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A Multi-Phase Segmentation Approach to the Electrical Impedance Tomography Problem

Dissertation

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Abstract

In Electrical Impedance Tomography (EIT), different current patterns are injected to the unknown object through the electrodes attached at the boundary $\partial\Omega$ of Ω. The corresponding voltages V are then measured on its boundary surface. Based on these measured voltages, the image reconstruction of the conductivity distribution σ is done by solving an inverse problem of a generalized Laplace equation

$$-\nabla \cdot (\sigma \nabla \phi) = 0 \quad \text{on } \Omega = [0,1] \times [0,1]$$

subject to a homogeneous Neumann boundary condition. In other words, with known V, we seek to solve for the typically piecewise values of σ, from which the geometry of internal objects may be inferred. We approach this problem by using a *multi-phase segmentation* method. We express σ as

$$\sigma(x) = \sum_{m=1}^{M} \sigma_m(x)\chi_m(x),$$

where χ_m is the characteristic function of a subdomain Ω_m such that $\Omega_m \cap \Omega_n = \emptyset$, $m \neq n$ and $\Omega = \cup_{m=1}^{M} \Omega_m$. The expected number of segments of Ω is M. Using a calculated optimality condition, the conductivity value σ_m is expressed as a function of χ_m. The total variation is then introduced to regularize the resulting cost functional. Using a descent method, an update for χ_m is proposed. Examples using topological derivative to obtain an initial estimate for χ_m are also presented.

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Chapter 1

Introduction

The Electrical Impedance Tomography (EIT) is an imaging technique which tries to recover the spatial distribution of the conductivities in the interior of a body Ω based on electrical measurements from electrodes placed around its boundary $\partial\Omega$. The mathematical formulation of the EIT problem is due to Alberto Calderón [15]. This is the reason why the EIT problem is sometimes referred to in the literature of inverse problems as the *"Calderón's inverse problem"* or simply *"Calderón's problem"*.

EIT is a non-invasive technique which is proposed to have a wide range of applications. For example, it is proposed in [72] that EIT can be used to diagnose breast cancer on its early stage. This is because the conductivity of the malignant tumor is significantly higher compared to normal breast tissue. In [21], pulmonary functions are monitored using EIT. Other plausible applications of EIT in medical science can be found in [26] and [20]. Detection of leaks from buried pipes using EIT is studied in [43]. Other applications of EIT include determining the location of mineral deposits [61], tracing the spread of contaminants in the earth [1], and non-destructive evaluation of machine parts [56].

We now describe the mathematical formulation of the EIT problem . Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a smooth boundary. The problem of image reconstruction via EIT consists of two sub-problems: the forward problem and the inverse problem. In the forward EIT problem, the electric potential ϕ on Ω and the boundary voltage $V = \phi|_{\partial\Omega}$ are solved using the conductivity distribution $\sigma \in L^\infty(\Omega)$ with $\sigma(x) \geq \underline{\sigma} > 0$, $\forall x \in \Omega$ and the boundary currents $f \in L^2(\partial\Omega)$. This is done by solving a generalized Laplace equation:

$$\begin{aligned} \nabla \cdot (\sigma \nabla \phi) &= 0 \quad \text{on } \Omega \\ \sigma \frac{\partial \phi}{\partial n} &= f \quad \text{on } \partial\Omega \end{aligned}$$

where n is the outward normal vector to $\partial\Omega$. The boundary currents are chosen in such a way that the law of conservation of charge is preserved. That is, $\int_{\partial\Omega} f = 0$. Moreover, the electric potential must satisfy $\int_{\partial\Omega} \phi dS = 0$ which amounts to the choosing of the reference voltage. The term $\sigma\nabla\phi$ represents current flow. This partial differential equation is a well-posed boundary value problem with solution $\phi \in H^1(\Omega)$ which is unique up to a constant. This model is often referred to as the continuum model. For other EIT models, one can refer to [10, 52, 28].

Each boundary current $f \in L^2(\partial\Omega)$ applied to $\partial\Omega$ has a corresponding boundary voltage $\phi|_{\partial\Omega}$. These boundary voltages depend linearly on f . On the other hand, the inverse EIT problem (also known as the image reconstruction problem) is the recovery of the

conductivity distribution inside Ω given the boundary voltage V and current measurements at the boundary. Denote $\tilde{L}(\partial\Omega) := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega} f dS = 0 \right\}$. Then we can define the map $\Lambda_\sigma : \tilde{L}(\partial\Omega) \to \tilde{L}(\partial\Omega)$ by

$$\Lambda_\sigma(f) = \phi|_{\partial\Omega} \tag{1.0.1}$$

where $\phi \in H^1(\Omega)$ is the unique solution of (2.1.1) with $\int_{\partial\Omega} \phi dS = 0$. The operator Λ_σ is called the *Neumann-to-Dirichlet* map (compare, e.g., [22, 71]). This map depends nonlinearly on σ. The inverse EIT problem is the determination of σ with Λ_σ known. Unfortunately, the inverse EIT problem is severely ill-posed. However, it is proven that the inverse problem has a unique solution for $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$. In other words, if $\Lambda_{\sigma_1} f = \Lambda_{\sigma_2} f$ then $\sigma_1 = \sigma_2$ in $\bar{\Omega}$ provided that σ_1 and σ_2 are in a suitable space. We refer for the discussion of this result together with the proof to [71, 68, 10] for the case $n \geq 3$ and to [5, 45] for the case $n = 2$.

A number of algorithms have been proposed by finding solutison under certain assumptions or by doing few modifications. Various methods in solving the EIT problem are discussed in [10] and [52]. One approach in solving the inverse problem is by treating σ as a piecewise constant function. For example, one can refer to [40, 54, 70, 55, 17]. In these, EIT was solved and analyzed under the assumption that σ is piecewise constant. We will also make our analysis under this hypothesis. By doing so, we can treat the inverse problem as a segmentation problem. The inverse problem is usually solved by minimizing a functional of least square form. Usually, a regularization term is necessary (see, e.g., [40, 22]). In this work, we will proceed on a similar manner. We will find a regularization term by selecting the suitable segmentation technique.

The rest of the paper is organized as follows: In the next chapter, we formulate our proposed method. We begin by finding a suitable functional that when minimized will give the solution of the inverse problem. We will then choose which is the best segmentation method that we will incorporate with the said functional. We introduce a necessary regularization afterwards. At the end of the next chapter, we state our algorithm for solving the inverse problem. In Chapter 3, we will make the mathematical analysis of the proposed method. A finite element discretization of the proposed method will be discussed in Chapter 4. Furthermore, the existence and uniqueness of solutions in the discrete setting and the convergence to the respective quantities in the continuous case for successively smaller step sizes will be proved. Finally, numerical results will be presented in Chapter 5. Notations and known mathematical facts that are used throughout this paper will be summarized in the Appendix.

Chapter 2

The Proposed Method

In this chapter, we present our approach in solving the inverse electrical impedance tomography problem. A typical approach in solving this problem is by minimizing a functional of least square that is defined only on the boundary of Ω (see, e.g., [40, 22, 10]). That is, the L^2 distance between the given boundary voltage and the output of the Neumann-to-Dirichlet map given any arbitrary conductivity distribution is minimized. We will proceed in this manner as well but we will add additional terms to the least square distance. We will treat the recovery of the conductivity distribution as a segmentation problem. By doing this, we will be able to find the suitable regularization terms. Thus, we will be able to formulate a functional whose domain is the set of conductivity distributions. After formulating the functional, we proceed with the computation of its derivatives. Finally, with the help of these derivatives, our proposed algorithm will be presented. A rigorous mathematical analysis of this method will be discussed in the next chapter.

2.1 Formulation of the Functional

We already introduced the Neumann-to-Dirichlet map in (1.0.1) as the linear operator $\Lambda_\sigma : \tilde{L}(\partial\Omega) \to \tilde{L}(\partial\Omega)$ which maps any $f \in \tilde{L}^2(\Omega) := \left\{ f \in L^2(\partial\Omega) \right\}$ to a function $\phi|_{\partial\Omega} \in \tilde{L}^2(\Omega)$ by first solving the forward problem

$$\begin{aligned} \nabla \cdot (\sigma \nabla \phi) &= 0 \quad \text{on } \Omega \\ \sigma \frac{\partial \phi}{\partial n} &= f \quad \text{on } \partial\Omega \end{aligned} \tag{2.1.1}$$

Let us fix $f \in \tilde{L}^2(\Omega)$ and define the function $F : L^2(\Omega) \to \tilde{L}(\partial\Omega)$ by

$$F(\sigma) = \phi|_{\partial\Omega} \tag{2.1.2}$$

where ϕ is the solution of (2.1.1) given σ and f. Let $\tilde{\sigma}$ be the actual conductivity distribution in Ω. The inverse problem is recovering $\tilde{\sigma}$ given $\Lambda_{\tilde{\sigma}}$. From here onwards, we denote $\tilde{V} := \Lambda_{\tilde{\sigma}}(f) = F(\tilde{\sigma})$. We refer to $\tilde{V}$ as the measured boundary voltage or the known boundary voltage. In order to solve the inverse problem, what we can do is to minimize the L^2 distance between $F(\sigma)$ and $\tilde{V}$ in $\partial\Omega$, where σ is an arbitrary element of $L^2(\Omega)$. Hence, we introduce the functional $\tilde{J} : L^2(\Omega) \to \mathbb{R}$

$$\tilde{J}(\sigma) = \frac{1}{2} \int_{\partial\Omega} \left| F(\sigma) - \tilde{V} \right|^2 dS. \tag{2.1.3}$$

Ideally, we want to make $\tilde{J}$ as close to 0 as possible. Minimizing this functional as it is can be difficult. Hence, regularization is needed (see, e.g., [40, 22]). The rest of the section will be devoted in finding the appropriate regularization.

If we take a look at Table (2.1), we will see that the conductivity values of healthy tissues show great contrast. This contrast makes imaging in electrical impedance tomography possible. This supports the hypothesis we mentioned in the first chapter that the actual conductivity distribution $\tilde{\sigma}$ is piecewise constant. By assuming that $\tilde{\sigma}$ is piecewise constant, the inverse problem becomes a segmentation problem. Hence, we are now looking for a minimizer of (2.1.3) which is piecewise constant in Ω. Because we are now dealing with a segmentation problem, we need an a priori knowledge on the number of segments M. Thus, we would like σ to be expressed in the terms

$$\sigma(x) = \sum_{m=1}^{M} \sigma_m(x)\chi_m(x). \tag{2.1.4}$$

Here, χ_m is the characteristic function of a subdomain Ω_m, where $\Omega_{m_1} \cap \Omega_{m_2} = \emptyset$ whenever $m_1 \neq m_2$ and $\Omega = \cup_{i=1}^{M} \Omega_i$. We illustrate this in Figure (2.1.1) for the case when $M = 4$. Observe that Ω is segmented, in which the segments or clusters are the supports of χ_1, χ_2, χ_3 and χ_4.

Tissue	Conductivity (mS/cm)
C.S.F	15.4
blood	6.7
liver	2.8
skeletal muscle (longitudinal)	8.0
skeletal muscle (transverse)	0.6
cardiac muscle (longitudinal)	6.3
cardiac muscle (transverse)	2.3
neural tissue	1.7
grey matter	3.5
white matter	1.5
lung (expiration)	1.0
lung (inspiration)	0.4
fat	0.36
bone	0.06

Table 2.1: Conductivities of Healthy Tissues [36, 7].

We are now left with the problem of selecting the suitable segmentation method that we are going to use to solve the inverse problem. In this work, we will explore the use of a multi-phase segmentation approach proposed by S. Fürtinger in his PhD Dissertation [32]. We will discuss in summary how he came up with this method. We will justify along the way how this method is the suitable method for our problem. We will proceed the same way he did: we will first discuss segmentation methods that are already found in literature. The method suggested in Fürtinger's dissertation will be a hybrid of these segmentation techniques.

One of the simplest methods of segmentation is the method of *K-means clustering*. Given an image $\tilde{I}$, the method of K-means assumes that Ω is segmented into K clusters, where

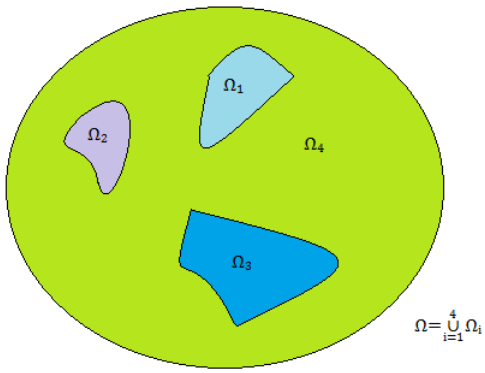


Figure 2.1.1: An example of the segmentation of Ω when $M = 4$.

K is known a priori. That is, $\Omega = \cup_{k=1}^{K} \Omega_k$ where $\{\Omega_k\}_{k=1}^{K}$ is a collection of disjoint sets. The image $\tilde{I}$ is approximated by $\bar{I}_K$ such that $\bar{I}_K = I_k$ on Ω_k for some $I_k \in \mathbb{R}$. Hence, $\bar{I}_K = \sum_{k=1}^{K} I_k \chi_k$, where χ_k is the characteristic function of Ω_k. The approximation $\bar{I}_K$ is a solution of the minimization problem (compare, [32]):

$$\min_{\{I_k\}\{\chi_k\}} \int_\Omega \left| \sum_{k=1}^{K} I_k \chi_k - \tilde{I} \right|^2 dV \quad \text{such that } \Omega = \cup_{k=1}^{K} \Omega_k.$$

The algorithm for this method is fairly simple. A pixel of $\tilde{I}$ is assigned to the kth cluster based on its intensity value that has the closest mean value I_k. This is referred to as the *Assignment Step*. The second step is called the *Update Step*. In this step, the new means are calculated to be the centroids of the observations in the new clusters. If the clusters are not changing anymore, the algorithm terminates. Although this segmentation technique is easy to implement, it is purely heuristic and global convergence cannot be guaranteed [8]. Moreover, the final result of this algorithm has strong dependence on the initialization of segments and associated means. This method works well with images that are piecewise constant. Of course, in general, $\tilde{I}$ is not piecewise constant. In Figure (2.1.2), a one dimensional example is illustrated. The image $\tilde{I}$ (red) is piecewise linear but obviously not piecewise constant. The image I (blue) is the approximation of $\tilde{I}$ on the support of χ (cyan) obtained by implementing this method. Although the right side of $\tilde{I}$ was segmented properly, the other side was not. This is sometimes referred to as "staircasing" (for obvious reason). Although the method of K-means encounters a number of mathematical difficulty, we will not dismiss this method right away because the solution $\tilde{\sigma}$ is piecewise constant. Though not directly, we will make use of this method. We come back to this method after we discuss the next segmentation technique.

Contrary to the method of K-means, there is another segmentation technique in which images are modeled as piecewise smooth functions. We are referring to the segmentation method that minimizes the *Mumford-Shah functional* introduced by Mumford and Shah in 1989 in their paper [57]. The said functional is given by

$$J_{MS}(I, \Gamma) = \frac{\alpha}{2} \int_\Omega \left| I - \tilde{I} \right|^2 dV + \frac{\kappa}{2} \int_{\Omega \setminus \Gamma} |\nabla I|^2 \, dV + \nu \mathcal{H}(\Gamma)$$

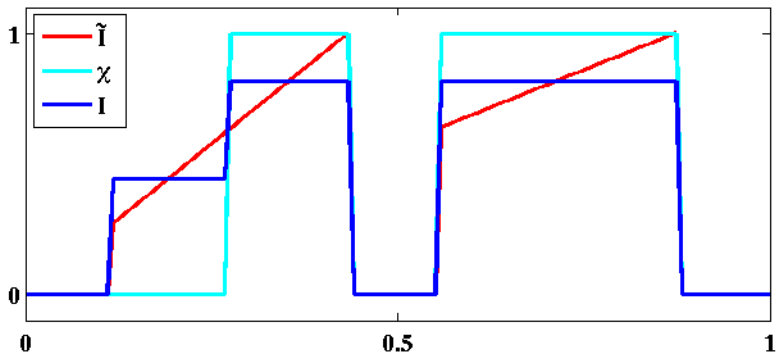


Figure 2.1.2: The *"staircasing"* problem of the method of K-means ($K = 2$). The given image $\tilde{I}$ (red) is approximated by the product of I (blue) and χ (cyan) [32].

where $\mathcal{H}$ denotes the one-dimensional Hausdorff measure (see, e.g., [39, 31]). The appearance of the geometric quantity Γ as an independent variable makes the analysis of the Mumford-Shah functional very hard. The existence of the minimum of this functional is not even guaranteed if there are no additional assumptions on the regularity of Γ (compare, e.g., [6]). To address the difficulty of handling J_{MS}, one approach is to approximate it by a functional that is not dependent on Γ but is defined on standard Sobolev spaces. One popular approximation is the so-called *Ambrosio-Tortorelli functional*. This functional was first introduced in 1990 in [3]. In order to give the formulation of this functional, we first need to define a so-called *phase functions* ψ_ϵ by

$$\psi_\epsilon(x) := \begin{cases} \frac{d_\Gamma(x)}{\epsilon}, & \text{if } d_\Gamma(x) \leq \epsilon \\ 1, & \text{otherwise} \end{cases}$$

for $\epsilon > 0$, where d_Γ is given by

$$d_\Gamma(x) := \inf_{y \in \Gamma} |x - y| .$$

We also consider the energy associated to ψ_ϵ [64], which is given by

$$L_\epsilon(\psi_\epsilon) := \int_\Omega \epsilon |\nabla \psi_\epsilon|^2 + \frac{1}{4\epsilon}(1 - \psi_\epsilon)^2 \, dV.$$

Using these, we can now formally give the formulation of the Ambrosio-Tortorelli functional:

$$J_{AT}(I, \psi) = \frac{\alpha}{2} \int_\Omega \left| I - \tilde{I} \right|^2 dV + \frac{\kappa}{2} \int_\Omega |\nabla I|^2 \psi dV + L_\epsilon(\psi) .$$

The first term of the this functional is for the fidelity of I to $\tilde{I}$. Observe that in the second term, if $|\nabla I|$ is large, then ψ will be forced to be small. On the other hand, the second term of L_ϵ induces ψ to be close to 1. Therefore, ψ is given incentive to be close to 0 near the edges and 1 otherwise. If a minimizer $\tilde{\psi}$ of J_{AT} is obtained, then the edges of $\tilde{I}$ can be located on places where $\tilde{\psi}$ is small. As we stated, the Ambrosio-Tortorelli functional J_{AT} is an approximation of the Mumford-Shah functional J_{MS}. In [35], it is proved that J_{AT} converges to J_{MS} as $\epsilon \to 0$.

Recall that the method of K-means gives a binary edge map. On the contrary, an Ambrosio-Tortorelli segmentation results to a fuzzy phase function. We illustrate this in

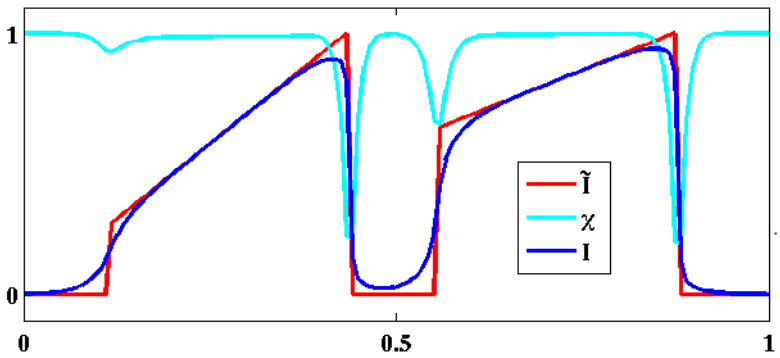


Figure 2.1.3: A demonstration of the fuzziness of the Ambrosio-Tortorelli phase function χ (cyan) in one dimension. The image $\tilde{I}$ (red) is approximated by I (blue) [32].

Figure (2.1.3). One might think that a solution to this problem is by assigning a point to be an edge if the phase function is less than 1. This might be an obvious choice but with the presence of noise, an edge might be detected on places where there is none. So although segmentation using the Ambrosio-Tortorelli might gives good results, it has its own demerit. Thus, choosing this technique as it is might not be the best suitable choice for our problem.

We now present a multi-phase segmentation approach that S. Fürtinger proposed in [32]. His motivation was to take advantage of the merits of the method of K-means and the method by Ambrosio-Tortorelli. He first let $\tilde{I}$ be estimated by $\sum_{k=1}^{K} I_k \chi_k$ (from method of K-means). He then replaced I in the Ambrosio-Tortorelli functional J_{AT} with $\sum_{k=1}^{K} I_k \chi_k$. Finally, upon simplification he obtained the functional

$$J_S\left(I_k, \chi_k\right) = \sum_{k=1}^{K} \int_{\Omega} \left[\left| I_k - \tilde{I} \right|^2 \chi_k + \alpha \left(\chi_k + \epsilon\right) \left|\nabla I_k\right|^2 \right] dV$$

for $0 < \epsilon \ll 1$ and $\alpha \gg \epsilon$. If I_k has more regularity, then this functional can be adjusted to

$$J_S\left(I_k, \chi_k\right) = \sum_{k=1}^{K} \int_{\Omega} \left[\left| I_k - \tilde{I} \right|^2 \chi_k + \alpha \left(\chi_k + \epsilon\right) \left|\nabla^m I_k\right|^2 \right] dV. \tag{2.1.5}$$

Observe that because of the term $\alpha \chi_k \left|\nabla^m I_k\right|^2$, I_k has an incentive to be in $\mathcal{P}^{m-1}$ on the support Ω_k of χ_k. For this particular method, the segmentation is done by first expressing χ_k in terms of I_k and then minimizing the resulting functional that is only dependent on I_k. Then χ_k is expressed in terms of I_k via

$$\chi_k\left(x\right) = \begin{cases} 1 & \text{if } \left| I_k\left(x\right) - \tilde{I}\left(x\right) \right| < \left| I_l\left(x\right) - \tilde{I}\left(x\right) \right| \text{ for } l \neq k \\ 0 & \text{otherwise} \end{cases}.$$

An illustration of the segmentation done using this method can be seen in Figure (2.1.4). The image $\tilde{I}$ (red) is approximated by $\sum_{i=1}^{2} I_i \chi_i$. Since $K = 2$, we can simply let $\chi := \chi_1$ so that $\chi_2 = 1 - \chi$. Notice that the staircasing observed in Figure (2.1.2) is not present here. Moreover, χ is binary and is not a fuzzy phase function compared to the result obtained in Figure (2.1.3).

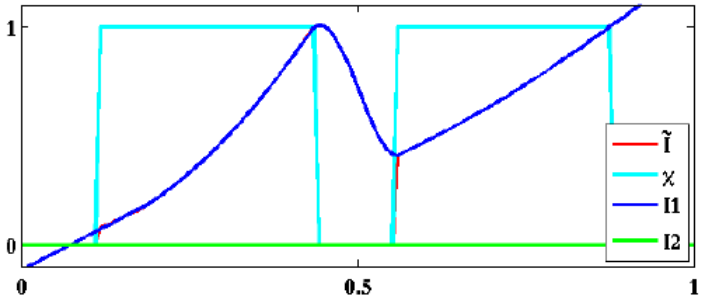


Figure 2.1.4: A multi-phase segmentation approach in one dimension. The image $\tilde{I}$ (red) is approximated by the sum of the restriction of I_1 (blue) on the support of χ (cyan) and the restriction of I_2 (green) on the support of $1-\chi$ [32].

We can now incorporate this segmentation technique to our proposed method. But first, we will set $m = 1$ on the second term of (2.1.5). With $m = 1$, I has an incentive to be constant on Ω_k. This agrees with our problem because we want the images to be piecewise constant. Obviously, we cannot just use this segmentation technique directly because we do not know what exactly $\tilde{\sigma}$ is. Nevertheless, if we replace $\tilde{I}$ in (2.1.5) with $\tilde{\sigma}$ and use i instead of k for our index, we get

$$J_S(\sigma_m, \chi_m) = \sum_{i=1}^{M} \int_\Omega \left[|\sigma_i - \tilde{\sigma}|^2 \chi_i + \alpha (\chi_i + \epsilon) |\nabla \sigma_i|^2 \right] dV. \tag{2.1.6}$$

Take note that we only use the first-order derivative in the second term because we want to make σ_m constant on Ω_m. How do we incorporate this to our problem? The first term of (2.1.6) is a least square distance that assures $\tilde{\sigma}$ to be close to σ_m in Ω_m. Although we do not have the knowledge of $\tilde{\sigma}$ on the whole Ω, we have available boundary data. Instead of using the least square distance in the whole Ω, we can just consider the L^2 least square distance at the boundary. Hence, we can simply replace the first term in (2.1.6) with (2.1.3). We also take into consideration the decomposition of σ introduced in (2.1.4). Therefore, our new functional becomes

$$\begin{aligned} \overline{J}(\sigma_m, \chi_m) = \frac{1}{2} \int_{\partial\Omega} \left| F\left(\sum_{m=1}^{M} \sigma_m \chi_m \right) - \tilde{V} \right|^2 dS \\ + \int_\Omega \sum_{m=1}^{M} \int_\Omega \alpha (\chi_m + \epsilon) |\nabla \sigma_m|^2 dV. \end{aligned} \tag{2.1.7}$$

To guarantee uniqueness of the solution, we will add another regularization. To accomplish this, we explore the use of the *Levenberg-Marquadt algorithm*. This algorithm is a quasi-Newton type method that is widely used in practice (see, e.g., [53, 50, 14, 58]). Local convergence of this method applied to the electrical impedance tomography is studied in [22]. The Levenberg-Marquadt algorithm was first suggested by Levenberg (1944) and Marquadt (1963). A detailed discussion of this technique in a more general setting can be found in [23]. Suppose $\bar{\sigma}_m$ is an estimate of σ_m. In accordance to this algorithm, we add the term $\int_\Omega \theta (\sigma_m - \bar{\sigma}_m)^2 dV$ to the second term of $\bar{J}$ in (2.1.6). And

so we can now formulate the functional that we are going to use throughout this work:

$$J(\sigma_m, \chi_m) = \frac{1}{2}\int_{\partial\Omega}\left|F\left(\sum_{m=1}^{M}\sigma_m\chi_m\right) - \tilde{V}\right|^2 dS + \sum_{m=1}^{M}\int_{\Omega}\alpha\left|\nabla\sigma_m\right|^2(\chi_m+\epsilon) + \theta\left(\sigma_m - \bar{\sigma}_m\right)^2 dV \tag{2.1.8}$$

for some $\theta > 0$. Take note that θ has to be chosen with care. Making it small might make σ_m deviate from $\bar{\sigma}_m$. Making it big will assure that σ_m will stay close to $\bar{\sigma}_m$. But what if $\bar{\sigma}_m$ is still far away from the actual solution $\tilde{\sigma}$? We address this problem when we do the analysis of the numerical approximation of our proposed method.

The addition of $\int_\Omega \theta\left(\sigma_m - \bar{\sigma}_m\right)^2 dV$ to our functional might not seem apparent at the moment. But in the next section, we will show why this additional regularization is necessary for our problem.

2.2 An Optimality Condition for the Conductivities

Our goal is of course to minimize J in (2.1.8), that is, we wish to solve the problem

$$\min_{\{\sigma_m\}\{\chi_m\}} J(\sigma_m, \chi_m).$$

This section aims to express σ_m as a function of χ_m so that J will become a functional dependent on one quantity only. How are we going to do this? We will fix χ_m and differentiate J with respect to σ_m. We wish that by equating the derivative to zero gives us the necessary optimality condition that relates σ_m to χ_m.

We begin by understanding how a perturbation $\delta\sigma$ affects ϕ. That is, we wish to establish the relationship between $\delta\sigma$ and the resulting perturbation $\delta\phi$ of ϕ. Recall that the EIT problem is given by

$$\nabla\cdot(\sigma\nabla\phi) = 0 \quad \text{on } \Omega \tag{2.2.1}$$

$$\sigma\frac{\partial\phi}{\partial n} = f \quad \text{on } \partial\Omega. \tag{2.2.2}$$

Let $\eta > 0$ and let $\delta\sigma$ be a perturbation of σ. Let $\phi(\sigma+\eta\delta\sigma)$ be the solution of (2.2.1)-(2.2.2) given $\sigma+\eta\delta\sigma$. Denote

$$\delta\phi(\sigma;\eta\delta\sigma) := \phi(\sigma+\eta\delta\sigma) - \phi(\sigma). \tag{2.2.3}$$

In (2.2.1), we replace ϕ and σ by $\phi(\sigma+\eta\delta\sigma)$ and $\sigma+\eta\delta\sigma$, respectively. Thus,

$$\nabla\cdot((\sigma+\eta\delta\sigma)\nabla\phi(\sigma+\eta\delta\sigma)) = 0.$$

We will simply write ϕ instead of $\phi(\sigma)$ and use (2.2.3) to get

$$\nabla\cdot((\sigma+\eta\delta\sigma)\nabla(\phi+\delta\phi(\sigma;\eta\delta\sigma))) = 0.$$

Using the fact that σ and ϕ satisfy (2.2.1), the above equation becomes

$$\nabla\cdot(\sigma\nabla\delta\phi(\sigma;\eta\delta\sigma)) = -\nabla\cdot(\eta\delta\sigma\nabla\phi(\sigma+\eta\delta\sigma)). \tag{2.2.4}$$

Dividing (2.2.4) by η and taking the limit as $\eta \to 0$, we obtain

$$\nabla \cdot \left(\sigma \nabla \frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) \right) = -\nabla \cdot (\delta \sigma \nabla \phi) . \tag{2.2.5}$$

Doing the same with the boundary condition (2.2.2) and conisdering the fact that $\delta\sigma$ vanishes at the boundary, we get

$$\sigma \frac{\partial \phi}{\partial n} + \sigma \frac{\partial \delta \phi}{\partial n} = f.$$

Furthermore, using the boundary condition (2.2.2), the above equation becomes

$$\sigma \frac{\partial \delta \phi}{\partial n} = 0. \tag{2.2.6}$$

Dividing by η and then taking the limit as $\eta \to 0$, we obtain

$$\sigma \frac{\partial}{\partial n} \left(\nabla \frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) \right) = 0. \tag{2.2.7}$$

Using (2.2.5) and (2.2.7), we can infer that $\delta\phi$ satisfies (compare, e.g., [22, 59]):

$$\nabla \cdot \left(\sigma \nabla \left(\frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) \right) \right) = -\nabla \cdot (\delta \sigma \nabla \phi) \quad \text{on } \Omega \tag{2.2.8}$$

$$\sigma \frac{\partial}{\partial n} \nabla \left(\frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) \right) = 0 \quad \text{on } \partial\Omega. \tag{2.2.9}$$

We do the same to (2.1.2) and make the association

$$\frac{\delta F}{\delta \sigma} (\sigma; \delta \sigma) = \frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) |_{\partial \Omega}$$

where $\frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma)$ satisfies (2.2.8) and (2.2.9). Moreover, because of the linearity on $\delta\sigma$, we just denote

$$F' (\sigma) (\delta \sigma) = \frac{\delta F}{\delta \sigma} (\sigma; \delta \sigma) = \frac{\delta \phi}{\delta \sigma} (\sigma; \delta \sigma) |_{\partial \Omega}. \tag{2.2.10}$$

In order to get the derivative of J with respect to σ_m for a fixed χ_m, we first need to define the *forward solution* and the *adjoint solution* of the EIT problem.

Definition 1. We define the *forward solution* ϕ as the solution of (2.1.1) given $f \in \tilde{L}^2 (\Omega)$ and $\sigma \in L^\infty (\Omega)$. Moreover, let $\tilde{V}$ be the known boundary voltage. We define the adjoint problem as

$$\nabla \cdot (\sigma \nabla \phi^*) = 0 \quad \text{on } \Omega \tag{2.2.11}$$

$$\sigma \frac{\partial \phi^*}{\partial n} = F (\sigma) - \tilde{V}. \tag{2.2.12}$$

We refer the solution ϕ^* of this problem as the *adjoint solution.*

Before we proceed with the computation of the derivative of J, we first need to prove the following result.

Lemma 2. *Let $\delta\sigma$ be a perturbation of σ and $\delta\phi$ be the corresponding perturbation ϕ. Then*

$$\int_{\partial\Omega} F'(\sigma)(\delta\sigma)\left(F(\sigma)-\tilde{V}\right)dS = -\int_{\Omega}\delta\sigma\nabla\phi\cdot\nabla\phi^* dV \tag{2.2.13}$$

(compare, e.g., [59, 22]).

Proof. With the help of one of the Green's identities (see Appendix), we get

$$\begin{aligned}\int_{\Omega}\nabla\cdot\left(\sigma\nabla\frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\right)\phi^* dV - \int_{\Omega}\frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\nabla\cdot(\sigma\nabla\phi^*)\,dV \\ = \int_{\partial\Omega}\sigma\left(\frac{\partial}{\partial n}\left(\frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\right)\phi^* - \frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\frac{\partial\phi^*}{\partial n}\right)dS.\end{aligned}$$

Using (2.2.8), (2.2.9), (2.2.11) and (2.2.12), the above equation becomes

$$-\int_{\Omega}\nabla\cdot(\delta\sigma\nabla\phi)\,\phi^* dV = -\int_{\partial\Omega}\frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\left(F(\sigma)-\tilde{V}\right)dS.$$

And because $\delta\sigma$ vanishes at the boundary, we can apply integration by parts to get

$$\int_{\Omega}\delta\sigma\nabla\phi\cdot\nabla\phi^* dV = -\int_{\partial\Omega}\frac{\delta\phi}{\delta\sigma}(\sigma;\delta\sigma)\left(F(\sigma)-\tilde{V}\right)dS.$$

Finally, combining the above equation and (2.2.10) proves our claim. □

Remark 3. If we define $(F'(\sigma))^*(g)$ as follows

$$\int_{\partial\Omega}F'(\sigma)(\delta\sigma)\,g dS = \int_{\Omega}\delta\sigma\left(F'(\sigma)\right)^*(g)\,dV$$

or in inner product notation

$$\left\langle F'(\sigma)(\delta\sigma), g\right\rangle_{L^2(\partial\Omega)} = \left\langle\delta\sigma, \left(F'(\sigma)\right)^*(g)\right\rangle_{L^2(\Omega)}$$

where $g\in\tilde{L}^2(\partial\Omega)$, then making the substitution $g = F(\sigma)-\tilde{V}$ gives

$$\left(F'(\sigma)\right)^*\left(F(\sigma)-\tilde{V}\right) = -\nabla\phi\cdot\nabla\phi^*.$$

This is the reason why the term $-\nabla\phi\cdot\nabla\phi^*$ is usually referred to as the L^2 *adjoint derivative* of $F'(\sigma)$ evaluated at $F(\sigma)-\tilde{V}$ (see, e.g., [22, 59]).

Observe that J is a function of σ_m and χ_m. If we wish to get the derivative of J with respect to σ_m, we should fix $\sigma_i\ \forall i\in\{1,2,\ldots,M\}$, $i\neq m$ and $\chi_m\ \forall m\in\{1,2,\ldots,M\}$. We now investigate the change in σ if we make a perturbation of σ_m. Let $m\in\{1,2,\ldots,M\}$ and $\delta\sigma_m$ be a perturbation of σ_m. Then the corresponding perturbation of σ in (2.1.4) is

$$\sigma+\delta\sigma = (\sigma_m+\delta\sigma_m)\chi_m + \sum_{i\neq m}\sigma_i\chi_i = \delta\sigma_m\chi_m + \sum_{i=1}^{M}\sigma_i\chi_i = \sigma+\delta\sigma_m\chi_m.$$

Thus, the corresponding perturbation $\delta\sigma$ of σ given $\delta\sigma_m$ is $\delta\sigma_m\chi_m$. Using this information, we can now finally compute the directional derivative (see Appendix) of J.

