Image Registration and Interpolation by Optical Flow with Maximal Rigidity

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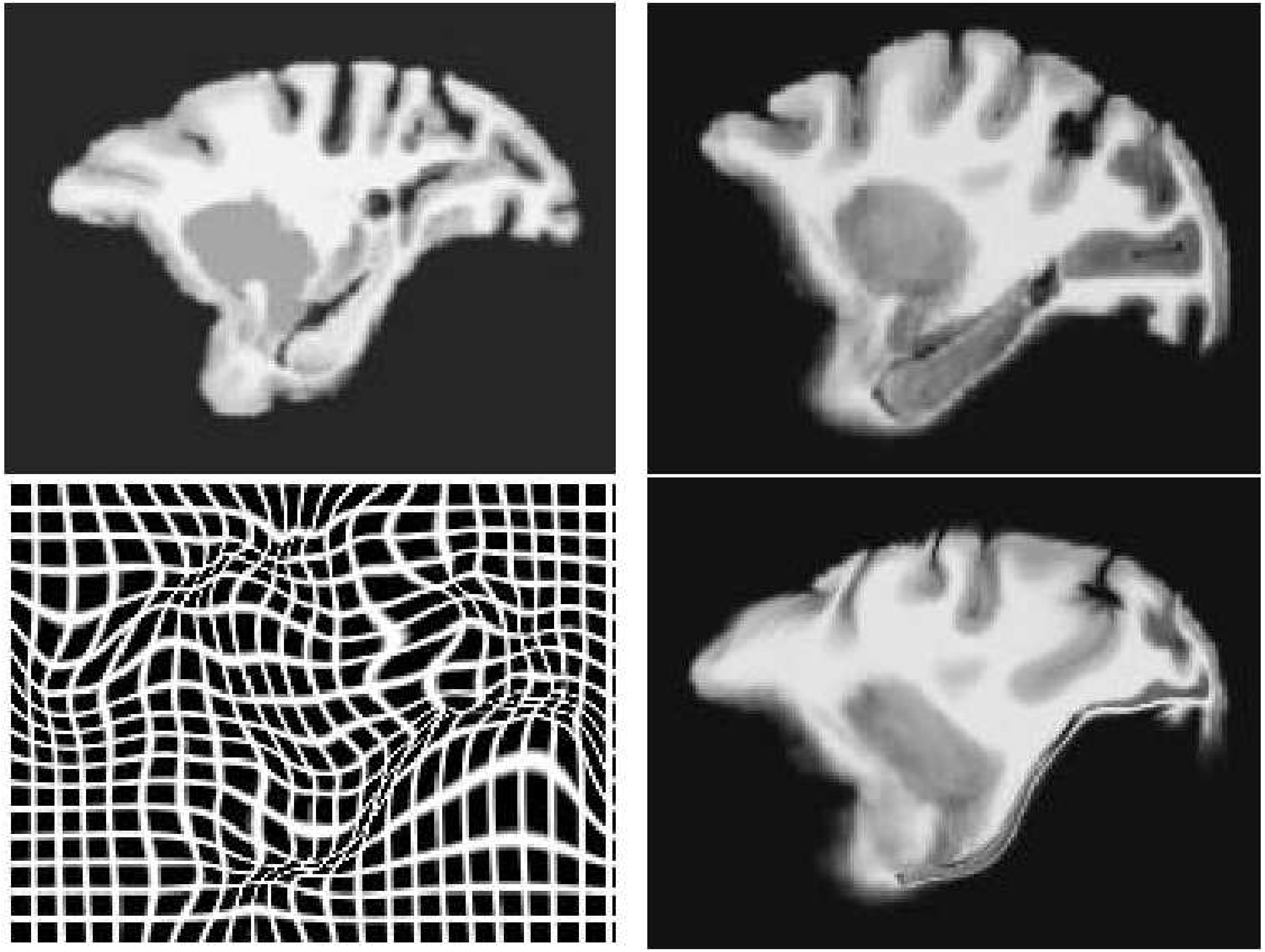
Outline

- What is Image Registration?
- How should image similarity be measured?
- How should registration be regularized?
- Variational formulation: optical flow with maximal rigidity.
- Numerical solution of optimality system.
- Results and directions.

Image Registration Defined

Definition: Coordinate transformation identifying like points in images.



L. Younes, 2000

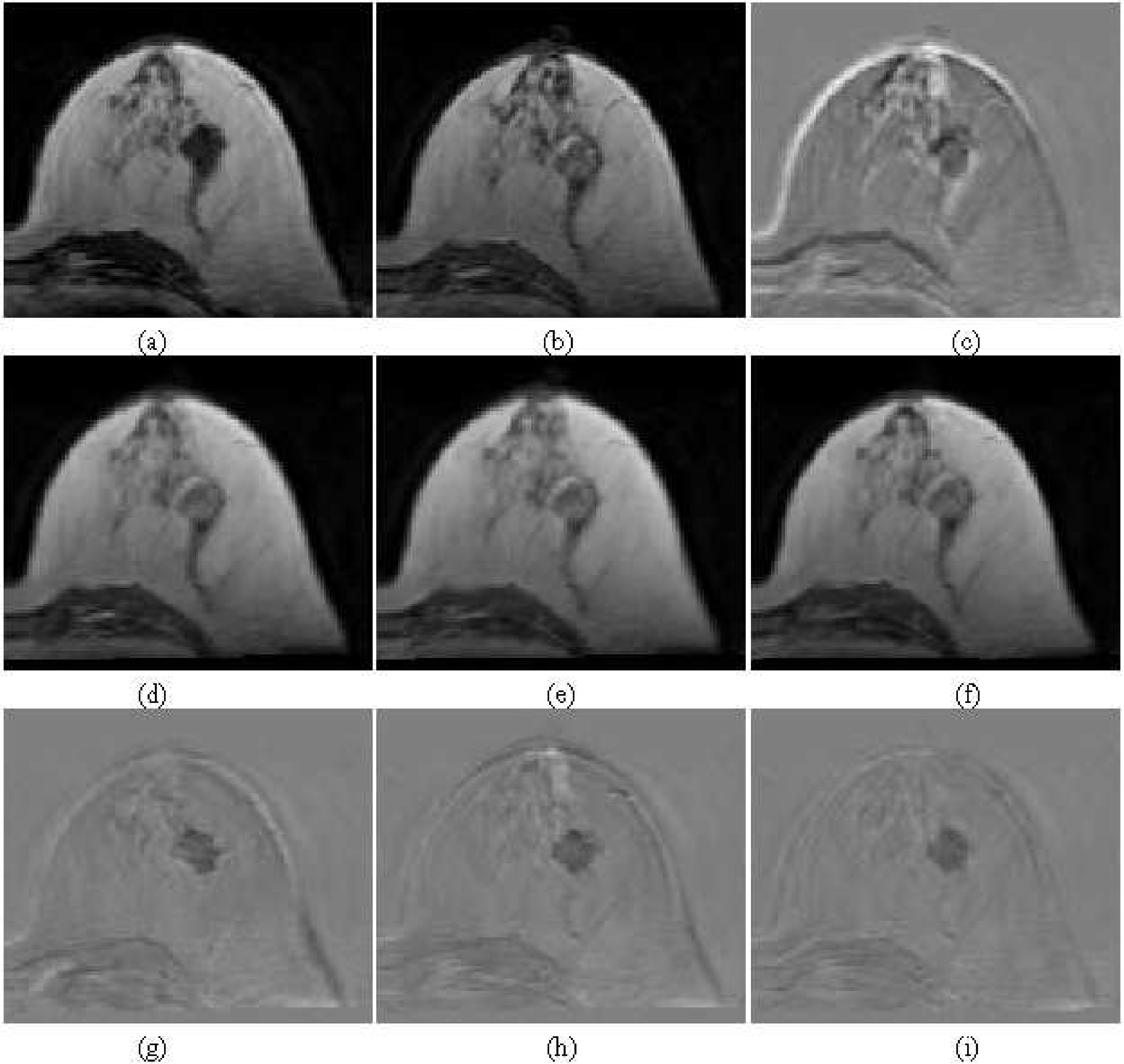
Top left: $I_0(\boldsymbol{\xi})$. Top right: $I_1(\boldsymbol{\xi})$.
Bottom left:
grid deformation, $G(\boldsymbol{x}(\boldsymbol{\xi}))$.
Bottom right:
morphing, $I_1(\boldsymbol{x}(\boldsymbol{\xi}))$.

Related processes:

- Image warping.
- Interpolation of images.
- Superposition of modalities.

