Active Control of Propeller Induced Noise Fields*

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Abstract. The active suppression of cabin noise due to an advanced turbo-prop design is considered within the framework of optimal control theory. A sinusoidal pressure field due to offending noise sources is assumed to be known throughout the cabin interior. Then the pressure field due to controlling sources within the cabin is assumed to be governed by a nonhomogeneous wave equation in the interior and by a special boundary condition designed to conform with the experimentally observed frequency dependent reflection properties of the cabin boundary. The form of the controlling sources is determined by considering the steady state behavior of the system, and the control strategy is asymptotically optimal.

Key Words. acoustics, optimal control, wave equations, sinusoidal control.

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1 Introduction.

This work is concerned with an acoustic problem which has arisen in connection with an advanced turbo-prop aircraft design offering very high fuel efficiency [12], [14]. Unfortunately, the propulsion mechanism creates such a noise that it cannot be tolerated by passengers within the cabin. Also, if the mass of the fuselage were increased to damp the offending noise, this would offset the fuel efficiency. Therefore, the active noise suppression problem consists in the determination of interior sound sources required to generate a secondary pressure wave which destructively interferes with the offending pressure wave in an optimal way.

First, suppose that p_1 is a pressure field in the cabin $\Omega \subset \mathbf{R}^3$, due to exterior noise sources in the absence of control sources. Specifically, p_1 has the form:

$$p_1(\mathbf{x}, t) = \sum_{m=-M}^{M} \hat{p}_{n,m}(\mathbf{x}) e^{im\omega t}$$

but for simplicity, it is assumed for now to be:

$$p_1(\mathbf{x}, t) = \hat{p}_1(\mathbf{x}) e^{i\omega t}.$$

Then let p_2 represent the pressure field due to control sources in the absence of the offending noise. With α, β, $\gamma > 0$, it is assumed to be governed by the following:

$$\left\{ \begin{array}{rcll} \partial_t^2 p_2 & = & \gamma^2 \Delta p_2 + F & \text{in} \quad \Omega \times [0, \infty) \\ 0 & = & \alpha p_2 + \beta \partial_t p_2 + \partial_n p_2 & \text{on} \quad \partial\Omega \times [0, \infty) \\ p_2(0) & = & 0 & \text{in} \quad \Omega \\ \partial_t p_2(0) & = & 0 & \text{in} \quad \Omega \end{array} \right. \tag{1.1}$$

where F represents (approximately monopolar) control sources. Specifically, for fixed t, $F \in \mathcal{F} \equiv [\{F_l\}_{l=1}^d]$, the span of simple functions $\{F_l\}_{l=1}^d$ defined by:

$$F_l \equiv \left\{ \begin{array}{ll} 1 & \mathbf{x} \in B_l \\ 0 & \text{otherwise} \end{array} \right.$$

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where the balls $\{B_l\}_{l=1}^d$ are fixed in Ω. While questions concerning the optimal positioning of these control sources are natural at this point, the study of such questions has been postponed for a later time. Next, note that the boundary condition in (1.1) has been carefully selected so that certain crucial conditions are satisfied. For example, it is shown below that this boundary condition conforms with the experimentally observed frequency dependent reflection properties of the cabin boundary.

It may now seem appropriate to attempt to compute the control F as the limit of say controls F^n which are optimal in the sense that each minimizes a functional involving $|p_1 + p_2|$ on an interval $[0, t_n]$, where $t_n \to \infty$. In fact, it was learned late in this investigation that one of the approaches mentioned below actually leads to such a globally optimal control strategy, even though it is only developed to be asymptotically optimal [3], [4]. Specifically, from a theoretical as well as intuitive point of view, it seemed most reasonable to proceed as follows. First, it is anticipated that after the control mechanism is activated at $t = 0$, p_2 will approach a steady periodic state p_3, in a stable fashion. With $\tau \equiv 2\pi/\omega$, the steady state p_3 is expected to be governed by:

$$\left\{ \begin{array}{rcll} \partial_t^2 p_3 & = & \gamma^2 \Delta p_3 + F & \text{in} \quad \Omega \times [0,\tau] \\ 0 & = & \alpha p_3 + \beta \partial_t p_3 + \partial_n p_3 & \text{on} \quad \partial\Omega \times [0,\tau] \\ p_3(0) & = & p_3(\tau) & \text{in} \quad \Omega \\ \partial_t p_3(0) & = & \partial_t p_3(\tau) & \text{in} \quad \Omega. \end{array} \right. \tag{1.2}$$

Then with $\theta > 0$, the control F is selected so that it minimizes the following over an appropriate class of functions:

$$J(F) \equiv \int_0^\tau \left\{ \|H(p_1 + p_3)\|^2 + \theta \|F\|^2 \right\} dt \tag{1.3}$$

subject to (1.2). Here, H is defined by:

$$Hp \equiv \left\{ \begin{array}{ll} p & \mathbf{x} \in \hat{\Omega} \\ 0 & \text{otherwise} \end{array} \right. \tag{1.4}$$

where $\hat{\Omega} \subseteq \Omega$ represents the subset of the cabin where minimal pressure is required. Finally, the control strategy used for p_2 in (1.1) is inspired by the form of the solution $F^\star$ to this minimization problem.

In the approach considered in section 4, $J(F)$ is minimized over sinusoidal controls of the form $F(\mathbf{x}, t) = \hat{F}(\mathbf{x}) e^{i\omega t}$ with $\hat{F}(\mathbf{x}) \in \mathcal{F}$. Then, in the approach of section 5, $J(F)$ is minimized over $L_2([0,\tau], \mathcal{F})$. However, it is shown in section 5 that the latter approach leads also to a sinusoidal optimal control. Nevertheless, in both sections the respective control strategies are found to be stable in the sense that when they are applied in (1.1) and (1.2), and p_3 is extended for all time by periodicity, then $[p_2 - p_3] \to 0$ as $t \to \infty$. In other words, even though the control is only designed to be asymptotically optimal, when it is applied at some finite time, the resulting state settles into an optimal steady state in a stable fashion.

The difference between the two approaches lies in the following. First, approach 2 involves feedback while approach 1 does not. Also, the first approach is less expensive to implement but is only asymptotically optimal. On the other hand, the second is globally optimal in the sense suggested above. However, only a casual explanation is offered in section 5 in support of this claim. For additional details, see [3] and [4].

Now, in the next two sections, some basic features of the model are elucidated and some important preliminary results are established.

2 Preliminary Results.

The first objective in this section is to support the above claim that the boundary condition in (1.1) leads to frequency dependent reflection at $\partial\Omega$. For this, consider scrutinizing an infinitesimal portion of $\partial\Omega$ so that in very rough terms, Ω becomes the right half space. Assuming such a geometry, the following lemma deals with the relation between a monotone incident wave and its reflection.

Lemma 2.1 *Suppose that in (1.1), $\Omega \equiv \{(x,y,z) \in \mathbf{R}^3 : x > 0\}$, and $F = 0$. Then for every ω and ϕ, the following solves (1.1):*

$$p = \exp\left\{i\omega\left[t + \gamma^{-1}(x\cos(\phi) - y\sin(\phi))\right]\right\} + R\exp\left\{i\omega\left[t - \gamma^{-1}(x\cos(\phi) + y\sin(\phi))\right]\right\}$$

provided:

$$R \equiv \frac{(\gamma^{-1}\cos(\phi) - \beta)i\omega - \alpha}{(\gamma^{-1}\cos(\phi) + \beta)i\omega + \alpha}.$$

Proof: Clearly, p satisfies the differential equation. For the boundary condition, set $p \equiv A + B$ so that on $\partial\Omega$, $B = RA$. Then observing that $\partial_n p = -\partial_x p$ gives:

$$\begin{aligned} 0 &= \alpha(A+B) + i\omega\beta(A+B) - i\omega\gamma^{-1}\cos(\phi)(A-B) \\ &= \alpha(1+R) + i\omega\beta(1+R) - i\omega\gamma^{-1}\cos(\phi)(1-R) \\ &= \{\alpha - [\gamma^{-1}\cos(\phi) - \beta]i\omega\} + R\{\alpha + [\gamma^{-1}\cos(\phi) + \beta]i\omega\} \end{aligned}$$

and the indicated expression for R follows. ■

Note that the first term in p represents an incident wave impinging on the boundary at an angle $\pi - \phi$, while the second term represents the reflected wave with reflection coefficient R. Also z_n, given by:

$$R \equiv \frac{z_n\cos(\phi) - \gamma}{z_n\cos(\phi) + \gamma}, \qquad z_n \equiv \frac{i\omega}{\alpha + \beta i\omega}$$

is referred to as the normal specific impedance of the material beyond Ω.

Next, to become aligned with well-established results in control theory, (1.1) is written in the following first order form:

$$\left\{\begin{array}{rcll} \partial_t\begin{pmatrix} p_2 \\ \partial_t p_2 \end{pmatrix} &=& \begin{pmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{pmatrix}\begin{pmatrix} p_2 \\ \partial_t p_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix} & \text{in} \quad \Omega \times [0,\infty) \\ 0 &=& \alpha p_2 + \beta\partial_t p_2 + \partial_n p_2 & \text{on} \quad \partial\Omega \times [0,\infty) \\ \begin{pmatrix} p_2(0) \\ \partial_t p_2(0) \end{pmatrix} &=& 0 & \text{in} \quad \Omega. \end{array}\right. \tag{2.1}$$

In order to establish some useful machinery for this problem, define the state space $\mathcal{H} \equiv H^1(\Omega) \times L_2(\Omega)$, which is clearly a Hilbert space when equipped with:

$$(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix})_{\mathcal{H}} \equiv (u_1, u_2) + (\nabla u_1, \nabla u_2) + (v_1, v_2) \qquad \text{and} \qquad \|\cdot\|_{\mathcal{H}} \equiv (\cdot,\cdot)_{\mathcal{H}}^{\frac{1}{2}}$$

but also when equipped with:

$$\langle\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\rangle_{\mathcal{H}} \equiv \alpha(u_1, u_2)_{0,\partial\Omega} + (\nabla u_1, \nabla u_2) + \gamma^{-2}(v_1, v_2) \qquad \text{and} \qquad |\cdot|_{\mathcal{H}} \equiv \langle\cdot,\cdot\rangle_{\mathcal{H}}^{\frac{1}{2}}$$

as shown in Lemma 2.2 below. Note that here and in the sequel, $(\cdot,\cdot)$ is used to denote the standard inner product on $L_2(\Omega)$, or a Cartesian product of the latter, depending upon the context. Similarly, $\|\cdot\|$ denotes the associated norm. Further, $(\cdot,\cdot)_m$ and $\|\cdot\|_m$ denote the usual inner product and norm on the Sobolev space $H^m(\Omega)$. Also, deviations from these cases will be represented explicitly with appropriate subscripts. For example, $\|\cdot\|_{s,D}$ denotes the norm on the (possibly fractional order) Sobolev space $H^s(D)$.

Next, let the operator A be defined with domain $\mathcal{D}(A)$ as follows:

$$A \equiv \begin{pmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{pmatrix}, \qquad \mathcal{D}(A) \equiv \left\{ \begin{array}{c} (u,v) \in \mathcal{H}: \quad u \in H^2(\Omega), \quad v \in H^1(\Omega) \\ 0 = \alpha u + \beta v + \partial_n u \end{array} \right\}.$$

Further, $B \in \mathcal{B}(\mathcal{F},\mathcal{H})$ and its adjoint are given by:

$$BF \equiv \begin{pmatrix} 0 \\ F \end{pmatrix}, \qquad B^* \begin{pmatrix} u \\ v \end{pmatrix} \equiv P_{\mathcal{F}} v$$

where $P_{\mathcal{F}}$ is the orthogonal projection of $L_2(\Omega)$ onto $\mathcal{F}$:

$$([P_{\mathcal{F}} - I]f, F) \equiv 0 \qquad \forall f \in L_2(\Omega), \quad \forall F \in \mathcal{F}. \tag{2.2}$$

Now, taking $P_2 \equiv (p_2, \partial_t p_2)^T$, (2.1) can be written as:

$$\left\{ \begin{array}{rcll} P_2' & = & AP_2 + BF & \quad t \geq 0 \\ P_2(0) & = & 0. & \end{array} \right. \tag{2.3}$$

Before proceeding, the following standard lemma [6] shows that the state space is a Hilbert space as claimed above.

Lemma 2.2 *For $\alpha > 0$:*

$$c_1\|v\|_1 \leq \{\alpha\|v\|^2_{0,\partial\Omega} + \|\nabla v\|^2\}^{\frac{1}{2}} \leq c_2\|v\|_1 \tag{2.4}$$

and the norms $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent. Hence $\mathcal{H}$ is a Hilbert space when equipped with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

The next two lemmas work together with the Lumer-Phillips Theorem [13] to show that A generates a contraction semigroup $S(t) \in \mathcal{C}([0,\infty),\mathcal{H})$.