Theorem 4. *The derivative of J with respect to σ_m in the direction of $\delta\sigma_m$ is given by*

$$\frac{\delta J}{\delta \sigma_m}(\sigma_m;\delta\sigma_m) = -2\int_\Omega \chi_m \delta\sigma_m \nabla\phi\cdot\nabla\phi^* dV - \int_\Omega 2\alpha\delta\sigma_m \nabla\cdot\left[(\chi_m+\epsilon)\nabla\sigma_m\right] - 2\theta(\sigma_m-\bar{\sigma}_m)\delta\sigma_m dV \tag{2.2.14}$$

provided that

$$\frac{\partial\sigma_m}{\partial n} = 0 \quad on\ \partial\Omega.$$

Proof. For the ease of calculation, we first need to denote

$$J(\sigma_m) = J_1(\sigma_m) + J_2(\sigma_m) + J_3(\sigma_m)$$

where

$$J_1(\sigma_m) := \frac{1}{2}\int_{\partial\Omega}\left|F\left(\sum_{m=1}^{M}\sigma_m\chi_m\right) - \tilde{V}\right|^2 dS$$

$$J_2(\sigma_m) := \sum_{m=1}^{M}\int_\Omega \alpha\left|\nabla\sigma_m\right|^2(\chi_m+\epsilon)\,dV$$

$$J_3(\sigma_m) := \sum_{m=1}^{M}\int_\Omega \theta(\sigma_m-\bar{\sigma}_m)^2\,dV.$$

We first compute the derivative of J_1 with respect to σ_m. We use chain rule, (2.2.13) and the fact that $\delta\sigma = \delta\sigma_m\chi_m$ to obtain

$$\begin{aligned}\frac{\delta J_1}{\delta\sigma_m}(\sigma_m;\delta\sigma_m) &= \int_{\partial\Omega} 2\left(F(\sigma)-\tilde{V}\right)F'(\sigma)(\delta\sigma_m\chi_m)\,dS\\ &= 2\int_{\partial\Omega} F'(\sigma)(\delta\sigma)\left(F(\sigma)-\tilde{V}\right)dS\\ &= -2\int_\Omega \delta\sigma\nabla\phi\cdot\nabla\phi^* dV\\ &= -2\int_\Omega \delta\sigma_m\chi_m\nabla\phi\cdot\nabla\phi^* dV.\end{aligned} \tag{2.2.15}$$

To calculate the derivative of J_2 with respect to σ_m, note that whenever $i \neq m$, the derivative is 0. Thus, it is sufficient to get the derivative of $\int_\Omega \alpha\left|\nabla\sigma_m\right|^2(\chi_m+\epsilon)\,dV$. Because χ_m is fixed, we apply chain rule again and obtain

$$\frac{\delta J_2}{\delta\sigma_m}(\sigma_m;\delta\sigma_m) = \int_\Omega 2\alpha(\chi_m+\epsilon)\nabla\sigma_m\cdot\nabla\delta\sigma_m dV.$$

Because of the assumption that $\frac{\partial\sigma_m}{\partial n} = 0$ on $\partial\Omega$, then a direct application of integration by parts implies

$$\frac{\delta J_2}{\delta\sigma_m}(\sigma_m;\delta\sigma_m) = -\int_\Omega 2\alpha\delta\sigma_m\nabla\cdot\left[(\chi_m+\epsilon)\nabla\sigma_m^k\right]dV. \tag{2.2.16}$$

The derivative of J_3 with respect to σ_m can be computed directly:

$$\frac{\delta J_3}{\delta\sigma_m}(\sigma_m;\delta\sigma_m) = 2\theta(\sigma_m-\tilde{\sigma}_m)\delta\sigma_m dV. \tag{2.2.17}$$

By the linearity of the derivatives, (2.2.15), (2.2.16) and (2.2.17) complete our proof. □

By equating (2.2.14) to 0, we can deduce that the necessary optimality condition for the minimization of J with respect to σ_m is

$$\begin{cases} -\alpha\nabla\cdot\left[\left(\chi_m+\epsilon\right)\nabla\sigma_m\right]+\theta\left(\sigma_m-\bar{\sigma}_m\right)=\chi_m\nabla\phi\cdot\nabla\phi^* & \text{on } \Omega \\ \dfrac{\partial\sigma_m}{\partial n}=0 \quad \text{on } \partial\Omega \end{cases}. \tag{2.2.18}$$

Again, the goal of this section is to express σ_m as a function of χ_m. Given χ_m, if we can find the solution σ_m of (2.2.18), then we are done. Numerically, obtaining a solution solely based on this partial differential equation can be a problem. That is why, when we solve (2.2.18) numerically, we will explore the use of the *Finite Element Method (FEM)*. The FEM is a widely used technique in solving partial differential equations. This method is characterized by making use of the variational formulation of the given partial differential equation. A lot of books about FEM can be found in literature (see, e.g., [67, 11]). Thus, we are interested in the variational solution of (2.2.18). If we multiply $v\in H^1(\Omega)$ to the first equation in (2.2.18) and integrating both sides, we will get

$$\int_\Omega\left(-\alpha\nabla\cdot\left[\left(\chi_m+\epsilon\right)\nabla\sigma_m\right]+\theta\left(\sigma_m-\bar{\sigma}_m\right)\right)vdV=\int_\Omega\chi_m\nabla\phi\cdot\nabla\phi^*vdV \quad \forall v\in H^1(\Omega). \tag{2.2.19}$$

Using the formula of integration by parts, the left hand side becomes

$$\int_\Omega\alpha\left(\chi_m+\epsilon\right)\nabla\sigma_m\cdot\nabla vdV-\int_{\partial\Omega}\alpha\left(\chi_m+\epsilon\right)v\frac{\partial\sigma_m}{\partial n}dS+\int_\Omega\theta\left(\sigma_m-\bar{\sigma}_m\right)vdV.$$

This expression, together with the boundary condition of (2.2.18), implies that (2.2.18) can be reformulated as

$$\begin{aligned}&\textstyle\int_\Omega\alpha\left(\chi_m+\epsilon\right)\nabla\sigma_m\cdot\nabla vdV+\int_\Omega\theta\sigma_m vdV\\ &=\textstyle\int_\Omega\chi_m\nabla\phi\cdot\nabla\phi^*vdV+\int_\Omega\theta\bar{\sigma}_m vdV.\end{aligned} \tag{2.2.20}$$

This is the the variational formulation of (2.2.18). Given χ_m, σ_m is calculated via this equation. Thus, we are able to express σ_m as a function of χ_m implicitly. We will prove existence of the solution of (2.2.20) via Lax-Milgram Theorem (see Appendix). To apply this theorem, one must formulate an equation of the form

$$a(u,v)=b(v)$$

that holds for all v in a particular space. In our case, we will choose this space to be the Sobolev space $H^1(\Omega)$. One of the necessary conditions to prove the existence and uniqueness of the solution is the coercivity of a, that is $a(v,v)\geq C\left\|v\right\|^2_{H^1(\Omega)}$for all $v\in\Omega$. This cannot be guaranteed if the parameter θ in (2.2.20) is 0. This justifies the addition of the regularity term $\theta\int_\Omega\left(\sigma_m-\bar{\sigma}_m\right)^2dV$ to $\bar{J}$ in (2.1.7) and thus obtaining J in (2.1.8). We will give a detailed discussion of this matter on the next chapter when we do the mathematical analysis of our proposed method.

If we think of χ_m as a function in $L^2(\Omega)$ and if we can show that the solution $\sigma_m\in H^1(\Omega)$ of (2.2.20) can be uniquely determined, then we can think of $\sigma_m(\chi_m)$ as a mapping from $L^2(\Omega)$ to $H^1(\Omega)$. And because $H^1(\Omega)\subset L^2(\Omega)$ then we can think of $\sigma_m(\chi_m)$ as a map from $L^2(\Omega)$ to $L^2(\Omega)$.

2.3 Updating χ_m

After implicitly expressing σ_m as a function of χ_m, we can now try to find a possible update for χ_m. The functional J in (2.1.8) can now be treated as a functional that depends only on χ_m. What we are going to do is to find the gradient of this functional and get the update using the *method of steepest descent.* The method of steepest descent is implemented by first solving the gradient of the functional and then making an update by taking a step in the direction proportional to the negative of the gradient. For the discussion of this and other gradient-based methods, we refer the reader to [65].

Thus, in order to calculate an update for χ_m, we first need to compute the gradient of J. To compute the derivative, we first need to calculate the derivative of J at χ_m in an arbitrary direction $\delta\chi_m$ and then apply the Riesz-Representation Theorem (see Appendix) to get the explicit formulation of the gradient (see, e.g., [26]).

We can now calculate the derivative of J. But first, to make our notations consistent, we will denote $\sigma_m^k := \bar{\sigma}$. The index k denotes the number of steps in the algorithm. Hence, σ_m^{k+1} denotes the new update for σ_m. We modify (2.2.20) as an adaptation to these new notations:

$$\int_\Omega \alpha\left(\chi_m+\epsilon\right)\nabla\sigma_m^{k+1}\cdot\nabla v dV+\int_\Omega \theta\left(\sigma_m^{k+1}-\sigma_m^k\right)v dV=\int_\Omega \chi_m\nabla\phi\cdot\nabla\phi^* v dV.$$

We also denote

$$\sigma^k=\sum_{m=1}^M \sigma_m^k\chi_m,\quad \sigma^{k+1}=\sum_{m=1}^M \sigma_m^{k+1}\chi_m^k. \tag{2.3.1}$$

Because of the changes in notations and because J is now only dependent on χ_m, we can now express J as

$$J(\chi_m) \quad =: \quad J_1(\chi_m)+J_2(\chi_m)+J_3\left(\chi_m\right) \tag{2.3.2}$$

where

$$J_1(\chi_m)=\int_{\partial\Omega}\left[F(\sigma^{k+1})-\tilde{V}\right]^2 dS \tag{2.3.3}$$

$$J_2(\chi_m)=\sum_{i=m}^M\int_\Omega \alpha|\nabla\sigma_m^{k+1}|^2(\chi_m+\epsilon)dV \tag{2.3.4}$$

$$J_3(\chi_m)=\sum_{i=m}^M\int_\Omega \theta\left(\sigma_m^{k+1}-\sigma_m^k\right)^2 dV$$

for $\alpha,\theta>0$ and $\epsilon\in(0,1)$.

In the previous section, we have established that we can view $\sigma_m^{k+1}\left(\chi_m\right)$ as a mapping from $L^2\left(\Omega\right)$ to $L^2\left(\Omega\right)$. To calculate the derivative of J at χ_m in the direction $\delta\chi_m$, it is necessary that the derivative of σ_m^{k+1} with respect to χ_m in the direction of $\delta\chi_m$ exists in $H^1\left(\Omega\right)$. We will just assume this for now and we will prove this in the next chapter (see Theorem (36)). We state and prove the formulation of the derivative of J at χ_m in the direction $\delta\chi_m$ in the following theorem.

Theorem 5. *Suppose the derivative of σ_m^{k+1} with respect to χ_m in the direction of $\delta\chi_m$ exists in $H^1\left(\Omega\right)$, that is,*

$$\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}\left(\chi_m;\delta\chi_m\right)\in H^1\left(\Omega\right)$$

for all $m \in \{1, 2, \ldots, M\}$. *Then the explicit formulation of the derivative of* J *at* χ_m *in the direction* $\delta\chi_m$ *is given by*

$$\begin{aligned}\frac{\delta J}{\delta \chi_m}(\chi_m; \delta\chi_m) = -2\left\{\int_\Omega \sigma_m^{k+1}\delta\chi_m \nabla\phi\cdot\nabla\phi^* dV + \int_\Omega \frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,\chi_m\nabla\phi\cdot\nabla\phi^* dV\right\}\\ + \int_\Omega \alpha|\nabla\sigma_m^{k+1}|^2\delta\chi_m + 2(\chi_m+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla\left(\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\right)dV\\ + \int_\Omega 2\theta\left(\sigma_m^{k+1}-\sigma_m^k\right)\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,dV. \qquad (2.3.5)\end{aligned}$$

Proof. Recall from (2.2.13) that

$$\int_{\partial\Omega} F'\left(\sigma^{k+1}\right)(\delta\sigma)\left[F(\sigma^{k+1})-\tilde{V}\right]dS = \int_\Omega \delta\sigma\left(F'(\sigma)\right)^*\left[F(\sigma^{k+1})-\tilde{V}\right]dV. \qquad (2.3.6)$$

Since this holds for any perturbation $\delta\sigma$, we can select $\delta\sigma$ to be $\frac{\delta\sigma^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)$ which can be calculated using (2.3.1) as

$$\frac{\delta\sigma^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m) = \sigma_m^{k+1}\delta\chi_m + \frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,\chi_m. \qquad (2.3.7)$$

Using chain rule and (2.3.6), we can compute the derivative of J_1 in (2.3.3) at χ_m in the direction $\delta\chi_m$ as follows:

$$\begin{aligned}\frac{\delta J_1}{\delta\chi_m}(\chi_m;\delta\chi_m) &= 2\int_{\partial\Omega}\left[F(\sigma^{k+1})-\tilde{V}\right]F'(\sigma^{k+1})\frac{\delta\sigma^{k+1}}{\delta\chi_m}\left(\chi_m^k;\delta\chi_m^k\right)dS\\ &= 2\int_\Omega \frac{\delta\sigma^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\left(F'(\sigma^{k+1})\right)^*\left[F(\sigma^{k+1})-\tilde{V}\right]dV\\ &= -2\int_\Omega \frac{\delta\sigma^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,\nabla\phi\cdot\nabla\phi^* dV.\end{aligned}$$

We can further simplify this using (2.3.7):

$$\begin{aligned}\frac{\delta J_1}{\delta\chi_m}(\chi_m;\delta\chi_m) &= -2\int_\Omega\left[\sigma_m^{k+1}\delta\chi_m + \frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,\chi_m\right]\nabla\phi\cdot\nabla\phi^* dV\\ &= -2\left\{\int_\Omega \sigma_m^{k+1}\delta\chi_m\nabla\phi\cdot\nabla\phi^* dV\right.\\ &= \left.+\int_\Omega \frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,\chi_m\nabla\phi\cdot\nabla\phi^* dV\right\}. \qquad (2.3.8)\end{aligned}$$

Let $i \neq m$. Then

$$\frac{\delta\left(\alpha|\nabla\sigma_i^{k+1}|^2(\chi_i+\epsilon) + \theta\left(\sigma_i^{k+1}-\sigma_i^k\right)^2\right)}{\delta\chi_m}(\chi_m;\delta\chi_m) = 0.$$

Therefore, when we compute the directional derivative of J_2 and J_3 in (2.3.4) at χ_m in the direction $\delta\chi_m$, we can ignore the terms when $i \neq m$. This means that we can drop

the summation sign altogether. By the application of product rule and chain rule, the derivative of J_2 and J_3 can be calculated directly:

$$\frac{\delta J_2}{\delta \chi_m}(\chi_m;\delta\chi_m) = \int_\Omega \alpha|\nabla\sigma_m^{k+1}|^2\delta\chi_m dV + \int_\Omega 2(\chi_m+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla\left(\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\right)dV \tag{2.3.9}$$

$$\frac{\delta J_3}{\delta \chi_m}(\chi_m;\delta\chi_m) = \int_\Omega 2\theta\left(\sigma_m^{k+1}-\sigma_m^k\right)\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m)\,dV. \tag{2.3.10}$$

Combining (2.3.8), (2.3.9) and (2.3.10) justifies our assertion. Furthermore, because $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m) \in H^1(\Omega)$ then $\nabla\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m) \in L^2(\Omega)$ and so $\frac{\delta J}{\delta\chi_m}(\chi_m;\delta\chi_m)$ is well-defined. □

In the next chapter, we will do a rigorous mathematical analysis of the proposed method. As already said, we will prove that $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m}(\chi_m;\delta\chi_m) \in H^1(\Omega)$ under certain assumptions. We will also use the derivative (2.3.5) of J to get an explicit formulation of the gradient of J . In fact,

$$\nabla(J(\chi_m)) = -2\sigma_m^{k+1}\nabla\phi\cdot\nabla\phi^* + \alpha|\nabla\sigma_m^{k+1}|^2.$$

We verify this result in the next chapter (see Theorem (37)) in our mathematical analysis. We can now formulate the update for χ_m by first defining the mapping $G : L^2(\Omega) \to L^2(\Omega)$ by

$$G(\chi_m) := \chi_m - \omega\left\{-2\sigma_m^{k+1}\nabla\phi\cdot\nabla\phi^* + \alpha|\nabla\sigma_m^{k+1}|^2\right\} \tag{2.3.11}$$

where $\omega \in (0,1)$ is the step size. Notice that G is exactly the update one will get if the method of steepest descent is used.

To give the update for χ_m some smoothness, we can incorporate a regularization called Total Variation (TV) in our update. TV regularization was first conjectured in [62] and introduced in [51] as an image restoration tool. Since then, TV-regularization has been used in solving different inverse problems. TV regularization has been used to solve the ill-posed EIT problem. For instance, one can refer to [25, 29, 41, 47]. The method suggested by Dobson and Santosa [25] works for the linearized problem of the EIT problem but is not numerically efficient. Somersalo and Kolehmainen made use of the Markov Chain Monte Carlo methods to solve the TV regularized inverse EIT problem. TV is the measure of the total amplitude of the oscillations of a function [41]. Given a function f, its total variation is given by

$$TV(f) = \int_\Omega |\nabla f|\,dV.$$

Our work intends to incorporate TV regularization with the proposed multi-phase segmentation approach to solve the inverse EIT problem. Hence, we add a regularization term on the functional J in (2.3.2) :

$$J_\gamma(\chi_m) = J(\chi_m) + \gamma\int_\Omega |\nabla\chi_m|\,dV$$

for some $\gamma \in (0,1)$. In actual computation, the gradient of the above functional may lead to a singularity when $\nabla\chi_m = 0$ and so we replace the above functional with a smooth approximation (compare, e.g., [69, 24])

$$J_{\gamma\beta}(\chi_m) = J(\chi_m) + \gamma J_\beta(\chi_m)$$

with

$$J_\beta(\chi_m) = \int_\Omega \sqrt{|\nabla\chi_m|^2 + \beta^2} dV$$

for some $\beta \in (0,1)$. Let n be the outward normal vector to $\partial\Omega$ and suppose $\frac{\partial\chi_m}{\partial n} = 0$ on $\partial\Omega$. The directional derivative of J_β at χ_m in the direction of $\delta\chi_m$ can be calculated using the Divergence Theorem:

$$\begin{aligned}
\frac{\delta J_\beta}{\delta\chi_m}(\chi_m;\delta\chi_m) &= \int_\Omega \frac{\nabla\chi_m\cdot\nabla\delta\chi_m}{\sqrt{|\nabla\chi_m|^2+\beta^2}} dV \\
&= -\int_\Omega \delta\chi_m \nabla\cdot\left(\frac{\nabla\chi_m}{\sqrt{|\nabla\chi_m|^2+\beta^2}}\right) + \int_{\partial\Omega}\delta\chi_m\frac{\partial\chi_m}{\partial n}dS \\
&= -\int_\Omega \delta\chi_m \nabla\cdot\left(\frac{\nabla\chi_m}{\sqrt{|\nabla\chi_m|^2+\beta^2}}\right).
\end{aligned}$$

The assumption that $\frac{\partial\chi_m}{\partial n} = 0$ on $\partial\Omega$ makes sense in our problem because χ_m is assumed to be the characteristic function of Ω_m, which is completely inside Ω for $m = 1, 2, \ldots, M-1$. Using Riesz-Representation Theorem, the gradient of $J_\beta(\chi_m)$ is given by (compare, e.g., [69, 24, 23])

$$\nabla J_\beta(\chi_m) = -\nabla\cdot\left(\frac{\nabla\chi_m}{\sqrt{|\nabla\chi_m|^2+\beta^2}}\right).$$

We incorporate the above gradient with (2.3.11) to find a possible update for χ_m:

$$(I + \omega\gamma L(\chi_m))\Theta = G(\chi_m) \quad \text{on } \Omega \tag{2.3.12}$$

$$\frac{\partial\Theta}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{2.3.13}$$

where the operator L is defined by

$$L(u)v := -\nabla\cdot\left(\frac{\nabla v}{\sqrt{|\nabla u|^2+\beta^2}}\right).$$

The solution Θ of (2.3.12)-(2.3.13) is our candidate for the update for χ_m. Again, because all our numerical simulations will be done using FEM, we are interested in the variational solution of (2.3.12)-(2.3.13). Multiplying v to both sides of (2.3.12) and with the help of the Divergence Theorem, we calculate the variational formulation of (2.3.12)-(2.3.13) to be

$$\int_\Omega \omega\gamma\frac{\nabla\Theta\cdot\nabla v}{\sqrt{|\nabla\chi_m|^2+\beta^2}} + \Theta v dV = \int_\Omega G(\chi_m) v dV \tag{2.3.14}$$

for all $v \in H^1(\Omega)$. We will show in the next chapter that the above variational formulation has a solution in $H^1(\Omega)$. We define the function $\Theta(\chi_m) : L^2(\Omega) \to L^2(\Omega)$ to be the solution of the above equation for a given $\chi_m \in L^2(\Omega)$. Obviously, Θ cannot be used as the update for χ_m because it is not binary. Thus, we introduce a thresholding step. Let χ_m^k be the current approximation of χ_m. The update χ_m^{k+1} is obtained by

$$\chi_m^{k+1}(x) = \begin{cases} 1 & \text{if } \Theta\left(\chi_m^k(x)\right) \geq \zeta \\ 0 & \text{otherwise} \end{cases}$$

for some $\zeta \in (0,1)$.

2.4 The Proposed Algorithm

The previous sections were devoted in finding an equation that relates σ_m and χ_m, as well as looking for an update for χ_m. In this section, we will state our proposed algorithm in solving the inverse electrical impedance tomography problem. But first we have to compute the variational formulation of the forward and the adjoint EIT problem. This is necessary because similar to how we would like to solve (2.2.18), we will also solve the numerical solution of the forward and the adjoint problem using the Finite Element Method.

Let $v \in H^1(\Omega)$ such that $\int_{\partial\Omega} v dS = 0$. If we multiply v to (2.2.2) and integrate over $\partial\Omega$, we get

$$\int_{\partial\Omega} \sigma \frac{\partial \phi}{\partial n} v dS = \int_{\partial\Omega} f v dS \tag{2.4.1}$$

where n is the the outward pointing normal vector on the boundary. The left-hand side of the above equation can be transformed as an integral in the whole Ω using the Divergence Theorem. Indeed,

$$\int_{\partial\Omega} \sigma \frac{\partial \phi}{\partial n} v dS = \int_{\partial\Omega} v\sigma\nabla\phi \cdot n dS = \int_{\Omega} \nabla \cdot (v\sigma\nabla\phi)\, dV. \tag{2.4.2}$$

Furthermore, the right-hand side of the above equation can be expanded using the product rule for divergence:

$$\begin{aligned} \int_{\Omega} \nabla \cdot (v\sigma\nabla\phi)\, dV &= \int_{\Omega} \nabla v \cdot (\sigma\nabla\phi)\, dV + \int_{\Omega} v\nabla \cdot (\sigma\nabla\phi)\, dV \\ &= \int_{\Omega} \sigma\nabla\phi \cdot \nabla v dV + \int_{\Omega} v\nabla \cdot (\sigma\nabla\phi)\, dV. \end{aligned} \tag{2.4.3}$$

Because ϕ and σ satisfy (2.2.1) on Ω, the right-hand side of the above equation becomes

$$\int_{\Omega} \sigma\nabla\phi \cdot \nabla v dV + \int_{\Omega} v\nabla \cdot (\sigma\nabla\phi)\, dV = \int_{\Omega} \sigma\nabla\phi \cdot \nabla v dV. \tag{2.4.4}$$

Thus, the variational formulation of the forward problem can be deduced from (2.4.1)-(2.4.4) as (compare, e.g., [46])

$$\int_{\Omega} \sigma\nabla\phi \cdot \nabla v dV = \int_{\partial\Omega} f v dS. \tag{2.4.5}$$

Similarly, by replacing f with $F(\sigma) - \tilde{V}$, the variational formulation of the adjoint problem is

$$\int_{\Omega} \sigma\nabla\phi^* \cdot \nabla v dV = \int_{\partial\Omega} \left(F(\sigma) - \tilde{V}\right) v dS. \tag{2.4.6}$$

Since $F(\sigma) = \phi|_{\partial\Omega}$ then (2.4.6) becomes

$$\int_{\Omega} \sigma\nabla\phi^* \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi - \tilde{V}\right) v dS. \tag{2.4.7}$$

We prove in the next chapter that (2.4.5) and (2.4.7) have unique solutions in $H^1(\Omega)$. The value of σ_M is assumed to be the conductivity at $\partial\Omega$ and thus, known. The corresponding χ_M of σ_M can be computed using the values of χ_m where $m \in \{1, 2, \ldots, M-1\}$ by

$$\chi_M = 1 - \sum_{m=1}^{M-1} \chi_m. \tag{2.4.8}$$

With these, we can now present a proposed algorithm in solving the Electrical Impedance Tomography Problem.

ALGORITHM 1.

1. *Initialization.* Given f and $\tilde{V}$. Choose parameters $\alpha, \epsilon, \theta, \omega, \gamma, \beta$ and ζ. Select the appropriate value of M, the maximum number of iterations K and the tolerance ρ. Set $k = 1$ and choose the initial σ_m^k. Select an initial guess χ_m^k, $m \in \{1, 2, \ldots, M-1\}$ for χ_m such that $\operatorname{supp}(\chi_{m_1}) \cap \operatorname{supp}(\chi_{m_1}) = \emptyset$ for any $m_1, m_2 \in \{1, 2, \ldots, M-1\}$ with $m_1 \neq m_2$. For $m = M$, σ_M^k is known and $\chi_M^k := 1 - \sum_{m=1}^{M-1} \chi_m^k$.

2. *Solving the forward and the adjoint problem.* Take $\sigma^k = \sum_{m=1}^{M} \sigma_m^k \chi_m^k$ and solve for ϕ^k and ϕ^{*k} through

$$\int_\Omega \sigma^k \nabla\phi^k \cdot \nabla v dV = \int_{\partial\Omega} f v dS$$

and

$$\int_\Omega \sigma^k \nabla\phi^{*k} \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi^k - \tilde{V}\right) v dS.$$

3. *Solving for σ_m^{k+1}.* Given χ_m^k for $m = 1, 2, \ldots, M-1$, σ_m^{k+1} is solved via

$$\int_\Omega \alpha\left(\chi_m^k + \epsilon\right) \nabla\sigma_m^{k+1} \cdot \nabla v dV + \int_\Omega \theta\sigma_m^{k+1} v dV = \int_\Omega \chi_m^k \nabla\phi^k \cdot \nabla\phi^{*k} v dV + \int_\Omega \theta\sigma_m^k v dV.$$

For $m = M$, set $\sigma_M^{k+1} = \sigma_M^k$.

4. *Updating χ_m.* Given χ_m^k for $m = 1, 2, \ldots, M-1$, we solve for

$$G\left(\chi_m^k\right) = \chi_m^k - \omega\left\{-2\sigma_m^{k+1}\nabla\phi \cdot \nabla\phi^* + \alpha|\nabla\sigma_m^{k+1}|^2\right\}$$

and

$$\int_\Omega \omega\gamma \frac{\nabla\Theta \cdot \nabla v}{\sqrt{|\nabla\chi_m^k|^2 + \beta^2}} + \Theta v dV = \int_\Omega G\left(\chi_m^k\right) v dV$$

and χ_m is updated by

$$\chi_m^{k+1}(x) = \begin{cases} 1 & \text{if } \Theta\left(\chi_m^k(x)\right) \geq \zeta \\ 0 & \text{otherwise} \end{cases}.$$

For $m = M$, set $\chi_M^{k+1} = 1 - \sum_{m=1}^{M-1} \chi_m^{k+1}$.

5. *Stopping Criteria.* If $k = K$ or $\sum_{m=1}^{M} \left\|\chi_m^{k+1} - \chi_m^k\right\|_{L^2(\Omega)} < \rho$, the algorithm terminates. Otherwise, $k \leftarrow k+1$ and go back to step 2.

2.5 Equal Conductivities on the Inclusions

In the first step of Algorithm (1), an initial guess for σ_m and χ_m, $m \in \{1, 2, \ldots, M-1\}$ must be specified. Because σ_m is assumed to be constant on Ω_m, the selection of the initial guess for this quantity does not pose a big problem. For example, if we want to reconstruct an image of the internal organs, we can give a good guess for σ_m by looking at Table (2.1). So as long as the value is not "too far" from the actual value, this should not be a problem. On the contrary, an initial guess for χ_m might pose a little problem. When we initialize χ_m, we are looking for a geometry in Ω and not just a constant value. When doing an image reconstruction in EIT, for example when reconstructing image of a body organ, an estimate of the location of this particular organ can somehow be estimated. It does not need to be extremely close to the actual solution but just a very rough estimate. This is the reason why we will work under the assumption that an initial guess for χ_m is given. We will explain in our numerical experimentation how we are going to choose these initial updates.

Suppose $M = 2$ in (2.1.4). This means that the conductivity σ can be formulated as $\sigma = \sigma_1\chi_1 + \sigma_2\chi_2$. And because we want the support of χ_1 and χ_2 to be disjoint, then $\chi_2 = 1 - \chi_1$. Moreover, σ_2 is known because the value at the boundary is known. Hence, we are only tasked to find χ_1 and σ_1. This also means that the support of χ_1 can be a union of disjoint sets in Ω. This is illustrated in Figure (2.5.1). Compared with Figure (2.1.1), Figure (2.5.1) assumes equal conductivities on the inclusions. This particular case has been studied in [17, 40, 54]. In these papers, the use of topological derivative in finding an initial estimate for χ_1 was introduced. Thus, we also adapt this particular technique.

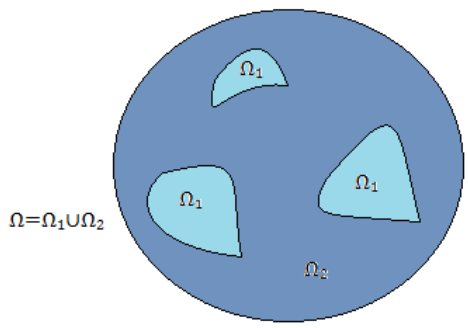


Figure 2.5.1: A segmentation of Ω with equal conductivity on the inclusion.

A shape functional is a functional whose domain is the set of all shapes in Ω. It maps a domain $\Omega_d \subset \Omega$ to a function value in $\mathbb{R}$. Observe that for the case when $M = 2$ in (2.1.4), the functional $\tilde{J}$ in (2.1.3) becomes a function of χ_1 and σ_1 only, because $\chi_2 = 1 - \chi_1$ and σ_2 are known. In the second section of this chapter, we were able to express σ_1 as a function of χ_1 implicitly. Essentially, $\tilde{J}$ depends only on χ_1 . Because χ_1 is a characteristic function, we can identify Ω_1 as the support of χ_1. Thus, we can reformulate the functional $\tilde{J}$ as the shape functional

$$\mathcal{J}(\Omega_1) = \frac{1}{2}\int_{\partial\Omega}\left(F\left(\sigma_1\chi_1 + \sigma_2\left(1-\chi_1\right)\right) - \tilde{V}\right)^2 dS \tag{2.5.1}$$

where Ω_1 is the support of χ_1, σ_1 implicitly depends on χ_1 and σ_2 is known.

The topological derivative of a given shape functional is the derivative with respect to infinitesimal changes in the topology [66]. A discussion of topological derivative in a more general context can be found in [60]. We will now present the topological derivative of $\mathcal{J}$ in (2.5.1). Let $\eta > 0$ and $x \in \Omega_1$ such that $B(x,\eta) \subset \Omega_1$. Furthermore, let $\Omega_1^\eta := \Omega_1 \backslash B(x,\eta)$. The topological derivative of $\mathcal{J}$ is therefore given by

$$D_T(x;\Omega_1) = \lim_{\eta \to 0} \frac{\mathcal{J}(\Omega_1^\eta) - \mathcal{J}(\Omega_1)}{|B(x,\eta)|}.$$

Using this, for η small enough, we can therefore expand $\mathcal{J}$ as (compare, e.g., [17])

$$\mathcal{J}(\Omega_1^\eta) = \mathcal{J}(\Omega_1) + |B(x,\eta)| D_T(x) + o(|B(x,\eta)|).$$

Meaning, $\mathcal{J}(\Omega_1^\eta) < \mathcal{J}(\Omega_1)$ if for some $\eta > 0$ and $x \in \Omega_1$, $|B(x,\eta)| D_T(x) < 0$. How do we get an initial update using this? We can simply let $\Omega_1 := \Omega$ and search the whole Ω where there is a possible negative topological derivative. Thus, we compute for $D_T(x;\Omega)$ and find the values of x for which $D_T(x;\Omega) < -C$ for some $C > 0$ large enough. We assign $\chi_1 = 1$ to all the values of x satisfying this and set $\chi_1 = 0$ otherwise. This is the same method adapted in [17]. A detailed explanation on how to estimate C can be found in [16].

Given $\Omega_1 \subset \Omega$, the topological derivative of $\mathcal{J}$ is given by

$$D_T(x,\Omega_1,\sigma_1) = \begin{cases} \dfrac{2\sigma_1(\sigma_1-\sigma_2)}{\sigma_1+\sigma_2} \nabla\phi(x)\cdot\nabla\phi^*(x) & x \in \Omega_1 \\ \dfrac{2\sigma_2(\sigma_1-\sigma_2)}{\sigma_2+\sigma_1} \nabla\phi(x)\cdot\nabla\phi^*(x) & x \in \Omega\backslash\overline{\Omega}_1 \end{cases}. \tag{2.5.2}$$

For the proof, we refer the reader to [17]. Higher order topological derivative for the electrical impedance tomography problem is discussed and proved in [40, 54] but we will only make use of the first order derivative discussed in [17, 16].

Going back to Algorithm (1), we can now give a modification on its first step. The changes are transcribed in bold characters.

1. *Initialization.* Given f and $\tilde{V}$. Choose parameters $\alpha, \epsilon, \theta, \omega, C$ and ζ. **Set** $M = 2$, the maximum number of iterations K and the tolerance ρ. Set $k = 1$ and choose the constant σ_1^k . **Compute** $D_T\left(x,\Omega_1,\sigma_1^k\right)$ **where** $\Omega_1 = \emptyset$. **Set** $\chi_1^k = 1\ \forall x \in \Omega$ **such that** $D_T\left(x,\Omega_1,\sigma_1^k\right) < -C$ **and** $\chi_1^k = 0$ **otherwise.** The quantity σ_2^k **is given and** $\chi_2^k = 1 - \chi_1^k$**.**

Chapter 3

Analysis of the Proposed Method

In this chapter, we will try to analyze the convergence of the proposed method. We will start with few results if we assume that $\sigma \in L^{\infty}(\Omega)$ and that χ_m is a characteristic function. We will then proceed with the introduction of the mollification of χ_m. Instead of using the characteristic function χ_m, we will use its smooth approximation instead. This is necessary because some of the essential results require χ_m to have higher regularity. This might seem like a deviation from our proposed method but we will argue that these modifications can be justified. Note that the smoothing of χ_m is only included for technical purposes and will not be used in practice. We will then introduce a modification of Algorithm (1) to adapt with the mollification of χ_m. Towards the end of this chapter, we prove that the modified version of Algorithm (1) has a fixed point via Schauder's Fixed Point Theorem.

3.1 Preliminary Estimates

In this section, we will try to understand how a perturbation on σ^k affects ϕ and ϕ^*. Recall that σ^k depends on χ_m for any $m \in \{1, 2, \ldots, M\}$. Therefore, ϕ and ϕ^* depend on χ_m as well. Working under the assumption that $\sigma \in L^{\infty}(\Omega)$ and χ_m is a characteristic function, we will show that ϕ and ϕ^* continuously depend on σ^k and on χ_m. We begin this section by showing that the variational forward problem and the variational adjoint problem both have unique solutions in $H^1(\Omega)$ when $\sigma \in L^{\infty}(\Omega)$. In the succeeding sections, we will study the behavior of ϕ and ϕ^* when we give additional regularity to σ^k.

Theorem 6. *Let $\sigma^k \in L^{\infty}(\Omega)$ such that $0 < \underline{\sigma} \leq \sigma^k(x)$ for all $x \in \Omega$ and $f \in \tilde{L}^2(\partial\Omega)$. Then (**the variational forward EIT problem**)*

$$\int_{\Omega} \sigma^k \nabla\phi \cdot \nabla v dV = \int_{\partial\Omega} \sigma^k \frac{\partial\phi}{\partial n} v dS \tag{3.1.1}$$

has a unique solution $\phi \in H^1(\Omega)$ with $\int_{\partial\Omega} \phi dS = 0$. Similarly, let

$$F\left(\sigma^k\right) = \phi|_{\partial\Omega}$$

*and $\tilde{V} \in \tilde{L}^2(\partial\Omega)$ be the known boundary voltage. Then (**the variational adjoint problem**)*

$$\int_{\Omega} \sigma^k \nabla\phi^* \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi - \tilde{V}\right) v dS \tag{3.1.2}$$

has a unique solution $\phi^* \in H^1(\Omega)$ *with* $\int_{\partial\Omega} \phi^* dS = 0$.

Proof. Define $I := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u dS = 0\right\}$. Define

$$a(u,v) := \int_\Omega \sigma^k \nabla u \cdot \nabla v dV$$

$$b(v) := \int_{\partial\Omega} f v dS.$$

Clearly, a is bilinear and b is linear. We now show that a is bounded and coercive:

$$\begin{aligned}
|a(u,v)| &\leq \int_\Omega \left|\sigma^k \nabla u \cdot \nabla v\right| dV \\
&\leq \left\|\sigma^k\right\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\leq \left\|\sigma^k\right\|_{L^\infty(\Omega)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

and because $u \in I$ then $\int_{\partial\Omega} u dS = 0$. Therefore, using a generalized Friedrich's inequality (see Appendix), we get

$$\begin{aligned}
|a(u,u)| &= \int_\Omega \sigma^k |\nabla u|^2 dV \\
&\geq \underline{\sigma} \|\nabla u\|^2_{L^2(\Omega)} \\
&= \frac{\sigma}{2} \|\nabla u\|^2_{L^2(\Omega)} + \frac{\sigma}{2} \|\nabla u\|^2_{L^2(\Omega)} \\
&\geq \frac{\sigma}{2}\left(\frac{1}{C}\|u\|^2_{L^2(\Omega)} - \left(\int_{\partial\Omega} u dS\right)^2\right) + \frac{\sigma}{2}\|\nabla u\|^2_{L^2(\Omega)} \\
&= \frac{\sigma}{2C}\|u\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|\nabla u\|^2_{L^2(\Omega)} \\
&\geq \underline{\sigma} \min\left\{\frac{1}{C}, 1\right\} \|u\|^2_{H^1(\Omega)}
\end{aligned}$$

for some $C > 0$. Finally, by the Trace Theorem (see Appendix)

$$\begin{aligned}
|b(v)| &\leq \int_\Omega \|fv\| dS \\
&\leq \|f\|_{\tilde{L}^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\
&\leq c\|f\|_{\tilde{L}^2(\partial\Omega)} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

Hence, by Lax-Milgram Theorem $\exists$ a unique $\phi \in H^1(\Omega)$ such that $\int_{\partial\Omega} \phi dS = 0$ satisfying the forward EIT problem.

Let $u, v \in I$.For the adjoint problem, we will use the same bilinear form $a(u,v)$. Instead of $b(v)$, we use

$$\bar{b}(v) = \int_{\partial\Omega} \left(\phi - \tilde{V}\right) v dS.$$

Clearly, b is linear and so we only need to show continuity. We will apply the Cauchy-Schwarz inequality first then use the Trace theorem. Thus,

$$\begin{aligned}\left|\bar{b}(v)\right| &\leq \int_{\Omega}\left\|\left(\phi-\tilde{V}\right)v\right\|dS\\ &\leq \left\|\left(\phi-\tilde{V}\right)\right\|_{\tilde{L}^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}\\ &\leq c\left\|\left(\phi-\tilde{V}\right)\right\|_{\tilde{L}^2(\partial\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

Therefore, by applying Lax-Milgram Theorem one more time, there exists a unique $\phi^* \in I$ satisfying the adjoint problem. □

Corollary 7. *Let ϕ and ϕ^* satisfy the variational forward and the variational adjoint problem stated in the previous theorem. Then we have the following estimates:*

$$\|\phi\|_{H^1(\Omega)} \leq C_1\|f\|_{\tilde{L}^2(\partial\Omega)} \tag{3.1.3}$$

$$\|\phi^*\|_{H^1(\Omega)} \leq C_2\left\|\left(\phi-\tilde{V}\right)\right\|_{\tilde{L}^2(\partial\Omega)} \tag{3.1.4}$$

for some $C_1, C_2 > 0$.