Registration within Temporal Sequences

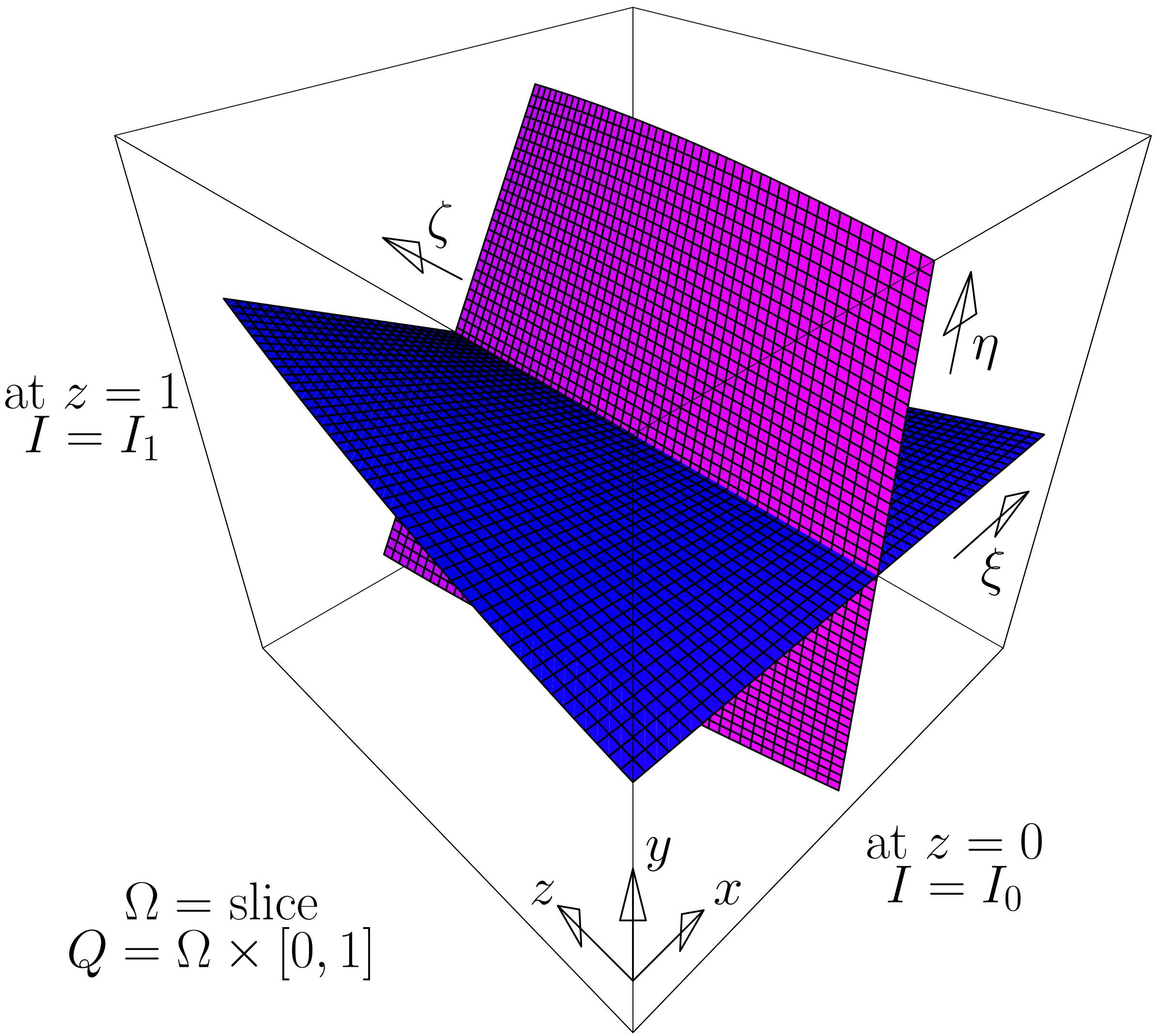
Leading application: pathology tracking in mammography.



D. Rueckert, 1998

(a) before injection,
(b) after injection,
(c) difference without registration.
Image (a) after
(d) rigid,
(e) affine, and
(f) non-rigid registration,
with corresponding differences from (b) in (g), (h), and (i).

The Coordinate System



- Physical coordinates:
  (x, y, z), $\boldsymbol{x} = (x, y)$.
- Curvilinear coordinates:
  (ξ, η, ζ), $\boldsymbol{\xi} = (\xi, \eta)$.
- Initially, $\boldsymbol{\xi}(\boldsymbol{x}, 0) = \boldsymbol{x}$.
  Otherwise, $\zeta(\boldsymbol{x}, z) = z$.
- Point $(\boldsymbol{x}(\boldsymbol{\xi}, \zeta), z)$ on trajectory with
  $\boldsymbol{x}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi}$.
- Displacement: $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{\xi}$.
- Optical flow: $\boldsymbol{u} = (u, v) = \boldsymbol{x}_\zeta(\boldsymbol{\xi}, \zeta)$.
- Trajectory tangent: $(u, v, 1)$.

Proposed Formulation

Minimize:

$$J(I,\sigma_0,\sigma_1,\boldsymbol{u}) = \int_0^1\int_\Omega |\nabla I\cdot\boldsymbol{u} + I_z|^2 d\boldsymbol{x}dz + \int_0^1\int_\Omega \phi(|\nabla\boldsymbol{u}^T + \nabla\boldsymbol{u}|^2) + \alpha|\boldsymbol{u}_z|^2 d\boldsymbol{x}dz$$

subject to:

$$I(\boldsymbol{x},0) = \sigma_0(I_0(\boldsymbol{x})), \qquad I(\boldsymbol{x},1) = \sigma_1(I_1(\boldsymbol{x})),$$

and possible landmark constraints:

$$\boldsymbol{x}(\boldsymbol{\xi}_j,1) = \boldsymbol{x}_j, \qquad j = 1,\dots,\hat{j} \qquad \text{where:} \qquad \boldsymbol{x}(\boldsymbol{\xi},\zeta) = \boldsymbol{\xi} + \int_0^\zeta \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi},\rho),\rho)d\rho.$$

- Similarity: $|\nabla I\cdot\boldsymbol{u} + I_z|^2$, and $I(\boldsymbol{x},0) = \sigma_0(I_0(\boldsymbol{x}))$, $I(\boldsymbol{x},1) = \sigma_1(I_1(\boldsymbol{x}))$.

- Rigidity: $[\nabla_{\boldsymbol{x}}\boldsymbol{u}]^T + [\nabla_{\boldsymbol{x}}\boldsymbol{u}] = 0 \Rightarrow [\nabla_{\boldsymbol{\xi}}\boldsymbol{x}]^T[\nabla_{\boldsymbol{\xi}}\boldsymbol{x}] = I$.

- Excision: $\phi(s) = \sqrt{s+\varepsilon}$ (reg TV).

- Depth regularization: $|\boldsymbol{u}_z|^2$.

- Independent of image order.

Image Similarity Measures

- The engineering standard: mutual information.
- For example [Rueckert et al],
  Seek (spline-based) transformation $\boldsymbol{x}$
  minimizing:

$$J(\boldsymbol{x}) = S(I_0, I_1(\boldsymbol{x})) + \mu R(\boldsymbol{x})$$

where:

$$R(\boldsymbol{x}) = \int_\Omega \sum_{|\alpha|=2} |D^\alpha \boldsymbol{x}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$

and similarity measure S is mutual information:

$$S(A,B) = H(A,B) - H(A) - H(B).$$

$H(A)$ is entropy of image A with intensities $\{a_i\}$:

$$H(A) = -\sum_i p(a_i) \log p(a_i)$$

$H(A,B)$ is joint entropy of (A,B) with pairs $\{(a_i, b_j)\}$:

$$H(A,B) = -\sum_{i,j} p(a_i, b_j) \log p(a_i, b_j)$$

- Simple example:

$$A : \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array} \quad B : \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\{a_i\} = \{0,0,1,1\} = \{b_j\}$$

$$\{(a_i, b_j)\} = \{(1,0),(1,1),(0,0),(0,1)\}$$

$$H(A) = H(B) = -2 \cdot \tfrac{1}{2} \log \tfrac{1}{2}$$

$$H(A,B) = -4 \cdot \tfrac{1}{4} \log \tfrac{1}{4} >$$
$$\text{-}2 \cdot \tfrac{1}{2} \log \tfrac{1}{2} = H(A,A)$$

Proposed Image Similarity Measure

Simplest:

$$\int_\Omega |I_0(\boldsymbol{\xi}) - I_1(\boldsymbol{x}(\boldsymbol{\xi},1))|^2 d\boldsymbol{\xi} = \int_0^1 \int_\Omega |dI/d\zeta|^2 d\boldsymbol{\xi} d\zeta$$

subject to:

$$I(\boldsymbol{x},0) = I_0(\boldsymbol{x}), \quad I(\boldsymbol{x},1) = I_1(\boldsymbol{x}).$$

Lagrangian to Eulerian:

$$\int_0^1 \int_\Omega |dI/d\zeta|^2 d\boldsymbol{\xi} d\zeta \longrightarrow \int_0^1 \int_\Omega |\nabla I \cdot \boldsymbol{u} + I_z|^2 d\boldsymbol{x} dz$$

without $1/\det(\nabla_{\boldsymbol{\xi}} \boldsymbol{x})$. With scaling:

$$\int_\Omega |\sigma_0(I_0(\boldsymbol{\xi})) - \sigma_1(I_1(\boldsymbol{x}(\boldsymbol{\xi},1)))|^2 d\boldsymbol{\xi} \longrightarrow \int_0^1 \int_\Omega |\nabla I \cdot \boldsymbol{u} + I_z|^2 d\boldsymbol{x} dz$$

subject to:

$$I(\boldsymbol{x},0) = \sigma_0(I_0(\boldsymbol{x})), \quad I(\boldsymbol{x},1) = \sigma_1(I_1(\boldsymbol{x}))$$

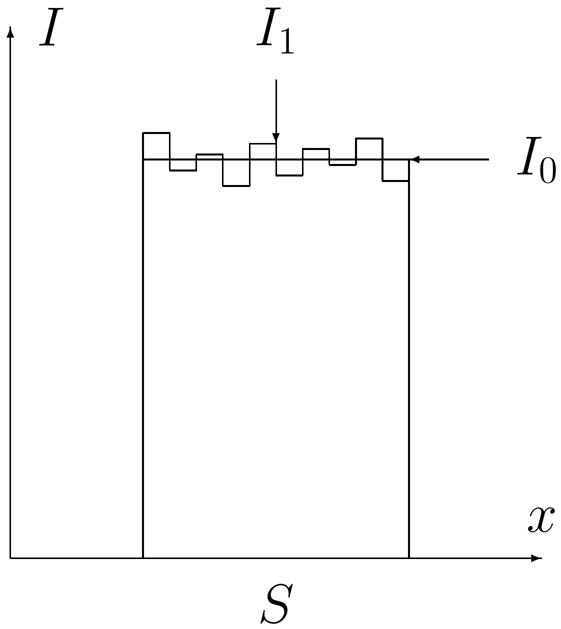
Alternatively differential operator to match level curves:

$$D_{\boldsymbol{\xi}} I_0(\boldsymbol{\xi}) - D_{\boldsymbol{\xi}} I_1(\boldsymbol{x}(\boldsymbol{\xi},1))$$

or else Gauss Maps (level set normal vector fields).