Lemma 2.3 *If $\mathcal{H}$ is equipped with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, A is $\mathcal{H}$-Dissipative on $\mathcal{D}(A)$.*

Proof: If $(u,v) \in \mathcal{D}(A)$, then:

$$\begin{aligned} \Re\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle_{\mathcal{H}} &= \Re\langle\begin{pmatrix} v \\ \gamma^2\Delta u \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle_{\mathcal{H}} = \Re\{\alpha(v,u)_{0,\partial\Omega} + (\nabla v, \nabla u) + (\Delta u, v)\} \\ &= \Re\{\alpha(v,u)_{0,\partial\Omega} - (\alpha u + \beta v, v)_{0,\partial\Omega}\} = -\beta(v,v)_{0,\partial\Omega} \leq 0. \end{aligned}$$

■

Lemma 2.4 *$\forall\lambda > 0$, $[\lambda - A]^{-1} \in \mathcal{B}(\mathcal{H})$.*

Proof: With:

$$f \in H^1(\Omega), \qquad g \in L_2(\Omega)$$

chosen arbitrarily, it will be shown that there exists:

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A) \qquad \text{such that} \qquad [\lambda - A]\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.5}$$

while:

$$\|u\|_2 \leq c(\|f\|_1 + \|g\|) \qquad \text{and} \qquad \|v\|_1 \leq c(\|f\|_1 + \|g\|). \tag{2.6}$$

By [11], there exists a unique $u \in H^2(\Omega)$ such that:

$$\begin{cases} -\gamma^2 \Delta u + \lambda^2 u &= g + \lambda f \quad \text{in} \quad \Omega \\ (\alpha + \beta\lambda)u + \partial_n u &= \beta f \quad \text{on} \quad \partial\Omega \end{cases}$$

and:

$$\|u\|_2 \leq c\|g + \lambda f\| + c\|f\|_{\frac{1}{2},\partial\Omega} \leq c\|g\| + c\|f\|_1$$

where the last inequality follows from the trace estimate [1],

$$\|\phi\|_{s-\frac{1}{2},\partial\Omega} \leq c\|\phi\|_{s,\Omega} \qquad \forall \phi \in H^s(\Omega). \tag{2.7}$$

Then setting $v \equiv \lambda u - f$ gives (2.5) and (2.6). ∎

Returning to the analysis of (1.1), the theorem below requires the following definitions. Given $X \in \mathcal{H}$ and $V \in L_2([0, t^\star], \mathcal{H})$, the abstract Cauchy problem:

$$\begin{cases} U'(t) &= AU(t) + V(t) \qquad 0 \leq t \leq t^\star \\ U(0) &= U^0 \end{cases} \tag{2.8}$$

is said to have a *weak solution* $U(t)$, if $U(t) \in \mathcal{C}([0, t^\star], \mathcal{H})$, and:

$$\int_0^{t^\star} \langle X, U\rangle_{\mathcal{H}} dt + \int_0^{t^\star} \langle Y, V\rangle_{\mathcal{H}} dt + \langle Y(0), U^0\rangle_{\mathcal{H}} = 0$$

$$\forall X \in \mathcal{C}([0, t^\star], \mathcal{H}), \qquad \text{and} \qquad Y(t) \equiv -\int_t^{t^\star} S^*(s-t)X(s)ds$$

where A is assumed to generate a semigroup $S(t) \in \mathcal{C}([0, t^\star], \mathcal{H})$. Also, it is well-known [7] that under the stated assumptions, the unique weak solution to (2.8) is given by the so-called *mild solution*:

$$U(t) \equiv S(t)U^0 + \int_0^t S(t-s)V(s)ds.$$

Theorem 2.1 *A generates a semigroup $S(t) \in \mathcal{C}([0, \infty), \mathcal{H})$, and $|S(t)|_{\mathcal{H}} \leq 1$, while $\|S(t)\|_{\mathcal{H}} \leq M_0$, $M_0 > 1$. Furthermore, for every $F(t) \in L_2([0, t^\star], L_2(\Omega))$, the unique weak solution to (2.3) on $[0, t^\star]$ is given by:*

$$P_2(t) = \int_0^t S(t-s)BF(s)ds.$$

Proof: From Lemmas 2.3 and 2.4, and the Lumer-Phillips Theorem, A generates a semigroup $S(t) \in \mathcal{C}([0, \infty), \mathcal{H})$, with $|S(t)|_{\mathcal{H}} \leq 1$. That $\|S(t)\|_{\mathcal{H}} \leq M_0$, $M_0 > 1$, follows from Lemma 2.2. The rest of the theorem follows as discussed above [7]. ∎

The next lemma shows that A has only discrete eigenvalues in its spectrum, as this conforms with physical intuition. It is also a first step toward showing that A is a spectral operator. Unfortunately though, the eigenfunctions of A do not form an orthogonal basis for $\mathcal{H}$ as an explicit computation shows.

Lemma 2.5 *A has a compact resolvent.*

Proof: Assume that:

$$f_k \underset{H^1(\Omega)}{\overset{k\to\infty}{\rightharpoonup}} f_0, \qquad \text{and} \qquad g_k \underset{L_2(\Omega)}{\overset{k\to\infty}{\rightharpoonup}} g_0,$$

and with $\lambda > 0$ set:

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} \equiv [\lambda - A]^{-1} \begin{pmatrix} f_k \\ g_k \end{pmatrix} \qquad k = 0, 1, 2 \ldots$$

First, by Lemma 2.4, $([\lambda - A]^{-1})^* = [\lambda - A^*]^{-1}$, and $[\lambda - A^*]^{-1} \in \mathcal{B}(L_2(\Omega))$, while:

$$\begin{aligned} (\begin{pmatrix} u_k \\ v_k \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \end{pmatrix})_{\mathcal{H}} &= (\begin{pmatrix} f_k \\ g_k \end{pmatrix}, [\lambda - A^*]^{-1} \begin{pmatrix} u^* \\ v^* \end{pmatrix})_{\mathcal{H}} \stackrel{k\to\infty}{\longrightarrow} (\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}, [\lambda - A^*]^{-1} \begin{pmatrix} u^* \\ v^* \end{pmatrix})_{\mathcal{H}} \\ &= (\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \end{pmatrix})_{\mathcal{H}} \qquad \forall \begin{pmatrix} u^* \\ v^* \end{pmatrix} \in \mathcal{H}. \end{aligned}$$

Hence:

$$u_k \underset{H^1(\Omega)}{\overset{k\to\infty}{\rightharpoonup}} u_0, \qquad \text{and} \qquad v_k \underset{L_2(\Omega)}{\overset{k\to\infty}{\rightharpoonup}} v_0.$$

Now, using (2.6):

$$\|u_k\|_2 \le c\{\|g_k\| + \|f_k\|_1\} \le C \ne C(k)$$

and:

$$\|v_k\|_1 \le c\{\|g_k\| + \|f_k\|_1\} \le C \ne C(k).$$

Since the embeddings $H^1(\Omega) \hookrightarrow L_2(\Omega)$ and $H^2(\Omega) \hookrightarrow H^1(\Omega)$ are compact [1], it follows that:

$$v_k \underset{L_2(\Omega)}{\overset{k\to\infty}{\longrightarrow}} v_0 \qquad \text{and} \qquad u_k \underset{H^1(\Omega)}{\overset{k\to\infty}{\longrightarrow}} u_0.$$

Finally, by Lemma 2.2, $|\cdot|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent. Therefore, the result follows. ■

To complete this section, (1.2) is analyzed using the first order form:

$$\begin{cases} P_3' &= AP_3 + BF \qquad 0 \le t \le \tau \\ P_3(0) &= P_3(\tau) \end{cases} \tag{2.9}$$

where $P_3 \equiv (p_3, \partial_t p_3)^T$. As above, given $X \in \mathcal{H}$ and $V \in L_2([0, t^\star], \mathcal{H})$, the abstract problem:

$$\begin{cases} U'(t) &= AU(t) + V(t) \qquad 0 \le t \le t^\star \\ U(0) &= U(t^\star) \end{cases} \tag{2.10}$$

is said to have a *weak solution* $U(t)$, if $U(t) \in \mathcal{C}([0, t^\star], \mathcal{H})$, and:

$$\int_0^{t^\star} \langle X, U\rangle_{\mathcal{H}} dt + \int_0^{t^\star} \langle Y, V\rangle_{\mathcal{H}} dt + \langle Y(0), [U(t^\star) - U^0]\rangle_{\mathcal{H}} = 0$$

$$\forall X \in \mathcal{C}([0, t^\star], \mathcal{H}), \qquad \text{and}$$

$$Y(t) \equiv -S^*(t^\star - t)[I - S^*(t^\star)]^{-1} \int_0^{t^\star} S^*(s)X(s)ds - \int_t^{t^\star} S^*(s-t)X(s)ds$$

where A is assumed to generate a semigroup $S(t) \in \mathcal{C}([0, t^\star], \mathcal{H})$, and $1 \in \rho[S^*(t^\star)]$. Also, as alluded in [8], under the stated assumptions the unique weak solution to (2.10) is given by the so-called *mild solution*:

$$U(t) \equiv S(t)[I - S(t^\star)]^{-1} \int_0^{t^\star} S(t^\star - s)V(s)ds + \int_0^{t^\star} S(t-s)V(s)ds$$

provided $1 \in \rho[S(t^\star)]$. Hence, the following is obtained.

Theorem 2.2 *The unique weak solution to (2.9) is given by the mild solution:*

$$P_3(t) = S(t)[I - S(\tau)]^{-1} \int_0^{\tau} S(\tau - s)BF(s)ds + \int_0^{t} S(t-s)BF(s)ds.$$

Proof: It is first shown that the resolvent sets $\rho[S^*(\tau)]$ and $\rho[S(\tau)]$ contain unity. This can be seen from the fact shown in the next section, that $S(t)$ satisfies an exponential decay estimate. Hence, there exists an $n \in \mathbf{N}$ for which $1 \in \rho[S(n\tau)]$. So for every $W \in \mathcal{H}$, the equation:

$$[I - S(\tau)]U = W$$

can be solved with:

$$U = [I - S(n\tau)]^{-1} \sum_{m=0}^{n-1} S(m\tau)W.$$

A similar argument places $1 \in \rho[S^*(\tau)]$. Now the rest of the proof is similar to that of Theorem 2.1 [7]. ■

3 The Decay Estimate.

In this section, the techniques of Lagnese [9] are adapted to show that there exists an $M_0 > 0$ and a $\mu_0 > 0$ such that:

$$|S(t)|_{\mathcal{H}} \leq M_0 e^{-\mu_0 t}. \tag{3.1}$$

Then according to Lemma 2.2, it will follow that there is an $\tilde{M}_0 > 0$ such that:

$$\|S(t)\|_{\mathcal{H}} \leq \tilde{M}_0 e^{-\mu_0 t}. \tag{3.2}$$

For this, let $(\rho_1, \rho_2)^T \in \mathcal{D}(A)$ be chosen arbitrarily, and suppose p satisfies:

$$\left\{ \begin{array}{rcll} \partial_t^2 p &=& \gamma^2 \Delta p & \text{in} \quad \Omega \times [0, \infty) \\ 0 &=& \alpha p + \beta \partial_t p + \partial_n p & \text{on} \quad \partial\Omega \times [0, \infty) \\ p(0) &=& \rho_1 & \text{in} \quad \Omega \\ \partial_t p(0) &=& \rho_1 & \text{in} \quad \Omega \end{array} \right. \tag{3.3}$$

Now with $P_0 \equiv (\rho_1, \rho_2)^T$ and $P \equiv (p, \partial_t p)^T$ define:

$$E(p, t) \equiv |S(t)P_0|_{\mathcal{H}} = |P(t)|^2_{\mathcal{H}} = \alpha \|p\|^2_{0,\partial\Omega} + \|\nabla p\|^2 + \gamma^{-2} \|\partial_t p\|^2.$$

Using the semigroup property in addition to the contraction property guaranteed by Theorem 2.1, it follows that:

$$E(p, t) = |S(h)S(t - h)P_0|^2_{\mathcal{H}} \leq |S(t - h)P_0|^2_{\mathcal{H}} = E(p, t - h) \qquad \forall t \geq h \geq 0. \tag{3.4}$$

Thus, $E(p, t)$ is a nonincreasing function of t. Further, for (3.1), the strategy is to show that:

$$\int_0^\infty E(p, t)dt \leq \bar{c}E(p, 0) \tag{3.5}$$

for a constant $\bar{c} > 0$ which is independent of p. The first step is to estimate the time derivative of the functional:

$$Q(t) \equiv \tfrac{1}{2} tE(p, t) + 2\gamma^{-2}(\ell \cdot \nabla p, \partial_t p) + \gamma^{-2}((\nabla \cdot \ell - 1)p, \partial_t p).$$

For this, the following technical lemma is required.