Note that using the Trace Theorem and triangle inequality, ϕ^* can be further estimated by

$$\|\phi^*\|_{H^1(\Omega)} \leq C_3\left(\|f\|_{\tilde{L}^2(\partial\Omega)}+\left\|\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right) \tag{3.1.5}$$

for some $C_3 > 0$. Throughout this work, we will use the following notations.

Definition 8. We let $\delta\sigma^k$ and $\delta\chi_m$ be the denotations of the perturbations of σ^k and χ_m, respectively. For any $\sigma^k \in L^\infty$ such that $\sigma^k \geq \underline{\sigma} > 0$, we have shown that $\exists\phi$ and ϕ^* in $H^1(\Omega)$ satisfying the forward and the adjoint problem. Hence, we use the notation $\phi\left(\sigma^k\right)$ and $\phi^*\left(\sigma^k\right)$ whenever we want to emphasize that ϕ and ϕ^* are the solutions of the forward and the adjoint problems given σ^k.

Remark 9. Given a perturbation $\delta\sigma^k \in L^\infty(\Omega)$ and $\eta > 0$, do the forward and the adjoint problems have unique solutions if we use $\sigma^k + \eta\delta\sigma^k$? We know that the forward and adjoint problems have unique solutions given $\sigma^k \in L^\infty$ if $\sigma^k(x) \geq \underline{\sigma} > 0$ for all $x \in \Omega$. To make sure that $\phi\left(\sigma^k+\eta\delta\sigma^k\right)$ is unique, we can simply select η sufficiently small so that $\left(\sigma^k+\eta\delta\sigma^k\right)(x) \geq \underline{\sigma}^\tau > 0$ for all $x \in \Omega$ and $\eta \in (0,\tau)$ for some $\tau > 0$. This is possible because $\sigma^k(x) \geq \underline{\sigma} > 0$. Thus, the coercivity of the bilinear functional in the variational formulations of both the forward and the adjoint problems are guaranteed and the solvability of these problems are assured. Consequently, by (3.1.3) and (3.1.5) there exist $C_1, C_2 > 0$ such that

$$\left\|\phi\left(\sigma^k+\eta\delta\sigma^k\right)\right\|_{H^1(\Omega)} \leq C_1\|f\|_{\tilde{L}^2(\partial\Omega)} \tag{3.1.6}$$

$$\left\|\phi^*\left(\sigma^k+\eta\delta\sigma^k\right)\right\|_{H^1(\Omega)} \leq C_2\left(\|f\|_{\tilde{L}^2(\partial\Omega)}+\left\|\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right) \tag{3.1.7}$$

for any $\eta \in (0,\tau)$.

The following result shows how a perturbation $\delta\sigma^k$ of $\sigma^k \in L^\infty(\Omega)$ affects ϕ and ϕ^*.

Theorem 10. *There exist $C_1, C_2 > 0$ such that*

$$\left\|\phi\left(\sigma^k + \eta\delta\sigma^k\right) - \phi\left(\sigma^k\right)\right\|_{H^1(\Omega)} \leq C_1\eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \tag{3.1.8}$$

and

$$\left\|\phi^*\left(\sigma^k + \eta\delta\sigma^k\right) - \phi^*\left(\sigma^k\right)\right\|_{H^1(\Omega)} \leq C_2\eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \tag{3.1.9}$$

for any $\eta \in (0, \tau)$, where τ is sufficiently small and chosen according to Remark (9).

Proof. From (3.1.1), we have

$$\int_\Omega \sigma^k \nabla\phi \cdot \nabla v dV = \int_\Omega f v dV. \tag{3.1.10}$$

Similarly, for $\sigma^k + \eta\delta\sigma^k$

$$\int_\Omega \left(\sigma^k + \eta\delta\sigma^k\right) \nabla\left(\phi + \delta\phi\right) \cdot \nabla v dV = \int_\Omega f v dV \tag{3.1.11}$$

where we denote

$$\delta\phi = \phi\left(\sigma^k + \eta\delta\sigma^k\right) - \phi\left(\sigma^k\right).$$

Subtracting (3.1.10) from (3.1.11), we will obtain

$$\int_\Omega \sigma^k \nabla\delta\phi \cdot \nabla v dV = -\int_\Omega \eta\delta\sigma^k \nabla\phi\left(\sigma^k + \eta\delta\sigma^k\right) \cdot \nabla v dV. \tag{3.1.12}$$

Define

$$a\left(u, v\right) = \int_\Omega \sigma^k \nabla u \cdot \nabla v dV$$

$$b_1\left(v\right) = -\int_\Omega \eta\delta\sigma^k \nabla\phi\left(\sigma^k + \eta\delta\sigma^k\right) \cdot \nabla v dV.$$

Clearly, a and b_1 are bilinear and linear, respectively. Recall from Theorem (6) that for any $u, v \in I = \left\{w \in H^1\left(\Omega\right) : \int_{\partial\Omega} w dS = 0\right\}$, $a(u, v)$ is coercive and continuous. We will now find an estimate for b_1. Indeed, by the Cauchy-Schwarz inequality and (3.1.6), we have

$$\begin{aligned}
\left|b_1\left(v\right)\right| &\leq \eta \int \left|\delta\sigma^k \nabla\phi\left(\sigma^k + \eta\delta\sigma^k\right) \cdot \nabla v\right| dV \\
&\leq \eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left\|\nabla\phi\left(\sigma^k + \eta\delta\sigma^k\right)\right\|_{L^2(\Omega)} \left\|\nabla v\right\|_{L^2(\Omega)} \\
&\leq \eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left\|\nabla\phi\left(\sigma^k + \eta\delta\sigma^k\right)\right\|_{L^2(\Omega)} \left\|v\right\|_{H^1(\Omega)} \\
&\leq \bar{C}_1\eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left\|f\right\|_{\tilde{L}^2(\partial\Omega)} \left\|v\right\|_{H^1(\Omega)}
\end{aligned}$$

for some $\bar{C}_1 > 0$. Thus, if we take $u = v = \delta\phi$ and use the previous inequality and the coercivity of $a\left(u, v\right)$ we get

$$\left\|\delta\phi\right\|_{H^1(\Omega)} \leq \frac{1}{\bar{C}}\bar{C}_1\eta \left\|f\right\|_{\tilde{L}^2(\partial\Omega)} \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \tag{3.1.13}$$

where $\bar{C} > 0$ is the coercivity constant. This proves the first inequality. For the second part, we proceed in a similar manner. From (3.1.2), we have

$$\int_{\Omega} \sigma^k \nabla \phi^* \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi - \tilde{V}\right) v dS. \tag{3.1.14}$$

Similarly, for $\sigma^k + \eta\delta\sigma^k$

$$\int_{\Omega} \left(\sigma^k + \eta\delta\sigma^k\right) \nabla \left(\phi^* + \delta\phi^*\right) \cdot \nabla v dV = \int_{\partial\Omega} \left[\phi \left(\sigma^k + \eta\delta\sigma^k\right) - \tilde{V}\right] v dV \tag{3.1.15}$$

where we denote

$$\delta\phi^* = \phi^* \left(\sigma^k + \eta\delta\sigma^k\right) - \phi^* \left(\sigma^k\right).$$

Subtracting (3.1.14) from (3.1.15), we will obtain

$$\int_{\Omega} \sigma^k \nabla \delta\phi^* \cdot \nabla v dV =: A_1(v) + A_2(v) \tag{3.1.16}$$

where

$$A_1(v) = \int_{\Omega} \eta\delta\sigma^k \nabla \left(\phi^* \left(\sigma^k + \eta\delta\sigma^k\right)\right) \cdot \nabla v dV$$

$$A_2(v) = \int_{\partial\Omega} \delta\phi v dV.$$

Let

$$a(u, v) = \int_{\Omega} \sigma^k \nabla u \cdot \nabla v dV$$

$$b_2(v) = A_1(v) + A_2(v).$$

Similar to our proof of the first inequality, we now only need to show that $b_2(v)$ is bounded.

By using the Cauchy-Schwarz inequality and (3.1.6), we will obtain

$$\begin{aligned}
|A_1(v)| &\leq \int_{\Omega} \left|\eta\delta\sigma^k \nabla \left(\phi^* \left(\sigma^k + \eta\delta\sigma^k\right)\right) \cdot \nabla v\right| dV \\
&\leq \eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^* \left(\sigma^k + \eta\delta\sigma^k\right)\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\leq \eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^* \left(\sigma^k + \eta\delta\sigma^k\right)\right\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\
&\leq \eta\hat{C} \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \left(\|f\|_{\tilde{L}^2(\partial\Omega)} + \left\|\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right) \|v\|_{H^1(\Omega)}
\end{aligned}$$

for some $\hat{C} > 0$. Furthermore, using the Cauchy-Schwarz inequality, the Trace theorem and (3.1.13), we will have

$$\begin{aligned}
|A_2(v)| &\leq \int_{\partial\Omega} |\delta\phi v| dV \\
&\leq \|\delta\phi\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\
&\leq \tilde{C} \|\delta\phi\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\
&\leq \tilde{C}\eta \left\|\delta\sigma^k\right\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

for some $\tilde{C} > 0$. The bounds for $A_1(v)$ and $A_2(v)$ will imply that $b_2(v)$ is bounded in $H^1(\Omega)$.

Thus, by using the substitution $u = v = \delta\phi^*$, the coercivity of $a(u,v)$ and the bounds for $b_2(v)$, we obtain

$$\|\delta\phi^*\|_{H^1(\Omega)} \leq \eta\frac{1}{\bar{C}}\left(\hat{C}\left[\|f\|_{\tilde{L}^2(\partial\Omega)} + \left\|\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right] + \tilde{C}\right)\left\|\delta\sigma^k\right\|_{L^\infty(\Omega)}.$$

This completes our proof. □

Recall that

$$\sigma^k = \sum_{m=1}^{M} \sigma_m^k \chi_m.$$

Therefore, ϕ and ϕ^* depend implicitly on χ_m for an arbitrary $m \in \{1, 2, \ldots, M\}$. Hence, the following notation is necessary. Here, χ_m is a real-valued function whose value is between 0 and 1. This means that σ^k is a function of χ_m. We will denote

$$\sigma^k(\chi_m) := \sum_{m=1}^{M} \sigma_m^k \chi_m,$$

where $\chi_m \in L^2(\Omega)$ such that $\chi_m(x) \in [0,1]$ for all $x \in \Omega$ for any $m \in \{1, 2, \ldots, M\}$. The forward and the adjoint solutions ϕ and ϕ^* are dependent on χ_m. Hence, for brevity we denote

$$\phi(\chi_m) := \phi\left(\sigma^k(\chi_m)\right) \quad \text{and} \quad \phi^*(\chi_m) := \phi^*\left(\sigma^k(\chi_m)\right). \tag{3.1.17}$$

We now show that ϕ and ϕ^* depend continuously on χ_m.

Theorem 11. *$\exists \bar{C}_1, \bar{C}_2 > 0$ such that*

$$\|\phi(\chi_m + \eta\delta\chi_m) - \phi(\chi_m)\|_{H^1(\Omega)} \leq \bar{C}_1\eta\|\delta\chi_m\|_{L^\infty(\Omega)} \tag{3.1.18}$$

and

$$\|\phi^*(\chi_m + \eta\delta\chi_m) - \phi^*(\chi_m)\|_{H^1(\Omega)} \leq \bar{C}_2\eta\|\delta\chi_m\|_{L^\infty(\Omega)} \tag{3.1.19}$$

for any $\eta \in (0, \tau)$, where τ is sufficiently small and chosen according to Remark (9).

Proof. We will use the decomposition of $\sigma^k(\chi_m)$:

$$\sigma^k(\chi_m) = \sum_{i=1}^{M} \sigma_i^k \chi_i. \tag{3.1.20}$$

Let $m \in \{1, 2, \ldots, M\}$ and $\eta \in (0, \tau)$. Therefore, if we use $\chi_m + \eta\delta\chi_m$ instead of χ_m, we have

$$\sigma^k(\chi_m) + \delta\sigma^k = \sum_{i \neq m}^{M} \sigma_i^k \chi_i + \sigma_m^k(\chi_m + \eta\delta\chi_m) \tag{3.1.21}$$

where $\delta\sigma^k$ is the associated change in $\sigma^k(\chi_m)$ given a $\eta\delta\chi_m$ perturbation of χ_m. Subtracting (3.1.20) from (3.1.21), we get

$$\delta\sigma^k = \eta\sigma_m^k\delta\chi_m. \tag{3.1.22}$$

This equation, together with (3.1.8), gives us

$$\begin{aligned}\|\phi(\chi_m+\eta\delta\chi_m)-\phi(\chi_m)\|_{H^1(\Omega)} &\le C_1\eta\left\|\delta\sigma^k\right\|_{L^\infty(\Omega)}\\ &\le C_1\eta\left\|\sigma_m^k\right\|_{L^\infty(\Omega)}\|\delta\chi_m\|_{L^\infty(\Omega)}\end{aligned}$$

which proves the first inequality. The second inequality follows similarly. Finally, using (3.1.22) and (3.1.19), we obtain

$$\begin{aligned}\|\phi^*(\chi_m+\eta\delta\chi_m)-\phi^*(\chi_m)\|_{H^1(\Omega)} &\le C_2\eta\left\|\delta\sigma^k\right\|_{L^\infty(\Omega)}\\ &\le C_2\eta\left\|\sigma_m^k\right\|_{L^\infty(\Omega)}\|\delta\chi_m\|_{L^\infty(\Omega)}.\end{aligned}$$

□

3.2 Smooth Approximation of χ_m

Recall from (2.2.20) that σ_m^{k+1} is obtained by solving the variational formulation

$$\int_\Omega \alpha(\chi_m+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla v dV+\int_\Omega\theta\left(\sigma_m^{k+1}-\sigma_m^k\right)v dV=\int_\Omega\chi_m\nabla\phi\cdot\nabla\phi^* v dV. \quad (3.2.1)$$

The equation (3.2.1) can be interpreted as

$$a\left(\sigma_m^{k+1},v\right)=b(v)\quad\forall v\in H^1(\Omega) \quad (3.2.2)$$

where

$$a\left(\sigma_m^{k+1},v\right):=\int_\Omega\alpha(\chi_m+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla v dV+\int_\Omega\theta\sigma_m^{k+1}v dV$$

and

$$b(v)=\int_\Omega\theta\sigma_m^k v dV+\int_\Omega\chi_m\nabla\phi\cdot\nabla\phi^* v dV.$$

In order to solve (3.2.2), we can use the Lax-Milgram Theorem. One of the conditions of this theorem is that $b(v)$ must be bounded. In order to show that the second term $\int_\Omega\chi_m^k\nabla\phi\cdot\nabla\phi^* v dV$ of b is bounded, it is necessary that $\chi_m\nabla\phi\cdot\nabla\phi^*$ is in $L^2(\Omega)$. How can we prove this? Note that $\chi_m\in L^\infty(\Omega)$. So if we apply Hölder's inequality, we only need to show that $\nabla\phi\cdot\nabla\phi^*\in L^2(\Omega)$. In the previous section, we proved that both $\nabla\phi$ and $\nabla\phi^*$ are in $L^2(\Omega)$ because both ϕ and ϕ^* are in $H^1(\Omega)$. But that does not necessarily mean $\nabla\phi\cdot\nabla\phi^*\in L^2(\Omega)$. However, if either $\nabla\phi$ or $\nabla\phi^*$ is in $L^\infty(\Omega)$ then $\nabla\phi\cdot\nabla\phi^*\in L^2(\Omega)$. In [22], it is proved that $\|\nabla\phi\|_{L^\infty(\Omega')}$ is bounded by $\|\nabla\phi\|_{L^2(\Omega)}$ for some Ω' compactly embedded in Ω. This was proved under the assumption that $\sigma^k\in C^1(\bar\Omega)$. Recall that $\sigma^k=\sum_{m=1}^M\sigma_m^k\chi_m$. Clearly, σ^k is not necessarily in $C^1(\bar\Omega)$ because $\chi_1,\chi_2,\ldots,\chi_m$ are characteristic functions. In order to resolve this, we will introduce a mollification $\chi_m^\delta=\chi_m*\xi_\delta$ of χ_m, where we choose ξ_δ such that χ_m^δ will be infinitely differentiable and that as $\delta\to0$, $\chi_{m,\delta}\to\chi_m$ almost everywhere. Hence, if $\sigma_m^k\in C^\infty(\bar\Omega)$ for all $m\in\{1,2,\ldots,M\}$, then σ^k is not just in $C^1(\bar\Omega)$ but in $C^\infty(\bar\Omega)$ as well. This mollification is done purely for theoretical purposes and is not applied in practice. This might seem like a deviation from our proposed method but technically, we can choose δ to be extremely close to 0 so that χ_m^δ is a really close approximation

of χ_m. We will show that these smooth approximations of χ_m will be consistent with our proposed method. We will do this by showing that the mollification will affect the corresponding ϕ and ϕ^* to a very small extent as long as the distance between $\chi_{m,\delta}$ and χ_m is small enough. Thus, we begin by defining the appropriate function ξ_δ. We will use a mollification proposed in [32].

Definition 12. Define

$$\xi_\delta(x) := \frac{1}{4\pi\delta} e^{-\frac{|x|^2}{4\delta}} \tag{3.2.3}$$

and let

$$g_\delta(x) := (g * \xi_\delta)(x) = \int_{\mathbb{R}^2} \xi_\delta(x-y)\, g(y)\, dy$$

for any $g \in L^p(\Omega)$, $1 \le p < \infty$ and $g(x)$ is set to 0 if $x \notin \Omega$.

Remark 13. It is worth noting that

$$\int_{\mathbb{R}^2} \xi_\delta dV = 1. \tag{3.2.4}$$

Lemma 14. *g_δ is a real analytic function on Ω and $g_\delta \to g$ almost everywhere as $\delta \to 0$. Furthermore, suppose $0 \le g(x) \le 1$, $\forall x \in \Omega$. Then $0 \le g_\delta(x) \le 1$ [32].*

Proof. Note that $\xi_\delta(x-y)$ is the heat kernel in two dimensions. Therefore, ξ_δ and g_δ are functions that are real analytic on $\mathbb{R}^2$ and thus on Ω as well [12].
Let $Q(\delta, x) := g_\delta(x)$. Then Q is the solution of

$$\begin{cases} \dfrac{\partial}{\partial\delta} Q + \Delta Q = 0, & x \in \mathbb{R}^2, \delta > 0 \\ Q(0,x) = g(x), & x \in \mathbb{R}^2 \end{cases}$$

and thus,

$$\lim_{\delta \to 0} Q(\delta, x) = g(x)$$

almost everywhere (compare with [42]).
Let us now assume that $0 \le g(x) \le 1$ for any $x \in \Omega$. Because $\xi_\delta \ge 0$, then by definition of g_δ we have $g_\delta \ge 0$. Moreover, using (3.2.4), we obtain

$$\begin{aligned} |g_\delta(x)| &\le \int_\Omega |\xi_\delta(y)\, g(x-y)|\, dV \\ &\le \int_\Omega |\xi_\delta(y)|\, dV \\ &= 1 \end{aligned}$$

which completes the proof. □

Corollary 15. *Suppose $\Omega' \subset \Omega$ such that $\mu(\Omega') > 0$ and assume that $g_\delta \equiv \beta$ on Ω', for some $\beta \in \mathbb{R}$. Then $\xi_\delta \equiv \beta$ on Ω [32].*

Proof. This directly follows from the real analyticity of g_δ on Ω (compare with [48]). □

Lemma 16. *Let $1 \leq p < \infty$ and take $1 \leq r \leq \infty$ such that $\frac{1}{r} + 1 - \frac{1}{p} \in [0,1]$. Define the operator from $L^p(\Omega)$ to $L^r(\Omega)$ by*

$$T_\delta(g) := g * \xi_\delta. \tag{3.2.5}$$

Then T_δ is continuous and injective [32].

Proof. Let $1 \leq q \leq \infty$ and $1 \leq p < \infty$. From Lemma (14), ξ_δ is real analytic and hence an element of $L^q(\Omega)$. Take $1 \leq r \leq \infty$ such that $\frac{1}{r} + 1 - \frac{1}{p} = \frac{1}{q}$. Then we can use Young's inequality for convolutions which implies

$$\|T_\delta(g)\|_{L^r(\Omega)} \leq \|g\|_{L^p(\Omega)} \|\xi_\delta\|_{L^q(\Omega)}.$$

Therefore, T_δ is continuous. Injectivity follows from the fact that $\int_{\mathbb{R}^2} \xi_\delta dV = 1$ (see e.g. [44]). □

Lemma 17. *For any $g \in L^p(\Omega)$, $1 \leq p < \infty$*

$$\partial^\nu (g * \xi_\delta) = (\partial^\nu g) * \xi_\delta$$

for $|\nu| \leq 1$. Moreover, $\partial^\nu (g * \xi_\delta) \to \partial^\nu g$ almost everywhere as $\delta \to 0$.

Proof. The proof is rather straightforward,

$$\begin{aligned}
\partial^\nu (g * \xi_\delta)(x) &= \partial^\nu \int_{\mathbb{R}^2} \xi_\delta(x-y) g(y) dy \\
&= \int_{\mathbb{R}^2} \partial_x^\nu \xi_\delta(x-y) g(y) dy \\
&= (-1)^{|\nu|} \int_{\mathbb{R}^2} \partial_y^\nu \xi_\delta(x-y) g(y) dy \\
&= \int_{\mathbb{R}^2} \xi_\delta(x-y) \partial^\nu g(y) dy \\
&= (\partial^\nu g) * \xi_\delta.
\end{aligned}$$

The second assertion follows from Lemma (14) (compare with [30]). □

From here onwards, we let

$$\chi_m^\delta := \chi_m * \xi_\delta$$

to be the mollification of χ_m.

Theorem 18. *For a fixed δ, the solution of the forward problem given χ_m^δ and the solution of the forward problem given χ_m satisfy*

$$\left\|\phi\left(\chi_m^\delta\right) - \phi(\chi_m)\right\|_{H^1(\Omega)} \leq C_1 \left\|\chi_m^\delta - \chi_m\right\|_{L^\infty(\Omega)} \tag{3.2.6}$$

for some $C_1 > 0$. The same applies to the adjoint problem

$$\left\|\phi^*\left(\chi_m^\delta\right) - \phi^*(\chi_m)\right\|_{H^1(\Omega)} \leq C_2 \left\|\chi_m^\delta - \chi_m\right\|_{L^\infty(\Omega)} \tag{3.2.7}$$

for some $C_2 > 0$.

Proof. We proved in (3.1.18) that ϕ depends continuously on χ_m. Note that we proved this given the assumption that $\chi_m \in L^2(\Omega)$ and $0 \leq \chi_m \leq 1$. By Lemma (14), we have shown that $\chi_m^\delta \in L^2(\Omega)$ and $0 \leq \chi_m^\delta \leq 1$. Hence, we get

$$\left\|\phi\left(\chi_m^\delta\right) - \phi(\chi_m)\right\|_{H^1(\Omega)} \leq C_1 \left\|\chi_m^\delta - \chi_m\right\|_{L^\infty(\Omega)}$$

for some $C_1 > 0$. Our second claim is proved similarly. We make use of (3.1.19) to infer

$$\left\|\phi^*\left(\chi_m^\delta\right) - \phi^*(\chi_m)\right\|_{H^1(\Omega)} \leq C_2 \left\|\chi_m^\delta - \chi_m\right\|_{L^\infty(\Omega)}$$

for some $C_2 > 0$. □

We know that χ_m^δ converges to χ_m pointwise. Although it does not guarantee us that $\left\|\chi_m^\delta - \chi_m\right\|_{L^\infty(\Omega)}$ converges to 0, this can still be a gauge to measure the distance between the solution of the forward problem using χ_m and the solution using χ_m^δ. In the previous chapter, we introduced the function G which is defined in (2.3.11). The function Θ arising from the TV regularization was then introduced. A thresholding on Θ gives the update for χ_m. Observe that G and Θ are expressed in terms of the gradient of the functional J. In the next section, we will show that the implicit formulation of the gradient of the functional J can be calculated if we use the mollification χ_m^δ. Thus, we will use $(\Theta \circ G)\left(\chi_m^\delta\right)$ in the thresholding step instead of $(\Theta \circ G)(\chi_m)$. To make our modifications consistent, we will now try to find a new thresholding approach to adapt with the mollification of χ_m. But we will first define a space that will be important in our succeeding computations. Let $\mathcal{B}(\Omega)$ be the Borel $\sigma-$algebra over Ω and let $\mu(\cdot)$ denote the Lebesgue measure. For any $A_1, A_2 \in \mathcal{B}(\Omega)$, we define the *symmetric difference* of A_1 and A_2 to be

$$A_1 \triangle A_2 = (A_1 \backslash A_2) \cup (A_2 \backslash A_1).$$

Definition 19. Define the distance $d : \mathcal{B}(\Omega) \times \mathcal{B}(\Omega) \to \mathbb{R} \cup \{\infty\}$ to be $d(A_1, A_2) = \mu(A_1 \triangle A_2)$. We now define $\mathcal{M}(\Omega) = (\mathcal{B}(\Omega), d)/\ker(d)$.

The space $\mathcal{M}(\Omega)$ is, in fact, a metric space [39, 32]. In Algorithm (1), we did a thresholding on $(\Theta \circ G)$ in order to get an update for χ_m. In the following definition, we will modify this thresholding to make it coherent with the mollification of χ_m.

Definition 20. Let $z \in H^1(\Omega) \backslash L^2(\Omega)$. Define $H : L^2(\Omega) \to \mathcal{M}(\Omega)$ by

$$H(g) = \left\{x \in \Omega : \left(\left(g^\delta - \zeta + \delta z\right) * \xi_\delta\right)(x) \geq 0\right\} \tag{3.2.8}$$

for some $\zeta \in (0, 1)$ and

$$g^\delta = g * \xi_\delta.$$

Moreover, define $M : \mathcal{M}(\Omega) \to L^2(\Omega)$ to be the map that assigns $\omega \in \mathcal{M}(\Omega)$ to its characteristic function χ_m, that is,

$$M(\omega) = \chi_m. \tag{3.2.9}$$

Observe that if we let $\delta \to 0$ then

$$(g_\delta - \zeta + \delta z) * \xi_\delta \to g - \zeta \tag{3.2.10}$$

by Lemma (14). This means that as $\delta \to 0$, $\mathcal{H}(g)$ becomes the set of $x \in \Omega$ for which $g(x) \geq \zeta$. Thus, $(M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right)$ becomes equivalent to the thresholding step introduced in Algorithm (1) as δ becomes smaller.

We can now modify Algorithm (1) so we can incorporate the mollification of χ_m.

ALGORITHM 2.

1. *Initialization.* Given f and $\tilde{V}$. Choose parameters $\alpha, \epsilon, \theta, \omega, \delta, \beta, \gamma$ and ζ. Select the appropriate value of M, the maximum number of iterations K and the tolerance ρ. Set $k = 1$ and choose the initial guess σ_m^k such that $\sigma_m^k \in C^\infty\left(\bar{\Omega}\right)$. Select an initial guess $\bar{\chi}_m^k$, $m \in \{1, 2, \ldots, M-1\}$ such that $\operatorname{supp}(\bar{\chi}_{m_1}) \cap \operatorname{supp}(\bar{\chi}_{m_1}) = \emptyset$ for any $m_1, m_2 \in \{1, 2, \ldots, M-1\}$ with $m_1 \neq m_2$. For $m = M$, σ_M^k is known and $\bar{\chi}_M^k = 1 - \sum_{m=1}^{M-1} \bar{\chi}_m^k$. For the case when $M = 2$, the initialization is the same as the one discussed in the last section of the previous chapter. Set $\chi_m^k = \bar{\chi}_m^k * \xi_\delta$.

2. *Solving the forward and the adjoint problem.* Take $\sigma^k = \sum_{m=1}^M \sigma_m^k \chi_m^k$ and solve for ϕ and ϕ^* through
$$\int_\Omega \sigma^k \nabla\phi \cdot \nabla v dV = \int_{\partial\Omega} f v dS$$
and
$$\int_\Omega \sigma^k \nabla\phi^* \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi - \tilde{V}\right) v dS.$$

3. *Solving for* σ_m^{k+1}. Given χ_m^k for $m = 1, 2, \ldots, M-1$, σ_m^{k+1} is solved via
$$\int_\Omega \alpha\left(\chi_m + \epsilon\right) \nabla\sigma_m^{k+1} \cdot \nabla v dV + \int_\Omega \theta\sigma_m^{k+1} v dV = \int_\Omega \chi_m \nabla\phi \cdot \nabla\phi^* v dV + \int_\Omega \theta\sigma_m^k v dV.$$
For $m = M$, set $\sigma_M^{k+1} = \sigma_M^k$.

4. *Updating* χ_m. Given χ_m^k for $m = 1, 2, \ldots, M-1$, the update for χ_m is given by
$$\chi_m^{k+1} = \left(T_\delta \circ M \circ H \circ \Theta \circ G \circ T_\delta\right)\left(\chi_m^k\right).$$
For $m = M$, set $\chi_M^{k+1} = 1 - \sum_{m=1}^{M-1} \chi_m^{k+1}$.

5. *Stopping Criteria.* If $k = K$ or $\sum_{m=1}^M \left\|\chi_m^{k+1} - \chi_m^k\right\|_{L^2(\Omega)} < \rho$, the algorithm terminates. Otherwise, go back to step 2.

Remark 21. It is again emphasized that the introduction of $z \in L^2(\Omega) \backslash H^1(\Omega)$ in (3.2.8) and the modification of Algorithm (1) as a result of the mollification of χ_m are purely technical devices and are used only for theoretical purposes. This might seem like a deviation from Algorithm (1) but as shown in (3.2.6), (3.2.7) and (3.2.10), these changes are justified.

3.3 The Gradient of the Functional J

As the title suggests, this section is devoted in finding an explicit formulation for the gradient of $J\left(\chi_m^\delta\right)$. As stated in the previous chapter, this gradient is necessary to calculate $G\left(\chi_m^\delta\right)$, which is essential in finding the update for χ_m. Replacing χ_m with χ_m^δ in (3.2.1) implies

$$\int_\Omega \alpha\left(\chi_m^\delta+\epsilon\right)\nabla\sigma_m^{k+1}\cdot\nabla v dV+\int_\Omega \theta\left(\sigma_m^{k+1}-\sigma_m^k\right)v dV=\int_\Omega \chi_m^\delta\nabla\phi\cdot\nabla\phi^* v dV. \quad (3.3.1)$$

Now, let

$$v_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)=-2\int_\Omega \frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\chi_m^\delta\nabla\phi\left(\chi_m^\delta;\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta;\delta\chi_m^\delta\right)dV$$

$$v_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)=-2\int_\Omega \sigma_m^{k+1}\left(\chi_m^\delta\right)\delta\chi_m^\delta\nabla\phi\left(\chi_m^\delta;\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta;\delta\chi_m^\delta\right)dV$$

$$\begin{aligned}v_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\alpha\int_\Omega 2(\chi_m^\delta+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla\left(\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right)\\ +2\theta\left(\sigma_m^{k+1}-\sigma_m^k\right)\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m\right)dV\end{aligned}$$

$$v_4\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\alpha\int_\Omega |\nabla\sigma_m^{k+1}|^2\delta\chi_m^\delta dV.$$

From (2.3.8) and (2.3.9), we have

$$\frac{\delta J}{\delta\chi_m}\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\frac{\delta J_1}{\delta\chi_m}\left(\chi_m^\delta;\delta\chi_m^\delta\right)+\frac{\delta J_1}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\sum_{i=1}^4 v_i\left(\chi_m^\delta;\delta\chi_m^\delta\right).$$

Substituting $v=\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ in (3.3.1), we get

$$\begin{aligned}v_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)+v_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)&=-2\int_\Omega \frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\chi_m^\delta\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)dV\\ &+2\alpha\int_\Omega(\chi_m^\delta+\epsilon)\nabla\sigma_m^{k+1}\cdot\nabla\left(\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right)dV\\ &+2\theta\int_\Omega\left(\sigma_m^{k+1}-\sigma_m^k\right)\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)dV\\ &=0.\end{aligned}$$

Therefore,

$$\frac{\delta J}{\delta\chi_m}\left(\chi_m^\delta;\delta\chi_m^\delta\right)=v_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)+v_4\left(\chi_m^\delta;\delta\chi_m^\delta\right). \quad (3.3.2)$$

This simplifies our calculation. Notice that the substitution $v=\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ in (3.3.1) can only be done if $v\in H^1(\Omega)$. Therefore, if we wish to use (3.3.2) to compute for the gradient of $J\left(\chi_m^\delta\right)$, we first need to show that $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\in H^1(\Omega)$.

Before we start our calculations to prove this, we will first make few assumptions. These will be used throughout our analysis.

Assumption 1. Let $\alpha, \underline{\sigma}, \theta > 0$, $\epsilon \in (0,1)$ and $m \in \{1, 2, \ldots, M-1\}$. We assume that $\sigma_m^k \in C^\infty(\bar{\Omega})$ such that $\sigma^k(\chi_m^\delta) = \sum \sigma_m^k \chi_m^\delta \geq \underline{\sigma} > 0$. We also assume that $\partial\Omega$ is sufficiently smooth.

The assumption that $\sigma_m^k \in C^\infty(\bar{\Omega})$ might seem like a very bold assumption but if we can show that $\sigma_m^{k+1} \in C^\infty(\bar{\Omega})$ as well, then this assumption makes sense. Moreover, as stated in Algorithm (2), the initial guess for σ_m can be chosen to be constant so that it is in $C^\infty(\bar{\Omega})$.
We now investigate what happens to ϕ and ϕ^* if we use the above assumption.

Lemma 22. *Given Assumption (1) and let $\phi \in H^1(\Omega)$, such that $\int_{\partial\Omega} \phi dS = 0$, be the solution of*

$$\nabla \cdot \left(\sigma^k \nabla \phi\right) = 0 \quad \text{on } \Omega$$

subject to

$$\sigma^k \frac{\partial \phi}{\partial n} = f \quad \text{on } \partial\Omega$$

where $\sigma^k = \sum_{m=1}^M \sigma_m^k \chi_m^\delta$ for any $f \in \tilde{L}(\partial\Omega)$. Then

$$\left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \leq C \left\|\phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \tag{3.3.3}$$

for some $C > 0$ (compare, [22]). In fact, $\phi(\chi_m^\delta) \in C^\infty(\bar{\Omega})$.

Proof. By Assumption (1) and Lemma (14), $\sigma^k \in C^\infty(\bar{\Omega})$. Let $l \geq 1$. Obviously, $\sigma^k \in C^l(\bar{\Omega})$. Therefore, using standard regularity estimates (see, e.g., [34]), we get

$$\left\|\phi\left(\chi_m^\delta\right)\right\|_{H^{l+2}(\Omega)} \leq C_1 \left\|\phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \tag{3.3.4}$$

for some $C_1 > 0$. Furthermore, by the Sobolev-Imbedding theorem (see Appendix), we have

$$\left\|\phi\left(\chi_m^\delta\right)\right\|_{C^{l,\gamma}(\bar{\Omega})} \leq C_2 \left\|\phi\left(\chi_m^\delta\right)\right\|_{H^{l+2}(\Omega)}. \tag{3.3.5}$$

By the definition of $\|\cdot\|_{C^{l,\gamma}(\bar{\Omega})}$, the embedding

$$C^{1,\gamma}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega}) \tag{3.3.6}$$

is continuous (see, e.g., [13]). If we compare (3.3.4), (3.3.5) and (3.3.6) we can deduce that $\exists C > 0$ such that

$$\left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \leq C \left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}, \tag{3.3.7}$$

which completes the first part of the proof. Moreover, because $l \geq 1$, it follows that $\phi(\chi_m^\delta) \in C^\infty(\bar{\Omega})$. □

Lemma 23. *Given Assumption (1), $\exists C > 0$ such that*

$$\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \leq C \left\|\phi^*\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}. \tag{3.3.8}$$

Furthermore, $\phi^*(\chi_m^\delta) \in C^\infty(\bar{\Omega})$.

Proof. This follows directly from the previous theorem by replacing f with $\phi - \tilde{V}$. □

We will now analyze the dependence of ϕ and ϕ^* on χ_m^δ. From here onwards, we denote

$$\delta\chi_m^\delta := \delta\chi_m * \xi_\delta$$

so that

$$(\chi_m + \eta\delta\chi_m) * \xi_\delta = \chi_m^\delta + \eta\delta\chi_m^\delta.$$

Suppose we replace χ_m^δ with $\chi_m^\delta + \eta\delta\chi_m^\delta$. Again by Remark (9), η should be taken from the set $(0, \tau)$ where τ is chosen to be sufficiently smooth so that $\sigma^k + \eta\delta\sigma^k \geq \underline{\sigma}^\tau > 0$. Therefore, from (3.1.6), (3.1.7), (3.1.17), (3.3.3) and (3.3.8), the following estimates hold:

$$\left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \leq C_1 \|f\|_{\tilde{L}^2(\partial\Omega)} \tag{3.3.9}$$

$$\left\|\nabla\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \leq C_2 \left(\|f\|_{\tilde{L}^2(\partial\Omega)} + \left\|\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right) \tag{3.3.10}$$

for some $C_1, C_2 > 0$ and for all $\eta \in (0, \tau)$.

In Theorem (11), we have shown that ϕ and ϕ^* continuously depend on χ_m. Observe that this theorem holds with any χ_m whose value is between 0 and 1. Therefore, this also holds when we use χ_m^δ instead because $0 \leq \chi_m^\delta \leq 1$ as proven in Lemma (14). We state this in the following lemma.

Lemma 24. *Given Assumption (1), $\exists C_1, C_2 > 0$ such that*

$$\left\|\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq C_1\eta \|\delta\chi_m\|_{L^2(\Omega)} \tag{3.3.11}$$

and

$$\left\|\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m\right)\right\|_{H^1(\Omega)} \leq C_2\eta \|\delta\chi_m\|_{L^2(\Omega)} \tag{3.3.12}$$

for any $\eta \in (0, \tau)$, where τ is sufficiently small and chosen according to Remark (9).