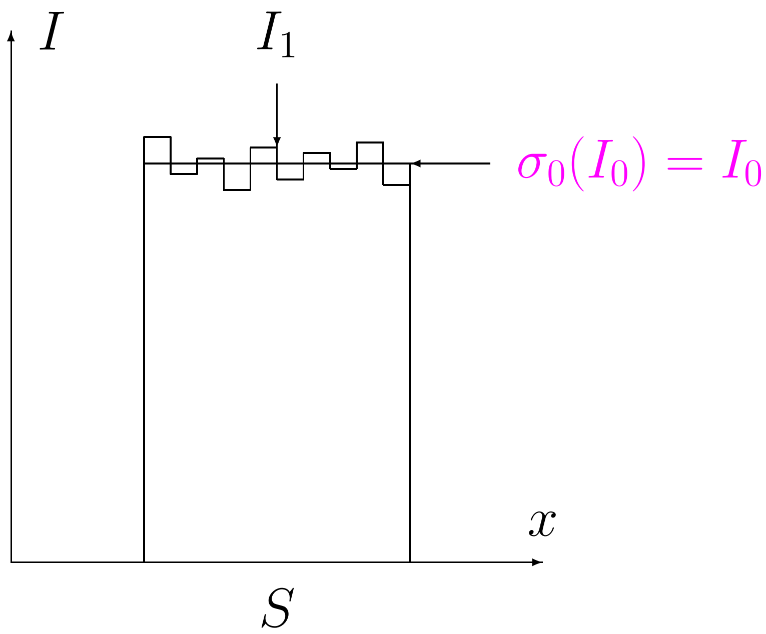
Mumford-Shaw formulation forthcoming.

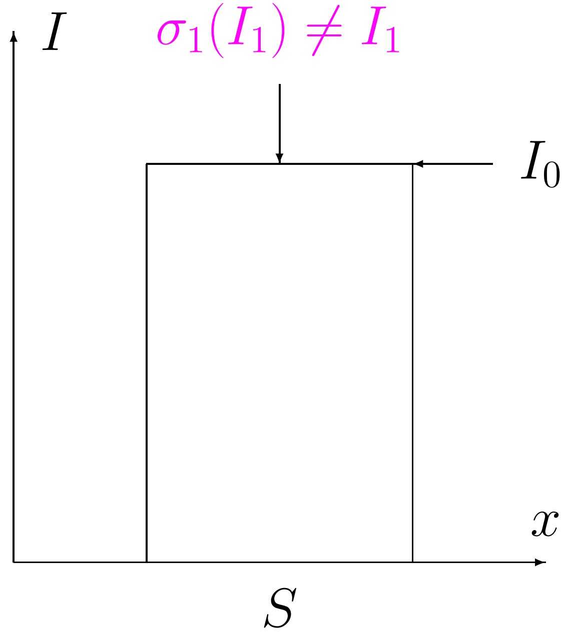
Reciprocal Scaling



$$I_0 = \chi_S$$
$$I_1 = (1 + \nu)\chi_S$$

Scaling I_0
not I_1





Scaling I_1
not I_0

Registration Regularization

- Rueckert, et al.

$$J(\boldsymbol{x}) = H(I_0, I_1(\boldsymbol{x})) - H(I_0) - H(I_1(\boldsymbol{x})) + \mu \int_\Omega \sum_{|\alpha|=2} |D^\alpha \boldsymbol{x}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$

- Modersitzki, Fischer.

$$J(\boldsymbol{d}) = \int_\Omega |I_1(\boldsymbol{\xi} + \boldsymbol{d}) - I_0(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} + \int_\Omega \lambda(\nabla \cdot \boldsymbol{d})^2 + \frac{1}{2}\mu|\nabla \boldsymbol{d}^T + \nabla \boldsymbol{d}|^2 d\boldsymbol{\xi}$$

- Hinterberger, Scherzer.

$$J(\boldsymbol{u}) = \int_0^1 \int_\Omega \alpha |I(\boldsymbol{x}, 1) - I(\boldsymbol{x}, z)|^2 d\boldsymbol{x} dz + \int_0^1 \int_\Omega |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} dz$$

where $I(\boldsymbol{x}, z)$ is subject to:

$$\beta[\Delta I + I_{zz}] + \nabla I \cdot \boldsymbol{u} + I_1 - I_0 = I_z$$

- Proposed Formulation: subject to $I(\boldsymbol{x}, 0) = \sigma_0(I_0(\boldsymbol{x}))$, $I(\boldsymbol{x}, 1) = \sigma_1(I_1(\boldsymbol{x}))$,

$$J(I, \sigma_0, \sigma_1, \boldsymbol{u}) = \int_0^1 \int_\Omega |\nabla I \cdot \boldsymbol{u} + I_z|^2 d\boldsymbol{x} dz + \int_0^1 \int_\Omega \phi(|\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}|^2) + \alpha |\boldsymbol{u}_z|^2 d\boldsymbol{x} dz$$

$$\boldsymbol{x}(\boldsymbol{\xi}, 1) = \boldsymbol{\xi} + \int_0^1 \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, \rho), \rho) d\rho$$

Lagrangian Regularization

$$J_{\mathrm{lin}}(\boldsymbol{d}) = \int_\Omega \left[I_0(\boldsymbol{\xi}) - I_1(\boldsymbol{\xi} + \boldsymbol{d}(\boldsymbol{\xi}, 1))\right]^2 d\boldsymbol{\xi}$$

$$+\int_0^1 \int_\Omega \left[\lambda \left(\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{d}\right)^2 + \frac{\mu}{2} \left|\nabla_{\boldsymbol{\xi}} \boldsymbol{d}^{\mathrm{T}} + \nabla_{\boldsymbol{\xi}} \boldsymbol{d}\right|^2\right] d\boldsymbol{\xi} d\zeta$$

$$+\int_0^1 \int_\Omega \nu \left(\boldsymbol{d}_\zeta \cdot \boldsymbol{d}_{\zeta\zeta}\right)^2 d\boldsymbol{\xi} d\zeta$$

$$J_{\mathrm{unl}}(\boldsymbol{x}) = \int_\Omega \left[I_0(\boldsymbol{\xi}) - I_1(\boldsymbol{x}(\boldsymbol{\xi}, 1))\right]^2 d\boldsymbol{\xi}$$

$$+\frac{\lambda}{4} \int_0^1 \int_\Omega \left[\sum_{i=1}^N \left(\|\nabla_\xi x_i\|^2 - 1\right)\right]^2 d\boldsymbol{\xi} d\zeta + \frac{\mu}{2} \int_0^1 \int_\Omega \left[\sum_{i=1}^N \left(\|\nabla_\xi x_i\|^2 - 1\right)^2 + \sum_{i \neq j} \left(\nabla_\xi x_i \cdot \nabla_\xi x_j\right)^2\right] d\boldsymbol{\xi} d\zeta$$

$$+\nu \int_0^1 \int_\Omega \left(\boldsymbol{x}_{\zeta\zeta} \cdot \boldsymbol{x}_{\zeta\zeta\zeta}\right)^2 d\boldsymbol{\xi} d\zeta$$

Rigid versus collapse:

$$\begin{aligned} \hat{\boldsymbol{x}}(\boldsymbol{\xi}, \zeta) &= R(\zeta)(\boldsymbol{\xi} - \boldsymbol{a}) + \boldsymbol{a}, & \hat{\boldsymbol{d}} &= \hat{\boldsymbol{x}} - \boldsymbol{\xi} \\ \tilde{\boldsymbol{x}}(\boldsymbol{\xi}, \zeta) &= (1 - 2\zeta)(\boldsymbol{\xi} - \boldsymbol{a}) + \boldsymbol{a}, & \tilde{\boldsymbol{d}} &= \tilde{\boldsymbol{x}} - \boldsymbol{\xi} \end{aligned}$$

$$J_{\mathrm{lin}}(\tilde{\boldsymbol{d}}) < J_{\mathrm{lin}}(\hat{\boldsymbol{d}}) \qquad J_{\mathrm{unl}}(\hat{\boldsymbol{x}}) = 0 < J_{\mathrm{unl}}(\tilde{\boldsymbol{x}}).$$