Lemma 3.1 *Given a smooth, real vector field ℓ, and p satisfying (3.3), the following identity holds in Ω:*

$$\begin{gathered} \partial_t \Big\{ t(|\nabla p|^2 + \gamma^{-2}|\partial_t p|^2) + 2\gamma^{-2}\Re\{(\ell \cdot \nabla p)\partial_t p^*\} + \gamma^{-2}(\nabla \cdot \ell - 1)\Re\{p \partial_t p^*\} \Big\} = \\ \nabla \cdot \Big\{ 2t\Re\{\partial_t p \nabla p^*\} + \gamma^{-2}\ell|\partial_t p|^2 + 2\Re\{(\ell \cdot \nabla p)\nabla p^*\} - \ell|\nabla p|^2 + (\nabla \cdot \ell - 1)\Re\{p \nabla p^*\} \Big\} \\ + 2\Re\{\nabla p : (I - D\ell) : \nabla p^*\} - \nabla(\nabla \cdot \ell) \cdot \Re\{p \nabla p^*\}. \end{gathered} \tag{3.6}$$

Also, on $\partial\Omega$:

$$\tfrac{1}{2}\alpha D_t \left\{ t|p|^2 \right\} = \tfrac{1}{2}\alpha(1 - 2t\alpha\beta^{-1})|p|^2 - \alpha\beta^{-1} t\Re\{p \partial_n p^*\}. \tag{3.7}$$

Proof: Using (3.3):

$$
\begin{aligned}
&\partial_t\Big\{t(|\nabla p|^2+\gamma^{-2}|\partial_t p|^2)+2\gamma^{-2}\Re\{(\ell\cdot\nabla p)\partial_t p^*\}+\gamma^{-2}(\nabla\cdot\ell-1)\Re\{p\partial_t p^*\}\Big\}\\
&\quad= (|\nabla p|^2+\gamma^{-2}|\partial_t p|^2)+2t\Re\{\nabla p\cdot\nabla\partial_t p^*+\gamma^{-2}\partial_t p\partial_t^2 p^*\}\\
&\qquad+2\gamma^{-2}\Re\{(\ell\cdot\nabla\partial_t p)\partial_t p^*\}+(\ell\cdot\nabla p)\partial_t^2 p^*\\
&\qquad+\gamma^{-2}(\nabla\cdot\ell-1)(|\partial_t p|^2+\Re\{p\partial_t^2 p^*\})\\
&\quad= |\nabla p|^2+2t\Re\{\nabla\cdot(\partial_t p\nabla p^*)\}+\gamma^{-2}\nabla\cdot(\ell|\partial_t p|^2)\\
&\qquad+2\Re\{(\ell\cdot\nabla p)\Delta p^*\}+(\nabla\cdot\ell-1)\Re\{p\Delta p^*\}.
\end{aligned}
$$

The last few terms can written as follows:

$$
\begin{aligned}
2(\ell\cdot\nabla p)\Delta p^* &= 2\nabla\cdot[(\ell\cdot\nabla p)\nabla p^*]-2\nabla(\ell\cdot\nabla p)\cdot\nabla p^*\\
&= 2\nabla\cdot[(\ell\cdot\nabla p)\nabla p^*]-2\nabla p:D\ell:\nabla p^*-\nabla\cdot(\ell|\nabla p|^2)+(\nabla\cdot\ell)|\nabla p|^2\\
(\nabla\cdot\ell)p\Delta p^* &= \nabla\cdot[(\nabla\cdot\ell)p\nabla p^*]-\nabla[(\nabla\cdot\ell)p]\cdot\nabla p^*\\
&= \nabla\cdot[(\nabla\cdot\ell)p\nabla p^*]-\nabla(\nabla\cdot\ell)\cdot(p\nabla p^*)-(\nabla\cdot\ell)|\nabla p|^2\\
-p\Delta p^* &= -\nabla\cdot(p\nabla p^*)+|\nabla p|^2.
\end{aligned}
$$

Combining the above:

$$
\begin{aligned}
&\partial_t\Big\{t(\gamma^{-2}|\partial_t p|^2+|\nabla p|^2)+2\gamma^{-2}\Re\{(\ell\cdot\nabla p)\partial_t p^*\}+\gamma^{-2}(\nabla\cdot\ell-1)\Re\{p\partial_t p^*\}\Big\}\\
&= |\nabla p|^2+2t\Re\{\nabla\cdot(\partial_t p\nabla p^*)\}+\gamma^{-2}\nabla\cdot(\ell|\partial_t p|^2)\\
&\quad+2\Re\{\nabla\cdot[(\ell\cdot\nabla p)\nabla p^*]\}-2\Re\{\nabla p:D\ell:\nabla p^*\}-\nabla\cdot(\ell|\nabla p|^2)\\
&\quad+\Re\{\nabla\cdot[(\nabla\cdot\ell)p\nabla p^*]\}-\nabla(\nabla\cdot\ell)\cdot\Re\{p\nabla p^*\}-\Re\{\nabla\cdot(p\nabla p^*)\}+|\nabla p|^2\\
&= \nabla\cdot\Big\{2t\Re\{\partial_t p\nabla p^*\}+\gamma^{-2}\ell|\partial_t p|^2+2\Re\{(\ell\cdot\nabla p)\nabla p^*\}-\ell|\nabla p|^2+(\nabla\cdot\ell)\Re\{p\nabla p^*\}-\Re\{p\nabla p^*\}\Big\}\\
&\quad+2\Re\{\nabla p:(I-D\ell)\nabla p^*\}-\nabla(\nabla\cdot\ell)\cdot\Re\{p\nabla p^*\}
\end{aligned}
$$

Hence, (3.6) is established. Finally, using the boundary condition in (3.3):

$$
\begin{aligned}
\tfrac{1}{2}\alpha D_t\Big\{t|p|^2\Big\} &= \tfrac{1}{2}\alpha\Big\{|p|^2+2t\Re\{p\partial_t p^*\}\Big\} = \tfrac{1}{2}\alpha\Big\{|p|^2+2t\Re\{p\beta^{-1}(-\alpha p^*-\partial_n p^*)\}\Big\}\\
&= \tfrac{1}{2}\alpha(1-2t\alpha\beta^{-1})|p|^2-\alpha\beta^{-1}t\Re\{p\partial_n p^*\}
\end{aligned}
$$

which gives (3.7). ■

Proposition 3.1 *Suppose that ℓ and p are as indicated in Lemma 3.1. Also, assume that $2I-[D\ell+D\ell^T]$ is uniformly negative definite throughout $\bar\Omega$ and that $\ell\cdot n\geq\ell_0>0$ on $\partial\Omega$. Then there exists a $t_0>0$ such that for $t\geq t_0$:*

$$\Re\{D_tQ(t)\}\leq-\tfrac{1}{4}E(p,t)+c_0\|p\|^2 \tag{3.8}$$

and:

$$\Re\{Q(t)\}\geq 0. \tag{3.9}$$

Proof: First, according to (3.6) and then (3.7):

$$
\begin{aligned}
\Re\{D_tQ(t)\} &= \tfrac{1}{2}\alpha(1-2t\alpha\beta^{-1})\|p\|^2_{0,\partial\Omega}-\alpha\beta^{-1}t\Re\{(p,\partial_n p)_{0,\partial\Omega}\}-\tfrac{1}{2}D_t[t(\|\nabla p\|^2+\gamma^{-2}\|\partial_t p\|^2)]\\
&\quad+D_t[t(\|\nabla p\|^2+\gamma^{-2}\|\partial_t p\|^2)+2\gamma^{-2}\Re\{(\ell\cdot\nabla p,\partial_t p)\}+\gamma^{-2}\Re\{((\nabla\cdot\ell-1)p,\partial_t p)\}]\\
&= \tfrac{1}{2}\alpha(1-2t\alpha\beta^{-1})\|p\|^2_{0,\partial\Omega}-\alpha\beta^{-1}t\Re\{(p,\partial_n p)_{0,\partial\Omega}\}\\
&\quad-\tfrac{1}{2}\|\nabla p\|^2-\tfrac{1}{2}\gamma^{-2}\|\partial_t p\|^2-t\Re\{(\nabla p,\nabla\partial_t p)+\gamma^{-2}(\partial_t p,\partial_t^2 p)\}\\
&\quad+2t\Re\{(\partial_t p,\partial_n p)_{0,\partial\Omega}\}+\gamma^{-2}((\ell\cdot n)\partial_t p,\partial_t p)_{0,\partial\Omega}\\
&\quad+2\Re\{(\ell\cdot\nabla p,\partial_n p)_{0,\partial\Omega}\}-((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega}+\Re\{((\nabla\cdot\ell-1)p,\partial_n p)_{0,\partial\Omega}\}\\
&\quad+2((I-D\ell)\nabla p,\nabla p)-\Re\{(p\nabla(\nabla\cdot\ell),\nabla p)\}.
\end{aligned}
$$

Using (3.3):

$$\begin{aligned}
\Re\{D_tQ(t)\} &= \tfrac{1}{2}\alpha\|p\|^2_{0,\partial\Omega} - \alpha^2\beta^{-1}t\|p\|^2_{0,\partial\Omega} - \alpha\beta^{-1}t\Re\{(p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad -\tfrac{1}{2}\|\nabla p\|^2 - \tfrac{1}{2}\gamma^{-2}\|\partial_t p\|^2 - t\Re\{(\partial_t p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad +2t\Re\{(\partial_t p,\partial_n p)_{0,\partial\Omega}\} + \beta^{-2}\gamma^{-2}((\ell\cdot n)(\alpha p+\partial_n p),(\alpha p+\partial_n p))_{0,\partial\Omega} \\
&\quad +2\Re\{(\ell\cdot\nabla p,\partial_n p)_{0,\partial\Omega}\} - ((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega} + \Re\{((\nabla\cdot\ell-1)p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad +((2I-[D\ell+D\ell^T])\nabla p,\nabla p) - \Re\{(p\nabla(\nabla\cdot\ell),\nabla p)\} \\
&= -\tfrac{1}{2}\{\alpha\|p\|^2_{0,\partial\Omega} + \|\nabla p\|^2 + \gamma^{-2}\|\partial_t p\|^2\} + \alpha\|p\|^2_{0,\partial\Omega} \\
&\quad -\alpha^2\beta^{-1}t\|p\|^2_{0,\partial\Omega} - \alpha\beta^{-1}t\Re\{(p,\partial_n p)_{0,\partial\Omega}\} - \beta^{-1}t\Re\{(\alpha p+\partial_n p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad +\beta^{-2}\gamma^{-2}((\ell\cdot n)(\alpha p+\partial_n p),(\alpha p+\partial_n p))_{0,\partial\Omega} \\
&\quad +2\Re\{(\ell\cdot\nabla p,\partial_n p)_{0,\partial\Omega}\} - ((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega} + \Re\{((\nabla\cdot\ell-1)p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad +((2I-[D\ell+D\ell^T])\nabla p,\nabla p) - \Re\{(p\nabla(\nabla\cdot\ell),\nabla p)\} \\
&= -\tfrac{1}{2}\{\alpha\|p\|^2_{0,\partial\Omega} + \|\nabla p\|^2 + \gamma^{-2}\|\partial_t p\|^2\} \\
&\quad -\beta^{-1}t\|\alpha p+\partial_n p\|^2_{0,\partial\Omega} - ((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega} \\
&\quad +\alpha\|p\|^2_{0,\partial\Omega} + \beta^{-2}\gamma^{-2}((\ell\cdot n)(\alpha p+\partial_n p),(\alpha p+\partial_n p))_{0,\partial\Omega} \\
&\quad +2\Re\{(\ell\cdot\nabla p,\partial_n p)_{0,\partial\Omega}\} + \Re\{((\nabla\cdot\ell-1)p,\partial_n p)_{0,\partial\Omega}\} \\
&\quad +((2I-[D\ell+D\ell^T])\nabla p,\nabla p) - \Re\{(p\nabla(\nabla\cdot\ell),\nabla p)\}.
\end{aligned}$$

These terms can be estimated as follows:

$$\begin{aligned}
\beta^{-2}\gamma^{-2}((\ell\cdot n)(\alpha p+\partial_n p),(\alpha p+\partial_n p))_{0,\partial\Omega} &\le c_1\|\alpha p+\partial_n p\|^2_{0,\partial\Omega} \\
2|(\ell\cdot\nabla p,\partial_n p)_{0,\partial\Omega}| &\le \tfrac{1}{2}\ell_0\|\nabla p\|^2_{0,\partial\Omega} + c\|\partial_n p\|^2_{0,\partial\Omega} \\
&\le \tfrac{1}{2}((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega} + c_2\|\alpha p+\partial_n p\|^2_{0,\partial\Omega} \\
&\qquad +c_3\|p\|^2_{0,\partial\Omega} \\
|(p\nabla(\nabla\cdot\ell),\nabla p)| &\le c_4\|p\|^2 + \tfrac{1}{8}\|\nabla p\|^2 \\
|((\nabla\cdot\ell-1)p,\partial_n p)_{0,\partial\Omega}| &\le c\|p\|^2_{0,\partial\Omega} + c\|\partial_n p\|^2_{0,\partial\Omega} \\
&\le c_5\|p\|^2_{0,\partial\Omega} + c_6\|\alpha p+\partial_n p\|^2_{0,\partial\Omega} \\
(\alpha+c_3+c_5)\|p\|^2_{0,\partial\Omega} &\le c\|p\|^2_{\frac{1}{2},\Omega} \;\le\; c\|p\|\|p\|_1 \\
&\le c_7\|p\|^2 + \tfrac{1}{8}\|\nabla p\|^2
\end{aligned}$$

where in the last estimate, the trace inequality (2.7) and the interpolation inequality [1]:

$$\|\phi\|^2_{\frac{1}{2},\Omega} \le c\|\phi\|\|\phi\|_1 \qquad \forall\phi\in H^1(\Omega) \tag{3.10}$$

were used. Now combining the above estimates and the negative definiteness of $2I-[D\ell+D\ell^T]$ gives:

$$\begin{aligned}
\Re\{D_tQ(t)\} &\le -\tfrac{1}{2}\{\alpha\|p\|^2_{0,\partial\Omega} + \|\nabla p\|^2 + \gamma^{-2}\|\partial_t p\|^2\} \\
&\quad +(c_1+c_2+c_6-\beta^{-1}t)\|\alpha p+\partial_n p\|^2_{0,\partial\Omega} - \tfrac{1}{2}((\ell\cdot n)\nabla p,\nabla p)_{0,\partial\Omega} \\
&\quad +\tfrac{1}{4}\|\nabla p\|^2 + (c_4+c_7)\|p\|^2.
\end{aligned}$$

Also, by (2.4):

$$\Re\{Q(t)\} \ge \tfrac{1}{2}tE(p,t) - c\Big[\|p\|^2 + \|\nabla p\|^2 + \|\partial_t p\|^2\Big] \ge (\tfrac{1}{2}t-c)E(p,t).$$

So with $c_0 \equiv c_4+c_7$, the result follows for $t_0>0$ sufficiently large. ■

Now in order to establish (3.5), (3.8) will be integrated over $[0,\infty)$. However, $\int_0^\infty \|p\|^2 dt$ must then be estimated in terms of $\int_0^\infty E(p,t)dt$ and $E(p,0)$, and it is this task which is undertaken next.