Proof. From (3.1.18) , we have

$$\left\|\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq \bar{C}_1\eta \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)}. \tag{3.3.13}$$

Using Young's inequality for convolution, we get

$$\left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} = \|\delta\chi_m * \xi_\delta\|_{L^\infty(\Omega)} \leq \|\delta\chi_m\|_{L^2(\Omega)} \|\xi_\delta\|_{L^2(\Omega)}. \tag{3.3.14}$$

If we compare (3.3.13) and (3.3.14), the first inequality follows. Similarly, from (3.1.19), we obtain

$$\left\|\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq \bar{C}_2\eta \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)}.$$

This, together with (3.3.14), gives the second inequality and thus completes our proof. □

Define

$$\psi\left(\chi_m^\delta\right) := \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right). \tag{3.3.15}$$

Observe that the right-hand side of (3.3.1) includes $\psi\left(\chi_m^\delta\right)$. In the next lemma, we show that ψ depends continuously on χ_m. This will be necessary when we analyze the solution of (3.3.1). Note that $\nabla\phi^*\left(\chi_m^\delta\right)\in L^2(\Omega)$ by Corollary (7) and $\nabla\phi\left(\chi_m^\delta\right)\in L^\infty(\Omega)$ by (3.3.3). Therefore, by Hölder's inequality,

$$\psi\left(\chi_m^\delta\right)\in L^2(\Omega). \tag{3.3.16}$$

This means that ψ is a map from $\chi_m\in L^2(\Omega)$ to $\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)\in L^2(\Omega)$. In the following lemma, we prove that this mapping is indeed continuous.

Lemma 25. *Given Assumption (1), $\exists C\geq 0$ such that*

$$\left\|\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\leq C\eta\left\|\delta\chi_m\right\|_{L^2(\Omega)} \tag{3.3.17}$$

for any $\eta\in(0,\tau)$, where τ is sufficiently small and chosen according to Remark (9).

Proof. Adding and subtracting $\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)$ to $\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)$, we get

$$\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)=: A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)+A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)$$

where

$$A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)$$

and

$$A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right).$$

Thus,

$$\begin{aligned}\left\|\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}&\leq\left\|A_1\left(\chi_m;\delta\chi_m\right)\right\|_{L^2(\Omega)}\\&\quad+\left\|A_2\left(\chi_m;\delta\chi_m\right)\right\|_{L^2(\Omega)}.\end{aligned} \tag{3.3.18}$$

We can estimate $A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ using the Hölder's inequality, (3.3.9) and (3.3.12):

$$\begin{aligned}\left\|A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}&=\left\|\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\left[\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)\right]\right\|_{L^2(\Omega)}\\&\leq\left\|\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\&\leq\left\|\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\delta\chi_m\right\|_{L^2(\Omega)}\\&\leq C_1\eta\left\|f\right\|_{\tilde{L}^2(\partial\Omega)}\left\|\delta\chi_m\right\|_{L^2(\Omega)}\end{aligned}$$

for some $C_1>0$. Similarly, using the Hölder's inequality, (3.3.8) and (3.3.11), $A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ can be estimated as follows:

$$\begin{aligned}\left\|A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}&=\left\|\left[\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)\right]\cdot\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\&\leq\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\&\leq C_2\eta\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\delta\chi_m\right\|_{L^2(\Omega)}\end{aligned}$$

for some $C_2 > 0$. The estimates for $A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ and $A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)$, together with (3.3.18), complete the proof. □

Remember that the goal of this section is to calculate an explicit formulation of the gradient of $J\left(\chi_m\right)$. We stated at the beginning of this section that we need to prove that $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right) \in H^1\left(\Omega\right)$. In order to accomplish this, we first need to show that both the derivatives of ϕ and ϕ^* converge in $H^1\left(\Omega\right)$. We start by looking for candidates for the derivatives.

Lemma 26. *Given Assumption (1), then* $\exists D_\phi\left(\chi_m;\delta\chi_m\right) \in H^1\left(\Omega\right)$ *satisfying*

$$\int_\Omega \sigma^k \nabla D_\phi\left(\chi_m;\delta\chi_m\right)\cdot\nabla v dV = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla v dV \tag{3.3.19}$$

with $\int_{\partial\Omega} D_\phi\left(\chi_m;\delta\chi_m\right) dS = 0$ *for all* $v \in H^1\left(\Omega\right)$ *such that* $\int_{\partial\Omega} v dS = 0$.

Proof. Let $u, v \in H^1\left(\Omega\right)$ such that $\int_{\partial\Omega} u dS = \int_{\partial\Omega} v dS = 0$. Define

$$a\left(u,v\right) = \int_\Omega \sigma^k \nabla u\cdot\nabla v dV$$

$$b(v) = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla v dV.$$

From the proof of Theorem (6), a is bilinear, coercive and bounded. Obviously, b is linear. We wish to employ the Lax-Milgram theorem so it is sufficient to show that b is bounded. Indeed, using the Cauchy-Schwarz inequality and Hölder's inequality we get

$$\begin{aligned}
\left|b\left(v\right)\right| &\leq \int_\Omega \left|\sigma_m^k \delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla v\right| dV \\
&\leq \left\|\sigma_m^k \delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \left\|\nabla v\right\|_{L^2(\Omega)} \\
&\leq \left\|\sigma_m^k \delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \left\|v\right\|_{L^2(\Omega)}.
\end{aligned}$$

The right-hand side of the last inequality is bounded because of Corollary (7) and the fact that $\sigma_m^k \delta\chi_m^\delta \in C^\infty\left(\bar{\Omega}\right)$. □

Lemma 27. *Given Assumption (1), then* $\exists D_{\phi^*}\left(\chi_m;\delta\chi_m\right) \in H^1\left(\Omega\right)$ *satisfying*

$$\int_\Omega \sigma^k \nabla D_{\phi^*}\left(\chi_m;\delta\chi_m\right)\cdot\nabla v dV = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi^*\left(\chi_m^\delta\right)\cdot\nabla v dV + \int_{\partial\Omega} D_\phi\left(\chi_m;\delta\chi_m\right) v dV. \tag{3.3.20}$$

with $\int_{\partial\Omega} D_{\phi^*}\left(\chi_m;\delta\chi_m\right) dS = 0$ *for all* $v \in H^1\left(\Omega\right)$ *such that* $\int_{\partial\Omega} v dS = 0$.

Proof. The proof is similar to the proof of the previous lemma. Let $u, v \in H^1\left(\Omega\right)$ such that $\int_{\partial\Omega} u dS = \int_{\partial\Omega} v dS = 0$. Define

$$a\left(u,v\right) = \int_\Omega \sigma^k \nabla u\cdot\nabla v dV$$

$$b(v) = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi^*\left(\chi_m^\delta\right)\cdot\nabla v dV + \int_{\partial\Omega} D_\phi\left(\chi_m;\delta\chi_m\right) v dV.$$

From the proof of Theorem (6), a is bilinear, coercive and bounded. Obviously, b is linear. Again, we wish to use the Lax-Milgram theorem so it is sufficient to show that b is bounded. Using the triangle inequality, the Cauchy-Schwarz inequality, Hölder's inequality and the trace theorem, we have

$$\begin{aligned}|b(v)| &\leq \int_\Omega \left|\sigma_m^k \delta\chi_m^\delta \nabla\phi^*\left(\chi_m^\delta\right)\cdot\nabla v\right| dV + \int_{\partial\Omega} \left|D_\phi\left(\chi_m;\delta\chi_m\right)v\right| dV \\ &\leq \left\|\sigma_m\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \left\|D_\phi\left(\chi_m;\delta\chi_m\right)\right\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\ &\leq \left\|\sigma_m\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + C\left\|D_\phi\left(\chi_m;\delta\chi_m\right)\right\|_{H^1(\partial\Omega)} \|v\|_{H^1(\partial\Omega)}\end{aligned}$$

for some $C > 0$. The right-hand side of the last inequality is bounded because of Corollary (7), $\sigma_m^k\delta\chi_m^\delta \in C^\infty\left(\bar{\Omega}\right)$ and the previous lemma. □

We will now prove that, in fact, $D_\phi\left(\chi_m;\delta\chi_m\right)$ and $D_{\phi^*}\left(\chi_m;\delta\chi_m\right)$ in the last two lemmas are the derivatives of ϕ and ϕ^* at χ_m in the direction of $\delta\chi_m$, respectively. We state and prove this in the following lemma.

Lemma 28. *Given Assumption (1), then*

$$\lim_{\eta\to 0}\left\|\frac{\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} = 0 \tag{3.3.21}$$

and

$$\lim_{\eta\to 0}\left\|\frac{\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\phi^*\left(\chi_m^\delta\right)}{\eta} - D_{\phi^*}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} = 0. \tag{3.3.22}$$

Thus, we can make the identifications $\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right) = D_\phi\left(\chi_m;\delta\chi_m\right)$ *and* $\frac{\delta\phi^*}{\delta\chi_m}\left(\chi_m^\delta;\delta\chi_m^\delta\right) = D_{\phi^*}\left(\chi_m;\delta\chi_m\right)$. *Furthermore, because* $D_\phi\left(\chi_m;\delta\chi_m\right), D_{\phi^*}\left(\chi_m;\delta\chi_m\right) \in H^1\left(\Omega\right)$ *we have*

$$\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right),\ \frac{\delta\phi^*}{\delta\chi_m}\left(\chi_m^\delta;\delta\chi_m^\delta\right) \in H^1\left(\Omega\right)$$

with $\int_{\partial\Omega}\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)dS = \int_{\partial\Omega}\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)dS = 0$.

Proof. From (3.1.1), we have

$$\int_\Omega \sigma^k\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla v dV = \int_\Omega f v dV$$

where

$$\sigma^k = \sum_{i=1}^M \sigma_i^k\chi_i^\delta.$$

Then for $m \in \{1,2,\ldots,M\}$, we will get

$$\int_\Omega\left(\sigma_m^k\chi_m^\delta + \sum_{i\neq m}\sigma_i^k\chi_i^\delta\right)\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla v dV = \int_\Omega f v dV. \tag{3.3.23}$$

Similarly, for $\chi_m^\delta + \eta\delta\chi_m^\delta$,

$$\int_\Omega \left(\sigma_m^k \left[\chi_m^\delta + \eta\delta\chi_m^\delta \right] + \sum_{i \neq m} \sigma_i^k \chi_i^\delta \right) \nabla\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) \cdot \nabla v dV = \int_\Omega f v dV. \tag{3.3.24}$$

Subtracting (3.3.23) from (3.3.24), we will obtain

$$\begin{aligned} &\int_\Omega \sigma^k \nabla \left(\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right) \right) \cdot \nabla v dV \\ &= \int_\Omega \eta \sigma_m^k \delta\chi_m^\delta \nabla\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) \cdot \nabla v dV. \end{aligned} \tag{3.3.25}$$

Dividing (3.3.25) by η, we will have

$$\begin{aligned} &\int_\Omega \sigma^k \nabla \left(\frac{\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right)}{\eta} \right) \cdot \nabla v dV \\ &= \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) \cdot \nabla v dV. \end{aligned} \tag{3.3.26}$$

Recall from (3.3.19) that

$$\int_\Omega \sigma^k \nabla D_\phi \left(\chi_m^\delta ; \delta\chi_m^\delta \right) \cdot \nabla v dV = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla\phi \left(\chi_m^\delta \right) \cdot \nabla v dV. \tag{3.3.27}$$

Subtracting (3.3.26) and (3.3.27), we will get

$$\int_\Omega \sigma^k \nabla \left(\frac{\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right)}{\eta} - D_\phi \left(\chi_m^\delta ; \delta\chi_m^\delta \right) \right) \cdot \nabla v dV =: A(v) \tag{3.3.28}$$

where

$$A(v) = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla \left[\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right) \right] \cdot \nabla v dV$$

which can be estimated using the Cauchy-Schwarz inequality and (3.1.18):

$$\begin{aligned} |A(v)| &\leq \int_\Omega \left| \sigma_m^k \delta\chi_m^\delta \nabla \left[\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right) \right] \cdot \nabla v \right| dV \\ &\leq \left\| \sigma_m^k \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \left\| \nabla \left[\phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right) \right] \right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\leq \left\| \sigma_m^k \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \left\| \phi \left(\chi_m^\delta + \eta\delta\chi_m^\delta \right) - \phi \left(\chi_m^\delta \right) \right\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq C_1 \eta \left\| \sigma_m^k \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{3.3.29}$$

It is worth noting that $\sigma_m^k \in L^\infty(\Omega)$ by Assumption (1) and that $\delta\chi_m(x) \in [0,1]$ for all $x \in \Omega$ which means that $\sigma_m^k \delta\chi_m \in L^\infty(\Omega)$ as well. Hence, $\sigma_m^k \delta\chi_m \in L^\infty(\Omega)$ which makes the last inequality bounded. Now observe that

$$a(u,v) := \int_\Omega \sigma^k \nabla u \cdot \nabla v dV$$

is coercive as demonstrated in the proof of Theorem (6), or in other words,

$$|a(u,u)| \geq \bar{C} \|u\|_{H^1(\Omega)}^2 \tag{3.3.30}$$

for some $\bar{C} > 0$, for any $u \in H^1(\Omega)$ such that $\int_{\partial\Omega} u dS = 0$. Hence, the left hand side of (3.3.28) is bounded from above if we set

$$u = \frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right).$$

Using this fact and comparing (3.3.28) and (3.3.29), we will obtain the estimate

$$\bar{C}\left\|\frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq C_1\eta\left\|\sigma_m^k\delta\chi_m^\delta\right\|_{L^\infty(\Omega)}\|\delta\chi_m\|_{L^2(\Omega)}.$$

Therefore, taking the limit of the last equality as $\eta \to 0$, we get

$$\lim_{\eta\to 0}\left\|\frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} = 0. \tag{3.3.31}$$

Now we only need to show convergence of the derivative of ϕ^* with respect to χ_m in $H^1(\Omega)$. The proof will be similar to the proof of the first limit. From (3.1.2), we have

$$\int_\Omega \sigma^k\nabla\phi^*\left(\chi_m^\delta\right)\cdot\nabla v dV = \int_{\partial\Omega}\left(\phi\left(\chi_m^\delta\right) - \tilde{V}\right) v dS.$$

For $m \in \{1, 2, \ldots, M\}$, we will get

$$\int_\Omega\left(\sigma_m^k\chi_m^\delta + \sum_{i\neq m}\sigma_i^k\chi_i^\delta\right)\nabla\phi^*\left(\chi_m^\delta\right)\cdot\nabla v dV = \int_{\partial\Omega}\left(\phi\left(\chi_m^\delta\right) - \tilde{V}\right) v dV. \tag{3.3.32}$$

Similarly, for $\chi_m^\delta + \eta\delta\chi_m^\delta$, $\eta > 0$,

$$\begin{aligned}\int_\Omega\left(\sigma_m^k\left[\chi_m^\delta + \eta\delta\chi_m^\delta\right] + \sum_{i\neq m}\sigma_i^k\chi_i^\delta\right)\nabla\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\cdot\nabla v dV \\ = \int_{\partial\Omega}\left[\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \tilde{V}\right] v dV.\end{aligned} \tag{3.3.33}$$

Subtracting (3.3.32) from (3.3.33), we will obtain

$$\int_\Omega \sigma^k\nabla\left(\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right)\cdot\nabla v dV =: A_1(v) + A_2(v) \tag{3.3.34}$$

where

$$A_1(v) = \int_\Omega \eta\sigma_m\delta\chi_m\nabla\phi^*\left(\chi_{m,\delta} + \eta\delta\chi_m\right)\cdot\nabla v dV$$

$$A_2(v) = \int_{\partial\Omega}\left[\phi\left(\chi_{m,\delta} + \eta\delta\chi_m\right) - \phi\left(\chi_{m,\delta}\right)\right] v dV.$$

Dividing (3.3.34) by η, we have

$$\int_\Omega \sigma^k\nabla\left(\frac{\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta}\right)\cdot\nabla v dV = \frac{A_1(v) + A_2(v)}{\eta}. \tag{3.3.35}$$

Recall from (3.3.20) that

$$\int_\Omega \sigma^k \nabla D_{\phi^*}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \cdot \nabla v dV = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla \phi^*\left(\chi_m^\delta\right) \cdot \nabla v dV + \int_{\partial\Omega} D_\phi v dV. \quad (3.3.36)$$

Subtracting (3.3.35) and (3.3.36), we get

$$\int_\Omega \sigma^k \nabla \left(\frac{\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)}{\eta} - D_{\phi^*}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right) \cdot \nabla v dV \\ =: B_1(v) + B_2(v) \quad (3.3.37)$$

where

$$B_1(v) = \int_\Omega \sigma_m^k \delta\chi_m^\delta \nabla \left[\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right] \cdot \nabla v dV$$

and

$$B_2(v) = \int_{\partial\Omega} \left[\frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right] v dV.$$

We can estimate $B_1(v)$ using the Cauchy-Schwarz inequality and (3.1.19):

$$\begin{aligned} |B_1(v)| &\leq \int_\Omega \left| \sigma_m^k \delta\chi_m^\delta \nabla \left[\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right] \cdot \nabla v \right| dV \\ &\leq \left\| \sigma_m \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \left\| \nabla \left[\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right] \right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\leq \left\| \sigma_m \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \left\| \nabla \left[\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)\right] \right\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq C_2 \eta \left\| \sigma_m \delta\chi_m^\delta \right\|_{L^\infty(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned} \quad (3.3.38)$$

for some $C_2 > 0$. On the other hand, we can estimate $B_2(v)$ using the Cauchy-Schwarz inequality and the Trace theorem:

$$\begin{aligned} |B_2(v)| &\leq \int_{\partial\Omega} \left| \left[\frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right] v \right| dV \\ &\leq \left\| \frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\ &\leq C_3 \left\| \frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned} \quad (3.3.39)$$

for some $C_3 > 0$. Again, using the coercivity of a, the left hand side of (3.3.37) is bounded from above if we make the substitution

$$v = \frac{\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)}{\eta} - D_{\phi^*}\left(\chi_m^\delta; \delta\chi_m^\delta\right)$$

in (3.3.30). Using this fact and comparing (3.3.37), (3.3.38) and (3.3.39), we will obtain the estimate

$$\bar{C} \left\| \frac{\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)}{\eta} - D_{\phi^*}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)} \leq B_3(\eta) + B_4(\eta) \quad (3.3.40)$$

where $\bar{C}$ comes from the coercivity of a,

$$B_3(\eta) := C_2\eta \left\|\sigma_m \delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)}$$

and

$$B_4(\eta) := C_3 \left\| \frac{\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right)}{\eta} - D_\phi\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)}.$$

Taking the limit as $\eta \to 0$, B_3 will clearly go to 0. B_4 will go to 0 as well because of (3.3.31). Thus,

$$\lim_{\eta\to 0} \left\| \frac{\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi^*\left(\chi_m^\delta\right)}{\eta} - \frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)} \to 0.$$

The last statements of the lemma can be inferred directly from the last two lemmas. □

Now that we have shown that both the derivatives of ϕ and ϕ^* converge in $H^1(\Omega)$, the next step is to show that ψ in (3.3.15) has a derivative with respect to χ_m^δ which converges in $L^2(\Omega)$. We will first prove a technical lemma then prove this convergence.

Lemma 29. *Given Assumption (1), and let*

$$\delta\phi := \phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \phi\left(\chi_m^\delta\right) \tag{3.3.41}$$

then $\exists C > 0$ *such that*

$$\|\nabla\delta\phi\|_{L^\infty(\Omega)} \le C\left\{\|\nabla\delta\phi\|_{L^2(\Omega)} + \eta\left\|\nabla\cdot\left(\sigma_m^k\delta\chi_m^\delta\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right)\right\|_{H^1(\Omega)}\right\} \tag{3.3.42}$$

for any $\eta \in (0,\tau)$, *where* τ *is sufficiently small and chosen according to Remark (9).*

Proof. Let $m \in \{1, 2, \ldots, M\}$ and set

$$\delta\sigma^k = \sigma^k\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma^k\left(\chi_m^\delta\right). \tag{3.3.43}$$

Then from (2.2.4) and (2.2.6), $\delta\phi$ and $\delta\sigma$ satisfy

$$\nabla\cdot\left(\sigma^k\left(\chi_m^\delta\right)\nabla\left(\delta\phi\right)\right) = -\nabla\cdot\left(\delta\sigma^k\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right) \quad \text{on } \Omega \tag{3.3.44}$$

$$\sigma\frac{\partial\left(\delta\phi\right)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3.3.45}$$

By Assumption (1) and the fact that $\delta\chi_m^\delta$ is a mollification of $\delta\chi_m$, we can deduce that $\nabla\cdot\left(\left(\delta\sigma^k\right)\nabla\phi\left(\chi_m^\delta\right)\right), \nabla\cdot\left(\delta\sigma^k\nabla\delta\phi\right) \in H^1(\Omega)$. Because $\chi_m^\delta, \sigma_m^k \in C^\infty\left(\bar{\Omega}\right)$ then $\sigma^k\left(\chi_m^\delta\right) \in C^\infty\left(\bar{\Omega}\right)$. Thus, $\sigma^k\left(\chi_m^\delta\right) \in C^1\left(\bar{\Omega}\right)$. Using standard regularity estimate (see, e.g., [34]),

$$\|\delta\phi\|_{H^3(\Omega)} \le C_1\left(\|\delta\phi\|_{H^1(\Omega)} + \left\|\nabla\cdot\left(\delta\sigma^k\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right)\right\|_{H^1(\Omega)}\right) \tag{3.3.46}$$

for some $C_1 > 0$. Furthermore, by the Sobolev-Imbedding theorem, we have

$$\|\delta\phi\|_{C^{1,\gamma}(\bar{\Omega})} \le C_2\left\|\sigma_m^{k+1}\right\|_{H^3(\Omega)}. \tag{3.3.47}$$

By the definition of $\|\cdot\|_{C^{l,\gamma}(\bar{\Omega})}$, the embedding

$$C^{1,\gamma}\left(\bar{\Omega}\right) \hookrightarrow C^{1}\left(\bar{\Omega}\right) \tag{3.3.48}$$

is continuous (see, e.g., [13]). If we compare (3.3.46), (3.3.47) and (3.3.48) we can deduce that $\exists \bar{C} > 0$ such that

$$\|\nabla \delta\phi\|_{L^{\infty}(\Omega)} \leq C_2 \left(\|\delta\phi\|_{H^1(\Omega)} + \left\| \nabla \cdot \left(\delta\sigma^k \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)} \right). \tag{3.3.49}$$

Observe that

$$\delta\sigma^k = \eta\sigma_m^k \delta\chi_m^{\delta}.$$

Therefore,

$$\begin{aligned} &\left\| \nabla \cdot \left(\delta\sigma^k \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)} \\ &= \eta \left\| \nabla \cdot \left(\sigma_m^k \delta\chi_m^{\delta} \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)}. \end{aligned} \tag{3.3.50}$$

Recall that all solutions of the forward problem has zero boundary integral. Thus,

$$\int_{\partial\Omega} \delta\phi dS = \int_{\partial\Omega} \phi\left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta}\right) - \phi\left(\chi_m^{\delta}\right) dS = 0 - 0 = 0.$$

Using this and the generalized Friedrich's inequality, we get

$$\begin{aligned} \|\nabla\delta\phi\|_{L^2(\Omega)}^2 &= \frac{1}{2}\|\nabla\delta\phi\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla\delta\phi\|_{L^2(\Omega)}^2 \\ &\geq \frac{C_3}{2}\|\delta\phi\|_{L^2(\Omega)}^2 - \frac{1}{2}\left(\int_{\partial\Omega}\delta\phi dS\right)^2 + \frac{1}{2}\|\nabla\delta\phi\|_{L^2(\Omega)}^2 \\ &= \frac{C_3}{2}\|\delta\phi\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla\delta\phi\|_{L^2(\Omega)}^2 \\ &\geq \min\left\{\frac{C_3}{2}, \frac{1}{2}\right\}\left\{\|\delta\phi\|_{L^2(\Omega)}^2 + \|\nabla\delta\phi\|_{L^2(\Omega)}^2\right\} \\ &= \min\left\{\frac{C_3}{2}, \frac{1}{2}\right\}\|\delta\phi\|_{H^1(\Omega)}^2 \end{aligned} \tag{3.3.51}$$

for some $C_3 > 0$. Using (3.3.51) and (3.3.50), the inequality (3.3.49) becomes

$$\begin{aligned} \|\nabla\delta\phi\|_{L^{\infty}(\Omega)} &\leq C_2 \left\{ \|\delta\phi\|_{H^1(\Omega)} + \left\| \nabla \cdot \left(\delta\sigma^k \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)} \right\} \\ &\leq C_2 \left\{ \frac{1}{\sqrt{\min\left\{\frac{C_3}{2}, \frac{1}{2}\right\}}} \|\nabla\delta\phi\|_{L^2(\Omega)} \right. \\ &\left. + \eta \left\| \nabla \cdot \left(\sigma_m^k \delta\chi_m^{\delta} \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)} \right\} \\ &\leq 2C_2 \max\left\{ \frac{1}{\sqrt{\min\left\{\frac{C_3}{2}, \frac{1}{2}\right\}}}, 1 \right\} \left\{ \|\nabla\delta\phi\|_{L^2(\Omega)} \right. \\ &\left. \leq +\eta \left\| \nabla \cdot \left(\sigma_m^k \delta\chi_m^{\delta} \nabla\phi \left(\chi_m^{\delta} + \eta\delta\chi_m^{\delta} \right) \right) \right\|_{H^1(\Omega)} \right\}. \end{aligned}$$

Because of (3.3.41) and (3.3.43), our claim immediately follows from the above inequality. □

Note that $\nabla\phi(\chi_m) \in L^\infty(\bar{\Omega})$, $\nabla\phi^*(\chi_m) \in L^\infty(\Omega)$ and $\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)$, $\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right) \in H^1(\Omega)$. Thus,

$$\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)+\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)\in L^2(\Omega).$$

We will show in the next lemma that this is in fact the derivative of ψ defined in (3.3.15).

Lemma 30. *Given Assumption (1), then*

$$\lim_{\eta\to 0}\left\|\frac{\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)}{\eta}-\frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}=0 \tag{3.3.52}$$

with

$$\frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)+\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right).$$

Proof. For any perturbation $\delta\chi_m$ of χ_m, we have

$$\begin{aligned}\frac{\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)}{\eta}&=\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)}{\eta}-\frac{\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}\\&=\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)}{\eta}\\&\quad-\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}\\&\quad+\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}-\frac{\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}\\&=\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\frac{\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}\\&\quad+\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)}{\eta}\cdot\nabla\phi^*\left(\chi_m^\delta\right).\end{aligned} \tag{3.3.53}$$

To continue with our proof we will first calculate few estimates. By adding and subtracting a term the following is true:

$$\begin{aligned}\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\frac{\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}-\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\\:=(A_1+A_2)\left(\chi_m^\delta;\delta\chi_m^\delta\right)\end{aligned} \tag{3.3.54}$$

where

$$\begin{aligned}A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)&=\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\frac{\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}\\&\quad-\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right).\end{aligned}$$

$$=$$

$$A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)=\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\frac{\delta\nabla\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right).$$

Using the Hölder's inequality and (3.3.9), we can estimate $A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)$:

$$\begin{aligned}\left\|A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}&\leq\left\|\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\\&\quad\cdot\left\|\frac{\nabla\phi^*\left(\chi_m+\eta\delta\chi_m\right)-\nabla\phi^*\left(\chi_m\right)}{\eta}-\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m;\delta\chi_m\right)\right\|_{L^2(\Omega)}\\&\leq C_1\left\|f\right\|_{\tilde{L}^2(\partial\Omega)}\\&\quad\cdot\left\|\frac{\nabla\phi^*\left(\chi_m+\eta\delta\chi_m\right)-\nabla\phi^*\left(\chi_m\right)}{\eta}-\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m;\delta\chi_m\right)\right\|_{L^2(\Omega)}\end{aligned}\tag{3.3.55}$$

for some $C_1>0$. Recall from (3.3.11) that

$$\left\|\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\phi\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}\leq C_2\eta\left\|\delta\chi_m\right\|_{L^2(\Omega)}\tag{3.3.56}$$

for some $C_2>0$. Using the Cauchy-Schwarz inequality, (3.3.22) and (3.3.42), we can estimate $A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)$:

$$\begin{aligned}\left\|A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}&\leq\left\|\nabla\left(\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\phi\left(\chi_m^\delta\right)\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\&\leq C_3\left\{\left\|\nabla\left(\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\phi\left(\chi_m^\delta\right)\right)\right\|_{L^2(\Omega)}\right.\\&\quad\left.+\eta\left\|\nabla\cdot\left(\sigma_m^k\delta\chi_m^\delta\nabla\left(\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right)\right)\right\|_{H^1(\Omega)}\right\}\\&\quad\cdot\left\|\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)}\\&\leq C_3\eta\left\{C_4\left\|\delta\chi_m\right\|_{L^2(\Omega)}+\left\|\nabla\cdot\left(\sigma_m^k\delta\chi_m^\delta\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right)\right\|_{H^1(\Omega)}\right\}\\&\quad\cdot\left\|\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)}\end{aligned}\tag{3.3.57}$$

for some $C_3,C_4>0$. Define

$$\begin{aligned}A_3\left(\chi_m^\delta;\delta\chi_m^\delta\right):=&\frac{\nabla\phi\left(\chi_m+\eta\delta\chi_m\right)-\nabla\phi\left(\chi_m\right)}{\eta}\cdot\nabla\phi^*\left(\chi_m\right)\\&-\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m;\delta\chi_m\right)\cdot\nabla\phi^*\left(\chi_m\right).\end{aligned}\tag{3.3.58}$$

Hence,

$$\frac{\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)}{\eta}=\sum_{i=1}^3A_i\left(\chi_m^\delta;\delta\chi_m^\delta\right).$$

If we can find an appropriate estimate for $A_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)$, then we are almost done. Indeed, this can easily be done using the Hölder's inequality and (3.3.8):

$$\begin{aligned}A_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)\leq&\left\|\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)}{\eta}-\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\&\cdot\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}.\end{aligned}$$

Comparing (3.3.53), (3.3.54) and (3.3.58) we obtain

$$\begin{aligned}\left\|\frac{\psi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\psi\left(\chi_m^\delta\right)}{\eta}-\frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} &\leq \left\|A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &+\left\|A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &+\left\|A_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}\end{aligned}$$

using the triangle inequality. Now we only need to show that all the terms on the right-hand side of this inequality converge to 0 as η goes to 0. From (3.3.57), $\left\|A_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}$ clearly goes to 0.

Recall from (3.3.21) and (3.3.22), we can deduce

$$\lim_{\eta\to 0}\left\|\frac{\nabla\phi\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi\left(\chi_m^\delta\right)}{\eta}-\nabla\frac{\delta\phi}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}=0$$

and

$$\lim_{\eta\to 0}\left\|\frac{\nabla\phi^*\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\phi^*\left(\chi_m^\delta\right)}{\eta}-\nabla\frac{\delta\phi^*}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}=0.$$

These imply

$$\lim_{\eta\to 0}\left\|A_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}=0$$

and

$$\lim_{\eta\to 0}\left\|A_3\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)}=0$$

which will complete the proof. □

Now that we have made an analysis of ϕ and ϕ^*, we can now begin our study on σ_m^{k+1}. We will first show that under Assumption (1), (3.3.1) has a unique solution σ_m^{k+1}. We then proceed with finding the regularity of the said solution. Observe that σ_m^{k+1} depends on ϕ and ϕ^*. Hence, we can use the results from our analysis of ϕ and ϕ^* to investigate how the mollification of χ_m affects σ_m^{k+1}. We will show that, in fact, σ_m^{k+1} continuously depends on χ_m. Furthermore, we prove that $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right)\in H^1(\Omega)$. We start by equipping $H^1(\Omega)$ with a suitable norm.

Proposition 31. *Given Assumption (1) and define* $|v|_{H^1(\Omega)}^2:=\alpha\int_\Omega(\chi+\epsilon)|\nabla v|^2\,dV+\theta\int_\Omega v^2dV$ *where* $\chi(x)\in[0,1]$ *for all* $x\in\Omega$, *and let* $\|\cdot\|_{H^1(\Omega)}$ *be the standard* $H^1(\Omega)$ *norm. Then* $|\cdot|_{H^1(\Omega)}$ *and* $\|\cdot\|_{H^1(\Omega)}$ *are equivalent.*

Proof. Observe that

$$\begin{aligned}\min\{\alpha\epsilon,\theta\}\|v\|_{H^1(\Omega)}^2 &\leq \alpha\int_\Omega\epsilon|\nabla v|^2\,dV+\theta\int_\Omega|v|^2\,dV\\ &\leq \alpha\int_\Omega(\chi+\epsilon)|\nabla v|^2\,dV+\theta\int_\Omega|v|^2\,dV\\ &=|v|_{H^1(\Omega)}^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}|v|^2_{H^1(\Omega)} &= \alpha \int_\Omega (\chi+\epsilon)\left|\nabla v\right|^2 dV + \theta \int_\Omega |v|^2 dV \\ &\leq \alpha \int_\Omega (1+\epsilon)\left|\nabla v\right|^2 dV + \theta \int_\Omega |v|^2 dV \\ &\leq \max\left\{\alpha\left(1+\epsilon\right),\theta\right\}\|v\|^2_{H^1(\Omega)}.\end{aligned}$$

□

We now establish the regularity of σ_m^{k+1} .

Lemma 32. *Given Assumption (1). The variational formulation*

$$\int_\Omega \alpha\left(\chi_m^\delta+\epsilon\right)\nabla\sigma_m^{k+1}\cdot\nabla v dV + \int_\Omega \theta\left(\sigma_m^{k+1}-\sigma_m^k\right) v dV = \int_\Omega \chi_m^\delta \nabla\phi\cdot\nabla\phi^* v dV \quad (3.3.59)$$

has a unique solution $\sigma_m^{k+1}\in H^1(\Omega)$ for all $v\in H^1(\Omega)$.
Furthermore,

$$\left\|\sigma_m^{k+1}\right\|_{H^1(\Omega)} \leq \frac{2\max\sqrt{\mu(\Omega)}\left\{C_1,C_2\right\}}{\min\left\{\alpha\epsilon,\theta\right\}} \quad (3.3.60)$$

where $C_1 = \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}$ and $C_2 = \theta\left\|\sigma_m^k\right\|_{L^\infty(\Omega)}$.

Proof. Let $u,v\in H^1(\Omega)$ and define

$$a(u,v) = \int_\Omega \alpha\left(\chi_m^\delta+\epsilon\right)\nabla u\cdot\nabla v dV + \int_\Omega \theta u v dV \quad (3.3.61)$$

$$b(v) = \int_\Omega \left(\chi_m^\delta\right)\nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right) v dV + \int_\Omega \theta\sigma_m^k v dV.$$

It is obvious that a is bilinear and b is linear. Using the Cauchy-Schwarz inequality, we can prove that $a(u,v)$ is continuous:

$$\begin{aligned}|a(u,v)| &\leq \alpha(1+\epsilon)\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \theta\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \max\left\{\alpha(1+\epsilon),\theta\right\}\left(\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \theta\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\right) \\ &\leq 2\max\left\{\alpha(1+\epsilon),\theta\right\}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

We can also easily show that $a(u,v)$ is coercive using the previous proposition:

$$\begin{aligned}|a(u,u)| &= \int_\Omega \alpha\left(\chi_m^\delta+\epsilon\right)|\nabla u|^2 V + \int_\Omega \theta u^2 dV \\ &= |u|^2_{H^1(\Omega)} \\ &\geq \min\left\{\alpha\epsilon,\theta\right\}\|v\|^2_{H^1(\Omega)}.\end{aligned}$$

Furthermore, the continuity of $b(v)$ can be proven using the Cauchy-Schwarz inequality, the bounds in (3.3.3) and (3.3.8):

$$\begin{aligned}
|b(v)| &\leq \int_\Omega \left|\chi_m^\delta \nabla\phi \cdot \nabla\phi^* v\right| dV + \int_\Omega \left|\theta \sigma_m^k v\right| dV \\
&\leq \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \theta \left\|\sigma_m^k\right\|_{L^\infty(\Omega)} \|v\|_{L^1(\Omega)} \\
&\leq \sqrt{\mu(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \\
&+ \theta\sqrt{\mu(\Omega)} \left\|\sigma_m^k\right\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq 2\sqrt{\mu(\Omega)} \max\left\{\left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}, \theta\left\|\sigma_m^k\right\|_{L^\infty(\Omega)}\right\} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

Hence, by Lax-Milgram Theorem there is a unique $\sigma_m^k \in H^1(\Omega)$ satisfying (3.3.59) for all $v \in H^1(\Omega)$. The $H^1(\Omega)$ bound for σ_m^{k+1} directly follows. □

Definition 33. Define $\sigma_m^{k+1} : L^2(\Omega) \to H^1(\Omega)$ to be the mapping from χ_m^δ to the solution σ_m^{k+1} of (3.3.59). In other words, we denote by $\sigma_m^{k+1}\left(\chi_m^\delta\right)$ the solution of (3.3.59) given χ_m^δ.

Remark 34. Given any perturbation $\delta\chi_m^\delta$ and $\eta > 0$, we have to make sure that the quantity $\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$ is well-defined. As shown in the proof of Lemma (32), replacing χ_m^δ with $\chi_m^\delta + \eta\delta\chi_m^\delta$ cannot be done with just any η. This is because to make sure that the bilinear functional a is coercive, $\chi_m^\delta + \eta\delta\chi_m^\delta + \epsilon > 0$. Since $\chi_m^\delta + \epsilon > 0$, then we can choose η small enough so that $\chi_m^\delta + \eta\delta\chi_m^\delta + \epsilon > 0$ is satisfied. Hence, similar to Remark (9), we will take η from the set $(0, \bar{\tau})$ for $\bar{\tau}$ sufficiently small so that $\sigma\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$ makes sense. Therefore, combining (3.3.9), (3.3.10) and (3.3.60), $\exists C > 0$ such that

$$\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} < \infty. \tag{3.3.62}$$

for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$.