Eulerian Regularization

Rigidity: $[\nabla_x \boldsymbol{u}]^T + [\nabla_x \boldsymbol{u}] = 0 \Rightarrow [\nabla_{\boldsymbol{\xi}} \boldsymbol{x}]^T [\nabla_{\boldsymbol{\xi}} \boldsymbol{x}] = I.$

Local $\boldsymbol{u} = \boldsymbol{x}_\zeta$ instead of global $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{\xi}$ displacements:

$$J(I, \boldsymbol{u}) = \int_Q (\nabla_{\boldsymbol{x}} I \cdot \boldsymbol{u} + I_z)^2 \, d\boldsymbol{x} dz + \int_Q \left[\phi \left(\left| \nabla_{\boldsymbol{x}} \boldsymbol{u}^{\mathrm{T}} + \nabla_{\boldsymbol{x}} \boldsymbol{u} \right|^2 \right) + \alpha \left| \boldsymbol{u}_z \right|^2 \right] d\boldsymbol{x} dz$$

With rigid field:

$$\hat{\boldsymbol{u}}(\boldsymbol{x}, z) = \hat{\boldsymbol{x}}_\zeta = W(\boldsymbol{x} - \boldsymbol{a}), \qquad R'(z) = W R(z)$$

$$\hat{I} = I_0(R(z)^T(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a}), \qquad I_0(\boldsymbol{\xi}) = I_1(\hat{\boldsymbol{x}}(\boldsymbol{\xi}, 1))$$

follows $J(\hat{I}, \hat{u}) = 0$. For the class:

$$\begin{aligned} \boldsymbol{x}(\boldsymbol{\xi}, \zeta) &= R(\zeta)(\boldsymbol{\xi} - \boldsymbol{a}) + (\boldsymbol{a} + \boldsymbol{b}\zeta) \\ \boldsymbol{u}(\boldsymbol{x}, z) &= W(\boldsymbol{x} - \boldsymbol{a} - \boldsymbol{b}z) + \boldsymbol{b} \end{aligned}$$

need $\varphi(|\nabla_{\boldsymbol{x}} \boldsymbol{u}_z|^2)$ and $|\boldsymbol{u}_{zz}|^2$ instead of $|\boldsymbol{u}_z|^2$.

For autonomous flows:

$$J(I, \boldsymbol{u}) = \int_Q (\nabla_{\boldsymbol{x}} I \cdot \boldsymbol{u} + I_z)^2 \, d\boldsymbol{x} dz + \int_\Omega \phi \left(\left| \nabla_{\boldsymbol{x}} \boldsymbol{u}^{\mathrm{T}} + \nabla_{\boldsymbol{x}} \boldsymbol{u} \right|^2 \right) d\boldsymbol{x}$$

Optimality Conditions for Intensity

$$(\nabla, \partial_z) \cdot [(\nabla I \cdot \boldsymbol{u} + I_z)(\boldsymbol{u}, 1)] =$$

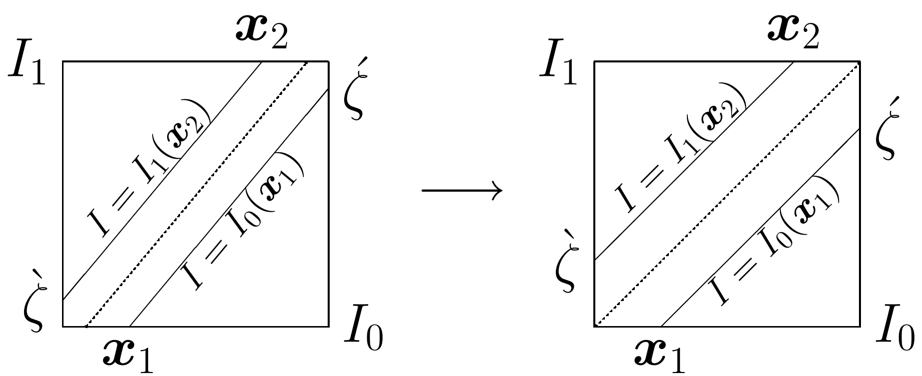
$$\frac{d^2 I}{d\zeta^2} + (\nabla \cdot \boldsymbol{u})\frac{dI}{d\zeta} = 0, \quad Q$$

Options:

$$I = 0,$$
$$\boldsymbol{n} \cdot \boldsymbol{u} = 0, \quad \text{or}$$
$$\nabla I \cdot \boldsymbol{u} + I_z = 0, \quad \Gamma$$

$\boldsymbol{n} \cdot \boldsymbol{u} = 0$ rules out rigidity.

$\nabla I \cdot \boldsymbol{u} + I_z = 0$ creates trivial solution:



Take $I = 0$ on Γ. Lagrangian solution:

$$I(\boldsymbol{x}(\boldsymbol{\xi}, \zeta), \zeta) =$$

$$\begin{cases} I_0(\boldsymbol{\xi})[1 - U(\boldsymbol{\xi}, \zeta, 1)] + I_1(\boldsymbol{x}(\boldsymbol{\xi}, 1))U(\boldsymbol{\xi}, \zeta, 1), \\ \qquad \boldsymbol{\xi} \in \Omega_0^{\mathrm{c}} \\ I_0(\boldsymbol{\xi})[1 - U(\boldsymbol{\xi}, \zeta, \acute{\zeta})], \\ \qquad \boldsymbol{x}(\boldsymbol{\xi}, \acute{\zeta}) \in \Gamma, \boldsymbol{\xi} \in \Omega_0^{\mathrm{i}} \end{cases}$$

or scaled with $\sigma_0(I_0)$ and $\sigma_1(I_1)$, and:

$$U(\boldsymbol{\xi}, \zeta, \acute{\zeta}) = \frac{\tilde{U}(\boldsymbol{\xi}, \zeta) - \tilde{U}(\boldsymbol{\xi}, 0)}{\tilde{U}(\boldsymbol{\xi}, \acute{\zeta}) - \tilde{U}(\boldsymbol{\xi}, 0)}, \boldsymbol{\xi} \in \Omega_0, \zeta \in [0, \acute{\zeta}]$$

$$\tilde{U}(\boldsymbol{\xi}, \zeta) = \int_0^{\zeta} \exp\left[- \int_0^{\varrho} \nabla \cdot \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}, \rho), \rho) d\rho\right] d\varrho$$

Optimality Conditions for Scaling

Lagrangian form of I inserted in J reveals optimal scaling:

$$\sigma_0(\iota) = \begin{cases} \dfrac{\int_{I_0(\boldsymbol{\xi})=\iota} \mathcal{I}_1(\boldsymbol{\xi}) \mathcal{U}(\boldsymbol{\xi}) d\boldsymbol{\xi}}{\int_{I_0(\boldsymbol{\xi})=\iota} \mathcal{U}(\boldsymbol{\xi}) d\boldsymbol{\xi}}, & \iota \in K_0 \\ \\ \dfrac{\int_{I_0(\boldsymbol{\xi})=\iota} \mathcal{I}_1(\boldsymbol{\xi}) \dfrac{\mathcal{U}(\boldsymbol{\xi})}{|\nabla I_0(\boldsymbol{\xi})|} d\boldsymbol{\xi}}{\int_{I_0(\boldsymbol{\xi})=\iota} \dfrac{\mathcal{U}(\boldsymbol{\xi})}{|\nabla I_0(\boldsymbol{\xi})|} d\boldsymbol{\xi}}, & \iota \in K_0^{\mathrm{c}} \end{cases}$$

where:

$$|\nabla I_0| = 0 \text{ in } \{\boldsymbol{\xi} : I_0(\boldsymbol{\xi}) \in K_0\}, \;\; K_0 \subset [0,1]$$

$$\mathcal{I}_1(\boldsymbol{\xi}) = \begin{cases} I_1(\boldsymbol{x}(\boldsymbol{\xi}, 1)), & \boldsymbol{\xi} \in \Omega_0^{\mathrm{c}}, \\ 0, \quad \boldsymbol{x}(\boldsymbol{\xi}, \acute{\zeta}) \in \Gamma, & \boldsymbol{\xi} \in \Omega_0^{\mathrm{i}} \end{cases}$$

$$\mathcal{U}(\boldsymbol{\xi}) = \begin{cases} \int_0^1 U_\zeta^2(\boldsymbol{\xi}, \zeta, 1) \det\left(\nabla_{\boldsymbol{\xi}} \boldsymbol{x}\right) d\zeta, & \boldsymbol{\xi} \in \Omega_0^{\mathrm{c}} \\ \\ \int_0^{\acute{\zeta}(\boldsymbol{\xi})} U_\zeta^2(\boldsymbol{\xi}, \zeta, \acute{\zeta}(\boldsymbol{\xi})) \det\left(\nabla_{\boldsymbol{\xi}} \boldsymbol{x}\right) d\zeta, & \boldsymbol{\xi} \in \Omega_0^{\mathrm{i}} \end{cases}$$

In words: $\sigma_0(\iota)$ is weighted average over $\{I_0(\boldsymbol{\xi}) = \iota\}$ of morphing $\mathcal{I}_1$