Let $T > 0$ be fixed and define the cut-off function $\phi \in \mathcal{C}^\infty(\mathbf{R})$ by $\phi(t) \equiv \exp\{T^{-2} - t^{-1}(2T - t)^{-1}\}$ for $t \in (0, T]$, and extended by zero to the left and by one to the right. Then let $u \equiv \phi p$, so that u satisfies:

$$\left\{ \begin{array}{rcll} \partial_t^2 u & = & \gamma^2 \Delta u + g & \text{in} \quad \Omega \times [0, \infty) \\ h & = & \alpha u + \beta \partial_t u + \partial_n u & \text{on} \quad \partial\Omega \times [0, \infty) \\ u(0) & = & 0 & \text{in} \quad \Omega \\ \partial_t u(0) & = & 0 & \text{in} \quad \Omega \end{array} \right. \tag{3.11}$$

where $g = 2\phi' \partial_t p + \phi'' p$ and $h = \beta \phi' p$. Since $g = h = 0$ for $t \geq T$, by (2.4) and (3.4):

$$\int_0^\infty \|g\|^2 dt \leq c \int_0^T \{\|\partial_t p\|^2 + \|p\|^2\} dt \leq c \int_0^T E(p, t) dt \leq c E(p, 0), \tag{3.12}$$

$$\int_0^\infty \|h\|_{0,\partial\Omega}^2 dt \leq c \int_0^T \|p\|_{0,\partial\Omega}^2 dt \leq c \int_0^T E(p, t) dt \leq c E(p, 0). \tag{3.13}$$

Now let $\omega \in \mathbf{C}$ with $\omega_r \equiv \Re\{\omega\}$ and $\omega_i \equiv \Im\{\omega\}$. Then for $\omega_i < 0$, define:

$$\hat{u}(\mathbf{x}, \omega) \equiv \int_0^\infty e^{-i\omega t} u(\mathbf{x}, t) dt, \qquad \hat{g}(\mathbf{x}, \omega) \equiv \int_0^T e^{-i\omega t} g(\mathbf{x}, t) dt, \qquad \hat{h}(\mathbf{x}, \omega) \equiv \int_0^T e^{-i\omega t} h(\mathbf{x}, t) dt$$

so that the following holds:

$$\left\{ \begin{array}{rcll} -\omega^2 \hat{u} & = & \gamma^2 \Delta \hat{u} + \hat{g} & \text{in} \quad \Omega \\ \hat{h} & = & (\alpha + i\omega\beta)\hat{u} + \partial_n \hat{u} & \text{on} \quad \partial\Omega \end{array} \right. \tag{3.14}$$

in the sense that:

$$b_\omega(\hat{u}, v) - \omega^2(\hat{u}, v) = (\hat{g}, v) + \gamma^2(\hat{h}, v)_{0,\partial\Omega} \qquad \forall v \in H^1(\Omega) \tag{3.15}$$

where:

$$b_\omega(w, v) \equiv \gamma^2(\nabla w, \nabla v) + \gamma^2(\alpha + i\omega\beta)(w, v)_{0,\partial\Omega}.$$

Steps are now taken to prove an estimate of $\hat{u}(\omega)$ in terms of $\hat{g}(\omega)$ and $\hat{h}(\omega)$, which holds for all ω in a certain neighborhood of the real axis.

Lemma 3.2 *If $\omega \in \mathbf{R}$, and:*

$$b_\omega(\hat{v}, v) = \omega^2(\hat{v}, v) \qquad \forall v \in H^1(\Omega) \tag{3.16}$$

then $\hat{v} = 0$.

Proof: First note that for $\omega \in \mathbf{R}$:

$$\Re\{b_\omega(v, v) - \omega^2(v, v)\} = \gamma^2\|\nabla v\|^2 - \omega^2\|v\|^2 + \gamma^2\alpha\|v\|_{0,\partial\Omega}^2 \tag{3.17}$$

and:

$$\Im\{b_\omega(v, v) - \omega^2(v, v)\} = \omega\beta\|v\|_{0,\partial\Omega}^2. \tag{3.18}$$

Hence, if $\omega = 0$, by (3.17), (3.16) and (2.4), $\hat{v} = 0$. On the other hand, if $\omega \in \mathbf{R}\backslash\{0\}$, it follows from (3.18) and (3.16) that $\hat{v} \in H_0^1(\Omega)$. So extend $\hat{v}$ by zero to the whole of $\mathbf{R}^3$ to obtain $\tilde{v} \in H^1(\mathbf{R}^3)$ which according to (3.16), satisfies:

$$(\tilde{v}, \gamma^2\Delta\phi + \omega^2\phi) = 0 \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathbf{R}^3).$$

It then follows that $\tilde{v}$ satisfies $\gamma^2\Delta\tilde{v} + \omega^2\tilde{v} = 0$ throughout $\mathbf{R}^3$ and is therefore entire [5]. So since $\tilde{v} = 0$ in $\mathbf{R}^3\backslash\Omega$, $\tilde{v} \equiv 0$, and thus $\hat{v} \equiv 0$. ∎

Lemma 3.3 *For $|\omega_i| \le \frac{1}{2}\alpha\beta^{-1}$ the following hold:*

$$c_1\|v\|_1^2 \le \Re\{b_\omega(v,v)\} \qquad \forall v \in H^1(\Omega) \tag{3.19}$$

$$|b_\omega(v,w)| \le c_2(1+|\omega|)\|v\|_1\|w\|_1 \qquad \forall v,w \in H^1(\Omega). \tag{3.20}$$

Further, there exists a unique $B_\omega \in \mathcal{B}(H^1(\Omega))$ with $B_\omega^{-1} \in \mathcal{B}(H^1(\Omega))$ such that:

$$b_\omega(v,w) = (B_\omega v, w)_1 \qquad \forall v,w \in H^1(\Omega) \tag{3.21}$$

while:

$$\|B_\omega^{-1}\|_1 \le c_1^{-1} \qquad \text{and} \qquad \|B_\omega\|_1 \le c_2(1+|\omega|). \tag{3.22}$$

Proof: First, note that:

$$\Re\{b_\omega(v,v)\} = \gamma^2(\nabla v, \nabla v) + \gamma^2(\alpha - \beta\omega_i)(v,v)_{0,\partial\Omega}$$

so (3.19) follows with (2.4). Then, (3.20) follows readily from the trace property (2.7). The remaining claims follow in a straightforward way with the Lax-Milgram lemma [6]. ■

Now, let the natural embedding of $H^1(\Omega)$ into $L_2(\Omega)$ be denoted by N so that:

$$(w,v) = (w, Nv) = (N^*w, v)_1 \qquad \forall w \in L_2(\Omega), \quad \forall v \in H^1(\Omega).$$

It is readily shown that:

$$\|N\|_{H^1(\Omega),L_2(\Omega)} = 1 \qquad \text{and} \qquad \|N^*\|_{L_2(\Omega),H^1(\Omega)} = 1. \tag{3.23}$$

Thus, for $|\omega_i| \le \frac{1}{2}\alpha\beta^{-1}$, define:

$$G_\omega \equiv B_\omega^{-1}N^* \in \mathcal{B}(L_2(\Omega), H^1(\Omega)) \qquad \text{and} \qquad \tilde{G}_\omega \equiv NG_\omega \in \mathcal{B}(L_2(\Omega)),$$

so that from (3.22) and (3.23), it follows that:

$$\|G_\omega\|_{L_2(\Omega),H^1(\Omega)} \le c_1^{-1} \qquad \text{and} \qquad \|\tilde{G}_\omega\| \le c_1^{-1}. \tag{3.24}$$

Moreover the following holds.

Lemma 3.4 *Given $\omega_0 > 0$, there exists a $\delta(\omega_0) \in (0, \frac{1}{2}\alpha\beta^{-1}]$ such that with:*

$$R_{\omega_0} \equiv \{\omega \in \mathbf{C} : |\omega_r| \le \omega_0, |\omega_i| \le \delta(\omega_0)\}$$

and $\omega \in R_{\omega_0}$, $(I - \omega^2\tilde{G}_\omega)^{-1} \in \mathcal{B}(L_2(\Omega))$.

Proof: First note that according to the continuity of N^* and B_ω^{-1}, and the compactness of N [1], $\tilde{G}_\omega$ is compact. Next it is shown that if $\omega^{-2} \in \mathbf{R}$, then $\omega^{-2} \notin \sigma(\tilde{G}_\omega)$. From this it follows with the Fredholm Alternative for compact operators, that $(\omega^{-2} - \tilde{G}_\omega)^{-1} \in \mathcal{B}(L_2(\Omega))$. So suppose that $\tilde{v} \in L_2(\Omega)$ satisfies $\tilde{G}_\omega\tilde{v} = \omega^{-2}\tilde{v}$, Then set $v \equiv \omega^2 G_\omega \tilde{v} \in H^1(\Omega)$ so that $Nv = \omega^2\tilde{G}_\omega\tilde{v} = \tilde{v}$ and $v = \omega^2 G_\omega N v$. Hence:

$$\begin{aligned} 0 &= ([I - \omega^2 G_\omega N]v, B_\omega^* w)_1 = (B_\omega v - \omega^2 N^* N v, w)_1 \\ &= (B_\omega v, w)_1 - \omega^2(Nv, Nw) \qquad \forall w \in H^1(\Omega). \end{aligned}$$

Therefore by (3.21):

$$b_\omega(v,w) = \omega^2(v,w) \qquad \forall w \in H^1(\Omega).$$

From Lemma 3.2, it follows that $v = 0$ and hence that $\tilde{v} = 0$. So the following is finite:

$$C_1(\omega_0) \equiv \max\{\|(I - \omega^2\tilde{G}_\omega)^{-1}\| : \omega \in R_{\omega_0} \cap \mathbf{R}\}$$

The next step is to show that with $\omega \in R_{\omega_0}$ that $C_1(\omega_0)\|\omega^2\tilde{G}_\omega - \omega_r^2\tilde{G}_{\omega_r}\|$ can be made less than unity provided $\delta(\omega_0)$ is sufficiently small. From this it follows with a Neumann Series argument that $(I-\omega^2\tilde{G}_\omega)^{-1} \in \mathcal{B}(L_2(\Omega))$. First note that by (3.22) and (3.24), the following are finite:

$$C_2(\omega_0) \equiv \max\{\|B_\omega^{-1}\|_1 : \omega \in R_{\omega_0}\}, \qquad C_3(\omega_0) \equiv \max\{\|\tilde{G}_\omega\| : \omega \in R_{\omega_0}\}.$$

Thus with (3.23), it follows that:

$$\begin{aligned}\|\omega^2\tilde{G}_\omega - \omega_r^2\tilde{G}_{\omega_r}\| &= \|(\omega^2-\omega_r^2)\tilde{G}_\omega + \omega_r^2 N B_\omega^{-1}[B_{\omega_r} - B_\omega]B_{\omega_r}^{-1}N^*\| \\ &\leq \operatorname{diam}[R_{\omega_0}]C_3(\omega_0)\delta(\omega_0) + \omega_0^2 C_2^2(\omega_0)\|B_\omega - B_{\omega_r}\|_1.\end{aligned}$$

Hence the result will follow if $\|B_\omega - B_{\omega_r}\|_1$ can be estimated in terms of $\delta(\omega_0)$. For this note that by (3.21) and (2.7):

$$\begin{aligned}\|B_\omega - B_{\omega_r}\|_1 &= \sup_{u,v\in H^1(\Omega)} \frac{|([B_\omega - B_{\omega_r}]u,v)_1|}{\|u\|_1\|v\|_1} = \sup_{u,v\in H^1(\Omega)} \frac{|b_\omega(u,v) - b_{\omega_r}(u,v)|}{\|u\|_1\|v\|_1} \\ &= \sup_{u,v\in H^1(\Omega)} \frac{|\beta\gamma^2(\omega-\omega_r)(u,v)_{0,\partial\Omega}|}{\|u\|_1\|v\|_1} = c|\omega-\omega_r| \leq c\delta(\omega_0).\end{aligned}$$

Hence, for $\delta(\omega_0)$ sufficiently small and $\omega \in R_{\omega_0}$, $(I-\omega^2\tilde{G}_\omega)^{-1} \in \mathcal{B}(L_2(\Omega))$. ∎

Now the following establishes a relation among $\hat{u}(\omega)$, $\hat{g}(\omega)$, $\hat{h}(\omega)$, and ω_0 which holds $\forall \omega \in R_{\omega_0}$.