Because χ_m^δ is a mollification of χ_m that is real analytic, then $\chi_m^\delta + \epsilon \in C^\infty\left(\bar{\Omega}\right)$. Moreover, from Lemma (22), $\nabla\phi\left(\chi_m^\delta\right), \nabla\phi^*\left(\chi_m^\delta\right) \in C^\infty\left(\bar{\Omega}\right)$. This means that $\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right) \cdot \nabla\phi^*\left(\chi_m^\delta\right)$, $\chi_m^\delta + \epsilon$, θ are all in $C^\infty\left(\bar{\Omega}\right)$. Then the strong solution of

$$\begin{cases} -\alpha\nabla\cdot\left[\left(\chi_m^\delta + \epsilon\right)\nabla\sigma_m^{k+1}\right] + \theta\left(\sigma_m^{k+1} - \sigma_m^k\right) = \chi_m^\delta \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right) & \text{on } \Omega \\ \dfrac{\partial\sigma_m^{k+1}}{\partial n} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.3.63}$$

is equivalent to the weak solution of (3.3.59). Using standard regularity estimates [34] , $\sigma_m^{k+1}\left(\chi_m^\delta\right) \in C^\infty\left(\bar{\Omega}\right)$. Consequently,

$$\left\|\nabla\sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} < \infty.$$

Furthermore,

$$\left\|\nabla\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} < \infty \tag{3.3.64}$$

for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$, τ and $\bar{\tau}$ are both sufficiently small and chosen according to Remark (9) and Remark (34), respectively.

Before we can show that $\frac{\delta \sigma_m^{k+1}}{\delta \chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \in H^1(\Omega)$, we first need to find a candidate derivative.

Lemma 35. *Given Assumption (1), $\exists D_\sigma(\chi_m; \delta\chi_m) \in H^1(\Omega)$ such that*

$$\begin{aligned} &\int_\Omega \alpha\left(\chi_m^\delta + \epsilon\right) \nabla D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right) \cdot \nabla v dV + \int_\Omega \theta D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right) v dV = \\ &\int_\Omega \chi_m^\delta \frac{\delta\psi}{\delta\chi_m^\delta}(\chi_m; \delta\chi_m) v dV + \int_\Omega \delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right) \cdot \nabla\phi^*\left(\chi_m^\delta\right) v dV \\ &\quad - \int_\Omega \alpha\delta\chi_m^\delta \nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta\right)\right) \cdot \nabla v dV \end{aligned} \tag{3.3.65}$$

for all $v \in H^1(\Omega)$.

Proof. Let $u, v \in H^1(\Omega)$. Define

$$a(u, v) = \int_\Omega \alpha\left(\chi_m^\delta + \epsilon\right) \nabla u \cdot \nabla v dV + \int_\Omega \theta u v dV$$

$$\begin{aligned} b(v) = &\int_\Omega \chi_m^\delta \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) v dV + \int_\Omega \delta\chi_m^\delta \psi\left(\chi_m^\delta\right) v dV \\ &- \int_\Omega \alpha\delta\chi_m^\delta \nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta\right)\right) \cdot \nabla v dV. \end{aligned}$$

We will again make use of the Lax-Milgram theorem. From the proof of Lemma (32), a is bilinear, coercive and bounded. Thus, it is sufficient to show that b is bounded. For the ease of calculation, denote

$$b(v) =: b_1(v) + b_2(v) + b_3(v)$$

where

$$\begin{aligned} b_1(v) &= \int_\Omega \chi_m^\delta \frac{\delta\psi}{\delta\chi_m^\delta}(\chi_m; \delta\chi_m) v dV \\ b_2(v) &= \int_\Omega \delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta\right) \cdot \nabla\phi^*\left(\chi_m^\delta\right) v dV \\ b_3(v) &= -\int_\Omega \alpha\delta\chi_m^\delta \nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta\right)\right) \cdot \nabla v dV. \end{aligned}$$

By showing that b_1, b_2 and b_3 are bounded, the triangle inequality guarantees continuity of b. To estimate b_1, we use the Cauchy-Schwarz inequality, (3.3.52) and the fact that $0 \le \chi_m^\delta \le 1$:

$$\begin{aligned} |b_1(v)| &\le \left\| \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le \left\| \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

The bound for b_2 can be obtained using the Cauchy-Schwarz inequality, the Hölder's inequality and (3.3.16):

$$\begin{aligned} |b_2(v)| &\le \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

Finally, the estimate for b_3 is calculated using the Cauchy-Schwarz inequality, the Hölder's inequality and Lemma (32):

$$\begin{aligned}|b_3(v)| &\leq \alpha \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta\right)\right)\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\leq \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta\right)\right)\right\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.\end{aligned}$$

□

We now have all the necessary tools to show that $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right) \in H^1(\Omega)$. We will prove that this is exactly $D_\sigma\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ computed in the previous lemma.

Theorem 36. *Given Assumption (1) , $\exists C > 0$ such that*

$$\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq C\eta \left\|\delta\chi_m\right\|_{L^2(\Omega)} \tag{3.3.66}$$

for any $\eta \in (0,\hat{\tau})$ where $\hat{\tau} = \min\{\tau,\bar{\tau}\}$, τ and $\bar{\tau}$ are both sufficiently small and chosen according to Remark (9) and Remark (34), respectively. Also,

$$\lim_{\eta\to 0}\left\|\frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\| = 0. \tag{3.3.67}$$

Furthermore, we make the identification $\frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta;\delta\chi_m^\delta\right) = D_\sigma\left(\chi_m^\delta;\delta\chi_m^\delta\right) \in H^1(\Omega)$.

Proof. Let $v \in H^1(\Omega)$ and $\eta > 0$. From (3.3.59), we have

$$\begin{aligned}\int_\Omega \alpha\left(\chi_m^\delta + \epsilon\right)\nabla\sigma_m^{k+1}\left(\chi_m^\delta\right)\cdot\nabla v dV + \int_\Omega \theta\left(\sigma_m^{k+1}\left(\chi_m^\delta\right) - \sigma_m^k\right) v dV \\ = \int_\Omega \chi_m^\delta \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right) v dV.\end{aligned} \tag{3.3.68}$$

Similarly, if we replace χ_m^δ in (3.3.59) with $\chi_m^\delta + \eta\delta\chi_m^\delta$, we will have

$$\begin{aligned}\int_\Omega \alpha\left(\chi_m^\delta + \eta\delta\chi_m^\delta + \epsilon\right)\nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right)\cdot\nabla v dV \\ + \int_\Omega \theta\left(\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^k\right) v dV =: A(v).\end{aligned} \tag{3.3.69}$$

where

$$A(v) = \int_\Omega \left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right) v dV.$$

Subtracting (3.3.68) from (3.3.69), we will get the above equations. We will get

$$A_1(v) + A_2(v) = B_1(v) + B_2(v) + B_3(v) \tag{3.3.70}$$

where

$$A_1(v) := \int_\Omega \alpha\left(\chi_m^\delta + \epsilon\right)\nabla\left[\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right]\cdot\nabla v dV$$

$$A_2(v) := \int_\Omega \theta \left(\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right) v dV$$

$$B_1(v) := \int_\Omega \chi_m^\delta \left[\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right) - \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)\right] v dV$$

$$B_2(v) := \int_\Omega \eta\delta\chi_m^\delta \nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right) v dV$$

$$B_3(v) := -\int_\Omega \alpha\eta\delta\chi_m^\delta \nabla\left(\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right)\cdot\nabla v dV$$

for all $v \in H^1(\Omega)$. Define

$$a\left(\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right), v\right) := A_1(v) + A_2(v)$$

$$b(v) := B_1(v) + B_2(v) + B_3(v)$$

for all $v \in H^1(\Omega)$. From Lemma (32), we have shown already that a is bilinear, coercive and continuous. Clearly, b is linear. Now we only need to show that b is continuous. So we need to estimate $B_1(v)$, $B_2(v)$ and $B_3(v)$. For all these, we will make use of the Cauchy-Schwarz inequality. To estimate $B_1(v)$, we will also use (3.3.17):

$$\begin{aligned}
|B_1(v)| &\le \left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right) - \nabla\phi\left(\chi_m^\delta\right)\cdot\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&= \left\|\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le C_1\eta \|\delta\chi_m\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

for some $C_1 > 0$. For $B_2(v)$ and $B_3(v)$, we will need Cauchy-Schwarz inequality, Hölder's inequality and Young's inequality for convolutions. Doing so, we obtain

$$\begin{aligned}
|B_2(v)| &\le \left\|\eta\delta\chi_m^\delta\right\|_{L^2(\Omega)} \left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le \eta \left\|\delta\chi_m^\delta\right\|_{L^2(\Omega)} \left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)} \\
&\le \eta \|\xi_\delta\|_{L^1(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \left\|\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
|B_3(v)| &\le \alpha\eta \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\le \alpha\eta \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} \\
&\le \alpha\eta \|\xi_\delta\|_{L^2(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

It is worth noting that $\left\|\nabla\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}$, $\left\|\nabla\phi^*\left(\chi_m^\delta + \delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}$ and the quantity $\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)}$ are independent of η as shown in (3.3.9), (3.3.10) and (3.3.62) . Combining these three estimates implies that $b(v)$ is bounded.
We make the substitution $u = v = \sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)$. Furthermore, using the coercivity of a and the boundedness of b, we conclude that

$$\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)} \le C\eta \|\delta\chi_m\|_{L^2(\Omega)}. \tag{3.3.71}$$

Hence, the proof of our first statement.
Subtracting (3.3.65) from (3.3.70), we will obtain

$$D_1(v) + D_2(v) = E_1(v) + E_2(v) + E_3(v) \tag{3.3.72}$$

where

$$D_1(v) = \int_\Omega \alpha\left(\chi_m^\delta + \epsilon\right)\nabla\left[\frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right]\cdot\nabla v dV$$

$$D_2(v) = \int_\Omega \theta\left[\frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right] v dV$$

$$E_1(v) = \int_\Omega \chi_m^\delta\left[\frac{\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)}{\eta} - \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right] v dV$$

$$E_2(v) = \int_\Omega \delta\chi_m^\delta\left[\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m\right)\right] v dV$$

$$E_3(v) = -\int_\Omega \alpha\delta\chi_m^\delta\nabla\left[\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right]\cdot\nabla v dV$$

and ψ is the function defined in (3.3.15).
Using the Cauchy-Schwarz inequality, the Hölder's inequality, (3.3.17) and (3.3.71), we can estimate $E_1(v)$, $E_2(v)$ and $E_3(v)$. Indeed,

$$\begin{aligned}
|E_1(v)| &\le \left\|\frac{\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)}{\eta} - \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le \left\|\frac{\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)}{\eta} - \frac{\delta\psi}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|E_2(v)| &\le \left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le C_2\eta\left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

for some $C_2 > 0$. Finally,

$$\begin{aligned}
|E_3(v)| &\le \alpha\left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\left[\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)\right]\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\le C\eta\left\|\delta\chi_m^\delta\right\|_{L^\infty(\Omega)} \|\delta\chi_m\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

If we make the substitution

$$v = \frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - \frac{\delta\sigma_m^{k+1}}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right)$$

then by Proposition (31), we get

$$|D_1(v) + D_2(v)| = \left| \frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right|_{H^1(\Omega)}^2$$
$$\geq \min\{\alpha\epsilon, \theta\} \left\| \frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)}^2 .$$

Hence, the above inequality and (3.3.72) imply

$$\min\{\alpha\epsilon, \theta\} \left\| \frac{\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \sigma_m^{k+1}\left(\chi_m^\delta\right)}{\eta} - D_\sigma\left(\chi_m^\delta; \delta\chi_m^\delta\right) \right\|_{H^1(\Omega)}$$
$$\leq |E_1(v)| + |E_2(v)| + |E_3(v)| .$$

Taking the limit as $\eta \to 0$, it is clear that

$$|E_2(v)| + |E_3(v)| \to 0.$$

Lastly, from (3.3.52), we have

$$|E_1(v)| \to 0$$

which then implies our second statement. The last statement immediately follows from the previous lemma. □

Now that we have established that $\frac{\delta\sigma}{\delta\chi_m^\delta}\left(\chi_m^\delta; \delta\chi_m^\delta\right) \in H^1(\Omega)$, (3.3.2) holds. Using this information, the gradient of the function $J(\chi_m)$ can be explicitly calculated. We state and prove this in the following theorem.

Theorem 37. *The gradient of J at χ_m^δ is given by*

$$\nabla\left(J\left(\chi_m^\delta\right)\right) = -2\sigma_m^{k+1}\left(\chi_m^\delta\right)\nabla\phi\left(\chi_m^\delta\right) \cdot \nabla\phi^*\left(\chi_m^\delta\right) + \alpha|\nabla\sigma_m^{k+1}\left(\chi_m^\delta\right)|^2. \tag{3.3.73}$$

Proof. Let $\delta\chi_m \in L^2(\Omega)$. From (3.3.2) and the definition of $v_2\left(\chi_m^\delta; \delta\chi_m^\delta\right)$ and $v_4\left(\chi_m^\delta; \delta\chi_m^\delta\right)$, we have

$$\frac{\delta J}{\delta\chi_m}\left(\chi_m^\delta; \delta\chi_m^\delta\right) = \frac{\delta J_1}{\delta\chi_m}\left(\chi_m^\delta; \delta\chi_m^\delta\right) + \frac{\delta J_1}{\delta\chi_m}\left(\chi_m^\delta; \delta\chi_m^\delta\right)$$
$$= \left\langle \delta\chi_m^\delta, -2\sigma_m^{k+1}\left(\chi_m^\delta\right)\nabla\phi\left(\chi_m^\delta\right) \cdot \nabla\phi^*\left(\chi_m^\delta\right) + \alpha|\nabla\sigma_m^{k+1}\left(\chi_m^\delta\right)|^2 \right\rangle_{L^2(\Omega)} .$$

Our claim follows directly using the Riesz-Representation theorem (see, e.g., [26]). □

3.4 Existence of a Fixed Point

In Algorithm (2), the update for χ_m was introduced. In this section, we will show that this update has a fixed point. In other words, we will show that

$$\Upsilon_m(\chi_m) := (T_\delta \circ M \circ H \circ \Theta \circ G \circ T_\delta)\left(\chi_m^\delta\right) \tag{3.4.1}$$

has a fixed point on some suitable space X. We will make use of the Schauder Fixed Point Theorem (see Appendix) to accomplish this. This means that we need to show

that Υ is continuous on a convex subset K of X and that $\Upsilon(K)$ is compact in K. In Theorem (37), we computed the explicit formulation of the gradient of J. This justifies the formulation of the function G defined in (2.3.11). We now show that

$$G(\chi_m) = \chi_m - \omega\left[-2\sigma_m^{k+1}(\chi_m)\psi(\chi_m) + \alpha|\nabla\sigma_m^{k+1}(\chi_m)|^2\right] \tag{3.4.2}$$

is continuous.

Lemma 38. *Given Assumption (1),*

$$\lim_{\eta\to 0}\left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} = 0. \tag{3.4.3}$$

Proof. Denote

$$A_1\left(\chi_m^\delta\right) := 2\omega\sigma_m^{k+1}\left(\chi_m^\delta\right)\psi\left(\chi_m^\delta\right)$$

and

$$A_2\left(\chi_m^\delta\right) := -\alpha\omega|\nabla\sigma_m^{k+1}(\chi_{m,\delta})|^2.$$

Thus, using (3.4.2), triangle inequality and the Young's inequality for convolution, we get

$$\begin{aligned}\left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} &\le \eta\left\|\delta\chi_m^\delta\right\|_{L^2(\Omega)} + \left\|A_1\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_1\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\quad + \left\|A_2\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_2\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\le \eta\|\xi_\delta\|_{L^1(\Omega)}\|\delta\chi_m\|_{L^2(\Omega)}\\ &\quad + \left\|A_1\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_1\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\quad + \left\|A_2\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_2\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}.\end{aligned} \tag{3.4.4}$$

Adding and subtracting $2\omega\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\psi\left(\chi_m^\delta\right)$ to $A_1\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_1\left(\chi_m^\delta\right)$ and using triangle inequality, we obtain

$$\left\|A_1\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - A_1\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \le \|B_1(\chi_m;\delta\chi_m)\|_{L^2(\Omega)} + \|B_2(\chi_m;\delta\chi_m)\|_{L^2(\Omega)}$$

where

$$B_1\left(\chi_m^\delta;\delta\chi_m^\delta\right) = -2\omega\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) + 2\omega\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\psi\left(\chi_m^\delta\right)$$

and

$$B_2\left(\chi_m^\delta;\delta\chi_m^\delta\right) = -2\omega\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\psi\left(\chi_m^\delta\right) + 2\omega\sigma_m^{k+1}\left(\chi_m^\delta\right)\psi\left(\chi_m^\delta\right).$$

By using the Hölder's inequality, (3.3.64), (3.3.17), (3.3.3), (3.3.8) and (3.3.66), we can estimate $B_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ and $B_2\left(\chi_m^\delta;\delta\chi_m^\delta\right)$ as follows:

$$\begin{aligned}\left\|B_1\left(\chi_m^\delta;\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} &\le 2\omega\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\psi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\le 2C_1\omega\eta\left\|\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\|\delta\chi_m\|_{L^2(\Omega)}\end{aligned} \tag{3.4.5}$$

for some $C_1 > 0$ and

$$\begin{aligned}\left\|B_2\left(\chi_m^\delta;\delta\chi_m\right)\right\|_{L^1(\Omega)} &\leq 2\omega\left\|\sigma_m^{k+1}\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\left\|\psi\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\leq C_2\eta\left\|\delta\chi_m\right\|_{L^2(\Omega)}\left\|\nabla\phi\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\phi^*\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \qquad (3.4.6)\end{aligned}$$

for some $C_2 > 0$. Also, $A_2\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-A_2\left(\chi_m^\delta\right)$ can be estimated using the Hölder's inequality, triangle inequality, (3.3.64) and (3.3.66) :

$$\begin{aligned}\left\|A_2\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-A_2\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} &\leq \alpha\omega\left\|\sigma_m^{k+1}\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\\ &\quad\cdot\left\|\sigma_m^{k+1}\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)+\sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\\ &\leq \alpha\omega C_3\eta\left\|\delta\chi_m\right\|_{L^2(\Omega)}\left\{\left\|\sigma_m^{k+1}\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\right.\\ &\quad\left.+\left\|\sigma_m^{k+1}\left(\chi_m^\delta\right)\right\|_{L^\infty(\Omega)}\right\} \qquad (3.4.7)\end{aligned}$$

for some $C_3 > 0$. Comparing (3.4.4), (3.4.5), (3.4.6) and (3.4.7) implies that $\exists C > 0$ such that

$$\left\|G\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \leq C\eta\left\|\delta\chi_m\right\|_{L^2(\Omega)}$$

for any $\eta \in (0,\hat{\tau})$, where $\hat{\tau} = \min\{\tau,\bar{\tau}\}$, τ and $\bar{\tau}$ are both sufficiently small and chosen according to Remark (9) and Remark (34), respectively. Taking the limit of the above inequality as $\eta \to 0$ gives us our desired result. □

Now that we have shown the continuity of the function G, we will now prove the continuity of the function Θ, which maps a function χ_m in $L^2(\Omega)$ to the solution of (2.3.14). Before proving continuity, we need to first show that given χ_m^δ, (2.3.14) has a solution in $H^1(\Omega)$. Note that because $\chi_m^\delta \in C^\infty(\Omega)$, $\nabla\chi_m^\delta$ is bounded in Ω for a fixed δ. Hence,

$$\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2} \leq \sqrt{\left\|\nabla\chi_m^\delta\right\|_{L^\infty(\Omega)}^2+\beta^2} =: \bar{K} < \infty.$$

Or equivalently,

$$\frac{1}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}} \geq \frac{1}{\bar{K}}. \qquad (3.4.8)$$

Obviously, for any $\beta > 0$,

$$\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2} \geq \beta$$

which can be expressed as

$$\frac{1}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}} \leq \frac{1}{\beta}. \qquad (3.4.9)$$

For $u, v \in H^1(\Omega)$, define

$$a(u,v) = \int_\Omega \omega\gamma\frac{\nabla u\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}} + uvdV \qquad (3.4.10)$$

and

$$b(v) = \int_\Omega G\left(\chi_m^\delta\right)vdV.$$

Clearly, a and b are bilinear and linear, respectively. Observe that using Cauchy-Schwarz inequality and (3.4.9), we get

$$\begin{aligned}|a(u,v)| &\leq \frac{\omega\gamma}{\beta}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq 2\max\left\{\frac{\omega\gamma}{\beta},1\right\}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\end{aligned}$$

for all $v \in H^1(\Omega)$, which makes a continuous. It can also be shown using Cauchy-Schwarz inequality that

$$|b(v)| \leq \left\|G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$$

and so $b(v)$ is also continuous. Using (3.4.9), a can be proven to be coercive. Indeed,

$$\begin{aligned}|a(u,u)| &\geq \frac{\omega\gamma}{\bar{K}}\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \\ &\geq \min\left\{\frac{\omega\gamma}{\bar{K}},1\right\}\|u\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.4.11}$$

Therefore, using Lax-Milgram theorem, there exists a unique $\Theta \in H^1(\Omega)$ satisfying

$$a(\Theta, v) = b(v)$$

for all $v \in H^1(\Omega)$. We state this in the following lemma.

Lemma 39. *Given Assumption (1), there exists a unique $\Theta \in H^1(\Omega)$ satisfying*

$$\int_\Omega \omega\gamma\frac{\nabla\Theta\cdot\nabla v}{\sqrt{|\nabla\chi_m^\delta|^2+\beta^2}} + \Theta v dV = \int_\Omega G\left(\chi_m^\delta\right) v dV \tag{3.4.12}$$

for any $\Theta \in H^1(\Omega)$ provided that $\frac{\partial\Theta}{\partial n} = 0$ on $\partial\Omega$. Furthermore, $\exists C > 0$ such that

$$\|\Theta\|_{H^1(\Omega)} \leq \frac{1}{\min\left\{\frac{\omega\gamma}{\bar{K}},1\right\}}\left\|G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}.$$

The above lemma tells us that given χ_m, a mollification can be done to obtain a unique solution $\Theta \in H^1(\Omega)$ to (3.4.12). Hence, we can think of $\Theta\left(\chi_m^\delta\right)$ as a function that maps an element $\chi_m \in L^2(\Omega)$ to an element $\Theta \in H^1(\Omega)$. Note that given a perturbation $\delta\chi_m^\delta$ of χ_m^δ, $\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$ is well-defined for any $0 < \eta < \infty$ because for coercivity we just need $\sqrt{\|\nabla\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\|^2_{L^\infty(\Omega)} + \beta^2}$ to be finite. Since $\chi_m^\delta + \eta\delta\chi_m^\delta \in C^\infty\left(\bar{\Omega}\right)$, this is not a problem. Just to make sure that $\eta < \infty$, we can choose $\eta \in (0,\hat{\tau})$, where $\hat{\tau} = \min\{\tau,\bar{\tau}\}$, τ and $\bar{\tau}$ are both sufficiently small and chosen according to Remark (9) and Remark (34), respectively. This way, $\phi\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$, $\phi^*\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$ and $\sigma_m^{k+1}\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)$ are all well-defined. From the definition of G and the inequalities (3.3.9), (3.3.10) and (3.3.64), we can infer that

$$\left\|\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} \leq C\left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} < \infty \tag{3.4.13}$$

for some $C > 0$.

We now prove that this map is continuous.

Lemma 40. *Given Assumption (1), then*

$$\lim_{\eta\to 0}\left\|\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\Theta\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}=0.$$

Proof. By the previous lemma, note that $\Theta\left(\chi_m^\delta\right)$ satisfies

$$\int_\Omega \omega\gamma\frac{\nabla\Theta\left(\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}}+\Theta\left(\chi_m^\delta\right)vdV=\int_\Omega G\left(\chi_m^\delta\right)vdV. \tag{3.4.14}$$

Similarly, $\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)$ satisfies

$$\begin{aligned}&\int_\Omega \omega\gamma\frac{\nabla\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right|^2+\beta^2}}+\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)vdV\\&\quad=\int_\Omega G\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)vdV.\end{aligned} \tag{3.4.15}$$

Subtracting (3.4.14) from (3.4.15), we get

$$A_1(v)+A_2(v)=B(v) \tag{3.4.16}$$

with

$$A_1(v):=\int_\Omega \omega\gamma\left[\frac{\nabla\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta+\eta\delta\chi_m^\delta\right|^2+\beta^2}}-\frac{\nabla\Theta\left(\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}}\right]dV$$

$$A_2(v):=\int_\Omega\left[\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\Theta\left(\chi_m^\delta\right)\right]vdV$$

$$B(v):=\int_\Omega\left[G\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-G\left(\chi_m^\delta\right)\right]vdV.$$

We subtract and add the term

$$\int_\Omega \omega\gamma\frac{\nabla\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}}dV$$

to $A_1(v)$ to obtain

$$A_1(v)=A_3(v)+A_4(v) \tag{3.4.17}$$

with

$$A_4:=\int_\Omega \omega\gamma D(\eta)\frac{\nabla\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\cdot\nabla v}{\sqrt{\left|\nabla\left(\chi_m^\delta\right)\right|^2+\beta^2}}$$

$$A_3(v):=\int_\Omega \omega\gamma\frac{\left[\nabla\Theta\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)-\nabla\Theta\left(\chi_m^\delta\right)\right]\cdot\nabla v}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}}dV$$

and

$$D(\eta):=\frac{1}{\sqrt{\left|\nabla\left(\chi_m^\delta+\eta\delta\chi_m^\delta\right)\right|^2+\beta^2}}-\frac{1}{\sqrt{\left|\nabla\chi_m^\delta\right|^2+\beta^2}}$$

From (3.4.16) and (3.4.17), we have

$$A_3(v)+A_2(v)=B(v)-A_4(v). \tag{3.4.18}$$

From the definition of the bilinear functional a in (3.4.10), we can deduce that

$$A_3(v) + A_2(v) = a\left(\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \Theta\left(\chi_m^\delta\right), v\right).$$

From the coercivity of a shown in (3.4.11), we get

$$\begin{aligned} &\left|(A_3 + A_2)\left(\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \Theta\left(\chi_m^\delta\right)\right)\right| \geq \\ &\min\left(\frac{\omega\gamma}{\bar{K}}, 1\right)\left\|\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \Theta\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}^2. \end{aligned} \tag{3.4.19}$$

On the other hand, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |B(v)| &\leq \left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq \left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{3.4.20}$$

Moreover, using Cauchy-Schwarz inequality, Hölder's inequality and (3.4.9), we estimate $A_4(v)$ as follows:

$$\begin{aligned} |A_4(v)| &\leq \frac{\omega\gamma}{\beta}\|D(\eta)\|_{L^\infty(\Omega)}\left\|\nabla\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\leq \frac{\omega\gamma}{\beta}\|D(\eta)\|_{L^\infty(\Omega)}\left\|\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned} \tag{3.4.21}$$

Comparing (3.4.18), (3.4.19), (3.4.20) and (3.4.21), we get

$$\begin{aligned} \min\left\{\frac{\omega\gamma}{\bar{K}}, 1\right\}\left\|\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - \Theta\left(\chi_m^\delta\right)\right\|_{H^1(\Omega)}^2 \leq &\left\|G\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \\ &+ \frac{\omega\gamma}{\beta}\|D(\eta)\|_{L^\infty(\Omega)} \\ &\cdot \left\|\Theta\left(\chi_m^\delta + \eta\delta\chi_m^\delta\right)\right\|_{H^1(\Omega)}. \end{aligned}$$

From (3.4.13), the right-hand side of the above inequality is bounded for any $\eta \in (0, \hat{\tau})$, where $\hat{\tau} = \min\{\tau, \bar{\tau}\}$, τ and $\bar{\tau}$ are both sufficiently small and chosen according to Remark (9) and Remark (34), respectively. Obviously,

$$\lim_{\eta\to 0}\|D(\eta)\|_{L^\infty(\Omega)} = 0.$$

The above equality, together with (3.4.3), asserts our claim. □

We now show that the function H defined in (3.2.8) is continuous. We obtained the proof from [32].

Lemma 41. *Let $z \in L^2(\Omega)\backslash H^1(\Omega)$ and $\{g_n\}_{n=1}^\infty \subset L^2(\Omega)$ that converges to g in $L^2(\Omega)$. Then*

$$\lim_{n\to\infty}\mu\left(H(g) \triangle H(g_n)\right) = 0 \tag{3.4.22}$$

where

$$H(g) = \left\{x \in \Omega : \left(\left(g^\delta - \zeta + \delta z\right) * \xi_\delta\right)(x) \geq 0\right\}$$

for some $\zeta \in (0,1)$ and $g^\delta = g * \xi_\delta$. In other words, H is continuous in $L^2(\Omega)$ (compare, e.g., [32]).

Proof. Let us denote g_n^δ and g^δ to be the mollifications of g_n and g, respectively. From the definition of symmetric difference, it is necessary to compute for

$$H(g)\backslash H(g_n) = \Big\{x \in \Omega : \Big(\Big(g^\delta - \zeta + \delta z\Big) * \xi_\delta\Big)(x) \geq 0 \wedge \Big(\Big(-g_n^\delta + \zeta - \delta z\Big) * \xi_\delta\Big)(x) > 0\Big\}. \tag{3.4.23}$$

Let $\epsilon > 0$ and define the set

$$U_n(\epsilon) := \Big\{x \in \Omega : \Big(\Big(g^\delta - g_n^\delta\Big) * \xi_\delta\Big)(x) \geq \epsilon\Big\}.$$

It follows directly that $H(g)\backslash H(g_n) \subset U_n(0)$ because of the distributive property of convolution. Define

$$V(\epsilon) := \Big\{x \in \Omega : 0 \leq \Big(\Big(g^\delta - \zeta + \delta z\Big) * \xi_\delta\Big)(x) < \epsilon \wedge \Big(\Big(-g_n^\delta + \zeta - \delta z\Big) * \xi_\delta\Big)(x) > 0\Big\}$$

and

$$W(\epsilon) := \Big\{x \in \Omega : \Big(\Big(g^\delta - \zeta + \delta z\Big) * \xi_\delta\Big)(x) \geq \epsilon \wedge \Big(\Big(-g_n^\delta + \zeta - \delta z\Big) * \xi_\delta\Big)(x) > 0\Big\}.$$

Hence, we can write (3.4.23) as

$$H(g)\backslash H(g_n) = (V(\epsilon) \cup W(\epsilon)). \tag{3.4.24}$$

Clearly,

$$\mu(W(\epsilon)) \leq \mu(U_n(\epsilon)). \tag{3.4.25}$$

We will now prove that as $\epsilon \to 0$, $\mu(V(\epsilon)) \to 0$. Consider the case when $H(g)\backslash H(g_n) = \emptyset$. Then (3.4.24) implies $V(\epsilon) = \emptyset$. Thus, $\mu(V(\epsilon)) = 0$ for any $\epsilon > 0$. Suppose $H(g)\backslash H(g_n) \neq \emptyset$ and let $\{\epsilon_n\}_{n=1}^\infty$ be a monotonically decreasing sequence of positive numbers converging to 0. By the definition of $V(\epsilon)$, $\{V(\epsilon_n)\}_{n=1}^\infty$ is a decreasing nested sequence of sets where $V_0 = \cap_{n=1}^\infty V(\epsilon_n)$. Thus, we wish to show that $\mu(V_0) = 0$. We will prove this by contradiction. Suppose $\mu(V_0) > 0$ then

$$\Big(\Big(g^\delta - \zeta + \delta z\Big) * \xi_\delta\Big)(x) = 0$$

for all $x \in V_0$. By Lemma (14), $\big(g^\delta - \zeta + \delta z\big) * \xi_\delta$ is real analytic and because $\Omega \supset \mu(V_0) > 0$, we can conclude from Lemma (15) that $\big(g^\delta - \zeta + \delta z\big) * \xi_\delta = 0$ on the whole Ω. Furthermore, $T_\delta(0) = 0$. Thus,

$$T_\delta(0) = 0 = \Big(g^\delta - \zeta + \delta z\Big) * \xi_\delta = T_\delta\Big(g^\delta - \zeta + \delta z\Big).$$

By Lemma (16), T_δ is injective and so

$$g^\delta - \zeta + \delta z = 0 \text{ on } \Omega$$

which can be rewritten as

$$\delta z = \zeta - g^\delta \text{ on } \Omega.$$

This is a contradition because $\zeta - g^\delta$ is analytic and z by assumption is not. Therefore, $\mu(V_0) = 0$ and $\mu(V(\epsilon)) \to 0$ as $\epsilon \to 0$. Thus, $\exists \bar{\epsilon} > 0$ such that

$$\mu(V(\hat{\epsilon})) < \frac{\epsilon}{4} \tag{3.4.26}$$

for any $\hat{\epsilon} \in (0, \bar{\epsilon})$.
Using the Chebyshev's inequality and the Young's inequality for convolution, we get

$$\begin{aligned}
\mu\left(U_n(\epsilon)\right) &\leq \frac{1}{\epsilon^2} \int_\Omega \left| \left(g^\delta - g_n^\delta\right) * \xi_\delta \right|^2 dV \\
&= \frac{1}{\epsilon^2} \left\| \left(g^\delta - g_n^\delta\right) * \xi_\delta \right\|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{\epsilon^2} \left\| g^\delta - g_n^\delta \right\|_{L^2(\Omega)}^2 \|\xi_\delta\|_{L^1(\Omega)}^2 \\
&= \frac{1}{\epsilon^2} \|(g - g_n) * \xi_\delta\|_{L^2(\Omega)}^2 \|\xi_\delta\|_{L^1(\Omega)}^2 \\
&\leq \frac{1}{\epsilon^2} \|g - g_n\|_{L^2(\Omega)}^2 \|\xi_\delta\|_{L^1(\Omega)}^4 .
\end{aligned}$$

Hence, for any $\epsilon > 0$, $\mu\left(U_n(\epsilon)\right) \to 0$ as $n \to \infty$ because $g_n \to g$ in $L^2(\Omega)$ by assumption.We can deduce that $\exists N_1 \in \mathbb{N}$ such that

$$\mu\left(U_n(\hat{\epsilon})\right) < \frac{\epsilon}{4} \tag{3.4.27}$$

for all $n \geq N_1$ and for all $\hat{\epsilon} \in (0, \bar{\epsilon})$.
Similarly for

$$H(g_n) \backslash H(g) = \left\{ x \in \Omega : \left(\left(g_n^\delta - \zeta + \delta z \right) * \xi_\delta \right)(x) \geq 0 \wedge \left(\left(-g^\delta + \zeta - \delta z \right) * \xi_\delta \right)(x) > 0 \right\},$$

we define the set

$$\bar{U}_n(\epsilon) := \left\{ x \in \Omega : \left(\left(g_n^\delta - g^\delta \right) * \xi_\delta \right)(x) \geq \epsilon \right\}.$$

It follows directly that $H(g) \backslash H(g_n) \subset U_n(0)$. Define

$$\bar{V}(\epsilon) := \left\{ x \in \Omega : \left(\left(g_n^\delta - \zeta + \delta z \right) * \xi_\delta \right)(x) \geq 0 \wedge 0 < \left(\left(-g^\delta + \zeta - \delta z \right) * \xi_\delta \right)(x) < \epsilon \right\}$$

and

$$\bar{W}(\epsilon) := \left\{ x \in \Omega : \left(\left(g_n^\delta - \zeta + \delta z \right) * \xi_\delta \right)(x) \geq 0 \wedge \left(\left(-g^\delta + \zeta - \delta z \right) * \xi_\delta \right)(x) \geq \epsilon \right\}.$$

Hence, we can write (3.4.23) as

$$H(g_n) \backslash H(g) = \left(\bar{V}(\epsilon) \cup \bar{W}(\epsilon) \right). \tag{3.4.28}$$

Clearly,

$$\mu\left(W(\epsilon)\right) \leq \mu\left(U_n(\epsilon)\right). \tag{3.4.29}$$

Using the same argument as above, we can conclude that $\mu\left(\bar{V}(\epsilon)\right) \to 0$ as $\epsilon \to 0$. That means $\exists \tilde{\epsilon} > 0$ such that

$$\mu\left(\bar{V}(\hat{\epsilon})\right) < \frac{\epsilon}{4} \tag{3.4.30}$$

for all $\hat{\epsilon} \in (0, \tilde{\epsilon})$. By again using the Chebyshev's inequality and the Young's inequality, we can obtain a similar inequality

$$\mu\left(\bar{U}_n(\epsilon)\right) \leq \frac{1}{\epsilon^2} \|g_n - g\|_{L^2(\Omega)}^2 \|\xi_\delta\|_{L^1(\Omega)}^4$$

which implies that $\mu\left(\bar{U}_n(\epsilon)\right) \to 0$ as $n \to \infty$. And so $\exists N_2 \in \mathbb{N}$ such that

$$\mu\left(\bar{U}_n(\hat{\epsilon})\right) < \frac{\epsilon}{4} \tag{3.4.31}$$

for all $n \geq N_2$ and for all $\hat{\epsilon} \in (0, \tilde{\epsilon})$. Therefore, for any $\hat{\epsilon} \in (0, \min\{\bar{\epsilon}, \tilde{\epsilon}\})$ and for any $n \geq \max(N_1, N_2)$, we compare (3.4.24), (3.4.25), (3.4.26), (3.4.27), (3.4.28), (3.4.29), (3.4.30) and (3.4.31) and use the definition of symmetric difference to get

$$\begin{aligned}
\mu(H(g) \triangle H(g_n)) &= \mu((H(g) \backslash H(g_n)) \cup (H(g_n) \backslash H(g))) \\
&= \mu(H(g) \backslash H(g_n)) + \mu(H(g_n) \backslash H(g)) \\
&= \mu(V(\hat{\epsilon}) \cup W(\hat{\epsilon})) + \mu\left(\bar{V}(\hat{\epsilon}) \cup \bar{W}(\hat{\epsilon})\right) \\
&= \mu(V(\hat{\epsilon})) + \mu(W(\hat{\epsilon})) + \mu\left(\bar{V}(\hat{\epsilon})\right) + \mu\left(\bar{W}(\hat{\epsilon})\right) \\
&\leq \mu(V(\hat{\epsilon})) + \mu(U_n(\hat{\epsilon}\epsilon)) + \mu\left(\bar{V}(\hat{\epsilon})\right) + \mu\left(\bar{U}_n(\hat{\epsilon})\right) \\
&< \epsilon.
\end{aligned}$$

Finally, because ϵ is arbitrary, our claim follows. □

In our next computations, we will prove the continuity of the function M defined in (3.2.9). Recall that M maps elements of $\mathcal{M}(\Omega)$ to their corresponding characteristic functions. We will now try to find a suitable space for these characteristic functions. Intuitively, convergence of these characteristic functions is co-dependent with the convergence of their associated supports. We will choose $L^2(\Omega)$ to be the space of the characteristic functions and select $\mathcal{M}(\Omega)$ to be the space of their associated supports. Recall that $\mathcal{M}(\Omega)$ is a metric space equipped with the measure of the symmetric difference. The following lemma proves how these two spaces are related.