Optimality Conditions for Optical Flow

$$B(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in C^\infty(\bar{Q})$$

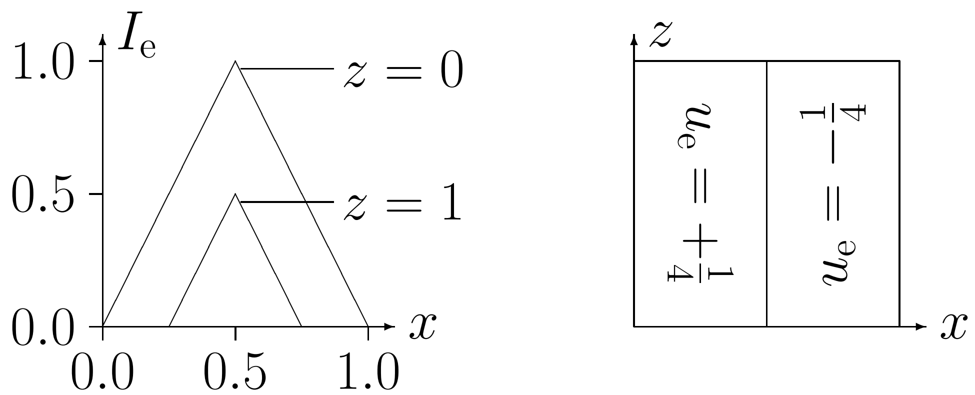
where:

$$\begin{aligned} B(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) &= \int_Q \left[(\nabla I \cdot \boldsymbol{u})(\nabla I \cdot \boldsymbol{v}) + \alpha\left(\boldsymbol{u}_z \cdot \boldsymbol{v}_z\right)\right] d\boldsymbol{x}dz \\ &+ \int_Q \phi'\left(\left|\nabla \boldsymbol{w}^{\mathrm{T}} + \nabla \boldsymbol{w}\right|^2\right)\left(\nabla \boldsymbol{u}^{\mathrm{T}} + \nabla \boldsymbol{u}\right) : \left(\nabla \boldsymbol{v}^{\mathrm{T}} + \nabla \boldsymbol{v}\right) d\boldsymbol{x}dz \\ F(\boldsymbol{v}) &= -\int_Q I_z \nabla I \cdot \boldsymbol{v} d\boldsymbol{x}dz \end{aligned}$$

Theorem. Suppose $I \in W^{1,\infty}(Q)$ manifests sufficiently few symmetries that for every $\boldsymbol{a} \in \boldsymbol{R}^2$ and $c \in \boldsymbol{R}$:

$$\int_Q |\nabla I \cdot (\boldsymbol{a} + cW\boldsymbol{x})|^2 d\boldsymbol{x}dz > 0$$

unless $\boldsymbol{a} = 0 = c$. Then with $\phi(s) = \beta(\boldsymbol{x},z)s$, $0 < \beta_0 \leq \beta(\boldsymbol{x},z) \leq \beta_1 < \infty$, there exists a unique $\boldsymbol{u} \in H^1(Q)$ such that $B(\cdot,\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{v})$, $\forall \boldsymbol{v} \in C^\infty(\bar{Q})$.



Here TV cost is finite (and stationary), Gaussian not.

Optimality Conditions for Landmarks

Lagrangian functional:

$$L(\boldsymbol{u}, \boldsymbol{\lambda}) = \tfrac{1}{2} J(\boldsymbol{u}) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{E}(\boldsymbol{u}),$$

where:

$$\boldsymbol{E}_{\jmath}(\boldsymbol{u}) \quad = \quad \boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, 1) - \boldsymbol{x}_{\jmath} \quad = \quad \boldsymbol{\xi}_{\jmath} - \boldsymbol{x}_{\jmath} + \textstyle\int_0^1 \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho), \rho) d\rho, \quad \jmath = 1, \ldots, \hat{\jmath}.$$

For:

$$\frac{\delta \boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \zeta)}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) = \textstyle\int_0^{\zeta} \left[\nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho), \rho) \dfrac{\delta \boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho)}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) + \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho), \rho) \right] d\rho.$$

define the solution operator:

$$\begin{aligned}
S_{\boldsymbol{u},\jmath}(\zeta, \varrho) S_{\boldsymbol{u},\jmath}(\varrho, \rho) &= S_{\boldsymbol{u},\jmath}(\zeta, \rho), \qquad 0 \le \rho \le \varrho \le \zeta \le 1 \\
S_{\boldsymbol{u},\jmath}(\zeta, \zeta) &= I, \qquad \zeta \in [0, 1] \\
\partial_{\zeta} S_{\boldsymbol{u},\jmath}(\zeta, \rho) &= \nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \zeta), \zeta) S_{\boldsymbol{u},\jmath}(\zeta, \rho) \\
\partial_{\rho} S_{\boldsymbol{u},\jmath}(\zeta, \rho) &= -S_{\boldsymbol{u},\jmath}(\zeta, \rho) \nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho), \rho)
\end{aligned}$$

Stationarity conditions:

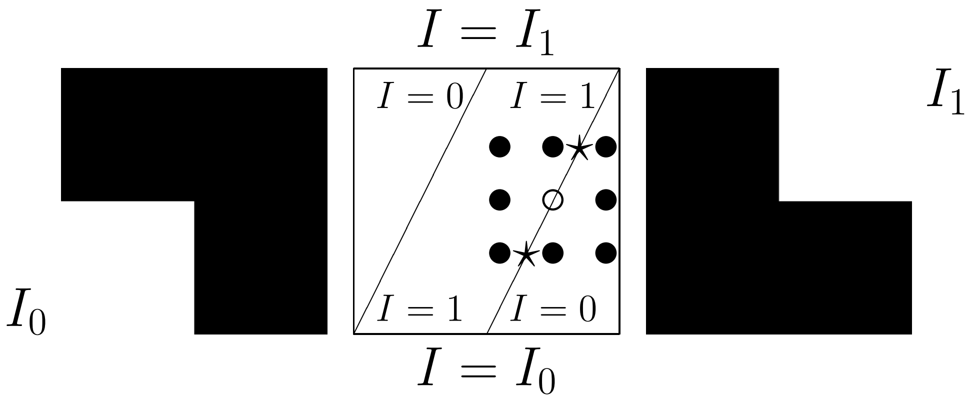
$$\frac{\delta L}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) = B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - F(\boldsymbol{v}) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in C^{\infty}(\bar{Q}), \qquad \frac{\partial L}{\partial \boldsymbol{\lambda}} = \boldsymbol{E}(\boldsymbol{u}) = 0$$

where:

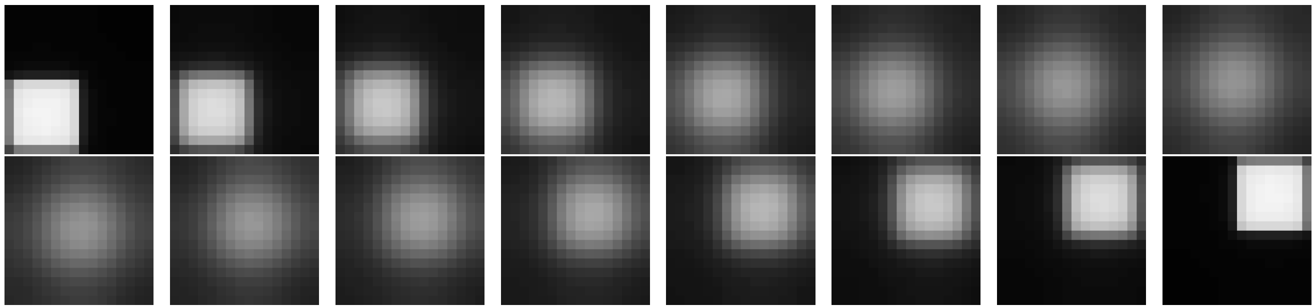
$$\boldsymbol{G}_{\jmath}(\boldsymbol{u}, \boldsymbol{v}) = \textstyle\int_0^1 S_{\boldsymbol{u},\jmath}(1, \rho) \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}_{\jmath}, \rho), \rho) d\rho$$

Numerical Treatment of Intensity

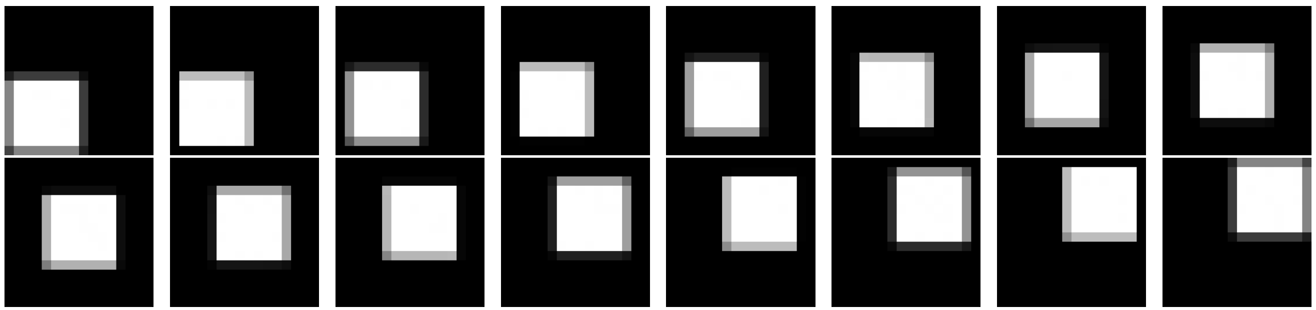
Simple translation problem:



with Eulerian discretization:

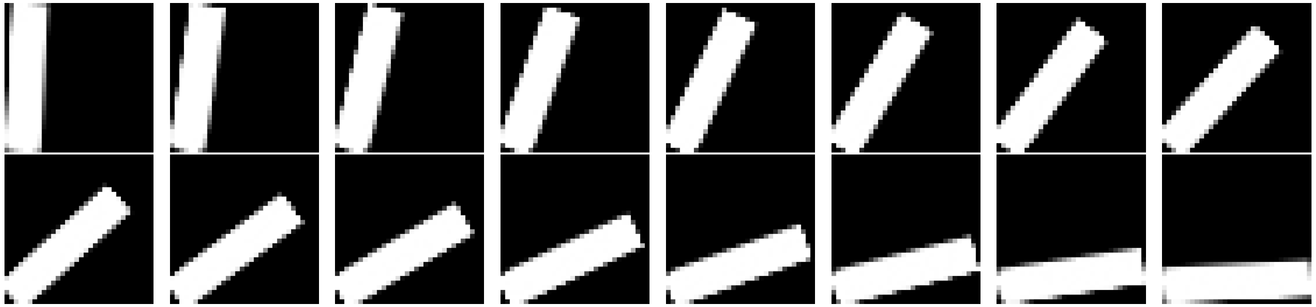


with Lagrangian discretization:



Numerical Treatment of Optical Flow

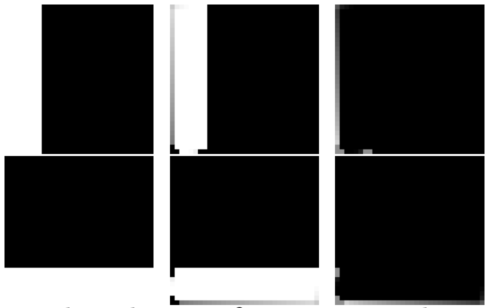
Lumped Finite Elements with Conjugate Gradient:



L^1-errors: 0.0094 on subdomains,
0.014 on full domains
L^∞-errors: 0.52 on subdomains,
1.0 on full domains

$I_0(\boldsymbol{\xi}), \mathcal{I}_1(\boldsymbol{\xi})$

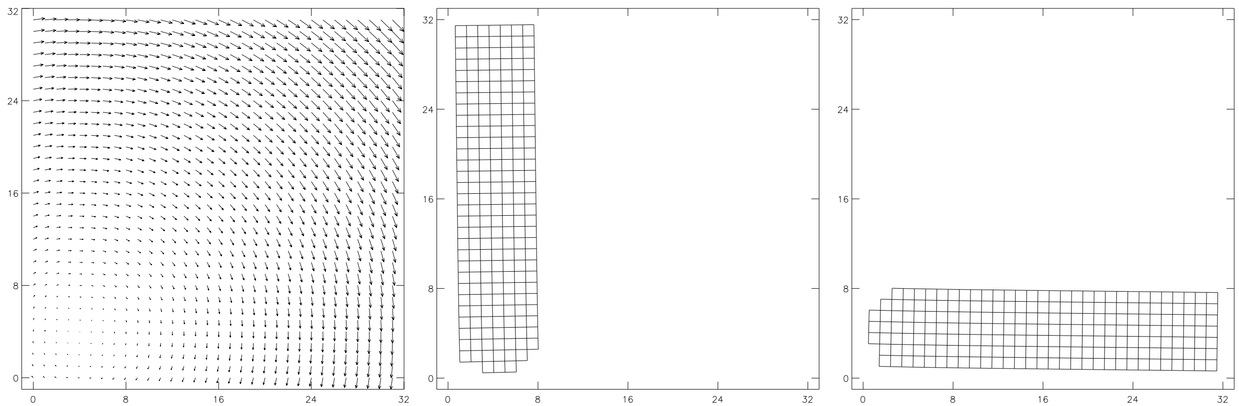
$I_1(\boldsymbol{\eta}), \mathcal{I}_0(\boldsymbol{\eta})$



$|I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$

$|I_1(\boldsymbol{\eta}) - \mathcal{I}_0(\boldsymbol{\eta})|$

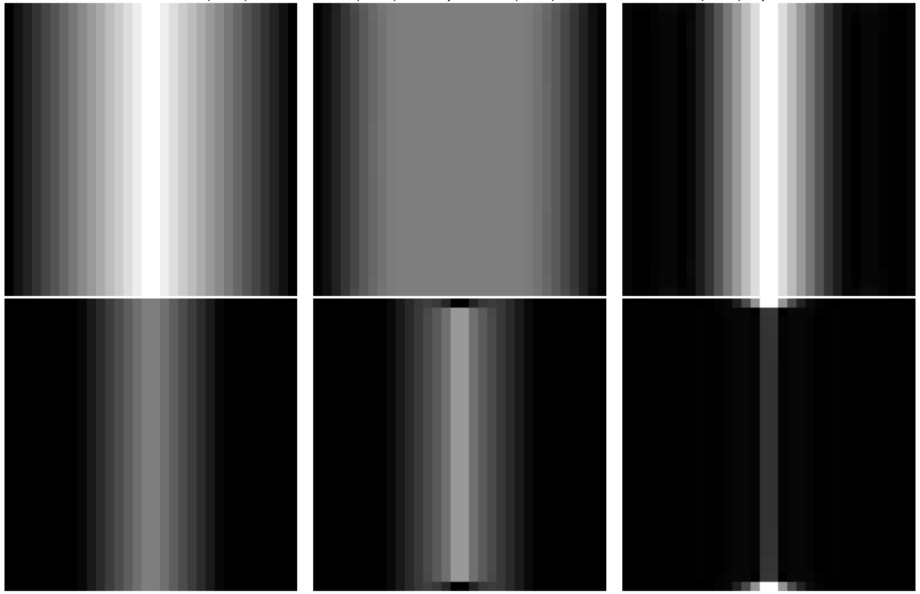
Optical flow and morphed uniform grids:



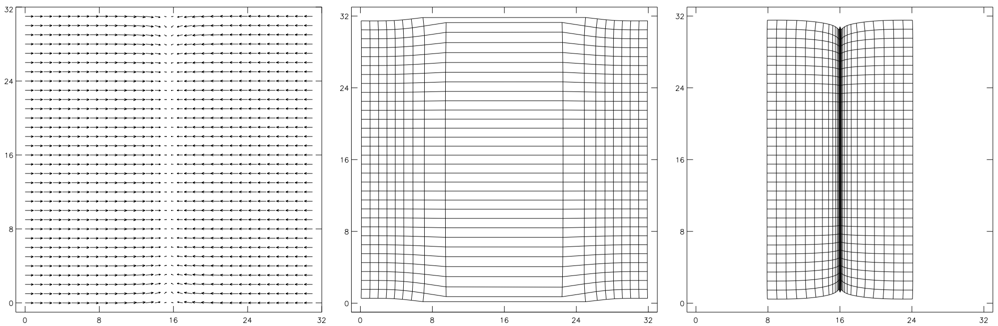
Here nontrivial pixels reach Γ. Extend images by zero to sufficiently large Ω.

Object Excision

$I_0(\boldsymbol{\xi}),\, \mathcal{I}_1(\boldsymbol{\xi}),\, |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$

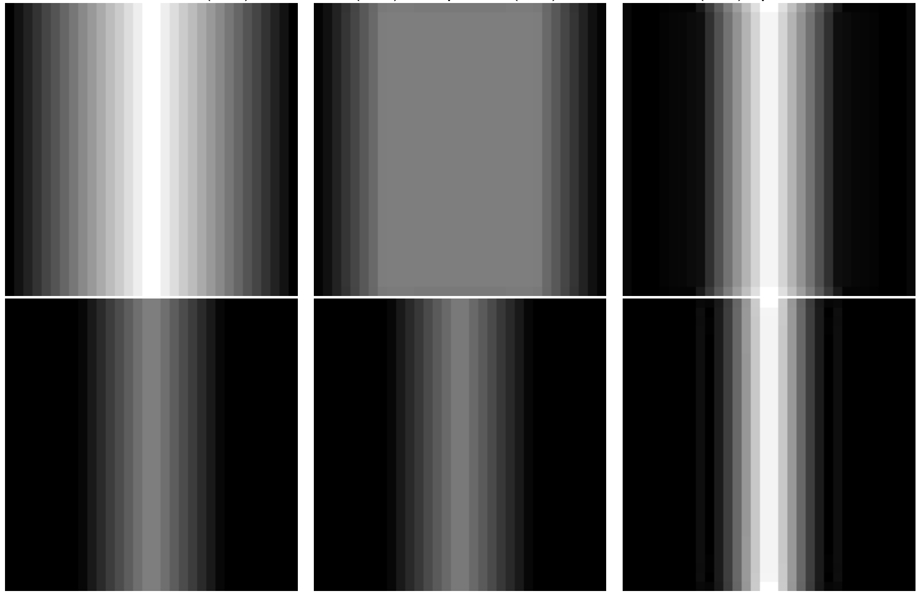


$I_1(\boldsymbol{\xi}),\, \mathcal{I}_0(\boldsymbol{\xi}),\, |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$