Theorem 3.1 *Let $\omega_0 > 0$. Then for every $\omega \in R_{\omega_0}$, for every $\hat{g}(\omega) \in L_2(\Omega)$, and for every $\hat{h}(\omega) \in L_2(\partial\Omega)$, there exists a unique $\hat{u}(\omega) \in H^1(\Omega)$ such that (3.15) holds together with:*

$$\|\hat{u}(\omega)\|_1 \leq c(\omega_0)(\|\hat{g}(\omega)\| + \|\hat{h}(\omega)\|_{0,\partial\Omega}). \tag{3.25}$$

Proof: It is first shown that there exists a $v_1 \in H^1(\Omega)$ such that:

$$b_\omega(v_1,w) = \gamma^2(\hat{h},w)_{0,\partial\Omega} \qquad \forall w \in H^1(\Omega). \tag{3.26}$$

By (2.7):

$$(h,w)_{0,\partial\Omega} \leq c\|h\|_{0,\partial\Omega}\|w\|_1 \qquad \forall h \in L_2(\partial\Omega), \quad \forall w \in H^1(\Omega)$$

so that by the Riesz Representation Theorem [1], there is a mapping $E \in \mathcal{B}(L_2(\partial\Omega), H^1(\Omega))$ for which:

$$(h,w)_{0,\partial\Omega} \equiv (Eh,w)_1 \qquad \forall h \in L_2(\partial\Omega), \quad \forall w \in H^1(\Omega).$$

From this and (3.21), it follows that $v_1 \equiv \gamma^2 B_\omega^{-1}E\hat{h}$ solves (3.26). Further, by (3.22):

$$\|v_1\|_1 \leq \gamma^2\|B_\omega^{-1}\|_1\|E\|_{L_2(\partial\Omega),H^1(\Omega)}\|\hat{h}\|_{0,\partial\Omega} \leq c_0\|\hat{h}\|_{0,\partial\Omega}. \tag{3.27}$$

Next, according to Lemma 3.4, the following are well-defined:

$$\tilde{v}_2 \equiv (I-\omega^2\tilde{G}_\omega)^{-1}\tilde{G}_\omega(\omega^2 N v_1 + \hat{g}) \in L_2(\Omega) \qquad \text{and} \qquad v_2 \equiv G_\omega(\omega^2\tilde{v}_2 + \omega^2 N v_1 + \hat{g}) \in H^1(\Omega).$$

Now since $Nv_2 = \tilde{G}_\omega(\omega^2\tilde{v}_2 + \omega^2 N v_1 + \hat{g}) = \tilde{v}_2$ it follows that:

$$\begin{aligned}b_\omega(v_2,w) &= (B_\omega v_2, w)_1 = (B_\omega G_\omega[\omega^2\tilde{v}_2 + \omega^2 N v_1 + \hat{g}], w)_1 \\ &= (\omega^2\tilde{v}_2 + \omega^2 N v_1 + \hat{g}, w) = (\omega^2 N v_2 + \omega^2 N v_1 + \hat{g}, w) \qquad \forall w \in H^1(\Omega).\end{aligned}$$

Hence:

$$b_\omega(v_2,w) - \omega^2(v_2,w) = \omega^2(v_1,w) + (\hat{g},w) \qquad \forall w \in H^1(\Omega). \tag{3.28}$$

Next, note that by Lemma 3.4, the following is finite:

$$C_1(\omega_0) \equiv \max\{\|\omega^2(I - \omega^2\tilde{G}_\omega)^{-1}\| : \omega \in R_{\omega_0}\}$$

as are the following according to (3.24):

$$C_2(\omega_0) \equiv \max\{\|G_\omega\|_{L_2(\Omega),H^1(\Omega)} : \omega \in R_{\omega_0}\}, \qquad C_3(\omega_0) \equiv \max\{\|\tilde{G}_\omega\| : \omega \in R_{\omega_0}\}.$$

Therefore with (3.27):

$$\begin{aligned} \|v_2\|_1 &\leq C_2(\omega_0)[C_1(\omega_0)C_3(\omega_0) + 1](|\omega|^2\|v_1\| + \|\hat{g}\|) \\ &\leq C_2(\omega_0)[C_1(\omega_0)C_3(\omega_0) + 1]([\omega_0^2 + \delta(\omega)^2]c_0\|\hat{h}\|_{0,\partial\Omega} + \|\hat{g}\|). \end{aligned} \tag{3.29}$$

From the sum of (3.26) and (3.28), it is seen that $\hat{u} \equiv v_1 + v_2$ satisfies (3.15). Also, uniqueness follows from Lemma 3.2. Finally, (3.25) follows from (3.27) and (3.29) with the indicated dependence upon ω_0. ∎

Finally, the next lemma gives the last estimate required for the inequality advertised earlier and proved in the proposition below.

Lemma 3.5 *For every $\varepsilon > 0$, there is a constant c_ε such that for every $\omega_i < 0$:*

$$\int_0^\infty e^{2\omega_i t}\|u\|^2 dt \leq c_\varepsilon E(p, 0) + \varepsilon \int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt. \tag{3.30}$$

Proof: Let $\varepsilon > 0$ and suppose $\omega_i \leq -\delta(\varepsilon^{-\frac{1}{2}})$. Then by (3.4):

$$\int_0^\infty e^{2\omega_i t}\|u\|^2 dt \leq \int_0^\infty e^{2\omega_i t} E(p, t) dt \leq E(p, 0)/|2\omega_i|.$$

So it suffices to establish (3.30) for $-\delta(\varepsilon^{-\frac{1}{2}}) < \omega_i < 0$. With this constraint, and $\omega \equiv \omega_r + i\omega_i$, note that:

$$e^{\omega_i t}u(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega_r}\hat{u}(\mathbf{x}, \omega)d\omega_r \qquad e^{\omega_i t}\partial_t u(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega_r}(i\omega)\hat{u}(\mathbf{x}, \omega)d\omega_r$$

$$e^{\omega_i t}g(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega_r}\hat{g}(\mathbf{x}, \omega)d\omega_r \qquad e^{\omega_i t}h(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\omega_r}\hat{h}(\mathbf{x}, \omega)d\omega_r.$$

By Parseval's equality:

$$\int_0^\infty e^{2\omega_i t}\|u\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{u}\|^2 d\omega_r \qquad \int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\omega\hat{u}\|^2 d\omega_r$$

$$\int_0^T e^{2\omega_i t}\|g\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{g}\|^2 d\omega_r \qquad \int_0^T e^{2\omega_i t}\|h\|_{0,\partial\Omega}^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{h}\|_{0,\partial\Omega}^2 d\omega_r.$$

For $|\omega_r| \leq \varepsilon^{-\frac{1}{2}}$ and $|\omega_i| \leq \delta(\varepsilon^{-\frac{1}{2}})$ it follows from Theorem 3.1 that:

$$\int_{-\varepsilon^{-\frac{1}{2}}}^{\varepsilon^{-\frac{1}{2}}} \|\hat{u}\|^2 d\omega_r \leq c(\varepsilon^{-\frac{1}{2}})\Big[\int_{-\infty}^\infty \|\hat{g}\|^2 d\omega_r + \int_{-\infty}^\infty \|\hat{h}\|_{0,\partial\Omega}^2 d\omega_r\Big].$$

Also from the above:

$$\begin{aligned} \int_{|\omega_r|>\varepsilon^{-\frac{1}{2}}} \|\hat{u}\|^2 d\omega_r &\leq \int_{|\omega_r|>\varepsilon^{-\frac{1}{2}}} \varepsilon\omega_r^2\|\hat{u}\|^2 d\omega_r \leq \varepsilon\int_{|\omega_r|>\varepsilon^{-\frac{1}{2}}} \|\omega\hat{u}\|^2 d\omega_r \\ &\leq 2\pi\varepsilon\int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt. \end{aligned}$$

Adding the last two inequalities gives:

$$\begin{aligned} 2\pi\int_0^\infty e^{2\omega_i t}\|u\|^2 dt &\leq c(\varepsilon^{-\frac{1}{2}})\Big[\int_{-\infty}^\infty \|\hat{g}\|^2 d\omega_r + \int_{-\infty}^\infty \|\hat{h}\|_{0,\partial\Omega}^2 d\omega_r\Big] + 2\pi\varepsilon\int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt \\ &\leq 2\pi c_\varepsilon\Big[\int_0^T e^{2\omega_i t}\|g\|^2 dt + \int_0^T e^{2\omega_i t}\|h\|_{0,\partial\Omega}^2 dt\Big] + 2\pi\varepsilon\int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt \end{aligned}$$

The result now follows from (3.12) and (3.13). ■

Proposition 3.2 *Suppose that p satisfies (3.3). Then for every $\varepsilon > 0$ there is a constant c_ε such that for every $\omega_i < 0$:*

$$\int_0^\infty e^{2\omega_i t}\|p\|^2 dt \leq c_\varepsilon E(p,0) + \varepsilon\int_0^\infty e^{2\omega_i t}\|\partial_t p\|^2 dt. \tag{3.31}$$

Proof: Recall that $p = u$ for $t \geq T$. Therefore with (2.4), (3.4) and (3.30):

$$\begin{aligned} \int_0^\infty e^{\omega_i t}\|p\|^2 dt &\leq c\int_0^T E(p,t)dt + \int_0^\infty e^{\omega_i t}\|u\|^2 dt \\ &\leq C_\varepsilon E(p,0) + \tfrac{1}{2}\varepsilon\int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt. \end{aligned}$$

Since $\partial_t u = \phi\partial_t p + \phi' p$ and $\phi' = 0$ for $t \geq T$, by (2.4) and (3.4):

$$\begin{aligned} \int_0^\infty e^{2\omega_i t}\|\partial_t u\|^2 dt &\leq c\int_0^T E(p,t)dt + 2\int_0^\infty e^{2\omega_i t}\|\partial_t p\|^2 dt \\ &\leq cE(p,0) + 2\int_0^\infty e^{2\omega_i t}\|\partial_t p\|^2 dt. \end{aligned}$$

Now (3.31) follows after combining the above. ■

Theorem 3.2 *Assume there is a smooth, real vector field ℓ such that $\ell \cdot n \geq \ell_0 > 0$ on $\partial\Omega$ and that $2I - [D\ell + D\ell^T]$ is uniformly negative definite on $\bar{\Omega}$. Then there exist positive constants M_0 and μ_0 such that (3.1) holds.*

Proof: With $\omega_i < 0$, it follows from Proposition 3.1 that for some $t_0 > 0$:

$$\int_{t_0}^\infty e^{2\omega_i t}\Re\{D_t Q(t)\}dt + \tfrac{1}{4}\int_{t_0}^\infty e^{2\omega_i t}E(p,t)dt \leq c\int_{t_0}^\infty e^{2\omega_i t}\|p\|^2 dt.$$

For the first term consider:

$$\int_{t_0}^\infty e^{2\omega_i t}D_t Q(t)dt = e^{2\omega_i t}Q(t)\Big|_{t_0}^\infty - 2\omega_i\int_{t_0}^\infty e^{2\omega_i t}Q(t)dt.$$

By (3.4) and (2.4):

$$|Q(t)| \leq ctE(p,t) + c\Big[\|p\|^2 + \|\nabla p\|^2 + \|\partial_t p\|^2\Big] \leq c(t+1)E(p,t) \leq c(t+1)E(p,0)$$

so it follows that:

$$|e^{2\omega_i t_0}Q(t_0)| \leq cE(p,0)$$

and:

$$\lim_{t\to\infty} e^{2\omega_i t}Q(t) = 0.$$

Combining the above now gives:

$$|2\omega_i|\int_{t_0}^{\infty} e^{2\omega_i t}\Re\{Q(t)\} + \tfrac{1}{4}\int_{t_0}^{\infty} e^{2\omega_i t}E(p,t) \leq cE(p,0) + c\int_{t_0}^{\infty} e^{2\omega_i t}\|p\|^2.$$

Further, by Proposition 3.1, $\Re\{Q(t)\} \geq 0$, for $t \geq t_0$. Hence by (3.4):

$$\int_0^{\infty} e^{2\omega_i t}E(p,t) \leq cE(p,0) + c\int_0^{\infty} e^{2\omega_i t}\|p\|^2$$

Then according to Proposition 3.2:

$$\int_0^{\infty} e^{2\omega_i t}\|p\|^2 \leq c_\varepsilon E(p,0) + \varepsilon\int_0^{\infty} e^{2\omega_i t}E(p,t) \qquad \forall\varepsilon > 0, \quad \forall\omega_i < 0.$$