Lemma 42. *Let $\hat{\chi}$ and χ be characteristic functions on Ω whose supports are given by $\Omega_{\hat{\chi}}$ and Ω_{χ}, respectively. Then*

$$\mu(\Omega_{\hat{\chi}} \triangle \Omega_{\chi}) = \|\hat{\chi} - \chi\|^2_{L^2(\Omega)}.$$

Proof. Because χ_m and $\bar{\chi}_m$ are characteristic functions, we have

$$\begin{aligned}
\Omega_{\hat{\chi}} \backslash \Omega_{\chi} &= \{x : x \in \Omega_{\hat{\chi}} \wedge x \notin \Omega_{\chi}\} \\
&= \{x : \hat{\chi}(x) = 1 \wedge \chi(x) = 0\}.
\end{aligned}$$

Similarly,

$$\Omega_{\chi} \backslash \Omega_{\hat{\chi}} = \{x : \chi(x) = 1 \wedge \hat{\chi}(x) = 0\}.$$

Thus, from the definition of symmetric difference and the fact that $\Omega_{\hat{\chi}} \backslash \Omega_{\chi}$ and $\Omega_{\chi} \backslash \Omega_{\hat{\chi}}$

are disjoint sets, we get

$$\begin{aligned}
\mu\left(\Omega_{\hat{\chi}} \triangle \Omega_{\chi}\right) &= \mu\left(\left(\Omega_{\hat{\chi}} \backslash \Omega_{\chi}\right) \cup \left(\Omega_{\chi} \backslash \Omega_{\hat{\chi}}\right)\right) \\
&= \mu\left(\left\{x : \hat{\chi}(x) = 1 \wedge \chi(x) = 0\right\}\right) + \mu\left(\left\{x : \chi(x) = 1 \wedge \hat{\chi}(x) = 0\right\}\right) \\
&= \int_{\Omega} \hat{\chi}(1-\chi)\, dV + \int_{\Omega} \chi(1-\hat{\chi})\, dV \\
&= \int_{\Omega} \hat{\chi} - \hat{\chi}\chi dV + \int_{\Omega} \chi - \chi\hat{\chi} dV \\
&= \int_{\Omega} \hat{\chi} - 2\hat{\chi}\chi + \chi dV \\
&= \int_{\Omega} \hat{\chi}^2 - 2\hat{\chi}\chi + \chi^2 dV \\
&= \int_{\Omega} |\hat{\chi} - \chi|^2\, dV \\
&= \|\hat{\chi} - \chi\|^2_{L^2(\Omega)}.
\end{aligned}$$

□

Now that we established a mode of convergence for the characteristic functions and their associated sets, we can now prove that M is continuous.

Lemma 43. *Suppose* $\{\omega_n\}_{n=1}^{\infty} \subset \mathcal{M}(\Omega)$ *such that* $\omega_n \to \omega$ *in* $\mathcal{M}(\Omega)$*, that is,*

$$\lim_{n\to\infty} \mu\left(\omega_n \triangle \omega\right) = 0.$$

Then

$$\lim_{n\to\infty} \|M(\omega_n) - M(\omega)\|_{L^2(\Omega)} = 0$$

where $M : \mathcal{M}(\Omega) \to L^2(\Omega)$ *is a function that maps* ω *to its corresponding characteristic function, that is,*

$$M(\omega) = \chi^{\omega}.$$

In other words, M *is continuous on* $\mathcal{M}(\Omega)$*.*

Proof. Denote

$$M(\omega_n) =: \chi^n \text{ and } M(\omega) := \chi^{\omega}.$$

Then by Lemma (42),

$$\begin{aligned}
\lim_{n\to\infty} \|M(\omega_n) - M(\omega)\|^2_{L^2(\Omega)} &= \lim_{n\to\infty} \|\chi^n - \chi^{\omega}\|^2_{L^2(\Omega)} \\
&= \lim_{n\to\infty} \mu\left(\omega_n \triangle \omega\right) \\
&= 0.
\end{aligned}$$

□

We have shown continuity of G, Θ, H and M. We can now prove that Υ_m has a fixed point for $m \in \{1, 2, \ldots, M-1\}$.

Theorem 44. *Given Assumption (1) and let $z \in L^2(\Omega) \backslash H^1(\Omega)$. Then the function $\Upsilon : L^2(\Omega) \to L^2(\Omega)$*

$$\Upsilon_m(\chi_m) := (T_\delta \circ M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right)$$

has a fixed point in the set

$$K := \left\{\chi_m \in L^2(\Omega) : 0 \leq \chi_m \leq 1 \ a.e. \ \Omega\right\}. \tag{3.4.32}$$

Proof. We will employ the Schauder Fixed Point Theorem. Hence, we first need to show that K is a convex set in some Banach space X. We will select $X = L^2(\Omega)$. Let $\chi_m, \bar{\chi}_m \in K$ and $\lambda \in (0,1)$. Obviously, $\lambda\chi_m + (1-\lambda)\bar{\chi}_m \in L^2(\Omega)$. We only need to show that $0 \leq \lambda\chi_m + (1-\lambda)\bar{\chi}_m \leq 1$. Because $\lambda, 1-\lambda > 0$ then $\lambda\chi_m + (1-\lambda)\bar{\chi}_m \geq 0$. Furthermore, $\lambda\chi_m + (1-\lambda)\bar{\chi}_m \leq \lambda + 1 - \lambda = 1$. Thus, $\lambda\chi_m + (1-\lambda)\bar{\chi}_m \in K$ and K is convex in $L^2(\Omega)$.

We show next that Υ is continuous. The functions G, Θ, H and M are continuous as proven in Lemma (38), Lemma (40), Lemma (41) and Lemma (43), respectively. From Lemma (16), T_δ is continuous as well by choosing $p = 2$, $r = 2$ and $q = 1$. Because composition of continuous functions is continuous, Υ is continuous.

We only need to show that $\Upsilon(K) \subset K$ and that $\Upsilon(K)$ is compact in K. Recall that $(M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right)$ is a characteristic function. Thus, $0 \leq (M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right) \leq 1$. By Lemma (14), $0 \leq (T_\delta \circ M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right) \leq 1$ and so $\Upsilon(\chi_m) \in K$. Let $\bar{\chi}_m$ be an arbitrary element of K. Let us denote $\omega := (H \circ \Theta \circ G)\left(\chi_m^\delta\right)$, $\chi^\omega := M(\omega)$, and $\chi_\delta^\omega := T_\delta(\chi^\omega)$. By Lemma (17), the Hölder's inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\nabla\chi_\delta^\omega(x)| &= \left|\int_\Omega \nabla\xi_\delta(x-y)\,\chi^\omega(y)\,dy\right| \\
&\leq \|\chi^\omega\|_{L^\infty(\Omega)} \int_\Omega |\nabla\xi_\delta(x-y)|\,dy \\
&\leq \sqrt{\mu(\Omega)}\,\|\nabla\xi_\delta\|_{L^2(\Omega)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\nabla\chi_\delta^\omega\|_{L^2(\Omega)}^2 &= \int_\Omega \left|\int_\Omega \nabla\xi_\delta(x-y)\,\chi^\omega(y)\,dy\right|^2 dx \\
&\leq \mu(\Omega)^2\,\|\nabla\xi_\delta\|_{L^2(\Omega)}^2.
\end{aligned}$$

From Lemma (14), χ_δ^ω is real analytic and so $\chi_\delta^\omega \in H^1(\Omega)$. For a fixed δ, we compute the $H^1(\Omega)$ norm of χ_δ^ω using the Young's inequality for convolutions and the Hölder's inequality:

$$\begin{aligned}
\|\Upsilon_m(\bar{\chi}_m)\|_{H^1(\Omega)}^2 &= \|\chi_\delta^\omega\|_{H^1(\Omega)}^2 \\
&= \|\chi_\delta^\omega\|_{L^2(\Omega)}^2 + \|\nabla\chi_\delta^\omega\|_{L^2(\Omega)}^2 \\
&\leq \|\chi^\omega * \xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\,\|\nabla\xi_\delta\|_{L^2(\Omega)}^2 \\
&\leq \|\chi^\omega\|_{L^1(\Omega)}^2\,\|\xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\,\|\nabla\xi_\delta\|_{L^2(\Omega)}^2 \\
&\leq \|\chi^\omega\|_{L^\infty(\Omega)}^2\,\mu(\Omega)^2\,\|\xi_\delta\|_{L^2(\Omega)}^2 + \mu(\Omega)^2\,\|\nabla\xi_\delta\|_{L^2(\Omega)}^2 \\
&\leq \mu(\Omega)^2\left(\|\xi_\delta\|_{L^2(\Omega)}^2 + \|\nabla\xi_\delta\|_{L^2(\Omega)}^2\right).
\end{aligned}$$

Since $\bar{\chi}_m$ is arbitrary, any sequence $\{\Upsilon_m(\chi_m^n)\}_{n=1}^{\infty}$ is bounded in the $H^1(\Omega)$ norm for a fixed δ. Because Ω is bounded, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $\{\Upsilon(\chi_m^n)\}_{n=1}^{\infty}$ has a convergent subsequence (see Appendix). Therefore $\Upsilon(K)$ is compact and using Schauder Fixed Point theorem, Υ_m has a fixed point on K. □

A fixed point is guaranteed for $m \in \{1, 2, \ldots .M-1\}$. We will now study the case when $m = M$. Let $\bar{\chi}_1^k, \bar{\chi}_1^k, \ldots, \bar{\chi}_{M-1}^k$ be the fixed points of $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{M-1}$, respectively. Recall from (2.4.8) that $\bar{\chi}_M^k = 1 - \sum_{m=1}^{M-1} \bar{\chi}_m^k$. In Algorithm (2), the output for $\bar{\chi}_M$ is $\bar{\chi}_M^{k+1} = 1 - \sum_{m=1}^{M-1} \Upsilon_m\left(\bar{\chi}_m^k\right)$. Because $\bar{\chi}_1^k, \bar{\chi}_1^k, \ldots, \bar{\chi}_{M-1}^k$ are the fixed points of $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{M-1}$, respectively, we get

$$\bar{\chi}_M^{k+1} = 1 - \sum_{m=1}^{M-1} \Upsilon_m\left(\bar{\chi}_m^k\right) = 1 - \sum_{m=1}^{M-1} \bar{\chi}_m^k = \bar{\chi}_M^k.$$

Thus, $\bar{\chi}_M^{k+1}$ is a fixed point as well. So as long as the fixed points of $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_{M-1}$ are attained, the fixed point when $m = M$ is guaranteed. The question is if $\bar{\chi}_M^{k+1}$ is in the convex set K defined in (3.4.32). We are solving an inverse problem and it is known a priori that the supports of the solutions $\bar{\chi}_1^k, \bar{\chi}_1^k, \ldots, \bar{\chi}_{M-1}^k$ are disjoint. Because $0 \leq \bar{\chi}_1^k, \bar{\chi}_1^k, \ldots, \bar{\chi}_{M-1}^k \leq 1$, then $0 \leq \bar{\chi}_{M-1}^k \leq 1$. This is true when $\bar{\chi}_1^k, \bar{\chi}_1^k, \ldots, \bar{\chi}_{M-1}^k$ are the fixed points (or the solutions). However, this might not be true for Algorithm (2). That is, if we take any arbitrary $\chi_m \in L^2(\Omega)$ such that $0 \leq \chi_m \leq 1$, we cannot guarantee that $0 \leq \Upsilon_m(\chi_m) \leq 1$. The following proposition will help us address this problem.

Proposition 45. *Let $m_1, m_2 \in \{1, 2, \ldots, M-1\}$ such that $m_1 \neq m_2$ and $supp(\chi_{m_1}) \cap supp(\chi_{m_2}) = \emptyset$. Then $supp\left(\chi_{m_1}^\delta\right) \cap supp\left(\chi_{m_2}^\delta\right) = \emptyset$, where $\chi_{m_1}^\delta = \chi_{m_1} * \xi_\delta$ and $\chi_{m_2}^\delta = \chi_{m_2} * \xi_\delta$.*

Proof. By the definition of symmetric difference of sets, it is sufficient to show that $\text{supp}\left(\chi_{m_1}^\delta\right) \triangle \text{supp}\left(\chi_{m_2}^\delta\right) = \emptyset$ or equivalently, $\mu\left(\text{supp}\left(\chi_{m_1}^\delta\right) \triangle \text{supp}\left(\chi_{m_2}^\delta\right)\right) = 0$. Indeed, by using Lemma (42) and the Young's inequality for convolutions, we obtain

$$\begin{aligned}
\mu\left(\text{supp}\left(\chi_{m_1}^\delta\right) \triangle \text{supp}\left(\chi_{m_2}^\delta\right)\right) &= \left\|\chi_{m_1}^\delta - \chi_{m_2}^\delta\right\|_{L^2(\Omega)}^2 \\
&= \|(\chi_{m_1} - \chi_{m_2}) * \xi_\delta\|_{L^2(\Omega)}^2 \\
&\leq \|\chi_{m_1} - \chi_{m_2}\|_{L^2(\Omega)}^2 \|\xi_\delta\|_{L^1(\Omega)}^2 \\
&= \mu\left(\text{supp}\left(\chi_{m_1}^\delta\right) \triangle \text{supp}\left(\chi_{m_2}^\delta\right)\right) \|\xi_\delta\|_{L^1(\Omega)}^2 \\
&= 0.
\end{aligned}$$

Because measures of any set is nonnegative, our assertion follows. □

The above proposition is equivalent to stating that as long as the supports of $\{\chi_m\}_{m=1}^{M-1}$ are disjoint, then the supports of $\left\{\chi_m^\delta\right\}_{m=1}^{M-1}$ are disjoint as well. Recall that the explicit formulation of G stated in (3.4.2) is

$$G\left(\chi_m^\delta\right) = \chi_m^\delta - \omega\left[-2\sigma_m^{k+1}\left(\chi_m^\delta\right)\psi\left(\chi_m^\delta\right) + \alpha|\nabla\sigma_m^{k+1}\left(\chi_m^\delta\right)|^2\right].$$

For example, if we take $\omega = 0$, then $G\left(\chi_m^\delta\right) = \chi_m^\delta$. In (3.2.10), we justified that $M \circ T$ is basically just a thresholding of $\Theta \circ G$. Hence, $(M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right) = \chi_m^\delta$. Again, using

the above proposition, $\Upsilon_m(\chi_m) = (T_\delta \circ M \circ H \circ \Theta \circ G)\left(\chi_m^\delta\right)$ gives a set of functions whose supports are disjoint. We are not saying that we should take $\omega = 0$ because that means we will get the same χ_m at the end of the iteration of Algorithm (2). What we can do is to employ a backtracking method. That is, if the resulting $\{\Upsilon_m(\chi_m)\}_{m=1}^{M-1}$ have supports that are not disjoint, we can choose ω to be smaller. This is why in the first step of Algorithm (2), we included a condition for the initial guess of $\{\chi_m\}_{m=1}^{M-1}$ to have disjoint supports. Thus, addressing our initial problem. This guarantees that $0 \leq \chi_M^k = 1 - \sum_{m=1}^{M-1} \chi_m^k \leq 1$, where k denotes the number of iterations of Algorithm (2).

We introduced Υ in (3.4.1) as the functional representation of the update of χ_m. In the proof of Theorem (44), we showed that Υ continuously depends on χ_m. Let $k \in \mathbb{N}$. We introduce the notation

$$\Upsilon_m^{(k)}(\chi_m) := \Upsilon_m\left(\Upsilon_m^{(k-1)}(\chi_m)\right) \tag{3.4.33}$$

where $\Upsilon^{(0)}(\chi_m) := \mathrm{id}$. Observe that (3.4.33) gives the update of χ_m in k-th iteration in Algorithm (2). We end this chapter with the following result.

Theorem 46. *Given Assumption (1) and set $\chi_m \in L^2(\Omega)$ such that $0 \leq \chi_m \leq 1$ to be the initial guess in Algorithm (2) for some $m \in \{1, 2, \ldots, M\}$. Then the result of Algorithm (2) after a finite number of iterations depends continuously on χ_m. In other words, $\Upsilon_m^{(k)}$ as given by (3.4.33) is continuous for every $k \in \mathbb{N}$.*

Proof. We prove this by induction. Suppose $k = 1$. This means that $\Upsilon_m^{(k)}$ is the same as Υ_m in (3.4.1) . Therefore, Υ_k is continuous as shown in the proof of Theorem (44). Suppose $\Upsilon_m^{(k-1)}$ is continuous then $\Upsilon_m^{(k-1)}(\chi_m) = \Upsilon_m\left(\Upsilon^{(k-1)}(\chi_m)\right)$ by definition. Because composition of continuous functions is continuous, the proof is complete. □

Remark 47. In Algorithm (2), the initial estimate for χ_M is calculated using the initial estimates for $\chi_1, \chi_2, \ldots, \chi_{M-1}$. Thus, the case when $m = M$ does not need to be tackled.

Chapter 4

Numerical Approximation

We have shown that the forward and the adjoint problems both have unique solutions in $H^1(\Omega)$. We have shown as well that the update σ_m^{k+1} is the solution in $H^1(\Omega)$ of our calculated optimality system. Furthermore, the function G was explicitly formulated and a variational formulation arising from TV regularization with solution $\Theta \in H^1(\Omega)$ was obtained. Four variational formulations of different variational formulations are included in the proposed algorithm. Thus, there is a need to find a suitable approximation space that we will use in our numerical calculations. But in order to proceed with finding discretized solutions of the partial differential equations in our proposed method, we have to show that we have consistent numerical approximations. We will establish this through the Finite Element Method (FEM).

We will begin by defining spline approximation spaces in one dimension and extending the notion in two dimensions using tensor product. Let $\Omega^{1D} := (0,1)$ and suppose $\Omega_i^{1D} := (x_{i-1}, x_i)$ is a grid on Ω^{1D} with stepsize $h = 1/N$ and nodes $x = ih$ for $i \in \{0, 1, \ldots, N\}$ for a fixed N. We now define the spline bases (compare with [32])

$$\mathcal{S}_h^{(q)}\left(\Omega^{1D}\right) := \left\{s \in \mathcal{P}^q\left([x_{i-1}, x_i]\right) : s \in C^{q-1}\left(\Omega^{1D}\right), i = 1, 2, \ldots, N\right\}$$

where $q \in \{0, 1, \ldots\}$ and $\mathcal{P}^q\left([x_{i-1}, x_i]\right)$ is the space of polynomials of degree q on $[x_{i-1}, x_i]$. Let

$$\pi_q(x) = (\pi_{q-1} * \pi_0)(x)$$

denote the canonical spline of order q where π_0 is the characteristic function of $[0,1]$. For the basis of $S_h^{(q)}\left(\Omega^{1D}\right)$, we use

$$\bar{s}_{i+q+1}^{(q)}(x) := \pi_q\left(\frac{x - x_i}{h}\right)$$

where $i \in \{-q, -q+1, \ldots, N-1\}$. In the two-dimensional case, $\Omega = (0,1)^2$ and the space $\mathcal{S}_h^{(q)}(\Omega)$ is regarded as tensor products of these bases. For extensive discussions on construction of approximation spaces on different dimensions, one can refer to [37]. Thus, $\mathcal{S}_h^{(q)}(\Omega)$ is spanned by

$$\left\{\pi_q\left(\frac{x - x_i}{h}\right)\pi_q\left(\frac{y - y_j}{h}\right)\right\}_{i,j=\{-q,-q+1,\ldots,N-1\}}.$$

We used x and y to distinguish the first variable from the second variable. Observe that $\mathcal{S}_h^{(q)}\left(\Omega^{1D}\right)$ has dimension $N+q$ and that $\mathcal{S}_h^{(q)}(\Omega)$ has dimension $(N+q)^2$. Using

lexicographic ordering, we denote the basis of $\mathcal{S}_h^{(q)}(\Omega)$ as $\left\{s_l^{(q)}\right\}_{l=1}^{(N+q)^2}$. We also denote

$$X_l := (x_{i-1}, x_i) \otimes (y_{j-1}, y_j)$$

and

$$\bar{X}_l := [x_{i-1}, x_i] \otimes [y_{j-1}, y_j]$$

where l is the corresponding index in lexicographic ordering. Notice that using this scheme, Ω is discretized with $(N+q)^2$ nodes. For any real-valued function g, we define $g_{h,l}$ to be the function value of g at the $l-$th node. Thus, we can now define the approximation of f to be

$$g_h := \sum_{l=1}^{(N+q)^2} g_{h,l} s_l^{(q)}.$$

Throughout this chapter, we will use the following lemma. These are standard results on spline theory and so the proofs will be omitted. For example, one can find a detailed discussion of spline theory in [63] and [11].

Lemma 48.

1. *For any* $l \in \left\{1, \ldots, (N+q)^2\right\}$, $s_l^{(q)}(x) > 0$ *for all* $x \in X_l$ *and* $s_l^{(q)} = 0$ *for all* $x \notin \bar{X}_l$. *Furthermore,* $\sum_{l=1}^{(N+q)^2} s_l^{(q)}(x) = 1$ *for all* $x \in X_l$ *(see, e.g. [63]).*

2. *Let* $u \in H^m(\Omega)$ *and* u_h *be its spline interpolation. Then*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 h^{m-1} \left(\sum_{i=1}^{2} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

where $C_1 > 0$ *depends only on* m *(see, e.g. [11]).*

3. *Let* $0 \leq k \leq 1$. *Suppose* $u \in H^k(\Omega)$ *and* u_h *is its spline interpolation. We have*

$$\|u - u_h\|_{H^k(\Omega)} \leq C_2 h^{m-k} \left(\sum_{i=1}^{2} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

where $C_2 > 0$ *depends only on* m *(see, e.g. [11]).*

4. $S_h^{(q)}(\Omega)$ *is dense in* $H^q(\Omega)$. *If* $q = 0$, $H^q(\Omega)$ *is identified to be* $L^2(\Omega)$ *(see, e.g. [63] and [32]).*

Remark 49. The third statement in Lemma (48) also holds in Ω^{1D} for $k = 0$ and $m = 1$. That is, if $g \in H^1\left(\Omega^{1D}\right)$ then

$$\|g - g_h\|_{L^2(\Omega^{1D})} \leq \bar{C} h \|Dg\|_{L^2(\Omega^{1D})}$$

for some $\bar{C} > 0$ independent of g and where

$$g_h = \sum_{l=1}^{N+1} g_{h,l} \bar{s}_l^{(1)}.$$

Recall that the forward problem is solving for ϕ given σ^k and f. To approximate σ_m^k, we will use $\mathcal{S}_h^{(1)}(\Omega)$. Hence,

$$\sigma_{m,h}^k = \sum_{l=1}^{(N+1)^2} \sigma_{h,l}^k s_l^{(1)}.$$

In Algorithm (2), we used χ_m^δ instead of χ_m. Hence, σ^k is a function of χ_m^δ. To emphasize this dependence, we are going to use the notation

$$\sigma^{k,\delta} := \sigma^k\left(\chi_m^\delta\right) = \sum_{m=1}^{M} \sigma_m^k \chi_m^\delta. \tag{4.0.1}$$

We will do the same for ϕ, ϕ^* and σ_m^{k+1}. That is, we denote

$$\phi^\delta := \phi\left(\chi_m^\delta\right), \quad \phi^{\delta *} := \phi^*\left(\chi_m^\delta\right), \quad \sigma_m^{k+1,\delta} := \sigma_m^{k+1}\left(\chi_m^\delta\right). \tag{4.0.2}$$

To approximate σ_m^k, we will use $\mathcal{S}_h^{(1)}(\Omega)$. Hence,

$$\sigma_{m,h}^k = \sum_{l=1}^{(N+1)^2} \sigma_{h,l}^k s_l^{(1)}.$$

Because f is a function defined only on $\partial\Omega$, we need to modify the basis functions by getting their restrictions on $\partial\Omega$:

$$s_{l,r}^{(1)}(x) := s_l^{(1)}|_{\partial\Omega}.$$

Note that the restriction will make the basis functions as functions in one dimension. Define $\mathcal{I} := \left\{l \in \left\{1, 2, \ldots, (N+1)^2\right\} : s_l^{(1)}(x) \neq 0 \text{ for some } x \in \partial\Omega\right\}$. In other words, $\mathcal{I}$ is the set of nodes on $\partial\Omega$. With this, we can now define the approximation of f by

$$f_h := \sum_{l\in\mathcal{I}} f_{h,l} s_{l,r}^{(1)}. \tag{4.0.3}$$

Because $\partial\Omega$ is simply a union of one-dimensional line segments, $s_{l,r}^{(1)}$ becomes a one-dimensional spline basis. Hence, we can view f_h as a union of one-dimensional approximations of f. Similarly, for the adjoint problem, we also need to discretize $\tilde{V}$:

$$\tilde{V}_h := \sum_{l\in\mathcal{I}} \tilde{V}_{h,l} s_{l,r}^{(1)}. \tag{4.0.4}$$

In solving for the update of σ_m in Algorithm (2), recall that we need to solve an optimality system given χ_m^δ. Because we view χ_m^δ as a characteristic function, it is fitting to take $\mathcal{S}_h^{(0)}(\Omega)$. Thus,

$$\chi_{m,h}^\delta := \sum_{l=1}^{N^2} \chi_{m,h,l}^\delta s_l^{(0)}.$$

We know that $0 \leq \chi_m^\delta \leq 1$. Observe that $\chi_{m,h}^\delta(x) = \sum_{l=1}^{N^2} \chi_{m,h,l}^\delta s_l^{(0)}(x) \leq \sum_{l=1}^{N^2} s_l^{(0)}(x) = 1$ for all $x \in X_l$, $l \in \left\{1, \ldots, N^2\right\}$ by Lemma (48). On the other hand, $\chi_{m,h}^\delta(x) = \sum_{l=1}^{N^2} \chi_{m,h,l}^\delta s_l^{(0)}(x) \geq 0$. Because $\Omega \backslash \cup_{l=1}^{N^2} X_l$ has zero measure, then

$$0 \leq \chi_{m,h}^\delta \leq 1 \quad \text{almost everywhere in } \Omega. \tag{4.0.5}$$

Similar to (4.0.1), we use the notation

$$\sigma_h^{k,\delta} := \sum_{m=1}^{M} \sigma_{m,h}^k \chi_{m,h}^\delta \tag{4.0.6}$$

to be the discretization of $\sigma^{k,\delta}$. By Assumption (1), $\sigma^{k,\delta}(x) \geq \underline{\sigma} > 0$. Therefore, using a similar argumentation as (4.0.5), we obtain

$$\sigma_h^{k,\delta}(x) \geq \underline{\sigma} > 0 \quad \text{almost everywhere in } \Omega. \tag{4.0.7}$$

Furthermore, because $\sigma_h^{k,\delta}$ is just a sum of products of essentially bounded functions, then

$$\sigma_h^{k,\delta} \in L^\infty(\Omega). \tag{4.0.8}$$

The following will be essential in showing that the discrete solutions of the partial differential equations converge to the actual solutions as h tends to zero.

Lemma 50. *Given Assumption (1), the discretizations $\sigma_{m,h}^k$, f_h, $\tilde{V}_h$ and $\chi_{m,h}^\delta$ satisfy*

$$\lim_{h\to 0} \left\| \sigma_{m,h}^k - \sigma_m^k \right\|_{L^\infty(\Omega)} = 0 \tag{4.0.9}$$

$$\lim_{h\to 0} \| f_h - f \|_{L^2(\partial\Omega)} = 0 \tag{4.0.10}$$

$$\lim_{h\to 0} \left\| \tilde{V}_h - \tilde{V} \right\|_{L^2(\partial\Omega)} = 0 \tag{4.0.11}$$

$$\lim_{h\to 0} \left\| \chi_{m,h}^\delta - \chi_m^\delta \right\|_{L^\infty(\Omega)} = 0. \tag{4.0.12}$$

Proof. These results directly follow from Lemma (48) and Remark (49). □

Corollary 51. *Given Assumption (1), then*

$$\lim_{h\to 0} \left\| \sigma_h^{k,\delta} - \sigma^{k,\delta} \right\|_{L^\infty(\Omega)} = 0. \tag{4.0.13}$$

Proof. By (4.0.1) and (4.0.6), we get

$$\sigma_h^{k,\delta} - \sigma^{k,\delta} = \sum_{m=1}^{M} \left(\sigma_{m,h}^k \chi_{m,h}^\delta - \sigma_m^k \chi_m^\delta \right).$$

Let $m \in \{1, 2, \ldots, M\}$. It is sufficient to show that $\sigma_{m,h}^k \chi_{m,h}^\delta - \sigma_m^k \chi_m^\delta$ converges to 0 in $L^\infty(\Omega)$ because of the triangle inequality. Indeed, by adding and subtracting $\sigma_m^k \chi_{m,h}^\delta$ and with the help of triangle inequality and the Hölder's inequality, we obtain

$$\begin{aligned} \left\| \sigma_{m,h}^k \chi_{m,h}^\delta - \sigma_m^k \chi_m^\delta \right\|_{L^\infty(\Omega)} &\leq \left\| \sigma_{m,h}^k \chi_{m,h}^\delta - \sigma_m^k \chi_{m,h}^\delta \right\|_{L^\infty(\Omega)} + \left\| \sigma_m^k \chi_{m,h}^\delta - \sigma_m^k \chi_m^\delta \right\|_{L^\infty(\Omega)} \\ &\leq \left\| \sigma_{m,h}^k - \sigma_m^k \right\|_{L^\infty(\Omega)} + \left\| \sigma_m^k \right\|_{L^\infty(\Omega)} \left\| \chi_{m,h}^\delta - \chi_m^\delta \right\|_{L^\infty(\Omega)}. \end{aligned}$$

Note that $\sigma_m^k \in L^\infty(\Omega)$ because by Assumption (1), $\sigma_m^k \in C^\infty(\bar{\Omega})$. Finally, taking the limit of the right-hand side as $h \to 0$, (4.0.9) and (4.0.12) imply our claim. □

Remark 52. Let $g \in \mathcal{S}_h^{(1)}(\Omega)$ and $l \in \{1, 2, \ldots, (N+1)^2\}$. By definition, g is a tensor product of two linear polynomials in X_l. Therefore, the derivative of g with respect to the first variable is a linear function on the second variable and vice versa. Thus, $\nabla g \in L^\infty(X_l)$. Furthermore, because $\Omega \setminus \cup_{l=1}^{(N+1)^2} X_l$ is a set of measure zero, then $\nabla g \in L^\infty(\Omega)$.

Now that we have the necessary tools from spline theory, we can now analyze the numerical approximation of our proposed method. We begin by showing the well-posedness and then the convergence of the finite element formulations of the forward and the adjoint problems. We then proceed with the same analysis of the update for σ_m, the function G and the *pre-tresholding* update Θ. Finally, we will present the discretization of Algorithm (2).

4.1 Analysis of the Finite Element Approaches for the Proposed Method

Given a $\sigma^k \in L^\infty(\Omega)$, recall from Theorem (6) that the forward problem has a unique solution $\phi \in H^1(\Omega)$. Because of the third statement of Lemma (48), it is logical to find a solution ϕ_h of the discretized forward problem in $\mathcal{S}_h^{(1)}(\Omega)$. Hence, the finite element discretization of the variational formulation of the forward problem is

$$a_h(\phi_h, v_h) = b_h(v_h), \quad \forall v_h \in \mathcal{S}_h^{(1)}(\Omega) \tag{4.1.1}$$

where $a_h : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$a_h(u_h, v_h) := \int_\Omega \sigma_h^{k,\delta} \nabla u_h \cdot \nabla v_h dV \tag{4.1.2}$$

where $\sigma_h^{k,\delta}$ is the discretization of $\sigma^{k,\delta}$ and $\int_{\partial\Omega} u_h dS = 0$ and $b_h(v) : H^1(\Omega) \to \mathbb{R}$ is defined by

$$b_h(v) := \int_{\partial\Omega} f_h v_h dS. \tag{4.1.3}$$

In this section, we aim to show that the solution ϕ_h^δ of (4.1.1) exists and that ϕ_h^δ converges to ϕ^δ in $H^1(\Omega)$ as we make the mesh size h approach zero. A similar analysis will be done to the other variational formulations involved in our proposed method. We use the notation ϕ_h^δ as the solution of (4.1.1) because we want to emphasize its dependence on χ_m^δ and to indicate that it is a discretized solution. We will make use of the same notation when we make the analysis of the finite element approach for the adjoint problem and the update for σ_m.

Theorem 53. *Given Assumption (1), there exists a unique $\phi_h^\delta \in \mathcal{S}_h^{(1)}(\Omega)$ satisfying (4.1.1) with $\int_{\partial\Omega} \phi_h^\delta dS = 0$.*

Proof. The proof is very similar to the proof of Theorem (6). Clearly, a_h is bilinear and b_h is linear. As a consequence of the continuous case, a_h is continuous:

$$|a_h(u_h, v_h)| \leq \left\|\sigma_h^{k,\delta}\right\|_{L^\infty(\Omega)} \|u_h\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)}. \tag{4.1.4}$$

Observe that the right-hand side of (4.1.4) is bounded because of (4.0.8). Observe that $f_h \in L^2(\partial\Omega)$ because it is a linear combination of $\left\{s_{l,r}^{(1)}\right\}_{l\in\mathcal{I}}$ which are functions in $C^0\left(\bar{\Omega}\right)$. Thus, we can show the continuity of $b_h(v)$ by

$$|b_h(v)| \leq \|f_h\|_{L^2(\partial\Omega)} \|v_h\|_{H^1(\Omega)}. \tag{4.1.5}$$

Coercivity of a_h in $H^1(\Omega)$ can be proven using the same argumentation as in Theorem (6) because of (4.0.7). Thus,

$$|a_h(u_h, u_h)| \geq \frac{\sigma}{2} \min\left\{\frac{1}{C}, 1\right\} \|u_h\|_{H^1(\Omega)}^2$$

for some $C > 0$. Since $\mathcal{S}_h^{(1)}(\Omega) \subset H^1(\Omega)$ by Lemma (48), the Lax-Milgram Theorem guarantees the unique solution of (4.1.1). □

In order to proceed with proving the convergence of the discretization of the forward problem, we will need the following lemma.

Lemma 54. *Let u be the solution of the variational equations $a(u,v) = b(v)$ for all $v \in H^1(\Omega)$ and u_h be the solution of the variational equation $a_h(u_h, v_h) = b_h(v_h)$ for all $v_h \in \mathcal{S}_h^{(1)}(\Omega)$. Suppose there exists $C_1 > 0$ independent of h such that*

$$C_1 \|u_h - w_h\|_{H^1(\Omega)} \leq \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{a_h(u_h - w_h, v_h)}{\|v_h\|_{H^1(\Omega)}} \tag{4.1.6}$$

and $C_2 > 0$ also independent of h, such that

$$a(u, v_h) \leq C_2 \|u\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} \tag{4.1.7}$$

for all $u \in H^1(\Omega)$ and $v_h \in \mathcal{S}_h^{(1)}(\Omega)$. Then u and u_h satisfy

$$\begin{aligned}\|u - u_h\|_{H^1(\Omega)} \leq \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} + \\ \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{ \left(1 + \frac{C_2}{C_1}\right) \|u - w_h\|_{H^1(\Omega)} \right. \\ \left. + \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}} \right\}.\end{aligned} \tag{4.1.8}$$

Proof. Let $w_h \in \mathcal{S}_h^{(1)}(\Omega)$. Because $\mathcal{S}_h^{(1)}(\Omega) \subset H^1(\Omega)$ then $a(u, v_h) = b(v_h)$ for all $v_h \in \mathcal{S}_h^{(1)}(\Omega)$. Moreover, $a_h(u_h, v_h) = b_h(v_h)$ for all $v_h \in \mathcal{S}_h^{(1)}(\Omega)$. Hence,

$$a_h(u_h - w_h, v_h) = a(u - w_h, v_h) + a(w_h, v_h) - a_h(w_h, v_h) + b_h(v_h) - b(v_h).$$

Dividing the above equation by $\|v_h\|_{H^1(\Omega)}$ and taking the supremum over all $v_h \in \mathcal{S}_h^{(1)}(\Omega)$ will give us

$$\begin{aligned}\sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{a_h(u_h - w_h, v_h)}{\|v_h\|_{H^1(\Omega)}} \leq \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{a_h(u - w_h, v_h)}{\|v_h\|_{H^1(\Omega)}} \\ + \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} + \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}}.\end{aligned}$$

Assumptions (4.1.6) and (4.1.7) will then give us

$$\begin{aligned} C_1 \left\| u_h - w_h \right\|_{H^1(\Omega)} \leq C_2 \left\| u - w_h \right\|_{H^1(\Omega)} & \\ + \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} & \frac{\left| b_h\left(v_h\right) - b\left(v_h\right) \right|}{\left\| v_h \right\|_{H^1(\Omega)}} \\ + \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} & \frac{\left| a\left(w_h, v_h\right) - a_h\left(w_h, v_h\right) \right|}{\left\| v_h \right\|_{H^1(\Omega)}}. \end{aligned} \tag{4.1.9}$$

Observe that using triangle inequality, we have

$$\left\| u - u_h \right\|_{H^1(\Omega)} \leq \left\| u - w_h \right\|_{H^1(\Omega)} + \left\| u_h - w_h \right\|_{H^1(\Omega)}. \tag{4.1.10}$$

Combining (4.1.9) and (4.1.10) gives us

$$\begin{aligned} \left\| u - u_h \right\|_{H^1(\Omega)} \leq \left\| u - w_h \right\|_{H^1(\Omega)} + \frac{C_2}{C_1} \left\| u - w_h \right\|_{H^1(\Omega)} & \\ + \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{\left| a\left(w_h, v_h\right) - a_h\left(w_h, v_h\right) \right|}{\left\| v_h \right\|_{H^1(\Omega)}} + \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} & \frac{\left| b_h\left(v_h\right) - b\left(v_h\right) \right|}{\left\| v_h \right\|_{H^1(\Omega)}}. \end{aligned}$$

Finally, taking the infimum of the above inequality over all $w_h \in \mathcal{S}_h^{(1)}(\Omega)$ proves our claim. □

The above lemma is also known as the *First Strang Lemma*. One can compare the proof of the lemma with the proof in [18] and [19]. This lemma works particularly well with non-consistent finite element approaches. Non-consistent methods are techniques that use different variational formulation for the approximation. That is, $a_h \neq a$. Our finite element approaches fit to this category. Thus, we employ the above lemma to prove convergence of the discretizations of our finite element approaches.