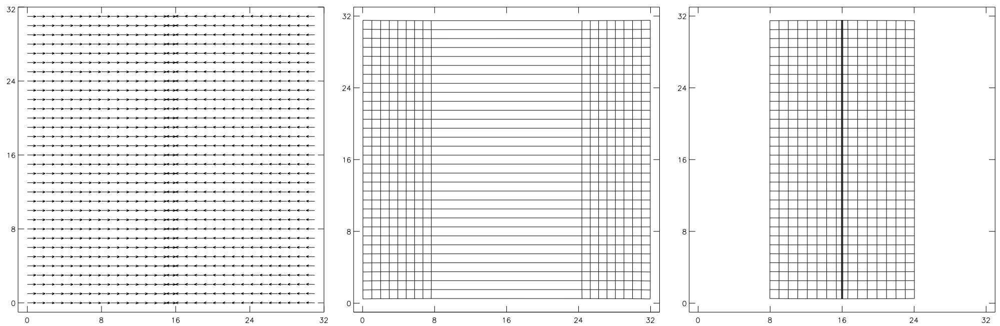


Gaussian: $\phi(s) = s$

$I_0(\boldsymbol{\xi}),\, \mathcal{I}_1(\boldsymbol{\xi}),\, |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$



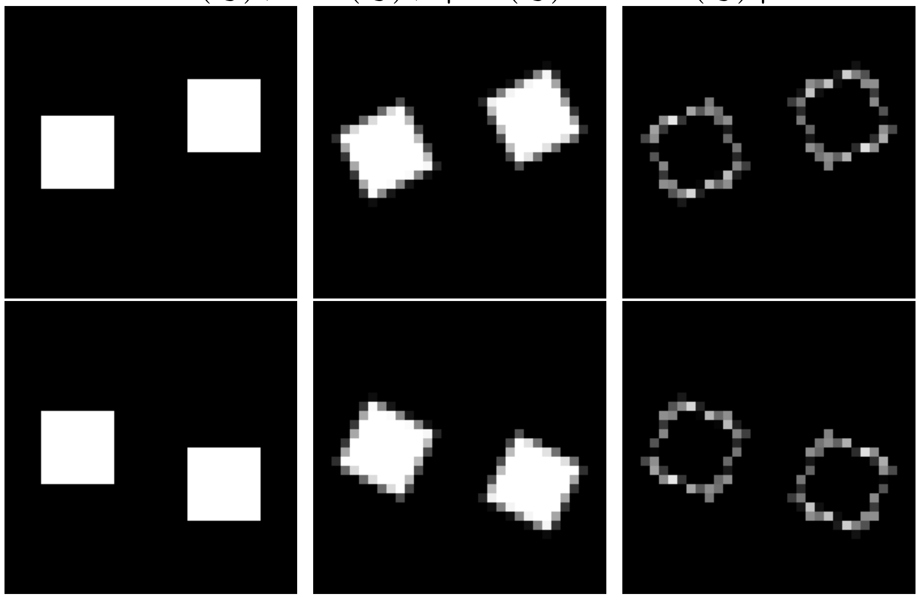
$I_1(\boldsymbol{\xi}),\, \mathcal{I}_0(\boldsymbol{\xi}),\, |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$



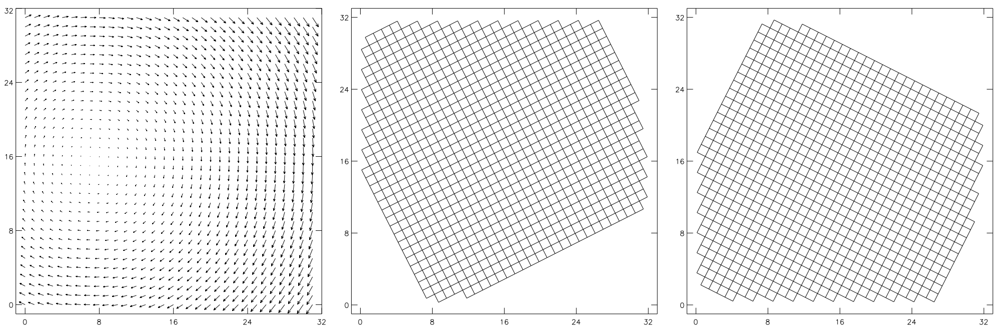
Reg TV: $\phi(s) = \sqrt{s + \varepsilon}$

Strongly and Weakly Rigid Registration

$I_0(\boldsymbol{\xi}),\ \mathcal{I}_1(\boldsymbol{\xi}),\ |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$

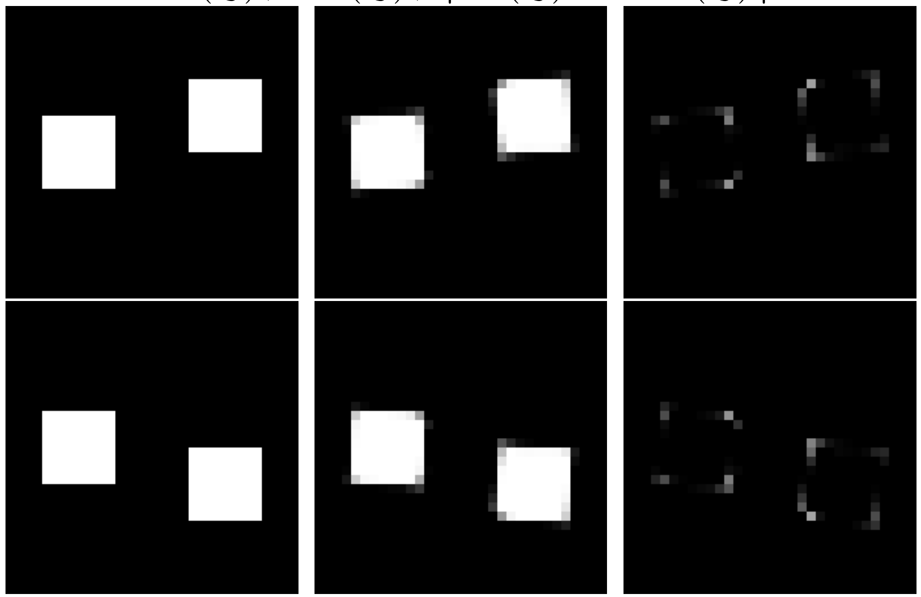


$I_1(\boldsymbol{\xi}),\ \mathcal{I}_0(\boldsymbol{\xi}),\ |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$

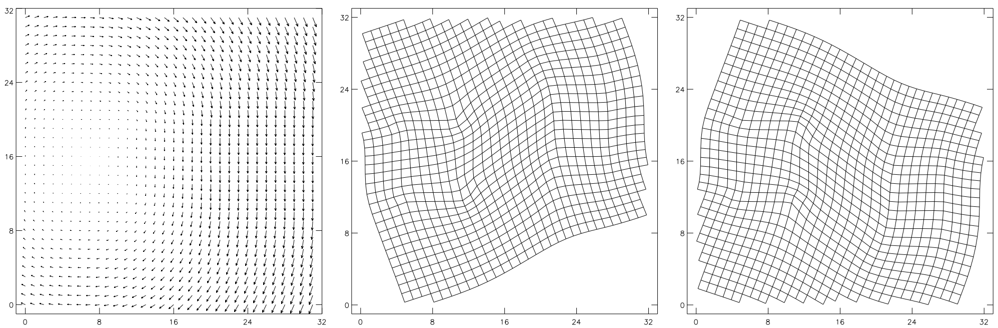


strongly rigid

$I_0(\boldsymbol{\xi}),\ \mathcal{I}_1(\boldsymbol{\xi}),\ |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$



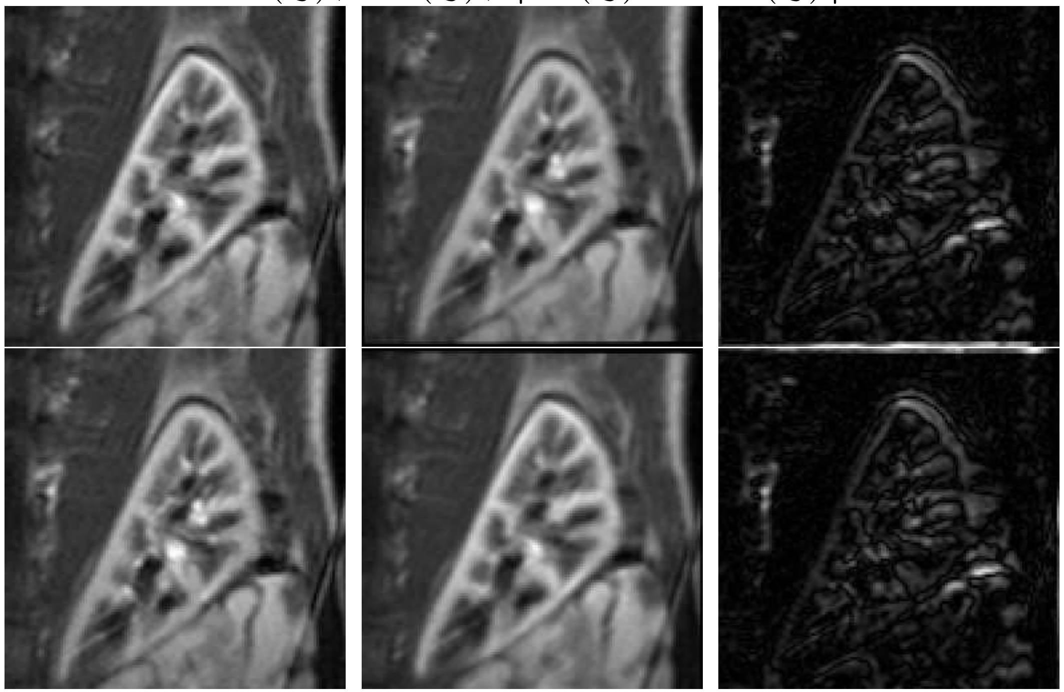
$I_1(\boldsymbol{\xi}),\ \mathcal{I}_0(\boldsymbol{\xi}),\ |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$



