$$\prod_{\chi \in X^-} (1 - \chi(p) z) = \frac{\prod_{\chi \in X} (1 - \chi(p) z)}{\prod_{\chi \in X^+} (1 - \chi(p) z)} = \frac{(1 - z^{f_p})^{g_p}}{(1 - z^{f_p^+})^{g_p^+}} .$$

Using ψ as defined above, the assertion follows. ∎

THEOREM 2. *Let $R^\pm$ be any sublattice of $\mathbf{Q}[G]^\pm = (1 \pm \sigma_{-1})\, \mathbf{Q}[G]$ or of $K^\pm = (1 \pm \sigma_{-1})\, K$ such that $R^\pm$ has maximal rank.*

If $m \in \mathbf{N}$ is even

$$(R^+ : \Theta_m\, R^+) = \Big| \prod_{p \in \mathbf{P},\ p|n} \left(1 - (p^{m-1} + p - 1)^{f_p^+}\right)^{g_p^+} \prod_{\chi \in X^+} \frac{B_{m,\chi}}{m} \Big| .$$

If $m \in \mathbf{N}$ is odd and $K \neq K^+$

$$(R^- : \Theta_m\, R^-) = \Big| \prod_{p \in \mathbf{P},\ p|n} \left(1 - \psi(p)\, (p^{m-1} + p - 1)^{f_p^+}\right)^{g_p^+} \prod_{\chi \in X^-} \frac{B_{m,\chi}}{m} \Big| .$$

PROOF: Extend each character $\chi \in X$ to $\chi : \mathbf{Q}[G] \to \mathbf{C}$ by $\mathbf{Q}$-linearity. Then Lemma 1.2.b) of [7], p.188, yields $(R^\pm : \Theta_m R^\pm) = |\prod_{\chi \in X^\pm} \chi(\Theta_m)|$ for any lattice $R^\pm \subset \mathbf{Q}[G]^\pm$, and by Leopoldt's theorem for any lattice $R^\pm \subset K^\pm$ as well. So we only have to compute $\chi(\Theta_m)$.

Let $\chi \in X$ and put $I_0 = \{i \mid 1 \le i \le r,\ p_i | f_\chi\}$. Then we have $\chi\big(\vartheta_m(K, n_I)\big) = \frac{1}{[\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n_I)}]} \chi\big(\vartheta_m(\mathbf{Q}^{(n)}, n_I)\big) = \frac{1}{[\mathbf{Q}^{(n)} : K\mathbf{Q}^{(n_I)}]} \frac{n_I^{m-1}}{m} \sum_{\substack{1 \le t \le n_I \\ (t, n_I)=1}} B_m(\langle \frac{t}{n_I} \rangle)\ \chi(\mathrm{cor}_{\mathbf{Q}^{(n)}/\mathbf{Q}^{(n_I)}}\, \sigma_t)$. Now observe that

$$\chi(\mathrm{cor}_{\mathbf{Q}^{(n)}/\mathbf{Q}^{(n_I)}}\, \sigma_t) = \begin{cases} 0 & \text{if } \chi|_{\mathrm{Gal}(\mathbf{Q}^{(n)}/\mathbf{Q}^{(n_I)})} \ne 1 \\ [\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_I)}]\, \chi(t) & \text{if } \chi \in X^{(n_I)},\text{ i.e. } I_0 \subset I \end{cases}$$

and $[\mathbf{Q}^{(n)} : \mathbf{Q}^{(n_I)}] = \prod_{i \notin I} (p_i - 1)\, p_i^{e_i - 1}$. If $I_0 \subset I$ we have furthermore $\frac{n_I^{m-1}}{m} \sum_{\substack{1 \le t \le n_I \\ (t, n_I)=1}} B_m(\langle \frac{t}{n_I} \rangle)\ \chi(t) = \prod_{i \in I \setminus I_0} (1 - p_i^{m-1}\, \chi(p_i))\, \frac{B_{m,\chi}}{m}$ (except for $m = n = 1$ and χ trivial, which will not be needed). Putting all together, we obtain $\chi(c_I\, \vartheta_m(K, n_I)) =$

$$= \begin{cases} 0 & \text{if } I_0 \not\subset I \\ \prod_{i \notin I} (1 - p_i)\, \chi(p_i) \prod_{i \in I \setminus I_0} (1 - p_i^{m-1}\, \chi(p_i))\, \frac{B_{m,\chi}}{m} & \text{if } I_0 \subset I \end{cases}$$

This yields $\chi(\Theta_m) = \chi\Big(\sum_{\substack{I \\ I_0 \subset I \subset \{1,\dots,r\}}} c_I\, \vartheta_m(K, n_I) \Big) =$

$$\frac{B_{m,\chi}}{m} \sum_{\substack{I \\ I_0 \subset I \subset \{1,\dots,r\}}} \Big(\prod_{i \notin I} (1 - p_i)\, \chi(p_i) \prod_{i \in I \setminus I_0} (1 - p_i^{m-1}\, \chi(p_i)) \Big) =$$

$$\frac{B_{m,\chi}}{m} \prod_{j \in \{1,\dots,r\} \setminus I_0} \Big((1 - p_j)\, \chi(p_j) + (1 - p_j^{m-1}\, \chi(p_j)) \Big) =$$

$\frac{B_{m,\chi}}{m} \prod_{j=1}^{r} \Big(1 - \chi(p_j)\, (p_j^{m-1} + p_j - 1) \Big)$. Now taking the product over all $\chi \in X^+$ if m is even, or over all $\chi \in X^-$ if m is odd, and using Lemma 6 gives the formulas of the theorem. ∎

Using the analytic class number formula for h_K^- ([8], Theorem 4.17) and Theorem 2 with $m = 1$, immediately yields the following

COROLLARY 1. *Assume that* $K \neq K^+$. *Then*

$$(R^- : \Theta_1 R^-) = h_K^- \frac{2^{\frac{[K:\mathbf{Q}]}{2}}}{Qw} \prod_{p\in \mathrm{P},\ p|n} |\, 1 - \psi(p)\, p^{f_p^+} \,|^{g_p^+}$$

Note that the above formula holds in the group ring as well as in the ring of integers. Thus we can choose for example $R^- = \{\alpha \in \mathbf{Z}[G] |\ \sigma_{-1}\,\alpha = -\alpha\}$ or $R^- = \{\alpha \in o_K |\ \sigma_{-1}\,\alpha = -\alpha\} \simeq \{\alpha \in \mathcal{O}_K |\ \sigma_{-1}\,\alpha = -\alpha\}$.

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Institut für Mathematik
Karl-Franzens-Universität
Halbärthgasse 1/1
A-8010 Graz