Theorem 55. *Given Assumption (1), then*

$$\lim_{h \to 0} \left\| \phi_h^\delta - \phi^\delta \right\|_{H^1(\Omega)} = 0. \tag{4.1.11}$$

Proof. Recall that ϕ^δ satisfies the variational formulation

$$a\left(\phi^\delta, v\right) = b\left(v\right), \quad \forall v \in H^1\left(\Omega\right) \tag{4.1.12}$$

where

$$a\left(\phi^\delta, v\right) = \int_\Omega \sigma^\delta \nabla \phi^\delta \cdot \nabla v dV$$

$$b\left(v\right) = \int_{\partial\Omega} f v dS$$

and $\int_{\partial\Omega} \phi^\delta dS = 0$. Furthermore, ϕ_h^δ satisfies the discretized variational formulation $a_h\left(\phi_h^\delta, v_h\right) = b_h\left(v_h\right)$ for all $v_h \in \mathcal{S}_h^{(1)}(\Omega)$ defined in (4.1.1). In order to utilize Lemma (54), we have to make sure that its conditions are satisfied. The first condition (4.1.6) is easily justified by the coercivity of a_h in the proof of Theorem (53). The second condition (4.1.7) is guaranteed by the continuity of a

$$a\left(\phi^\delta, v_h\right) \leq \left| a\left(\phi^\delta, v_h\right) \right| \leq \left\| \sigma^{k,\delta} \right\|_{L^\infty(\Omega)} \left\| \phi^\delta \right\|_{H^1(\Omega)} \left\| v_h \right\|_{H^1(\Omega)}$$

because $\left\|\sigma^{k,\delta}\right\|_{L^\infty(\Omega)}$ is clearly independent of h. Hence by Lemma (54), there exists $C_1, C_2 > 0$ both independent of h such that

$$\begin{aligned}\left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \leq \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} + \\ \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{ \left(1 + \frac{C_2}{C_1}\right) \left\|\phi^\delta - w_h\right\|_{H^1(\Omega)} \right. \\ \left. + \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}} \right\}. \end{aligned} \tag{4.1.13}$$

By the aid of the Cauchy-Schwarz inequality and the Trace theorem, we have

$$\begin{aligned}|b_h(v_h) - b(v_h)| &= \int_{\partial\Omega} (f_h - f) v_h dS \\ &\leq \|f_h - f\|_{L^2(\partial\Omega)} \|v_h\|_{L^2(\partial\Omega)} \\ &\leq C_4 \|f_h - f\|_{L^2(\partial\Omega)} \|v_h\|_{H^1(\Omega)}\end{aligned}$$

for some $C_4 > 0$. Hence,

$$\frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} \leq \frac{C_4}{C_1} \|f_h - f\|_{L^2(\partial\Omega)}. \tag{4.1.14}$$

We can also infer that

$$\begin{aligned}|a(w_h, v_h) - a_h(w_h, v_h)| &\leq \int_\Omega \left| \left(\sigma^{k,\delta} - \sigma_h^{k,\delta}\right) \nabla w_h \cdot \nabla v_h \right| dV \\ &\leq \left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \|\nabla w_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \\ &\leq \left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)}.\end{aligned}$$

Therefore,

$$\frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}} \leq \frac{1}{C_1} \left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)}.$$

The above inequality and (4.1.14) imply that (4.1.13) can be expressed as

$$\left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \leq \frac{C_4}{C_1} \|f_h - f\|_{L^2(\partial\Omega)} + A(h) \tag{4.1.15}$$

where

$$A(h) := \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{ \left(1 + \frac{C_2}{C_1}\right) \left\|\phi^\delta - w_h\right\|_{H^1(\Omega)} + \frac{1}{C_1} \left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)} \right\}.$$

Because ϕ^δ is in $H^1(\Omega)$, we can interpolate it by an element $\mathcal{I}_h\phi^\delta$ in $\mathcal{S}_h^{(1)}(\Omega)$. It is worth noting that $\mathcal{I}_h\phi^\delta$ is different from ϕ_h^δ. The quantity ϕ_h^δ is the solution of the discretized variational formulation while $\mathcal{I}_h\phi^\delta$ is obtained from ϕ^δ by interpolation. Now, it is obvious that

$$A(h) \leq \left(1 + \frac{C_2}{C_1}\right) \left\|\phi^\delta - \mathcal{I}_h\phi^\delta\right\|_{H^1(\Omega)} + \frac{1}{C_1} \left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \left\|\mathcal{I}_h\phi^\delta\right\|_{H^1(\Omega)}. \tag{4.1.16}$$

Using triangle inequality,

$$\left\|\mathcal{I}_h\phi^\delta\right\|_{H^1(\Omega)} \leq \left\|\phi^\delta - \mathcal{I}_h\phi^\delta\right\|_{H^1(\Omega)} + \left\|\phi^\delta\right\|_{H^1(\Omega)}. \tag{4.1.17}$$

So (4.1.16), (4.1.17) and Lemma (48) imply

$$\begin{aligned} A(h) \leq & \left(1+\frac{C_2}{C_1}\right) C_3 h \left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} \\ & + \frac{1}{C_1}\left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \left\{C_3 h \left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} + \left\|\phi^\delta\right\|_{H^1(\Omega)}\right\}. \end{aligned}$$

The above inequality, compared with (4.1.15), gives us

$$\begin{aligned} \left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \leq & \frac{C_4}{C_1}\left\|f_h - f\right\|_{L^2(\partial\Omega)} + \left(1+\frac{C_2}{C_1}\right) C_3 h \left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} \\ & + \frac{1}{C_1}\left\|\sigma_h^{k,\delta} - \sigma^{k,\delta}\right\|_{L^\infty(\Omega)} \left\{C_3 h \left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} + \left\|\phi^\delta\right\|_{H^1(\Omega)}\right\}. \end{aligned}$$

Because of (4.0.13) and (4.0.10), taking the limit as $h \to 0$ will give us our desired result. □

We will do a similar computation on the discretization of the adjoint problem. We will use the same $a_h(u_h, v_h)$ in (4.1.2) and a different $b_h(v_h)$. We will instead use

$$b_h(v_h) = \int_{\partial\Omega}\left(\phi_h^\delta - \tilde{V}_h\right) v_h dS \tag{4.1.18}$$

where $b_h(v_h) : H^1(\Omega) \to \mathbb{R}$ and $\tilde{V}_h$ is the discretization of the given boundary voltages. Hence, the discrete adjoint EIT problem is solving

$$\int_\Omega \sigma_h^{k,\delta}\nabla\phi_h^{\delta *}\cdot\nabla v_h dV = \int_{\partial\Omega}\left(\phi_h^\delta - \tilde{V}_h\right) v_h dS \tag{4.1.19}$$

for any $v_h \in \mathcal{S}_h^{(1)}(\Omega)$.

Theorem 56. *Given Assumption (1), there exists $\phi_h^{\delta *} \in \mathcal{S}_h^{(1)}(\Omega)$ satisfying (4.1.19) with $\int_{\partial\Omega}\phi_h^{\delta *} dS = 0$.*

Proof. The proof is very similar to the proof of Theorem (53). In Theorem (53), we already proved that a_h is bilinear, continuous and coercive. We only need to show that b_h as defined in (4.1.18) is continuous. Indeed,

$$\begin{aligned} |b_h(v_h)| &\leq \left\|\phi_h^\delta - \tilde{V}_h\right\|_{L^2(\partial\Omega)}\left\|v_h\right\|_{L^2(\partial\Omega)} \\ &\leq C\left\|\phi_h^\delta - \tilde{V}_h\right\|_{L^2(\partial\Omega)}\left\|v_h\right\|_{H^1(\Omega)} \end{aligned} \tag{4.1.20}$$

for some $C > 0$. By the Lax-Milgram Theorem, unique solvability of (4.1.19) is assured. □

We will now show the convergence of the discretization of the adjoint problem.

Theorem 57. *Given Assumption (1), then*

$$\lim_{h\to 0}\left\|\phi_h^{\delta *}-\phi^{\delta *}\right\|_{H^1(\Omega)}=0. \tag{4.1.21}$$

Proof. The proof will be similar to the proof of Theorem (55). Recall that $\phi^{\delta *}$ satisfies the variational formulation

$$a\left(\phi^{\delta *},v\right)=b\left(v\right),\quad \forall v\in H^1\left(\Omega\right)$$

where

$$a\left(\phi^{\delta *},v\right)=\int_\Omega \sigma^\delta\nabla\phi^{\delta *}\cdot\nabla v dV$$

$$b\left(v\right)=\int_{\partial\Omega}\left(\phi^\delta-\tilde{V}\right)v dS$$

and $\int_{\partial\Omega}\phi^{\delta *}dS=0$. Also, $\phi_h^{\delta *}$ satisfies (4.1.19). Since we are using the same bilinear functionals a and a_h, the conditions of Lemma (54) are satisfied as shown in the proof of Theorem (55). Therefore, there exist $C_1,C_2>0$ independent of h such that

$$\left\|\phi_h^{\delta *}-\phi^{\delta *}\right\|_{H^1(\Omega)}\leq\frac{1}{C_1}\sup_{v_h\in\mathcal{S}_h^{(1)}(\Omega)}\frac{\left|b_h\left(v_h\right)-b\left(v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}}+A(h) \tag{4.1.22}$$

where

$$A(h):=\inf_{w_h\in\mathcal{S}_h^{(1)}(\Omega)}\left\{\left(1+\frac{C_2}{C_1}\right)\left\|\phi^{\delta *}-w_h\right\|_{H^1(\Omega)}+\frac{1}{C_1}\sup_{v_h\in\mathcal{S}_h^{(1)}(\Omega)}\frac{\left|a\left(w_h,v_h\right)-a_h\left(w_h,v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}}\right\}.$$

Similar to the proof of Theorem (55), $A(h)$ can be estimated by

$$\begin{aligned}A(h)\leq&\left(1+\frac{C_2}{C_1}\right)C_3h\left(\sum_{i=1}^2\left\|\frac{\partial^2\phi^{\delta *}}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2}\\&+\frac{1}{C_1}\left\|\sigma_h^{k,\delta}-\sigma^{k,\delta}\right\|_{L^\infty(\Omega)}\left\{C_3h\left(\sum_{i=1}^2\left\|\frac{\partial^2\phi^{\delta *}}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2}+\left\|\phi^{\delta *}\right\|_{H^1(\Omega)}\right\}.\end{aligned} \tag{4.1.23}$$

Note that using Cauchy-Schwarz inequality, triangle inequality and the application of the Trace theorem, we have

$$\begin{aligned}\left|b_h\left(v_h\right)-b\left(v_h\right)\right|&\leq\left\|\left(\phi_h^\delta-\phi^\delta\right)-\left(\tilde{V}_h-\tilde{V}\right)\right\|_{\tilde{L}^2(\Omega)}\left\|v_h\right\|_{L^2(\partial\Omega)}\\&\leq\left(\left\|\phi_h^\delta-\phi^\delta\right\|_{\tilde{L}^2(\Omega)}+\left\|\tilde{V}_h-\tilde{V}\right\|_{\tilde{L}^2(\Omega)}\right)\left\|v_h\right\|_{L^2(\partial\Omega)}\\&\leq\left(C_4\left\|\phi_h^\delta-\phi^\delta\right\|_{H^1(\Omega)}+\left\|\tilde{V}_h-\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right)C_5\left\|v_h\right\|_{H^1(\Omega)}\end{aligned}$$

for some $C_4,C_5>0$. Therefore,

$$\frac{1}{C_1}\sup_{v_h\in\mathcal{S}_h^{(1)}(\Omega)}\frac{\left|b_h\left(v_h\right)-b\left(v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}}\leq C_5\left(C_4\left\|\phi_h^\delta-\phi^\delta\right\|_{H^1(\Omega)}+\left\|\tilde{V}_h-\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right). \tag{4.1.24}$$

Hence, (4.1.22), (4.1.23) and (4.1.24) give us

$$\begin{aligned}\left\|\phi_h^{\delta*}-\phi^{\delta*}\right\|_{H^1(\Omega)} &\leq C_5\left(C_4\left\|\phi_h^{\delta}-\phi^{\delta}\right\|_{H^1(\Omega)}+\left\|\tilde{V}_h-\tilde{V}\right\|_{\tilde{L}^2(\partial\Omega)}\right)\\ &+\left(1+\frac{C_2}{C_1}\right)C_3h\left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^{\delta*}}{\partial x_i^2}\right\|^2_{L^2(\Omega)}\right)^{1/2}\\ &+\frac{1}{C_1}\left\|\sigma_h^{k,\delta}-\sigma^{k,\delta}\right\|_{L^\infty(\Omega)}\left\{C_3h\left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^{\delta*}}{\partial x_i^2}\right\|^2_{L^2(\Omega)}\right)^{1/2}+\left\|\phi^{\delta*}\right\|_{H^1(\Omega)}\right\}.\end{aligned}$$

Using (4.0.13), (4.0.11) and Theorem (55), we can observe that taking the limit of the above inequality as $h\to 0$ means that the limit of $\|\phi_h^*-\phi^*\|_{H^1(\Omega)}\to 0$ as well. □

In Lemma (32), we proved that the update σ_m^{k+1} is in $H^1(\Omega)$ given Assumption (1) and using the mollification of χ_m. Hence, we choose $\mathcal{S}_h^{(1)}(\Omega)$ as the space of our approximation. Thus, the finite element discretization of (3.3.59) is given by

$$a_h(u_h,v_h)=b_h(v_h)\qquad\text{for all } v\in\mathcal{S}_h^{(1)}(\Omega)\tag{4.1.25}$$

where $a_h: H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is given by

$$a_h(u_h,v_h):=\int_\Omega\alpha\left(\chi_{m,h}^{\delta}+\epsilon\right)\nabla u_h\cdot\nabla v_h dV+\theta\int_\Omega u_h v_h dV\tag{4.1.26}$$

and $b_h(v_h): H^1(\Omega)\to\mathbb{R}$ is defined by

$$b_h(v_h):=\int_\Omega\chi_{m,h}^{\delta}\nabla\phi_h^{\delta}\cdot\nabla\phi_h^{\delta*}v_h dV+\int_\Omega\theta\sigma_{m,h}^{k}v_h dV.$$

Theorem 58. *Given Assumption (1), there exists a unique $\sigma_{m,h}^{k+1,\delta}\in\mathcal{S}_h^{(1)}(\Omega)$ satisfying (4.1.25).*

Proof. The proof is similar to the proof of the continuous case. Clearly, a_h is bilinear and b_h is linear. Using the Cauchy-Schwarz inequality, we can prove that a_h is continuous:

$$|a_h(u_h,v_h)|\leq 2\max\{\alpha(1+\epsilon),\theta\}\|u_h\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}.\tag{4.1.27}$$

Because of (4.0.5), the coercivity of $a_h(u_h,v_h)$ is also justified analogously:

$$|a_h(u_h,u_h)|\geq\min\{\alpha\epsilon,\theta\}\|u_h\|^2_{H^1(\Omega)}.$$

Furthermore, the continuity of $b_h(v_h)$ can be proven similarly:

$$|b_h(v_h)|\leq 2\sqrt{\mu(\Omega)}\max\left\{\left\|\nabla\phi_h^{\delta}\right\|_{L^\infty(\Omega)}\left\|\nabla\phi_h^{\delta*}\right\|_{L^\infty(\Omega)},\theta\left\|\sigma_{m,h}^{k}\right\|_{L^\infty(\Omega)}\right\}\|v_h\|_{H^1(\Omega)}.\tag{4.1.28}$$

The right-hand side of the last inequality is bounded because of Remark (52). Since $\mathcal{S}_h^{(1)}(\Omega)\subset H^1(\Omega)$ by Lemma (48), the Lax-Milgram Theorem implies that there is a unique $\sigma_m^k\in H^1(\Omega)$ satisfying (4.1.25) for all $v\in H^1(\Omega)$. □

Similar to what we did in the previous section, we will show that (4.1.25) has a unique solution that converges to $\sigma_m^{k+1,\delta}$ as $h \to 0$. But first we will show that $\left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)}$ does not diverge as h approaches 0.

Lemma 59. *Given Assumption (1),*

$$\lim_{h\to 0}\left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} < \infty \text{ and } \lim_{h\to 0}\left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} < \infty.$$

Proof. We will show this by contradiction. Suppose

$$\lim_{h\to 0}\left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} = \infty.$$

Let $R \in \mathbb{R}^+$. Then for h small enough, we have

$$\text{ess sup}\left|\nabla\phi_h^\delta\right| > R \tag{4.1.29}$$

in Ω. Recall that $\forall v_h \in \mathcal{S}_h^{(1)}(\Omega)$ such that $\int_{\partial\Omega} v_h dS = 0$, we have

$$\int_\Omega \sigma_h^{k,\delta}\nabla\phi_h^\delta \cdot \nabla v_h dV = \int_{\partial\Omega} f_h v_h dS.$$

Because $\phi_h^\delta \in \mathcal{S}_h^{(1)}(\Omega)$ and $\int_{\partial\Omega}\phi_h^\delta dS = 0$, then

$$\int_\Omega \sigma_h^{k,\delta}\left|\nabla\phi_h^\delta\right|^2 dV = \int_{\partial\Omega} f_h\phi_h^\delta dS.$$

Therefore, using (4.0.7) and (4.1.29), we get

$$\underline{\sigma}R^2\mu(\Omega) < \int_{\partial\Omega} f_h\phi_h^\delta dS.$$

By subtracting and adding f and ϕ^δ to f_h and ϕ_h^δ, respectively, using Cauchy-Schwarz inequality, triangle inequality and the Trace theorem, we obtain

$$\begin{aligned}
\underline{\sigma}R^2\mu(\Omega) &< \int_{\partial\Omega}(f_h - f + f)\left(\phi_h^\delta - \phi^\delta + \phi^\delta\right)dS \\
&\le \|f_h - f + f\|_{L^2(\partial\Omega)}\left\|\phi_h^\delta - \phi^\delta + \phi^\delta\right\|_{L^2(\partial\Omega)} \\
&\le C\|f_h - f + f\|_{L^2(\partial\Omega)}\left\|\phi_h^\delta - \phi^\delta + \phi^\delta\right\|_{H^1(\Omega)} \\
&\le C\left(\|f_h - f\|_{L^2(\partial\Omega)} + \|f\|_{L^2(\partial\Omega)}\right)\left(\left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} + \left\|\phi^\delta\right\|_{H^1(\Omega)}\right).
\end{aligned}$$

Since R is arbitrary, we can make it infinitely large so that h approaches 0. Therefore by (4.0.10) and (4.1.11), we will have $\|f\|_{L^2(\partial\Omega)}\left\|\phi^\delta\right\|_{H^1(\Omega)} \to \infty$ which is a contradiction. Hence,

$$\lim_{h\to 0}\left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} < \infty.$$

The other limit can be proved similarly. □

We again utilize the First Strang Lemma to prove convergence of the update as h approaches 0.

Theorem 60. *Given Assumption (1), then*

$$\lim_{h\to 0}\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_{m}^{k+1,\delta}\right\|_{H^1(\Omega)}=0. \tag{4.1.30}$$

Proof. Recall that $\sigma_m^{k+1,\delta}$ solves the variational formulation

$$a\left(\sigma_m^{k+1,\delta},v\right)=b\left(v\right)\qquad\text{for all } v\in H^1\left(\Omega\right)$$

where

$$a\left(\sigma_m^{k+1,\delta},v\right):=\int_\Omega\alpha\left(\chi_m^\delta+\epsilon\right)\nabla\sigma_m^{k+1,\delta}\cdot\nabla v dV+\theta\int_\Omega\sigma_m^{k+1,\delta}v dV$$

$$b\left(v\right):=\int_\Omega\chi_m^\delta\nabla\phi^\delta\cdot\nabla\phi^{\delta *}v dV+\int_\Omega\theta\sigma_m^k v dV.$$

Furthermore, $\sigma_{m,h}^{k+1,\delta}$ solves (4.1.25). Similar to how we prove convergence of the discretization of the forward and adjoint problems, we will also use Lemma (54). Thus, we need to show that the necessary conditions are satisfied. The first condition (4.1.6) is easily guaranteed by the coercivity of a_h in the proof of Theorem (58). The second condition (4.1.7) can be proved using the continuity of a, that is,

$$a\left(\sigma_m^{k+1,\delta},v_h\right)\le\left|a\left(\sigma_m^{k+1,\delta},v_h\right)\right|\le 2\max\left\{\alpha\left(1+\epsilon\right),\theta\right\}\left\|\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\left\|v_h\right\|_{H^1(\Omega)}.$$

Clearly, $2\max\left\{\alpha\left(1+\epsilon\right),\theta\right\}$ is independent of h and so by the First Strang Lemma, there exist $C_1,C_2>0$ both independent of h such that

$$\begin{aligned}\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\le&\frac{1}{C_1}\sup_{v_h\in\mathcal{S}_h^{(1)}(\Omega)}\frac{\left|b_h\left(v_h\right)-b\left(v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}}\\&+\inf_{w_h\in\mathcal{S}_h^{(1)}(\Omega)}\left\{\left(1+\frac{C_2}{C_1}\right)\left\|\sigma_m^{k+1,\delta}-w_h\right\|_{H^1(\Omega)}\right.\\&\left.+\frac{1}{C_1}\sup_{v_h\in\mathcal{S}_h^{(1)}(\Omega)}\frac{\left|a\left(w_h,v_h\right)-a_h\left(w_h,v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}}\right\}.\end{aligned} \tag{4.1.31}$$

Observe that by triangle inequality,

$$\left|b_h\left(v_h\right)-b\left(v_h\right)\right|\le\left|B\left(v_h\right)\right|+\left|D\left(v_h\right)\right| \tag{4.1.32}$$

where

$$B\left(v_h\right):=\int_\Omega\chi_{m,h}^\delta\nabla\phi_h^\delta\cdot\nabla\phi_h^{\delta *}v_h dV-\int_\Omega\chi_m^\delta\nabla\phi^\delta\cdot\nabla\phi^{\delta *}v_h dV$$

$$D\left(v_h\right):=\int_\Omega\theta\left(\sigma_{m,h}^k-\sigma_m^k\right)v dV.$$

We first estimate $D\left(v_h\right)$ using direct application of the Cauchy-Schwarz inequality:

$$\left|D\left(v_h\right)\right|\le\theta\left\|\sigma_{m,h}^k-\sigma_m^k\right\|_{L^2(\Omega)}\left\|v_h\right\|_{H^1(\Omega)}. \tag{4.1.33}$$

We need to subtract and add $\chi_m^\delta\nabla\phi_h^\delta\cdot\nabla\phi_h^{\delta *}v_h+\chi_{m,h}^\delta\nabla\phi_h^\delta\cdot\nabla\phi^{\delta *}v_h$ to $B\left(v_h\right)$ in order to find its bound. Doing so, we get

$$B\left(v_h\right)=B_1\left(v_h\right)+B_2\left(v_h\right)+B_3\left(v_h\right)$$

with

$$B_1(v_h) := \int_\Omega \left(\chi_{m,h}^\delta - \chi_m^\delta\right) \nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *} v_h dV$$

$$B_2(v_h) := \int_\Omega \chi_m^\delta \nabla\phi_h^\delta \cdot \left(\nabla\phi_h^{\delta *} - \nabla\phi^{\delta *}\right) v_h dV$$

and

$$B_3(v_h) := \int_\Omega \chi_m^\delta \left(\nabla\phi_h^\delta - \nabla\phi^\delta\right) \cdot \nabla\phi^{\delta *} v_h dV.$$

These can be estimated using the Hölder's inequality, the Cauchy-Schwarz inequality, Remark (52), (3.3.8) and the fact that $0 \le \chi_m^\delta \le 1$. Indeed,

$$\begin{aligned} |B_1(v_h)| &\le \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi_h^\delta\right\|_{L^2(\Omega)} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} \|v_h\|_{H^1(\Omega)}, \end{aligned} \tag{4.1.34}$$

$$\begin{aligned} |B_2(v_h)| &\le \left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla\phi_h^{\delta *} - \nabla\phi^{\delta *}\right\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le \left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^{\delta *} - \phi^{\delta *}\right\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} \end{aligned} \tag{4.1.35}$$

and

$$\begin{aligned} |B_3(v_h)| &\le \left\|\nabla\phi_h^\delta - \nabla\phi^\delta\right\|_{L^2(\Omega)} \left\|\nabla\phi^{\delta *}\right\|_{L^\infty(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le \left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi^{\delta *}\right\|_{L^\infty(\Omega)} \|v_h\|_{H^1(\Omega)}. \end{aligned} \tag{4.1.36}$$

Thus, from (4.1.32), (4.1.33), (4.1.34) , (4.1.35) and (4.1.36), we have

$$\begin{aligned} |b_h(v_h) - b(v_h)| \le \Big\{ &\theta \left\|\sigma_{m,h}^k - \sigma_m^k\right\|_{L^2(\Omega)} + \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} \\ &+ \left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^{\delta *} - \phi^{\delta *}\right\|_{H^1(\Omega)} \\ &+ \left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi^{\delta *}\right\|_{L^\infty(\Omega)} \Big\} \|v_h\|_{H^1(\Omega)}. \end{aligned}$$

So,

$$\begin{aligned} \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} \le \frac{1}{C_1} \Big\{ &\theta \left\|\sigma_{m,h}^k - \sigma_m^k\right\|_{L^2(\Omega)} \\ &+ \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} \\ &+ \left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^{\delta *} - \phi^{\delta *}\right\|_{H^1(\Omega)} \\ &+ \left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi^{\delta *}\right\|_{L^\infty(\Omega)} \Big\}. \end{aligned} \tag{4.1.37}$$

Note that,

$$\begin{aligned} \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \left\|\phi_h^\delta\right\|_{H^1(\Omega)} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)} \le \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^\infty(\Omega)} \Big\{ &\left\|\phi_h^\delta - \phi^\delta\right\|_{H^1(\Omega)} \\ &+ \left\|\phi^\delta\right\|_{H^1(\Omega)} \Big\} \left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)}. \end{aligned} \tag{4.1.38}$$

It can be shown using Cauchy-Schwarz inequality and Hölder's inequality that

$$\begin{aligned}
\left|a\left(w_h, v_h\right) - a_h\left(w_h, v_h\right)\right| &= \left|\int_\Omega \alpha\left(\chi_m^\delta - \chi_{m,h}^\delta\right) \nabla w_h \cdot \nabla v_h dV\right| \\
&\leq \alpha \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\|\nabla w_h\right\|_{L^2(\Omega)} \left\|\nabla v\right\|_{L^2(\Omega)} \\
&\leq \alpha \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\|w_h\right\|_{H^1(\Omega)} \left\|v\right\|_{H^1(\Omega)}.
\end{aligned}$$

Thus,

$$\frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{\left|a\left(w_h, v_h\right) - a_h\left(w_h, v_h\right)\right|}{\left\|v_h\right\|_{H^1(\Omega)}} \leq \frac{\alpha}{C_1} \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\|w_h\right\|_{H^1(\Omega)}. \quad (4.1.39)$$

From (4.1.31) and (4.1.39), there is a need to define

$$A\left(h\right) := \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{\left(1 + \frac{C_2}{C_1}\right) \left\|\sigma_m^{k+1,\delta} - w_h\right\|_{H^1(\Omega)} + \frac{\alpha}{C_1} \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\|w_h\right\|_{H^1(\Omega)}\right\}.$$

Because $\sigma_m^{k+1,\delta}$ is in $H^1\left(\Omega\right)$, we can interpolate it by an element $\mathcal{I}_h \sigma_m^{k+1,\delta}$ in $\mathcal{S}_h^{(1)}\left(\Omega\right)$. Now, it is obvious that

$$\begin{aligned}
A\left(h\right) \leq &\left(1 + \frac{C_2}{C_1}\right) \left\|\sigma_m^{k+1,\delta} - \mathcal{I}_h \sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)} \\
&+ \frac{\alpha}{C_1} \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\|\mathcal{I}_h \sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}. \qquad (4.1.40)
\end{aligned}$$

Using triangle inequality,

$$\left\|\mathcal{I}_h \sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)} \leq \left\|\sigma_m^{k+1,\delta} - \mathcal{I}_h \sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)} + \left\|\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}. \quad (4.1.41)$$

So (4.1.40), (4.1.41) and Lemma (48) imply

$$\begin{aligned}
A\left(h\right) \leq &\left(1 + \frac{C_2}{C_1}\right) C_3 h \left(\sum_{i=1}^{2} \left\|\frac{\partial^2 \phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} \\
&+ \frac{\alpha}{C_1} \left\|\chi_m^\delta - \chi_{m,h}^\delta\right\|_{L^\infty(\Omega)} \left\{C_3 h \left(\sum_{i=1}^{2} \left\|\frac{\partial^2 \phi^\delta}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2} + \left\|\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\right\}. \qquad (4.1.42)
\end{aligned}$$

Taking into consideration the inequalities (4.1.31), (4.1.37), (4.1.38), (4.1.39) and (4.1.42),

we get the estimate

$$\begin{aligned}\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_{m}^{k+1,\delta}\right\|_{H^1(\Omega)} \leq \frac{1}{C_1}\Bigg\{&\theta\left\|\sigma_{m,h}^{k}-\sigma_{m}^{k}\right\|_{L^2(\Omega)}\\ &+\left\|\chi_{m,h}^{\delta}-\chi_{m}^{\delta}\right\|_{L^\infty(\Omega)}\Bigg(\left\|\phi_h^{\delta}-\phi^{\delta}\right\|_{H^1(\Omega)}\\ &+\left\|\phi^{\delta}\right\|_{H^1(\Omega)}\Bigg)\left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)}\\ &+\left\|\nabla\phi_h^{\delta}\right\|_{L^\infty(\Omega)}\left\|\phi_h^{\delta *}-\phi^{\delta *}\right\|_{H^1(\Omega)}\\ &+\left\|\phi_h^{\delta}-\phi^{\delta}\right\|_{H^1(\Omega)}\left\|\nabla\phi^{\delta *}\right\|_{L^\infty(\Omega)}\Bigg\}\\ &+\left(1+\frac{C_2}{C_1}\right)C_3h\left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^{\delta}}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2}\\ &+\frac{\alpha}{C_1}\left\|\chi_{m}^{\delta}-\chi_{m,h}^{\delta}\right\|_{L^\infty(\Omega)}\Bigg\{C_3h\left(\sum_{i=1}^{2}\left\|\frac{\partial^2\phi^{\delta}}{\partial x_i^2}\right\|_{L^2(\Omega)}^2\right)^{1/2}\\ &+\left\|\sigma_{m}^{k+1,\delta}\right\|_{H^1(\Omega)}\Bigg\}.\end{aligned}$$

Observe that from Remark (52), $\left\|\nabla\phi_h^{\delta}\right\|_{L^\infty(\Omega)}$ and $\left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)}$ are bounded for any h. Also, they will not diverge as h approaches 0 as shown in Lemma (59). Furthermore, taking the limit as $h\to 0$, the right-hand side approaches 0 because of (4.0.9), (4.0.12), (4.1.11) and (4.1.21). Thus, $\sigma_{m,h}^{k+1,\delta}\to\sigma_{m}^{k+1,\delta}$ in $H^1(\Omega)$. □

Now that we have shown that the discretizations for ϕ,ϕ^* and σ_m^{k+1} are convergent, we can now analyze the convergence of the discretization of G defined in (3.4.2). But first, we need the following lemma which guarantees that $\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}$ will not diverge as h approaches 0.

Lemma 61. *Given Assumption (1),*

$$\lim_{h\to 0}\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}<\infty.$$

Proof. The proof will be similar to the proof of Lemma (59). We will also show this by contradiction. Suppose

$$\lim_{h\to 0}\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}=\infty.$$

Let $R\in\mathbb{R}^+$. Then for h small enough, we have

$$\text{ess sup}\left|\nabla\sigma_{m,h}^{k+1,\delta}\right|>R \tag{4.1.43}$$

in Ω. Recall that $\forall v_h\in\mathcal{S}_h^{(1)}(\Omega)$, we have

$$\int_\Omega\alpha\left(\chi_{m,h}^{\delta}+\epsilon\right)\nabla\sigma_{m,h}^{k+1,\delta}\cdot\nabla v_h dV+\theta\int_\Omega\sigma_{m,h}^{k+1,\delta}v dV=\int_\Omega\chi_{m,h}^{\delta}\nabla\phi_h^{\delta}\cdot\nabla\phi_h^{\delta *}v_h dV+\int_\Omega\theta\sigma_{m,h}^{k}v_h dV.$$

Because $\sigma_{m,h}^{k+1,\delta} \in \mathcal{S}_h^{(1)}(\Omega)$, then

$$\int_\Omega \alpha\left(\chi_{m,h}^\delta + \epsilon\right)\left|\nabla\sigma_{m,h}^{k+1,\delta}\right|^2 dV + \theta\int_\Omega \left|\sigma_{m,h}^{k+1,\delta}\right|^2 v dV = \int_{\partial\Omega} f_h \phi_h^\delta dS.$$

Therefore, using (4.0.7) and (4.1.43), we get

$$\alpha\epsilon R^2\mu(\Omega) < \theta\int_\Omega \left|\sigma_{m,h}^{k+1,\delta}\right|^2 dV - \int_\Omega \chi_{m,h}^\delta \nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *}\sigma_{m,h}^{k+1,\delta} dV + \int_\Omega \theta\sigma_{m,h}^k \sigma_{m,h}^{k+1,\delta} dV.$$

By subtracting and adding σ_m^{k+1} to $\sigma_{m,h}^{k+1,\delta}$, respectively, using Cauchy-Schwarz inequality, Hölder's inequality, triangle inequality, Lemma (59) and the fact that $\chi_{m,h}^\delta \leq 1$, we obtain

$$\begin{aligned}\alpha\epsilon R^2\mu(\Omega) < {} & C\left(\left\|\sigma_{m,h}^{k+1,\delta} - \sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)} + \left\|\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)}\right) \\ & + \theta\left(\left\|\sigma_{m,h}^{k+1,\delta} - \sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)} + \left\|\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)}\right)\left(\left\|\sigma_{m,h}^k - \sigma_m^k\right\|_{L^2(\Omega)} + \left\|\sigma_m^k\right\|_{L^2(\Omega)}\right)\end{aligned}$$

where $C = 2\max\left\{\theta, \mu(\Omega)\left\|\nabla\phi_h^\delta\right\|_{L^\infty(\Omega)}\left\|\nabla\phi_h^{\delta *}\right\|_{L^\infty(\Omega)}\right\}$. Since R is arbitrary, we can make it infinitely large so that $h \to 0$ and $\alpha\epsilon R^2\mu(\Omega) \to \infty$. Therefore by (4.0.9) and (4.1.30), we will have $C\left\|\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)} + \theta\left\|\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)}\left\|\sigma_m^k\right\|_{L^2(\Omega)} \to \infty$ which is a contradiction. Hence,

$$\lim_{h\to 0}\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)} < \infty.$$

□

Recall from Algorithm (2) that thresholding on $\Theta \circ G$ is implemented to get the update for χ_m. Hence, convergence of the discretization of G is necessary. Needless to say, we also need to show that the discretization of Θ is convergent. The proof of the convergence of these two functionals will be a direct application of Theorems (55), (57) and (60) because of their dependence on ϕ, ϕ^* and σ_m^{k+1}.

Theorem 62. *Given Assumption (1),*

$$\lim_{h\to 0}\left\|G\left(\chi_{m,h}^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} = 0 \tag{4.1.44}$$

where G is the function defined in (3.4.2).