Finally, the latter two inequalities give (3.5), and with (3.4), it follows that $\forall t > 0$:

$$tE(p,t) \leq \int_0^{\infty} E(p,s)ds \leq \bar{c}E(p,0).$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$ [13]:

$$|S(t)|_{\mathcal{H}} = \sup_{X\in\mathcal{H}} \frac{|S(t)X|_{\mathcal{H}}}{|X|_{\mathcal{H}}} = \sup_{P_0\in\mathcal{D}(A)} \frac{|S(t)P_0|_{\mathcal{H}}}{|P_0|_{\mathcal{H}}} = \sup_{P_0\in\mathcal{D}(A)} \left[\frac{E(p,t)}{E(p,0)}\right] \leq \bar{c}/t.$$

Hence, there is a $t^\star > 0$ and a $\mu_0 > 0$ such that:

$$|S(t^\star)|_{\mathcal{H}} = e^{-\mu_0 t^\star} < 1.$$

So with:

$$M_0 \equiv e^{\mu_0 t^\star} \max_{0\leq t\leq t^\star} |S(t)|_{\mathcal{H}},$$

let $t > 0$ be chosen arbitrarily. Then set $t = mt^\star + r$ with $r < t^\star$ and $m \in \mathbf{N}$, to obtain:

$$|S(t)|_{\mathcal{H}} = |S^m(t^\star)S(r)|_{\mathcal{H}} \leq |S(t^\star)|_{\mathcal{H}}^m M_0 e^{-\mu_0 t^\star} = M_0 e^{-\mu_0(m+1)t^\star} \leq M_0 e^{-\mu_0(mt^\star + r)} = M_0 e^{-\mu_0 t}$$

which gives (3.1). ■

4 Control Constrained to be Sinusoidal.

In this section, the following problem is considered. Among the functions $\hat{F} \in \mathcal{F}$, find $\hat{F}^\star$ which minimizes:

$$\hat{J}(\hat{F}) \equiv \|H(\hat{p}_1 + \hat{p}_3)\|^2 + \theta\|\hat{F}\|^2 \tag{4.1}$$

subject to:

$$\left\{\begin{array}{rcll} -\omega^2\hat{p}_3 &=& \gamma^2\Delta\hat{p}_3 + \hat{F} & \text{in} \quad \Omega \\ 0 &=& (\alpha + i\omega\beta)\hat{p}_3 + \partial_n\hat{p}_3 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{4.2}$$

This problem is motivated by considering the minimization of (1.3), subject to (1.2), over the class of sinusoidal functions of the form $F(\mathbf{x},t) = \hat{F}(\mathbf{x})e^{i\omega t}$. Note that with F in this form, and $\hat{p}_3$ satisfying (4.2), by Theorem 2.2, $\hat{p}_3 e^{i\omega t}$ is the unique solution to (1.2). Therefore, since $p_1 = \hat{p}_1(\mathbf{x})e^{i\omega t}$, the minimization of J over sinusoidal controls is equivalent to the minimization of $\hat{J}$ above.

According to Theorem 3.1, (4.2) is well-posed. In fact, from the proof of Theorem 3.1, it can be seen that with:

$$\mathcal{G}_\omega \equiv G_\omega[\omega^2(I - \omega^2\tilde{G}_\omega)^{-1}\tilde{G}_\omega + I] = G_\omega(I - \omega^2\tilde{G}_\omega)^{-1} \qquad \text{and} \qquad \tilde{\mathcal{G}}_\omega \equiv N\mathcal{G}_\omega.$$

$\hat{p}_3 = \mathcal{G}_\omega\hat{F}$ and $\hat{J}$ takes the form:

$$\hat{J}(\hat{F}) \equiv \|H(\hat{p}_1 + \tilde{\mathcal{G}}_\omega\hat{F})\|^2 + \theta\|\hat{F}\|^2 \qquad \hat{F} \in \mathcal{F}.$$

Questions concerning the unconstrained minimization of this functional are settled as follows.

Theorem 4.1 *Provided $\theta > 0$, there exists a unique $\hat{F}^\star \in \mathcal{F}$ such that:*

$$\hat{J}(\hat{F}^\star) = \inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F}).$$

Furthermore, it is given explicitly by:

$$\hat{F}^\star = -[\theta + P_{\mathcal{F}} \tilde{\mathcal{G}}_\omega^* H \tilde{\mathcal{G}}_\omega]^{-1} P_{\mathcal{F}} \tilde{\mathcal{G}}_\omega^* H \hat{p}_1. \tag{4.3}$$

Proof: First note that $\hat{J}(\hat{F})$ can be written in the form:

$$\begin{aligned} \hat{J}(\hat{F}) &= \{(H\tilde{\mathcal{G}}_\omega \hat{F}, H\tilde{\mathcal{G}}_\omega \hat{F}) + \theta(\hat{F}, \hat{F})\} + 2\Re\{(H\hat{p}_1, H\tilde{\mathcal{G}}_\omega \hat{F})\} + (H\hat{p}_1, H\hat{p}_1) \\ &\equiv X(\hat{F}, \hat{F}) + Y(\hat{F}) + Z \end{aligned}$$

where according to (3.24) and Lemma 3.4, X is a continuous, coercive sequilinear form:

$$X(U,U) \geq c_1 \|U\|^2 \geq \theta \|U\|^2 \qquad \forall U \in L_2(\Omega),$$

$$|X(U,V)| \leq c_2 \|U\| \|V\| \qquad \forall U, V \in L_2(\Omega),$$

and Y is a continuous linear form:

$$|Y(U)| \leq c_3 \|U\| \qquad \forall U \in L_2(\Omega).$$

Therefore, if $\{\hat{F}^n\}_{n=1}^\infty$ is a minimizing sequence:

$$\lim_{n \to \infty} \hat{J}(\hat{F}^n) = \inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F})$$

then it must be bounded according to:

$$c_1 \|\hat{F}^n\|^2 - c_3 \|\hat{F}^n\| + Z \leq \hat{J}(\hat{F}^n).$$

Now, since $\mathcal{F}$ is finite dimensional, this sequence (or a certain subsequence) can be assumed to converge to some $\hat{F}^\star \in \mathcal{F}$. Hence, by the continuity of $\hat{J}$:

$$\hat{J}(\hat{F}^\star) = \lim_{n \to \infty} \hat{J}(\hat{F}^n) = \inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F}).$$

Therefore, existence is established. For uniqueness, note that $X(U,U)$ is strictly convex:

$$\begin{aligned} &X((1-\eta)U + \eta V, (1-\eta)U + \eta V) \\ &\quad = (1-\eta)X(U,U) + \eta X(V,V) - \eta(1-\eta)X(U-V, U-V) \\ &\quad < (1-\eta)X(U,U) + \eta X(V,V) \qquad \forall \eta \in (0,1) \quad \forall U \neq V \in L_2(\Omega) \end{aligned}$$

and hence $\hat{J}(\hat{F})$ is as well. So if it is assumed that $U \neq V$ and:

$$\hat{J}(U) = \inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F}) = \hat{J}(V)$$

then:

$$\inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F}) \leq \hat{J}(\tfrac{1}{2}U + \tfrac{1}{2}V) < \tfrac{1}{2}\hat{J}(U) + \tfrac{1}{2}\hat{J}(V) = \inf_{\hat{F} \in \mathcal{F}} \hat{J}(\hat{F})$$

and the contradiction gives uniqueness. For (4.3), note that with some calculations:

$$\begin{aligned} \hat{J}(\hat{F}^\star + \varepsilon \hat{F}) &= \hat{J}(\hat{F}^\star) + 2\varepsilon \Re\{(H[\tilde{\mathcal{G}}_\omega \hat{F}^\star + \hat{p}_1], \tilde{\mathcal{G}}_\omega \hat{F}) \\ &\quad + \theta(\hat{F}^\star, \hat{F})\} + \varepsilon^2 \{\|H\tilde{\mathcal{G}}_\omega \hat{F}\|^2 + \theta \|\hat{F}\|^2\}. \end{aligned}$$

So a necessary condition that $\hat{F}^\star$ minimize $\hat{J}(\hat{F})$ is that:

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} D_\varepsilon \hat{J}(\hat{F}^\star + \varepsilon\hat{F}) = 2\Re\{(H[\hat{p}_1 + \tilde{\mathcal{G}}_\omega \hat{F}^\star], \tilde{\mathcal{G}}_\omega \hat{F}) + \theta(\hat{F}^\star, \hat{F})\} \\ &= 2\Re\{(P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H[\hat{p}_1 + \tilde{\mathcal{G}}_\omega \hat{F}^\star] + \theta\hat{F}^\star, \hat{F})\}. \end{aligned}$$

Letting $\hat{F} = P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H[\hat{p}_1 + \tilde{\mathcal{G}}_\omega \hat{F}^\star] + \theta\hat{F}^\star$ shows that $\hat{F}^\star$ is characterized by:

$$[\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]\hat{F}^\star = -P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\hat{p}_1.$$

For (4.3), note that by (2.2):

$$\begin{aligned} ([\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]U, V) &= \theta(U,V) + (\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega U, V) \\ = \theta(U,V) + (U, \tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega V) &= (U, [\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]V) \qquad \forall U, V \in \mathcal{F} \end{aligned}$$

and:

$$\begin{aligned} \theta(U,U) &\le \theta(U,U) + (H\tilde{\mathcal{G}}_\omega U, H\tilde{\mathcal{G}}_\omega U) \\ &= ([\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]U, U) \qquad \forall U \in \mathcal{F}. \end{aligned}$$

Hence, $[\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]$ is selfadjoint and positive definite on $\mathcal{F}$, and (4.3) follows. ■

Now note that according to (4.3) the optimal state p_3 satisfies:

$$\left\{ \begin{array}{rcll} \partial_t^2 p_3 &=& \gamma^2\Delta p_3 - [\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]^{-1} P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H p_1 & \text{in } \Omega\times[0,\tau] \\ 0 &=& \alpha p_3 + \beta\partial_t p_3 + \partial_n p_3 & \text{on } \partial\Omega\times[0,\tau] \\ p_3(0) &=& p_3(\tau) & \text{in } \Omega \\ \partial_t p_3(0) &=& \partial_t p_3(\tau) & \text{in } \Omega. \end{array} \right. \tag{4.4}$$

In fact, this inspires the following control strategy for p_2.

$$\left\{ \begin{array}{rcll} \partial_t^2 p_2 &=& \gamma^2\Delta p_2 - [\theta + P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H\tilde{\mathcal{G}}_\omega]^{-1} P_{\mathcal{F}}\tilde{\mathcal{G}}_\omega^* H p_1 & \text{in } \Omega\times[0,\infty) \\ 0 &=& \alpha p_2 + \beta\partial_t p_2 + \partial_n p_2 & \text{on } \partial\Omega\times[0,\infty) \\ p_2(0) &=& 0 & \text{in } \Omega \\ \partial_t p_2(0) &=& 0 & \text{in } \Omega. \end{array} \right. \tag{4.5}$$

However, in order for this strategy to be at least asymptotically optimal, it must be shown that $[p_2 - p_3] \to 0$ as $t \to \infty$. In other words, it must be shown that if $P \equiv (p, \partial_t p)^T$ satisfies:

$$\left\{ \begin{array}{rcll} P' &=& AP & t \ge 0 \\ P(0) &=& P_0 & \end{array} \right.$$

then regardless of the choice of $P_0 \in \mathcal{H}$, $p \to 0$ as $t \to \infty$. Actually, according to (3.2), this decay is guaranteed to occur at an exponential rate. Hence, the above control strategy is exponentially stable, but in contrast to the approach of the next section, it is only asymptotically optimal.

5 Control Without Sinusoidal Constraint.

In this section, the following problem is considered. Among the functions $F \in L_2([0,\tau], \mathcal{F})$, find $F^\star$ which minimizes:

$$J(F) \equiv \int_0^\tau \{(M[P_1 + P_3], [P_1 + P_3])_{\mathcal{H}} + \theta(F,F)\}dt \tag{5.1}$$

subject to (2.9). Here, M is given by:

$$M \equiv \begin{pmatrix} N^*H & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad P_1 \equiv \begin{pmatrix} p_1 \\ \partial_t p_1 \end{pmatrix} = \begin{pmatrix} \hat{p}_1 \\ i\omega\hat{p}_1 \end{pmatrix} e^{i\omega t} \equiv \hat{P}_1 e^{i\omega t}.$$

Note that:

$$\begin{aligned}(M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix})_{\mathcal{H}} &= (N^*Hu_1, v_1)_1 = (Hu_1, Nv_1) \\ = (Nu_1, Hv_1) = (u_1, N^*Hv_1)_1 &= (\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix})_{\mathcal{H}}\end{aligned}$$

and:

$$(M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix})_{\mathcal{H}} = (N^*Hu_1, u_1)_1 = (Hu_1, Nu_1) = \|Hu_1\|^2.$$

Hence, M is selfadjoint and nonnegative.