weakly rigid

MRI Application – Images

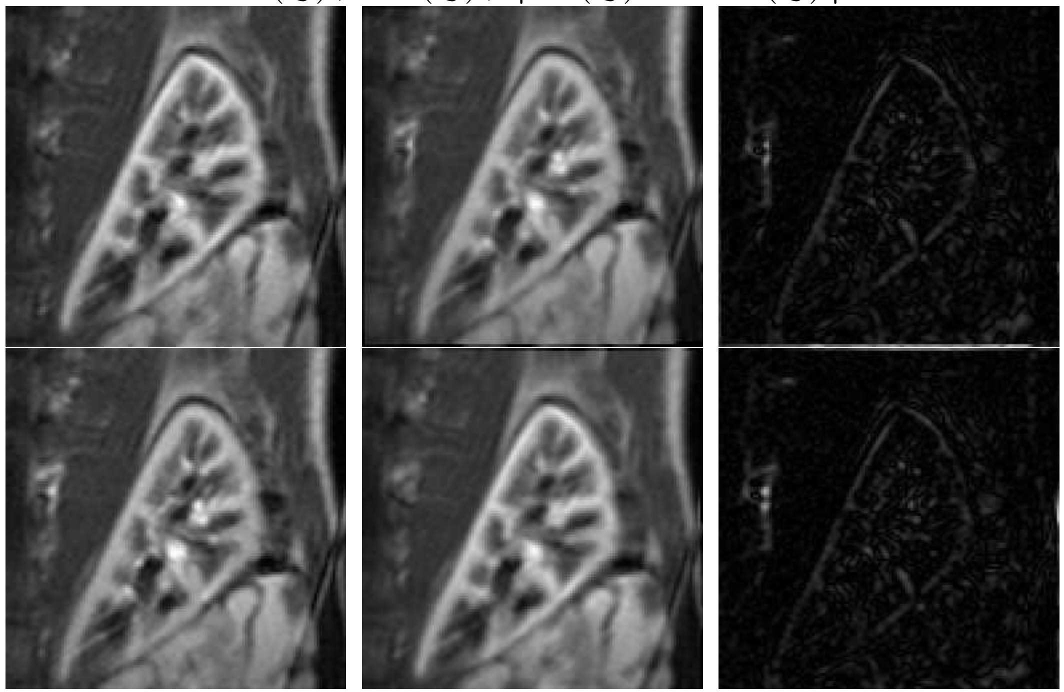
$I_0(\boldsymbol{\xi}),\ \mathcal{I}_1(\boldsymbol{\xi}),\ |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$



$I_1(\boldsymbol{\xi}),\ \mathcal{I}_0(\boldsymbol{\xi}),\ |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$

strongly rigid

$I_0(\boldsymbol{\xi}),\ \mathcal{I}_1(\boldsymbol{\xi}),\ |I_0(\boldsymbol{\xi}) - \mathcal{I}_1(\boldsymbol{\xi})|$



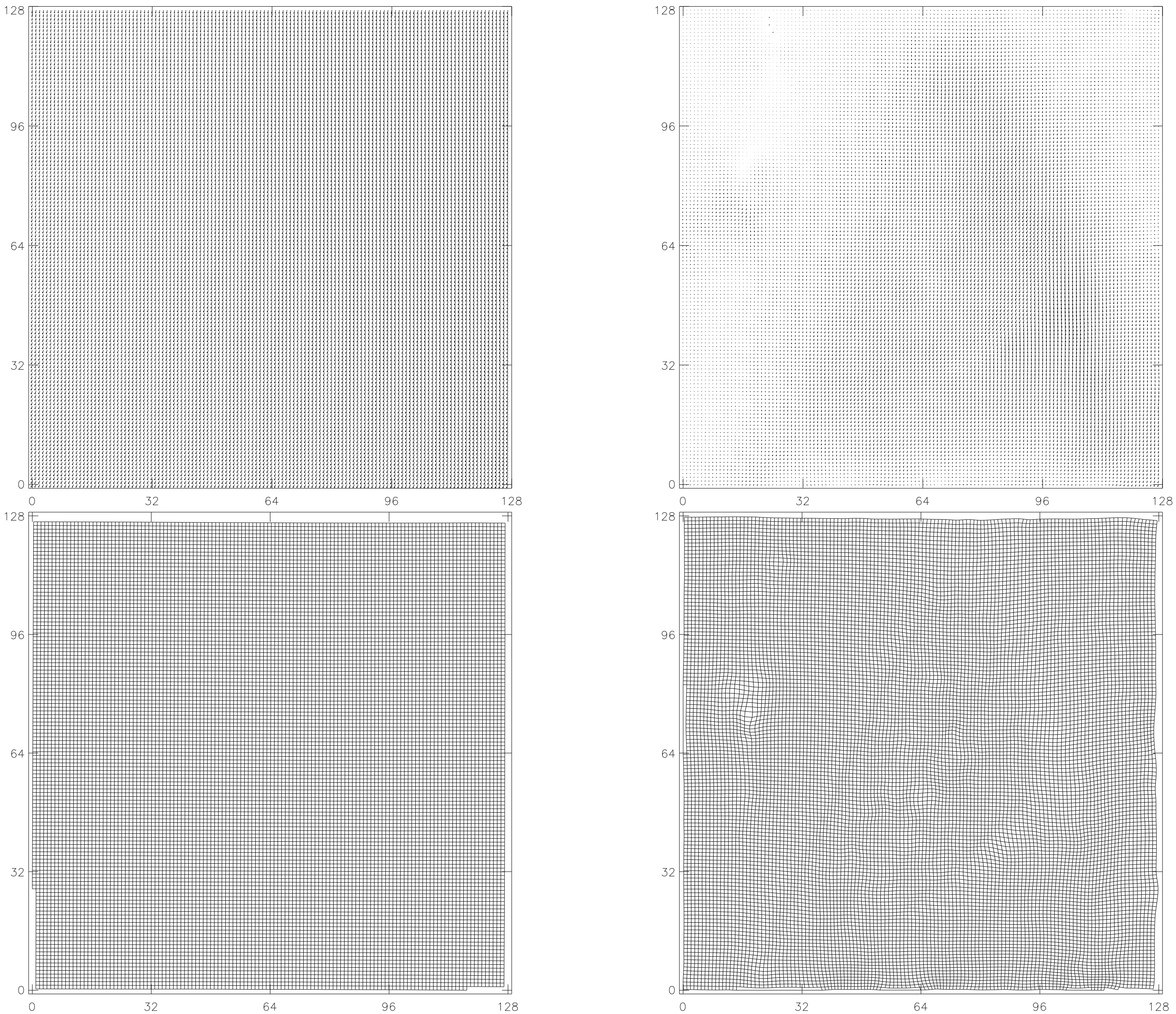
$I_1(\boldsymbol{\xi}),\ \mathcal{I}_0(\boldsymbol{\xi}),\ |I_1(\boldsymbol{\xi}) - \mathcal{I}_0(\boldsymbol{\xi})|$

weakly rigid

MRI Application – Fields

Optical flow field:

Morphed uniform grid:



strongly rigid

weakly rigid

Forthcoming Mumford-Shaw Formulation

Minimize:

$$
\begin{aligned}
J(I,\boldsymbol{u}) \;=\; & \tfrac{1}{2}\int_{\Omega_0} |I - I_0|^2 d\boldsymbol{x} + \tfrac{1}{2}\int_{\Omega_1} |I - I_1|^2 d\boldsymbol{x} \\
+ \; & \tfrac{1}{2}\mu \int_0^1 dz \int_{\Omega_z \setminus \Gamma(z)} |\nabla I|^2 d\boldsymbol{x} + \alpha \mathcal{H}^{N-1}(\Gamma) \\
+ \; & \tfrac{1}{2}\nu \int_0^1 dz \int_{\Omega_z} |\nabla \boldsymbol{u}^{\mathrm{T}} + \nabla \boldsymbol{u}|^2 d\boldsymbol{x} \\
+ \; & \tfrac{1}{2}\theta \int_0^1 dz \int_{\Omega_z} |\boldsymbol{u}_z|^2 d\boldsymbol{x}
\end{aligned}
$$

where:

$$\boldsymbol{x}(\boldsymbol{\xi},\zeta) = \boldsymbol{\xi} + \int_0^{\zeta} \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi},\rho),\rho)d\rho$$

and:

$$\Gamma(z) = \{(\boldsymbol{x},z) : \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi},z), \boldsymbol{\xi} \in \Gamma\}$$