The optimal control which solves this minimization problem will be shown to have the form:

$$F^\star(t) = -\theta^{-1}B^*QP_3(t) - \theta^{-1}B^*R(t) \tag{5.2}$$

where, in a sense made precise below, Q satisfies the algebraic Ricatti equation:

$$QA + A^*Q + M = \theta^{-1}QBB^*Q \tag{5.3}$$

and $R(\mathbf{x}, t) = \hat{R}(\mathbf{x})e^{i\omega t}$, with $\hat{R}(\mathbf{x})$ satisfying:

$$\hat{R} = -[i\omega + (A^* - \theta^{-1}QBB^*)]^{-1}M\hat{P}_1. \tag{5.4}$$

Furthermore, it will follow that the optimal state P_3 satisfies:

$$\left\{ \begin{array}{rcll} P_3' &=& (A - \theta^{-1}BB^*Q)P_3 - \theta^{-1}BB^*R & \qquad 0 \le t \le \tau \\ P_3(0) &=& P_3(\tau). & \end{array} \right. \tag{5.5}$$

In fact, this inspires the following control strategy for p_2:

$$\left\{ \begin{array}{rcll} P_2' &=& (A - \theta^{-1}BB^*Q)P_2 - \theta^{-1}BB^*R & \qquad t \ge 0 \\ P_2(0) &=& 0. & \end{array} \right. \tag{5.6}$$

However, in order for this strategy to be at least asymptotically optimal, it must be shown that $[p_2 - p_3] \to 0$ as $t \to \infty$. In other words, it must be shown that if $P \equiv (p, \partial_t p)^T$ satisfies:

$$\left\{ \begin{array}{rcll} P' &=& (A - \theta^{-1}BB^*Q)P & \qquad t \ge 0 \\ P(0) &=& P_0 & \end{array} \right. \tag{5.7}$$

then regardless of the choice of $P_0 \in \mathcal{H}$, $p \to 0$ as $t \to \infty$. This is obtained from Lemma 5.1 below.

Proposition 5.1 *There exists a unique selfadjoint, nonnegative $Q \in \mathcal{B}(\mathcal{H})$ such that:*

$$(AU, QV)_{\mathcal{H}} + (QU, AV)_{\mathcal{H}} + (MU, V)_{\mathcal{H}} = \theta^{-1}(QBB^*QU, V)_{\mathcal{H}} \qquad \forall U, V \in \mathcal{D}(A). \tag{5.8}$$

Also:

$$Q = T^*(\tau - t)QT(\tau - t) + \int_t^\tau T^*(s - t)[M + \theta^{-1}QBB^*Q]T(s - t)ds \qquad \forall t \in [0, \tau]. \tag{5.9}$$

Proof: By generalizing the techniques of [7] to the complex field,† the first part of the theorem can be established if operators K_1 and K_2 can be found such that $A^* - M^{\frac{1}{2}}K_1$ and $A - BK_2$ generate continuous semigroups which satisfy an exponential decay estimate. However with (3.2), this follows with $K_1 = K_2 = 0$. For the last part, note that according to (5.8), for every $U, V \in \mathcal{D}(A)$:

$$([A - \theta^{-1}BB^*Q]U, QV)_{\mathcal{H}} + (QU, [A - \theta^{-1}BB^*Q]V)_{\mathcal{H}} + ([M + \theta^{-1}QBB^*Q]U, V)_{\mathcal{H}} = 0.$$

With $U_0, V_0 \in \mathcal{D}(A)$ and $0 \le t \le s \le \tau$, set $U = T(s-t)U_0$ and $V = T(s-t)V_0$ to obtain [13]:

$$(T(\tau - t)U_0, QT(\tau - t)V_0)_{\mathcal{H}} - (U_0, QV_0)_{\mathcal{H}} + \int_t^\tau ([M + \theta^{-1}QBB^*Q]T(s-t)U_0, T(s-t)V_0)_{\mathcal{H}} = 0$$

or:

$$(U_0, T^*(\tau - t)QT(\tau - t)V_0 - QV_0 + \int_t^\tau T^*(s-t)[M + \theta^{-1}QBB^*Q]T(s-t)V_0 ds)_{\mathcal{H}} = 0$$

$$\forall U_0, V_0 \in \mathcal{D}(A).$$

Then since $\mathcal{D}(A)$ is dense in $\mathcal{H}$ [13], extending by continuity gives (5.9). ■

In addition to showing that $[p_2 - p_3]$ decays exponentially, the following lemma also helps to solve (5.4).

Lemma 5.1 *$A - \theta^{-1}BB^*Q$ generates a semigroup $T(t) \in \mathcal{C}([0,\infty), \mathcal{H})$ for which there exist constants $M_1 > 0$ and $\mu_1 > 0$ such that:*

$$\|T(t)\|_{\mathcal{H}} \le M_1 e^{-\mu_1 t} \qquad \forall t \ge 0. \tag{5.10}$$

Moreover, the unique solution to:

$$\left\{ \begin{array}{rcll} P_3' & = & (A - \theta^{-1}BB^*Q)P_3 + B\tilde{F} & \quad 0 \le t \le \tau \\ P_3(0) & = & P_3(\tau) & \end{array} \right. \tag{5.11}$$

is given by the mild solution:

$$P_3(t) = T(t)\hat{P}_3 + \int_0^t T(t-s)B\tilde{F}(s)ds \tag{5.12}$$

where:

$$\hat{P}_3 \equiv [I - T(\tau)]^{-1} \int_0^\tau T(\tau - s)B\tilde{F}(s)ds. \tag{5.13}$$

Proof: By generalizing the techniques of [7] to the complex field,‡ the first part of lemma can be established if operators K_1 and K_2 can be found such that $A^* - M^{\frac{1}{2}}K_1$ and $A - BK_2$ generate continuous semigroups which satisfy an exponential decay estimate. However with (3.2), this follows with $K_1 = K_2 = 0$. The rest of the lemma is obtained as with the proof of Theorem 2.2. ■

Proposition 5.2 *There exists a unique $\hat{R} \in \mathcal{H}$ such that:*

$$[i\omega + (A^* - \theta^{-1}QBB^*)]\hat{R} = -M\hat{P}_1. \tag{5.14}$$

Also:

$$R(t) = T^*(\tau - t)\hat{R} + \int_t^\tau T^*(s-t)MP_1(s)ds \qquad \forall t \in [0,\tau]. \tag{5.15}$$

†This amounts to understanding each (in)equality to hold in the sense of real parts only.

‡This amounts to understanding each (in)equality to hold in the sense of real parts only.

Proof: Since $T^*(t)$ is generated by $A^* - \theta^{-1}QBB^*$ [13], with (5.10) it follows that $T^*(t)$ satisfies an exponential decay estimate. Hence, there exists an $n \in \mathbf{N}$ for which $1 \in \rho[T^*(n\tau)]$. So for every $W \in \mathcal{H}$, the equation:

$$[I - T^*(\tau)]U = W$$

can be solved with:

$$U = [I - T^*(n\tau)]^{-1} \sum_{m=0}^{n-1} T^*(m\tau)W.$$

Hence, $\rho[T^*(\tau)]$ contains unity. Also, according to a well-known result [13]:

$$\exp\{\tau\sigma[A^* - \theta^{-1}QBB^*]\} \subset \sigma[T^*(\tau)].$$

Therefore, $-i\omega \in \rho[A^* - \theta^{-1}QBB^*]$ and (5.14) is obtained. For (5.15), note that by (5.14):

$$R' + (A^* - \theta^{-1}QBB^*)R = -MP_1$$

or:

$$(R'(s), U)_{\mathcal{H}} + (R(s), [A - \theta^{-1}BB^*Q]U)_{\mathcal{H}} + (MP_1(s), U)_{\mathcal{H}} = 0 \qquad \forall U \in \mathcal{D}(A).$$

With $U_0 \in \mathcal{D}(A)$ and $0 \le t \le s \le \tau$, set $U = T(s-t)U_0$ to obtain:

$$(R(\tau), T(\tau - t)U_0)_{\mathcal{H}} - (R(t), U_0)_{\mathcal{H}} + \int_t^\tau (MP_1(s), T(s-t)U_0)_{\mathcal{H}} ds = 0$$

or:

$$(T^*(\tau - t)R(\tau) - R(t) + \int_t^\tau T^*(s-t)MP_1(s), U_0)_{\mathcal{H}} = 0 \qquad \forall U_0 \in \mathcal{D}(A).$$

Then since $\mathcal{D}(A)$ is dense in $\mathcal{H}$ [13], (5.15) follows. ∎

Next, (5.2) requires the following two lemmas.

Lemma 5.2 *With P_3 given by (5.12) the following holds:*

$$D_1 \equiv \int_0^\tau (P_3(s), \tilde{M}P_3(s))_{\mathcal{H}} ds - 2\int_0^\tau \Re\{(P_3(s), QB\tilde{F}(s))_{\mathcal{H}}\} ds = 0 \tag{5.16}$$

where $\tilde{M} \equiv M + \theta^{-1}QBB^*Q$.

Proof: Substituting (5.12) and (5.9) gives:

$$\begin{aligned}
D_1 \quad = \quad & \int_0^\tau (T(s)\hat{P}_3, \tilde{M}T(s)\hat{P}_3)_{\mathcal{H}} ds \\
+ \quad & 2\int_0^\tau \int_0^s \Re\{(T(s)\hat{P}_3, \tilde{M}T(s-\rho)B\tilde{F}(\rho))_{\mathcal{H}}\} d\rho ds \\
+ \quad & \int_0^\tau \int_0^s \int_0^s (T(s-\eta)B\tilde{F}(\eta), \tilde{M}T(s-\rho)B\tilde{F}(\rho))_{\mathcal{H}} d\eta d\rho ds \\
- \quad & 2\int_0^\tau \Re\{(T(s)\hat{P}_3, T^*(\tau - s)QT(\tau - s)B\tilde{F}(s))_{\mathcal{H}}\} ds \\
- \quad & 2\int_0^\tau \int_s^\tau \Re\{(T(s)\hat{P}_3, T^*(\rho - s)\tilde{M}T(\rho - s)B\tilde{F}(s))_{\mathcal{H}}\} d\rho ds \\
- \quad & 2\int_0^\tau \int_0^s \Re\{(T(s-\eta)B\tilde{F}(\eta), T^*(\tau - s)QT(\tau - s)B\tilde{F}(s))_{\mathcal{H}}\} d\eta ds \\
- \quad & 2\int_0^\tau \int_0^s \int_s^\tau \Re\{(T(s-\eta)B\tilde{F}(\eta), T^*(\rho - s)\tilde{M}T(\rho - s)B\tilde{F}(s))_{\mathcal{H}}\} d\rho d\eta ds.
\end{aligned}$$

The fifth term can be rewritten as follows:

$$-2\int_0^\tau\int_s^\tau \Re\{(T(\rho)\hat{P}_3, \tilde{M}T(\rho-s)B\tilde{F}(s))_{\mathcal{H}}\}d\rho ds =$$

$$-2\int_0^\tau\int_s^\rho \Re\{(T(\rho)\hat{P}_3, \tilde{M}T(\rho-s)B\tilde{F}(s))_{\mathcal{H}}\}dsd\rho.$$

So, it cancels the second term. Next, with:

$$I_1(\eta,s) \equiv \Re\{(T(\tau-\eta)B\tilde{F}(\eta), QT(\tau-s)B\tilde{F}(s))_{\mathcal{H}}\} = I_1(s,\eta)$$

note that the sixth term can be rewritten as:

$$\begin{aligned}
-2\int_0^\tau\int_0^s I_1(\eta,s)d\eta ds &= -\int_0^\tau\int_0^s I_1(\eta,s)d\eta ds - \int_0^\tau\int_\eta^\tau I_1(\eta,s)dsd\eta \\
&= -\int_0^\tau\int_0^s I_1(\eta,s)d\eta ds - \int_0^\tau\int_s^\tau I_1(s,\eta)d\eta ds \;=\; -\int_0^\tau\int_0^\tau I_1(\eta,s)d\eta ds.
\end{aligned}$$

Hence, recalling (5.13), the fourth and sixth terms become:

$$\begin{aligned}
&-2\Re\{(T(\tau)\hat{P}_3, Q[I-T(\tau)]\hat{P}_3)_{\mathcal{H}}\} - ([I-T(\tau)]\hat{P}_3, Q[I-T(\tau)]\hat{P}_3)_{\mathcal{H}} \\
&\qquad = -(\hat{P}_3, Q\hat{P}_3)_{\mathcal{H}} + (T(\tau)\hat{P}_3, QT(\tau)\hat{P}_3)_{\mathcal{H}}
\end{aligned}$$

which, according to (5.9), cancels the first term. Finally, with:

$$I_2(\eta,\rho,s) \equiv \Re\{(T(s-\eta)B\tilde{F}(\eta), \tilde{M}T(s-\rho)B\tilde{F}(\rho))_{\mathcal{H}}\} = I_2(\rho,\eta,s)$$

the last term can be rewritten as:

$$\begin{aligned}
&-2\int_0^\tau\int_0^\rho\int_\rho^\tau I_2(\eta,\rho,s)dsd\eta d\rho \\
&= -2\int_0^\tau\int_\rho^\tau\int_0^\rho I_2(\eta,\rho,s)d\eta dsd\rho \;=\; -2\int_0^\tau\int_0^s\int_0^\rho I_2(\eta,\rho,s)d\eta d\rho ds \\
&= -\int_0^\tau\int_0^s\int_0^\rho I_2(\eta,\rho,s)d\eta d\rho ds - \int_0^\tau\int_0^s\int_0^\eta I_2(\rho,\eta,s)d\rho d\eta ds \\
&= -\int_0^\tau\int_0^s\int_0^\rho I_2(\eta,\rho,s)d\eta d\rho ds - \int_0^\tau\int_0^s\int_\rho^s I_2(\rho,\eta,s)d\eta d\rho ds \\
&= -\int_0^\tau\int_0^s\int_0^s I_2(\eta,\rho,s)d\eta d\rho ds.
\end{aligned}$$

Then since the third term is real, it is cancelled by the last term, and (5.16) is established. ■

Lemma 5.3 *With P_3 given by (5.12) the following holds:*

$$D_2 \equiv \int_0^\tau (P_3(s), MP_1(s))_{\mathcal{H}}ds - \int_0^\tau (B\tilde{F}(s), R(s))_{\mathcal{H}}ds = 0. \tag{5.17}$$

Proof: Substituting (5.12) and (5.15) gives:

$$
\begin{aligned}
D_2 &= \int_0^\tau (T(s)\hat{P}_3, MP_1(s))_{\mathcal{H}} ds + \int_0^\tau \int_0^s (T(s-\eta)B\tilde{F}(\eta), MP_1(s))_{\mathcal{H}} d\eta ds \\
&\quad - \int_0^\tau (T(\tau - s)B\tilde{F}(s), \hat{R})_{\mathcal{H}} ds - \int_0^\tau \int_s^\tau (T(\eta - s)B\tilde{F}(s), MP_1(\eta))_{\mathcal{H}} d\eta ds \\
&= (\hat{P}_3, R(0) - T^*(\tau)\hat{R})_{\mathcal{H}} - ([I - T(\tau)]\hat{P}_3, \hat{R})_{\mathcal{H}} + \int_0^\tau \int_0^s (T(s-\eta)B\tilde{F}(\eta), MP_1(s))_{\mathcal{H}} d\eta ds \\
&\quad - \int_0^\tau \int_0^\eta (T(\eta - s)B\tilde{F}(s), MP_1(\eta))_{\mathcal{H}} ds d\eta \quad = \quad 0.
\end{aligned}
$$

■

Finally, (5.2) is established as follows.

Theorem 5.1 *The unique optimal control $F^\star \in L_2([0,\tau], \mathcal{F})$ which minimizes (5.1) subject to (2.9) is given by (5.2).*

Proof: According to Theorem 2.2, with $\mathcal{E} \in \mathcal{B}(L_2([0,\tau], \mathcal{F}), L_2([0,\tau], \mathcal{H}))$ defined by:

$$
[\mathcal{E}F](t) \equiv S(t)[I - S(\tau)]^{-1} \int_0^\tau S(\tau - s)BF(s)ds + \int_0^t S(t-s)BF(s)ds
$$

$J(F)$ can be written in the form:

$$
\begin{aligned}
J(F) &= \int_0^\tau \{(\mathcal{E}F, M\mathcal{E}F)_{\mathcal{H}} + \theta(F,F)\}\, dt + 2\int_0^\tau \Re\{(P_1, M\mathcal{E}F)_{\mathcal{H}}\} dt + \int_0^\tau (P_1, MP_1)_{\mathcal{H}} dt \\
&\equiv X(F,F) + Y(F) + Z
\end{aligned}
$$

where X is a continuous, coercive sequilinear form:

$$
\theta\|U\|^2 \le c_1\|U\|^2 \le X(U,U) \qquad \forall U \in L_2([0,\tau], L_2(\Omega)),
$$

$$
|X(U,V)| \le c_2\|U\|\|V\| \qquad \forall U, V \in L_2([0,\tau], L_2(\Omega)),
$$

which is strictly convex:

$$
\begin{aligned}
&X((1-\eta)U + \eta V, (1-\eta)U + \eta V) = \\
&\quad (1-\eta)X(U,U) + \eta X(V,V) - \eta(1-\eta)X(U-V, U-V) \\
&\quad\quad < \quad (1-\eta)X(U,U) + \eta X(V,V) \qquad \forall \eta \in (0,1) \quad \forall U \neq V \in L_2([0,\tau], L_2(\Omega))
\end{aligned}
$$

and Y is a continuous linear form:

$$
|Y(U)| \le c_3\|U\| \qquad \forall U \in L_2([0,\tau], L_2(\Omega)).
$$

Therefore, if $\{F^n\}_{n=1}^\infty$ is a minimizing sequence:

$$
\lim_{n\to\infty} J(F^n) = \inf_{F \in L_2([0,\tau],\mathcal{F})} J(F)
$$

then it must be bounded according to:

$$
c_1\|F^n\|^2 - c_3\|F^n\| + Z \le J(F^n).
$$

Hence, this sequence (or a certain subsequence) can be assumed to converge weakly to some $F^\star \in L_2([0,\tau],\mathcal{F})$. Then since $J(F)$ is strictly convex, it is weakly lower semicontinuous [2] and:

$$\inf_{F\in L_2([0,\tau],\mathcal{F})} J(F) = \lim_{n\to\infty} J(F^n) \geq J(F^\star) \geq \inf_{F\in L_2([0,\tau],\mathcal{F})} J(F).$$

Therefore, existence is established. Next, suppose that $U \neq V$ and:

$$J(U) = \inf_{F\in L_2([0,\tau],\mathcal{F})} J(F) = J(V).$$

Then by the strict convexity of $J(F)$:

$$\inf_{F\in L_2([0,\tau],\mathcal{F})} J(F) \leq J(\tfrac{1}{2}U + \tfrac{1}{2}V) < \tfrac{1}{2}J(U) + \tfrac{1}{2}J(V) = \inf_{F\in L_2([0,\tau],\mathcal{F})} J(F)$$

and the contradiction gives uniqueness. For (5.2), it will be shown that:

$$J(\bar{F} - \theta^{-1}B^*(QP_3 + R)) = J(-\theta^{-1}B^*(QP_3 + R)) + \theta\int_0^\tau \|\bar{F}\|^2 dt \qquad \forall \bar{F} \in L_2([0,\tau], L_2(\Omega)). \tag{5.18}$$

According to Lemma 5.1, if F in (2.9) is replaced by $\tilde{F} - \theta^{-1}B^*QP_3$, then P_3 is given by (5.12). So, combining (5.16) and (5.17) gives:

$$\begin{aligned}
0 &= D_1 + 2\Re\{D_2\} \\
&= \int_0^\tau (P_3(s), [M + \theta^{-1}QBB^*Q]P_3(s))_{\mathcal{H}} ds - 2\int_0^\tau \Re\{(P_3(s), QB\tilde{F}(s))_{\mathcal{H}}\} ds \\
&\quad +2\int_0^\tau \Re\{(P_3(s), MP_1(s))_{\mathcal{H}}\} - 2\int_0^\tau \Re\{(B\tilde{F}(s), R(s))_{\mathcal{H}}\} \\
&= \int_0^\tau ([P_3(s) + P_1(s)], M[P_1(s) + P_3(s)])_{\mathcal{H}} ds - \int_0^\tau (P_1(s), MP_1(s))_{\mathcal{H}} ds \\
&\quad +\int_0^\tau (\theta[\theta^{-1}B^*QP_3(s)], [\theta^{-1}B^*QP_3(s)]) ds \\
&\quad +2\int_0^\tau \Re\{(-\theta^{-1}B^*QP_3(s), \theta\tilde{F}(s))\} ds + 2\int_0^\tau \Re\{(\theta\tilde{F}(s), -\theta^{-1}B^*R(s))\} ds.
\end{aligned}$$

Further, with $\tilde{F} = \bar{F} - \theta^{-1}B^*R$:

$$\begin{aligned}
0 &= \int_0^\tau ([P_1(s) + P_3(s)], M[P_1(s) + P_3(s)])_{\mathcal{H}} ds - \int_0^\tau (P_1(s), MP_1(s))_{\mathcal{H}} ds \\
&+ \int_0^\tau (\theta[\bar{F} - \theta^{-1}B^*(QP_3(s) + R(s))], [\bar{F} - \theta^{-1}B^*(QP_3(s) + R(s))]) - \int_0^\tau (\theta\bar{F}(s), \bar{F}(s)) ds \\
&+ \int_0^\tau (\theta[-\theta^{-1}B^*R(s)], [-\theta^{-1}B^*R(s)]) ds.
\end{aligned}$$

Hence:

$$\begin{aligned}
J(\bar{F}(s) - \theta^{-1}B^*(QP_3(s) + R(s))) &= \int_0^\tau (P_1(s), MP_1(s))_{\mathcal{H}} ds + \int_0^\tau (\theta\bar{F}(s), \bar{F}(s)) ds \\
&\quad -\int_0^\tau (R(s), \theta^{-1}BB^*R(s))_{\mathcal{H}} ds
\end{aligned}$$

and (5.18) follows. Finally, according to (5.18):

$$J(\bar{F}(s) - \theta^{-1}B^*(QP_3(s) + R(s))) \geq J(-\theta^{-1}B^*(QP_3(s) + R(s))) \qquad \forall \bar{F} \in L_2([0,\tau], \mathcal{F}).$$

Therefore, the unique optimal control is $F^\star = -\theta^{-1}B^*(QP_3(s) + R(s))$. ■

Finally, that the optimal control (5.2) is actually sinusoidal is established as follows.

Lemma 5.4 *There exists an $\hat{F}^\star \in \mathcal{H}$ such that the optimal control given in (5.2) is actually $F^\star = \hat{F}^\star e^{i\omega t}$.*

Proof: As in the proof of Proposition 5.2, it can be shown that $i\omega \in \rho[A - \theta^{-1}BB^*Q]$. So define:

$$\hat{P}_3 \equiv [i\omega - (A - \theta^{-1}BB^*Q)]^{-1}\theta^{-1}BB^*\hat{R}$$

and note that $P_3 \equiv \hat{P}_3 e^{i\omega t}$ satisfies (5.5). Furthermore, according to Lemma 5.1, this is the unique solution to (5.5). Therefore, with:

$$\hat{F}^\star \equiv -\theta^{-1}B^*Q\hat{P}_3 - \theta^{-1}B^*\hat{R}$$

the result follows. ■

Now, a brief discussion is offered in support of the claim that the control strategy seen in (5.6) is actually globally, as opposed to only asymptotically optimal. For a rigorous treatment, see [3] and [4]. It turns out that $F^\star = -\theta^{-1}B^*(QP_2 + R)$ actually minimizes the functional:

$$\tilde{J}(F) \equiv \lim_{t^\star \to \infty} J_{t^\star}(F), \qquad J_{t^\star}(F) \equiv \int_0^{t^\star} \{([P_1 + P_2], M[P_1 + P_2])_{\mathcal{H}} + \theta(F,F)\}dt.$$

subject to (2.3). First note that with $t_n \to \infty$, $J_{t_n}(F)$ can be shown to be minimized by:

$$F_n(t) \equiv -\theta^{-1}B^*[Q_n(t)P_2(t) + R_n(t)] \tag{5.19}$$

where:

$$\left\{ \begin{array}{rcl} Q_n(s) & = & T_n^*(t,s)Q_n(t)T_n(t,s) + \displaystyle\int_s^t T_n^*(\eta,s)[M + \theta^{-1}Q_n(\eta)BB^*Q_n(\eta)]T_n(\eta,s)d\eta \\ & & \hfill 0 \leq s \leq t \leq t_n \\ Q_n(t_n) & = & 0 \end{array} \right. \tag{5.20}$$

$$\left\{ \begin{array}{rcl} R_n(s) & = & T_n^*(t,s)R_n(t) + \displaystyle\int_s^t T_n^*(\eta,s)MP_1(\eta)d\eta \qquad 0 \leq s \leq t \leq t_n \\ R_n(t_n) & = & 0 \end{array} \right. \tag{5.21}$$

and $T_n(t,s)$ is the evolution operator corresponding to the perturbation of A by $-\theta^{-1}BB^*Q_n(t)$. In other words, $T_n(t,s)U_s$ solves:

$$\left\{ \begin{array}{rcl} U'(t) & = & [A - \theta^{-1}BB^*Q_n(t)]U(t) \qquad s \leq t \leq t_n \\ U(s) & = & U_s. \end{array} \right.$$

(See [7] for additional details.) Then using (5.20), it can be shown that $Q_n(t) \to Q$ which satisfies (5.9), and by formal differentiation, this operator satisfies (5.8). Also, the convergence $T_n(t,s) \to T(t-s)$ is obtained. Next, it can be shown that:

$$R_n(s) \to R(s) \equiv \int_s^\infty T^*(\eta - s)M\hat{P}_1 e^{i\omega\eta}d\eta.$$

By changing variables appropriately, it can be shown that $R(s+\tau)=R(s)$ and in fact, $R(s)=\hat{R}e^{i\omega s}$ where $\hat{R}\equiv R(0)$. Then (5.15) is obtained with (5.21). Also, some formal calculations lead to (5.14) as follows:

$$\begin{aligned}[i\omega+(A^*-\theta^{-1}QBB^*)]\hat{R} \;&\equiv\; [(A^*-\theta^{-1}QBB^*)+i\omega]\int_0^\infty T^*(\eta)M\hat{P}_1 e^{i\omega\eta}d\eta \\ &=\; \int_0^\infty D_\eta\Big[T^*(\eta)M\hat{P}_1 e^{i\omega\eta}\Big]d\eta \;=\; -M\hat{P}_1.\end{aligned}$$

Finally, the convergence $F_n^\star \to F^\star$ is obtained, but it remains to show that $F^\star$ actually minimizes $\tilde{J}$. For the details, see [3] and [4].

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