Proof. Let

$$A_1\left(\chi_{m,h}^\delta\right) := 2\omega\sigma_{m,h}^{k+1,\delta}\nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *}, \quad A_1\left(\chi_m^\delta\right) := 2\omega\sigma_m^{k+1,\delta}\nabla\phi^\delta \cdot \nabla\phi^{\delta *}$$

and

$$A_2\left(\chi_{m,h}^\delta\right) := -\alpha\omega|\nabla\sigma_{m,h}^{k+1,\delta}|^2, \quad A_2\left(\chi_m^\delta\right) := -\alpha\omega|\nabla\sigma_m^{k+1,\delta}|^2.$$

Hence, by using the definition of G in (3.4.2) and the triangle inequality, we get

$$\begin{aligned}\left\|G\left(\chi_{m,h}^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \leq {} & \left\|\chi_{m,h}^\delta - \chi_m^\delta\right\|_{L^2(\Omega)} + \left\|A_1\left(\chi_{m,h}^\delta\right) - A_1\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} \\ & + \left\|A_2\left(\chi_{m,h}^\delta\right) - A_2\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}.\end{aligned} \tag{4.1.45}$$

Subtracting and adding $2\omega\sigma_{m,h}^{k+1,\delta}\nabla\phi^{\delta}\cdot\nabla\phi^{\delta *}$ to $A_1\left(\chi_{m,h}^{\delta}\right)-A_1\left(\chi_m^{\delta}\right)$ and using triangle inequality, we obtain

$$\left\|A_1\left(\chi_{m,h}^{\delta}\right)-A_1\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}\leq 2\omega\left(\left\|B_1\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}+\left\|B_2\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}\right)$$

where

$$B_1\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)=\sigma_{m,h}^{k+1,\delta}\nabla\phi_h^{\delta}\cdot\nabla\phi_h^{\delta *}-\sigma_{m,h}^{k+1,\delta}\nabla\phi^{\delta}\cdot\nabla\phi^{*\delta}$$

and

$$B_2\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)=\sigma_{m,h}^{k+1,\delta}\nabla\phi^{\delta}\cdot\nabla\phi^{\delta *}-\sigma_{m}^{k+1,\delta}\nabla\phi^{\delta}\cdot\nabla\phi^{*\delta}.$$

Note that $\sigma_{m,h}^{k+1,\delta}\in S_h^{(1)}(\Omega)\subset C^0\left(\bar{\Omega}\right)\subset L^{\infty}(\Omega)$. Observe as well that since $\phi_h^{\delta}\in S_h^{(1)}(\Omega)$, then by Remark (52), $\nabla\phi_h^{\delta}\in L^{\infty}(\Omega)$. Moreover, $\nabla\phi^{\delta *}\in L^{\infty}(\Omega)$ by (3.3.8). Using these, we can estimate $B_1\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)$ by first subtracting and adding $2\omega\sigma_{m,h}^{k+1,\delta}\nabla\phi_h^{\delta}\cdot\nabla\phi^{\delta *}$ and then employing the Hölder's inequality and the triangle inequality:

$$\begin{aligned}\left\|B_1\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}&\leq\left\|\sigma_{m,h}^{k+1,\delta}\right\|_{L^{\infty}(\Omega)}\left\{\left\|\nabla\phi_h^{\delta}\cdot\left(\nabla\phi_h^{\delta *}-\nabla\phi^{\delta *}\right)\right\|_{L^2(\Omega)}\right.\\&\left.+\left\|\left(\nabla\phi_h^{\delta}-\nabla\phi^{\delta}\right)\cdot\nabla\phi^{\delta *}\right\|_{L^2(\Omega)}\right\}\\&\leq\left\|\sigma_{m,h}^{k+1,\delta}\right\|_{L^{\infty}(\Omega)}\left\{\left\|\nabla\phi_h^{\delta}\right\|_{L^{\infty}(\Omega)}\left\|\nabla\left(\phi_h^{\delta *}-\phi^{\delta *}\right)\right\|_{L^2(\Omega)}\right.\\&\left.+\left\|\nabla\left(\phi_h^{\delta}-\phi^{\delta}\right)\right\|_{L^2(\Omega)}\left\|\nabla\phi^{\delta *}\right\|_{L^{\infty}(\Omega)}\right\}\\&\leq\left\|\sigma_{m,h}^{k+1,\delta}\right\|_{L^{\infty}(\Omega)}\left\{\left\|\nabla\phi_h^{\delta}\right\|_{L^{\infty}(\Omega)}\left\|\phi_h^{\delta *}-\phi^{\delta *}\right\|_{H^1(\Omega)}\right.\\&\leq\left.+\left\|\phi_h^{\delta}-\phi^{\delta}\right\|_{H^1(\Omega)}\left\|\nabla\phi^{\delta *}\right\|_{L^{\infty}(\Omega)}\right\}.\end{aligned}\tag{4.1.46}$$

From (3.3.3) and (3.3.8), $\nabla\phi^{\delta *}\in L^{\infty}(\Omega)$ and $\nabla\phi^{\delta *}\in L^{\infty}(\Omega)$. Therefore, the estimate for $B_2\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)$ can be obtained by the direct application of the Hölder's inequality and the triangle inequality:

$$\begin{aligned}\left\|B_2\left(\chi_{m,h}^{\delta};\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}&\leq\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_{m}^{k+1,\delta}\right\|_{L^2(\Omega)}\left\|\nabla\phi^{\delta}\right\|_{L^{\infty}(\Omega)}\left\|\nabla\phi^{*}\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}\\&\leq\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_{m}^{k+1,\delta}\right\|_{H^1(\Omega)}\left\|\nabla\phi^{\delta}\right\|_{L^{\infty}(\Omega)}\left\|\nabla\phi^{*}\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}.\end{aligned}\tag{4.1.47}$$

Because $\sigma_{m,h}^{k+1,\delta}\in S_h^{(1)}(\Omega)$, we can again use Remark (52) to deduce that $\nabla\sigma_{m,h}^{k+1,\delta}\in L^{\infty}(\Omega)$. By (3.3.62), $\nabla\sigma_{m}^{k+1,\delta}\in L^{\infty}(\Omega)$ as well. Thus, the term $A_2\left(\chi_{m,h}^{\delta}\right)-A_2\left(\chi_m^{\delta}\right)$

can be estimated using the Hölder's inequality and the triangle inequality:

$$\begin{aligned}
\left\|A_2\left(\chi_{m,h}^{\delta}\right)-A_2\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)} &= \alpha\omega\left\||\nabla\sigma_{m,h}^{k+1,\delta}|^2-|\nabla\sigma_m^{k+1,\delta}|^2\right\|_{L^2(\Omega)} \\
&= \alpha\omega\left\|\left(\nabla\sigma_{m,h}^{k+1,\delta}-\nabla\sigma_m^{k+1,\delta}\right)\cdot\left(\nabla\sigma_{m,h}^{k+1,\delta}+\nabla\sigma_m^{k+1,\delta}\right)\right\|_{L^2(\Omega)} \\
&\leq \alpha\omega\left\|\nabla\sigma_{m,h}^{k+1,\delta}-\nabla\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)}\left\|\nabla\sigma_{m,h}^{k+1,\delta}+\nabla\sigma_m^{k+1,\delta}\right\|_{L^\infty(\Omega)} \\
&\leq \alpha\omega\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\left(\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}\right. \\
&\leq \left.+\left\|\nabla\sigma_m^{k+1,\delta}\right\|_{L^\infty(\Omega)}\right). \qquad (4.1.48)
\end{aligned}$$

Comparing (4.1.45), (4.1.46), (4.1.47) and (4.1.48) gives us

$$\begin{aligned}
\left\|G\left(\chi_{m,h}^{\delta}\right)-G\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)} \leq& \left\|\chi_{m,h}^{\delta}-\chi_m^{\delta}\right\|_{L^2(\Omega)} \\
&+2\omega\left\|\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}\left\{\left\|\nabla\phi_h^{\delta}\right\|_{L^\infty(\Omega)}\left\|\phi_h^{\delta*}-\phi^{\delta*}\right\|_{H^1(\Omega)}\right. \\
&\left.+\left\|\phi_h^{\delta}-\phi^{\delta}\right\|_{H^1(\Omega)}\left\|\nabla\phi^{\delta*}\right\|_{L^\infty(\Omega)}\right\} \\
&+2\omega\left\|\sigma_{m,h}^{k+1,\delta}-\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\left\|\nabla\phi^{\delta}\right\|_{L^\infty(\Omega)}\left\|\nabla\phi^*\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)} \\
&+\alpha\omega\left\|\nabla\sigma_{m,h}^{k+1,\delta}-\nabla\sigma_m^{k+1,\delta}\right\|_{L^2(\Omega)}\left(\left\|\nabla\sigma_{m,h}^{k+1,\delta}\right\|_{L^\infty(\Omega)}\right. \\
&\left.+\left\|\nabla\sigma_m^{k+1,\delta}\right\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

Finally, taking the limit as $h\to 0$, $\left\|G\left(\chi_{m,h}^{\delta}\right)-G\left(\chi_m^{\delta}\right)\right\|_{L^2(\Omega)}\to 0$ as supported by (4.0.12), (4.1.11), (4.1.21), (4.1.30) and Lemma (61). □

To finally conclude that our proposed method is consistent, we only need to show that the discretization of the finite element formulation of

$$\int_\Omega \omega\gamma\frac{\nabla\Theta\cdot\nabla v}{\sqrt{|\nabla\chi_m^{\delta}|^2+\beta^2}}+\Theta v dV=\int_\Omega G\left(\chi_m^{\delta}\right)v dV \quad \text{for all } v\in H^1(\Omega) \qquad (4.1.49)$$

is convergent. For brevity, we will use the notation

$$\Theta^{\delta}:=\Theta\left(\chi_m^{\delta}\right)$$

as the solution of (4.1.49). Similar to what we did to ϕ, ϕ^* and σ_m^{k+1}, the discretized solution of (4.1.49) satisfies

$$a_h\left(u_h,v_h\right)=b_h\left(v_h\right) \quad \text{for all } v_h\in\mathcal{S}_h^{(1)}(\Omega) \qquad (4.1.50)$$

where

$$a_h\left(u_h,v_h\right)=\int_\Omega \omega\gamma\frac{\nabla u_h\cdot\nabla v_h}{\sqrt{\left|\nabla\chi_{m,h}^{\delta}\right|^2+\beta^2}}+u_h v_h dV$$

$$b_h(v_h) = \int_\Omega G\left(\chi^\delta_{m,h}\right) v dV.$$

From (3.4.8) and Lemma (48), we can infer that

$$\frac{1}{\sqrt{\left|\nabla \chi^\delta_{m,h}\right|^2 + \beta^2}} \geq \frac{1}{\bar{K}}. \tag{4.1.51}$$

And obviously,

$$\frac{1}{\sqrt{\left|\nabla \chi^\delta_{m,h}\right|^2 + \beta^2}} \leq \frac{1}{\beta}. \tag{4.1.52}$$

The bounds above will guarantee uniqueness of the solution of (4.1.50). We state and prove this in the following theorem.

Theorem 63. *Given Assumption (1), there exists Θ^δ_h satisfying (4.1.50).*

Proof. The proof will be similar to the proof of the continuous case. Clearly, a_h is bilinear and b_h is linear. As a consequence of the continuous case and (4.1.52), a_h is continuous:

$$|a_h(u_h, v_h)| \leq 2\max\left\{\left(\frac{\omega\gamma}{\beta}, 1\right)\right\} \|u_h\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)}. \tag{4.1.53}$$

Furthermore, we can show the continuity of $b_h(v)$ using Cauchy-Schwarz inequality:

$$|b_h(v)| \leq \left\|G\left(\chi^\delta_{m,h}\right)\right\|_{L^2(\Omega)} \|v_h\|_{H^1(\Omega)}. \tag{4.1.54}$$

Coercivity of $a_h(u_h, v_h)$ in $H^1(\Omega)$ can be proven using the same argumentation as in (3.4.11). Thus,

$$|a_h(u_h, u_h)| \geq \min\left\{\frac{\omega\gamma}{\bar{K}}, 1\right\} \|u_h\|^2_{H^1(\Omega)}.$$

Since $\mathcal{S}^{(1)}_h(\Omega) \subset H^1(\Omega)$ by Lemma (48), the application of the Lax-Milgram theorem guarantees the unique solution of (4.1.50). □

Now that the existence of the solution in the discretized setting is established, we can now finally proceed with the proof of convergence of Θ^δ_h to Θ^δ in $H^1(\Omega)$.

Theorem 64. *Given Assumption (1),*

$$\lim_{h\to 0} \left\|\Theta^\delta_h - \Theta^\delta\right\|_{H^1(\Omega)} = 0.$$

Proof. Note that Θ^δ solves the variational formulation

$$a\left(\Theta^\delta, v\right) = b(v) \quad \text{for all } v \in H^1(\Omega)$$

where

$$a\left(\Theta^\delta, v\right) = \int_\Omega \omega\gamma \frac{\nabla\Theta^\delta \cdot \nabla v}{\sqrt{|\nabla\chi^\delta_m|^2 + \beta^2}} + \Theta^\delta v dV$$

$$b(v) = \int_\Omega G\left(\chi_m^\delta\right) v dV.$$

On the other hand, Θ_h^δ satisfies (4.1.50). We will again make use of the First Strang Lemma. Thus, we need to show that the necessary conditions are satisfied. The first condition (4.1.6) is easily guaranteed by the coercivity of a_h in the proof of the previous theorem. The second condition (4.1.7) can be proved using the continuity of a, that is,

$$a\left(\Theta^\delta, v_h\right) \leq \left|a\left(\Theta^\delta, v_h\right)\right| \leq 2\max\left\{\frac{\omega\gamma}{\beta}, 1\right\} \left\|\Theta^\delta\right\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)}.$$

Clearly, $2\max\left\{\frac{\omega\gamma}{\beta}, 1\right\}$ is independent of h and so by the First Strang Lemma, there exist $C_1, C_2 > 0$ both independent of h such that

$$\begin{aligned}\left\|\Theta_h^\delta - \Theta^\delta\right\|_{H^1(\Omega)} \leq \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} \\ + \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{\left(1 + \frac{C_2}{C_1}\right) \left\|\Theta^\delta - w_h\right\|_{H^1(\Omega)}\right. \\ \left. + \frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}}\right\}. \end{aligned} \tag{4.1.55}$$

By using Cauchy-Schwarz inequality, it can be deduced that

$$\frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|b_h(v_h) - b(v_h)|}{\|v_h\|_{H^1(\Omega)}} \leq \frac{1}{C_1} \left\|G\left(\chi_{m,h}^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)}. \tag{4.1.56}$$

On the other hand,

$$\begin{aligned}|a(w_h, v_h) - a_h(w_h, v_h)| &= \omega\gamma \left|\int_\Omega \left(\frac{1}{\sqrt{|\nabla\chi_m^\delta|^2 + \beta^2}} - \frac{1}{\sqrt{\left|\nabla\chi_{m,h}^\delta\right|^2 + \beta^2}}\right) \nabla w_h \cdot \nabla v_h dV\right| \\ &\leq \omega\gamma \|A(h)\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} \end{aligned} \tag{4.1.57}$$

where

$$A(h) := \frac{1}{\sqrt{|\nabla\chi_m^\delta|^2 + \beta^2}} - \frac{1}{\sqrt{\left|\nabla\chi_{m,h}^\delta\right|^2 + \beta^2}}.$$

Thus,

$$\frac{1}{C_1} \sup_{v_h \in \mathcal{S}_h^{(1)}(\Omega)} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_{H^1(\Omega)}} \leq \frac{\omega\gamma}{C_1} \|A(h)\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)}. \tag{4.1.58}$$

Denote

$$B(h) := \inf_{w_h \in \mathcal{S}_h^{(1)}(\Omega)} \left\{\left(1 + \frac{C_2}{C_1}\right) \left\|\Theta^\delta - w_h\right\|_{H^1(\Omega)} + \frac{\omega\gamma}{C_1} \|A(h)\|_{L^\infty(\Omega)} \|w_h\|_{H^1(\Omega)}\right\}.$$

Because Θ^δ is in $H^1(\Omega)$, we can interpolate it by an element $\mathcal{I}_h\Theta^\delta$ in $\mathcal{S}_h^{(1)}(\Omega)$. Now, it is obvious that

$$B(h) \leq \left(1+\frac{C_2}{C_1}\right)\left\|\Theta^\delta - \mathcal{I}_h\Theta^\delta\right\|_{H^1(\Omega)} + \frac{\omega\gamma}{C_1}\left\|A(h)\right\|_{L^\infty(\Omega)}\left\|\mathcal{I}_h\Theta^\delta\right\|_{H^1(\Omega)}. \tag{4.1.59}$$

Using triangle inequality,

$$\left\|\mathcal{I}_h\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)} \leq \left\|\Theta^\delta - \mathcal{I}_h\Theta^\delta\right\|_{H^1(\Omega)} + \left\|\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}. \tag{4.1.60}$$

Therefore, comparing (4.1.55), (4.1.56), (4.1.57), (4.1.58), (4.1.59), and (4.1.60) implies

$$\begin{aligned}\left\|\Theta_h^\delta - \Theta^\delta\right\|_{H^1(\Omega)} \leq \frac{1}{C_1}\left\|G\left(\chi_{m,h}^\delta\right) - G\left(\chi_m^\delta\right)\right\|_{L^2(\Omega)} + \left(1+\frac{C_2}{C_1}\right)\left\|\Theta^\delta - \mathcal{I}_h\Theta^\delta\right\|_{H^1(\Omega)} \\ + \frac{\omega\gamma}{C_1}\left\|A(h)\right\|_{L^\infty(\Omega)}\left(\left\|\Theta^\delta - \mathcal{I}_h\Theta^\delta\right\|_{H^1(\Omega)} + \left\|\sigma_m^{k+1,\delta}\right\|_{H^1(\Omega)}\right).\end{aligned} \tag{4.1.61}$$

Applying the second statement of Lemma (48) to $\nabla\chi_{m,h}^\delta$ means

$$\lim_{h\to 0}\left\|A(h)\right\|_{L^\infty(\Omega)} = 0. \tag{4.1.62}$$

Finally, the right-hand side of (4.1.61) goes to 0 as h approaches 0 because of (4.1.62), Lemma (48) and (4.1.44). This completes the proof. □

4.2 A Discrete Formulation of the Proposed Method

Now that we have established solvability and convergence of the finite element approaches involved in our proposed method, we can now modify Algorithm (2) so that it can be implemented numerically. Recall from the the first section that given a pre-selected $N \in \mathbb{N}$, the step size is computed to be $h = 1/N$. We then approximated σ_m^k and f by $\sigma_{m,h}^k$ and f_h, respectively. Given $\sigma_{m,h}^k$ and f_h, we wish to solve the forward problem numerically. That is, we wish to find the solution ϕ_h^δ of (4.1.1) in $\mathcal{S}_h^{(1)}(\Omega)$. Because $\mathcal{S}_h^{(1)}(\Omega)$ is spanned by $\left\{s_l^{(1)}\right\}_{l=1}^{(N+1)^2}$, ϕ_h^δ can be expressed as

$$\phi_h^\delta = \sum_{l=1}^{(N+1)^2}\phi_{h,l}^\delta s_l^{(1)}. \tag{4.2.1}$$

Hence, it is sufficient to find the coefficients $\phi_{h,l}^\delta$. Let

$$\boldsymbol{\phi} := \left\{\phi_{h,l}^\delta\right\}_{l=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2}. \tag{4.2.2}$$

Recall that ϕ_h^δ satisfies

$$\int_\Omega \sigma_h^{k,\delta}\nabla\phi_h^\delta\cdot\nabla v dV = \int_{\partial\Omega} f_h v dS \tag{4.2.3}$$

for any $v \in \mathcal{S}_h^{(1)}(\Omega)$. Let $j \in \left\{1,2,\ldots,(N+1)^2\right\}$, $s_j^{(1)} \in \mathcal{S}_h^{(1)}(\Omega)$ and so making the substitution $v = s_j^{(1)}$ and combining (4.2.1) and (4.2.3), we get

$$\sum_{l=1}^{(N+1)^2}\phi_{h,l}^\delta\int_\Omega \sigma_h^{k,\delta}\nabla s_l^{(1)}\cdot\nabla s_j^{(1)}dV = \int_{\partial\Omega} f_h s_j^{(1)}dS. \tag{4.2.4}$$

Thus, we introduce the following matrices

$$\boldsymbol{\mathcal{A}}[w] := \left\{ \int_{\Omega} w \nabla s_l^{(1)} \cdot \nabla s_j^{(1)} dV \right\}_{l,j=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2} \times \mathbb{R}^{(N+1)^2} \tag{4.2.5}$$

$$\boldsymbol{\mathcal{B}}[w] := \left\{ \int_{\partial\Omega} w s_j^{(1)} dS \right\}_{j=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2}. \tag{4.2.6}$$

Using these matrices and (4.2.2), the equation (4.2.4) becomes the system

$$\boldsymbol{\mathcal{A}}\left[\sigma_h^k\right] \boldsymbol{\phi} = \boldsymbol{\mathcal{B}}[f_h]. \tag{4.2.7}$$

Recall that to guarantee the uniqueness of the solution of the forward problem, it is necessary that $\int_{\partial\Omega} \phi_h^\delta dS = 0$. Thus, we need to modify (4.2.7) to make sure that a solution exists. Let $\mathcal{I} := \left\{ l \in \left\{1, 2, \ldots, (N+1)^2\right\} : s_l^{(1)}(x) \neq 0 \text{ for some } x \in \partial\Omega \right\}$ and define $\boldsymbol{p} \in \mathbb{R}^{(N+1)^2}$ such that

$$\boldsymbol{p}_l = \begin{cases} 1, & \text{if } l \in \mathcal{I} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, (4.2.7) is modified through

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_h^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}[f_h] \\ 0 \end{bmatrix}.$$

Hence, ϕ_h^δ can be calculated by solving the above matrix system. To calculate $\phi_h^{\delta *}$, we proceed in a similar manner using the obtained ϕ_h^δ. Indeed, because we also want $\phi_h^{\delta *}$ to be in $\mathcal{S}_h^{(1)}(\Omega)$, it can be represented as

$$\phi_h^{\delta *} = \sum_{l=1}^{(N+1)^2} \phi_{h,l}^{\delta *} s_l^{(1)}. \tag{4.2.8}$$

Moreover, we denote the coefficients of $\phi_h^{\delta *}$ as

$$\boldsymbol{\phi}^* := \left\{ \phi_{h,l}^{\delta *} \right\}_{l=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2}. \tag{4.2.9}$$

The approximation $\phi_h^{\delta *}$ satisfies

$$\int_{\Omega} \sigma_h^{k,\delta} \nabla \phi_h^{\delta *} \cdot \nabla v dV = \int_{\partial\Omega} \left(\phi_h^\delta - \tilde{V}_h \right) v dS \tag{4.2.10}$$

for any $v \in \mathcal{S}_h^{(1)}(\Omega)$. Again, by making the substitution $v = s_j^{(1)}$, (4.2.10) becomes

$$\sum_{l=1}^{(N+1)^2} \phi_{h,l}^{\delta *} \int_{\Omega} \sigma_h^k \nabla s_l^{(1)} \cdot \nabla s_j^{(1)} dV = \int_{\partial\Omega} \left(\phi_h^\delta - \tilde{V}_h \right) s_j^{(1)} dS. \tag{4.2.11}$$

By (4.2.5), (4.2.6), (4.2.9) and (4.2.11), the coefficients of the discretized $\phi_h^{\delta *}$ can be computed numerically through

$$\boldsymbol{\mathcal{A}}\left[\sigma_h^{k,\delta}\right] \boldsymbol{\phi}^* = \boldsymbol{\mathcal{B}}\left[\phi_h^\delta - \tilde{V}_h\right]. \tag{4.2.12}$$

Inserting the condition that $\int_{\partial\Omega} \phi_h^{\delta *} dS = 0$, (4.2.12) becomes

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_h^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi^*} \\ \hat{\boldsymbol{\phi}}\boldsymbol{*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}\left[\phi_h^\delta - \tilde{V}_h\right] \\ 0 \end{bmatrix}.$$

Now that we have devised a way of calculating ϕ_h^δ and $\phi_h^{\delta *}$ numerically, we can now try to formulate how we can solve for the update $\sigma_{m,h}^{k+1,\delta}$. As proved in Theorem (58), $\sigma_{m,h}^{k+1,\delta} \in \mathcal{S}_h^{(1)}(\Omega)$. Hence, it can be expressed as

$$\sigma_{m,h}^{k+1,\delta} = \sum_{l=1}^{(N+1)^2} \sigma_{m,h,l}^{k+1,\delta} s_l^{(1)}. \tag{4.2.13}$$

Thus, similar to how we solved for ϕ_h^δ and $\phi_h^{\delta *}$, we wish to solve for the coefficients

$$\boldsymbol{\sigma_m^{k+1}} := \left\{\sigma_{m,h,l}^{k+1,\delta}\right\}_{l=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2} \tag{4.2.14}$$

of $\sigma_{m,h}^{k+1,\delta}$. From (4.1.25) and Theorem (58), we have

$$\begin{aligned} \int_\Omega \alpha\left(\chi_{m,h}^\delta + \epsilon\right) \nabla\sigma_{m,h}^{k+1,\delta} \cdot \nabla v dV + \theta \int_\Omega \sigma_{m,h}^{k+1,\delta} v dV = \int_\Omega \chi_{m,h}^\delta \nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *} v dV \\ + \int_\Omega \theta\sigma_{m,h}^k v dV \end{aligned} \tag{4.2.15}$$

for any $v \in \mathcal{S}_h^{(1)}(\Omega)$. Letting $v = s_j^{(1)}$ and using (4.2.13), the equation (4.2.15) becomes

$$\begin{aligned} \sum_{l=1}^{(N+1)^2} \sigma_{m,h,l}^{k+1,\delta} \int_\Omega \alpha\left(\chi_{m,h}^\delta + \epsilon\right) \nabla s_l^{(1)} \cdot \nabla s_j^{(1)} dV + \theta \sum_{l=1}^{(N+1)^2} \sigma_{m,h,l}^{k+1,\delta} \int_\Omega s_l^{(1)} s_j^{(1)} dV \\ = \int_\Omega \chi_{m,h}^\delta \nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *} s_j^{(1)} dV + \int_\Omega \theta\sigma_{m,h}^k s_j^{(1)} dV. \end{aligned} \tag{4.2.16}$$

In order to solve for the coefficients of $\sigma_{m,h}^{k+1,\delta}$ using (4.2.16), we need to introduce the following matrices:

$$\boldsymbol{C}[w] := \left\{\int_\Omega w s_l^{(1)} s_j^{(1)} dV\right\}_{l,j=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2} \times \mathbb{R}^{(N+1)^2} \tag{4.2.17}$$

and

$$\boldsymbol{\mathcal{D}}[w] = \left\{\int_\Omega w s_j^{(1)} dS\right\}_{j=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2}. \tag{4.2.18}$$

Using the matrices (4.2.5), (4.2.17) and (4.2.18) and the vector representation of the coefficients of $\sigma_{m,h}^{k+1,\delta}$ (4.2.14), the equation (4.2.16) becomes

$$\left(\alpha\boldsymbol{\mathcal{A}}\left[\chi_{m,h}^\delta + \epsilon\right] + \theta\boldsymbol{C}[1]\right) \boldsymbol{\sigma_m^{k+1}} = \boldsymbol{\mathcal{D}}\left[\chi_{m,h}^\delta \nabla\phi_h^\delta \cdot \nabla\phi_h^{\delta *} + \theta\sigma_{m,h}^k\right].$$

We already established the solvability of $\sigma_{m,h}^{k+1,\delta}, \phi_h^\delta$ and $\phi_h^{\delta *}$. Now we need to discretize G and Θ because the updated χ_m is obtained by thresholding $\Theta \circ G$. Recall from (3.4.2) that the function G is given by

$$G\left(\chi_m^\delta\right) = \chi_m^\delta - \omega\left[-2\sigma_m^{k+1}\left(\chi_m^\delta\right) \nabla\phi^\delta \cdot \nabla\phi^{\delta *} + \alpha|\nabla\sigma_m^{k+1,\delta}|^2\right].$$

And so,

$$G\left(\chi_{m,h}^{\delta}\right) = \chi_{m,h}^{\delta} - \omega\left[-2\sigma_{m,h}^{k+1}\nabla\phi_h^{\delta}\cdot\nabla\phi_h^{\delta *} + \alpha|\nabla\sigma_{m,h}^{k+1,\delta}|^2\right]. \tag{4.2.19}$$

As proved in Theorem (63), $\Theta_h^{\delta} \in \mathcal{S}_h^{(1)}(\Omega)$. Hence, it can be expressed as

$$\Theta_h^{\delta} = \sum_{l=1}^{(N+1)^2} \Theta_{h,l}^{\delta} s_l^{(1)}. \tag{4.2.20}$$

Thus, we wish to solve for the coefficients

$$\boldsymbol{\Theta} = \left\{\Theta_{h,l}^{\delta}\right\}_{l=1}^{(N+1)^2} \in \mathbb{R}^{(N+1)^2} \tag{4.2.21}$$

of Θ_h^{δ}. From (4.1.50) and Theorem (63), we have

$$\int_{\Omega} \omega\gamma \frac{\nabla\Theta_h^{\delta}\cdot\nabla v_h}{\sqrt{\left|\nabla\chi_{m,h}^{\delta}\right|^2 + \beta^2}} + \Theta_h^{\delta} v_h dV = \int_{\Omega} G\left(\chi_{m,h}^{\delta}\right) v_h dV \tag{4.2.22}$$

for any $v_h \in \mathcal{S}_h^{(1)}(\Omega)$. Letting $v = s_j^{(1)}$ and using (4.2.20), the above equation (4.2.22) becomes

$$\begin{aligned}\sum_{l=1}^{(N+1)^2} \Theta_{h,l}^{\delta} \int_{\Omega} \omega\gamma \frac{\nabla s_l^{(1)}\cdot\nabla s_j^{(1)}}{\sqrt{\left|\nabla\chi_{m,h}^{\delta}\right|^2 + \beta^2}} dV + \sum_{l=1}^{(N+1)^2} \Theta_{h,l}^{\delta} \int_{\Omega} s_l^{(1)} s_j^{(1)} dV \\ = \int_{\Omega} G\left(\chi_{m,h}^{\delta}\right) s_l^{(1)} dV.\end{aligned} \tag{4.2.23}$$

Using the matrices (4.2.5), (4.2.17) and (4.2.18) and the vector representation of the coefficients of $\Theta_{h,l}^{\delta}$ (4.2.21), the equation (4.2.23) becomes

$$\left(\omega\gamma\boldsymbol{\mathcal{A}}\left[\frac{1}{\sqrt{\left|\nabla\chi_{m,h}^{\delta}\right|^2 + \beta^2}}\right] + \boldsymbol{C}[1]\right)\boldsymbol{\Theta} = \boldsymbol{\mathcal{D}}\left[G\left(\chi_{m,h}^{\delta}\right)\right].$$

To get the numerical update for χ_m, we need to threshold $\boldsymbol{\Theta}$. However, $\chi_{m,h}^{\delta} \in \mathcal{S}_h^{(0)}$ means $\boldsymbol{\chi_m} := \left\{\chi_{m,h,l}^{\delta}\right\}_{l=1}^{N^2} \in \mathbb{R}^{N^2}$. So, we need to introduce a projection matrix that maps linear splines to constants:

$$\boldsymbol{\mathcal{R}} := \left\{\int_{\Omega} s_i^{(0)} s_j^{(1)} dV\right\}_{\substack{i=1,2,\ldots,N^2 \\ j=1,2,\ldots,(N+1)^2}} \in \mathbb{R}^{N^2\times(N+1)^2}.$$

We can now compute $\boldsymbol{\chi_m}$ by thresholding the projection of $\boldsymbol{\Theta}$:

$$\boldsymbol{\chi_m} = \text{bool}\left(\boldsymbol{\mathcal{R}}\boldsymbol{\Theta} > \zeta\right)$$

for some $\zeta \in (0,1)$.

In Chapter 2, when we formulated the functional J defined in (2.1.8), we said that we have to choose the parameter θ with care. The last term $\int_\Omega \theta \left(\sigma_m^{k+1} - \sigma_m^k\right)^2 dV$ in J penalizes σ_m^{k+1} to stay in the neighborhood of σ_m^k. If we choose θ to be large then σ_m^{k+1} might stick too close to σ_m^k and in turn, stay away from the solution $\tilde{\sigma}_m$. If we select θ to be too small then σ_m^{k+1} might deviate too much from σ_m^k. So how do we choose θ? At the beginning of the iteration, we assume that the initial guess is still far from the true solution. Hence, we choose $\theta := \theta_1$ to be small enough. As the algorithm proceeds, we select $\theta_k := \nu\theta_{k-1}$ for some $\nu > 1$. This way, as the algorithm progresses, we presume that σ_m^k becomes closer and closer to the actual solution. How do we know if we are deviating away from the solution? We will do a backtracking step. If $J(\theta_k) < J(\theta_{k-1})$ then set $\theta_k := \nu\theta_{k-1}$. Otherwise, we revert back to the previous value of θ, that is, set $\theta_k := \theta_{k-1}$.

We want to again emphasize that the introduction of δ is for theoretical purposes only. But because our proposed method has a fixed point for any positive δ, we can choose it to be extremely small. We still include this in the numerical algorithm but in our numerical simulations, we will set δ to be 0.

Let $\boldsymbol{\chi_m}^k$ and $\boldsymbol{\chi_m}^{k+1}$ be the approximations of χ_m at the beginning and at the end of the iteration, respectively. We will use the stopping criterion $\sum_{m=1}^M \max\left(\boldsymbol{\chi_m}^{k+1} - \boldsymbol{\chi_m}^k\right) < \rho$, where ρ is the sufficiently small tolerance. We summarize this in the following algorithm. This is the pseudo code of the discretization of our proposed method and will be the basis of all our numerical experiments.

ALGORITHM 3.

1. *Initialization.* Set $k = 1$ and choose parameters $\alpha, \epsilon, \theta_k, \omega, \delta, \nu, \beta, \gamma$ and ζ. Select the appropriate value of M, the maximum number of iterations K, the natural number N so that $h = 1/N$ and the tolerance ρ. Set $\boldsymbol{\sigma_m^k}$ to be a constant vector in $\mathbb{R}^{(N+1)^2}$ as the initial guess for the coefficients of σ_m^k. Select an initial guess $\boldsymbol{\chi_m}^k \in \mathbb{R}^{N^2}$, $m \in \{1, 2, \ldots, M-1\}$ to be the initial guess for the coefficients of $\chi_{m,h}^\delta$. For $m = M$, $\boldsymbol{\sigma_M^k}$ is known and $\boldsymbol{\chi_M}^k = 1 - \sum_{m=1}^{M-1} \boldsymbol{\chi_m}^k$. Set f_h and $\tilde{V}_h$ according to (4.0.3) and (4.0.4), respectively.

2. *Solving the forward problem.* Set $\left\{\chi_{m,h,l}^{\delta,k}\right\}_{l=1}^{N^2} = \boldsymbol{\chi_m}^k$, $\chi_{m,h}^{\delta,k} = \sum_{l=1}^{N^2} \chi_{m,h,l}^{\delta,k} s_l^{(0)}$, $\left\{\sigma_{m,h,l}^k\right\}_{l=1}^{(N+1)^2} = \boldsymbol{\sigma_m^k}$, $\sigma_{m,h}^k = \sum_{l=1}^{(N+1)^2} \sigma_{m,h,l}^k s_l^{(1)}$ and $\sigma_h^{k,\delta} = \sum_{m=1}^M \sigma_{m,h}^k \chi_{m,h}^{\delta,k}$. The discretized solution $\boldsymbol{\phi}$ of the forward problem is solved via

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_h^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}\left[f_h\right] \\ 0 \end{bmatrix}.$$

3. *Solving the adjoint problem.* Set $\left\{\phi_{h,l}^\delta\right\}_{l=1}^{(N+1)^2} = \boldsymbol{\phi}$ and $\phi_h^\delta = \sum_{l=1}^{(N+1)^2} \phi_{h,l}^\delta s_l^{(1)}$. The discretized solution $\boldsymbol{\phi}^*$ of the adjoint problem is solved via

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_h^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}^* \\ \hat{\boldsymbol{\phi}}* \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}\left[\phi_h^\delta - \tilde{V}_h\right] \\ 0 \end{bmatrix}.$$

4. *Solving for the coefficients of* $\sigma_{m,h}^{k+1}$. Set $\left\{\phi_{h,l}^{*}\right\}_{l=1}^{(N+1)^2} = \boldsymbol{\phi}^{*}$and $\phi_h^{\delta *} = \sum_{l=1}^{(N+1)^2} \phi_{h,l}^{\delta *} s_l^{(1)}$. Take $\nabla\phi_h^{\delta} = \sum_{l=1}^{(N+1)^2} \phi_{h,l}^{\delta} \nabla s_l^{(1)}$ and $\nabla\phi_h^{\delta *} = \sum_{l=1}^{(N+1)^2} \phi_{h,l}^{*} \nabla s_l^{(1)}$. The coefficients of $\sigma_{m,h}^{k+1,\delta}$ is solved using

$$\left(\alpha\boldsymbol{\mathcal{A}}\left[\chi_{m,h}^{\delta,k}+\epsilon\right]+\theta\boldsymbol{C}\left[1\right]\right)\boldsymbol{\sigma_m^{k+1}} = \boldsymbol{\mathcal{D}}\left[\chi_{m,h}^{\delta,k}\nabla\phi_h^{\delta}\cdot\nabla\phi_h^{\delta *}+\theta_k\sigma_{m,h}^{k}\right].$$

For $m = M$, set $\boldsymbol{\sigma_m^{k+1}} = \boldsymbol{\sigma_M^{k}}$.

5. *Updating* θ. If $J\left(\nu\theta_{k-1}\right) < J\left(\theta_{k-1}\right)$, $\theta_k = \nu\theta_{k-1}$. Else, $\theta_k = \theta_{k-1}$.

6. *Calculating* Θ. The discretized solution $\boldsymbol{\Theta}$ of (4.1.50) is solved via

$$\left(\omega\gamma\boldsymbol{\mathcal{A}}\left[\frac{1}{\sqrt{\left|\nabla\chi_{m,h}^{\delta,k}\right|^2+\beta^2}}\right]+\boldsymbol{C}\left[1\right]\right)\boldsymbol{\Theta} = \boldsymbol{\mathcal{D}}\left[G\left(\chi_{m,h}^{\delta,k}\right)\right].$$

7. *Solving for the coefficients of* $\chi_{m,h}^{k+1}$. Set

$$\boldsymbol{\chi_m}^{k+1} = \mathrm{bool}\left(\boldsymbol{\mathcal{R}\Theta} > \zeta\right).$$

8. *Stopping Criteria.* If $k = K$ or $\sum_{m=1}^{M}\max\left(\boldsymbol{\chi_m}^{k+1}-\boldsymbol{\chi_m}^{k}\right) < \rho$, the algorithm terminates. Otherwise, $k \leftarrow k+1$ and go back to step 2.

Remark 65. In the last section of Chapter 2, we considered the case when the conductivities are the same on the inclusions. For this particular case, we have to add an additional step to include the topological derivative. Recall that

$$D_T\left(x,\Omega_1,\sigma_1\right) = \begin{cases} \dfrac{2\sigma_1\left(\sigma_1-\sigma_2\right)}{\sigma_1+\sigma_2}\nabla\phi(x)\cdot\nabla\phi^{*}(x), & x\in\Omega_1 \\ \dfrac{2\sigma_2\left(\sigma_1-\sigma_2\right)}{\sigma_2+\sigma_1}\nabla\phi(x)\cdot\nabla\phi^{*}(x), & x\in\Omega\backslash\overline{\Omega}_1 \end{cases}.$$

Note that the initial χ_1 is obtained by setting $\Omega_1 = \emptyset$. Thus, we only need to discretize

$$D_T\left(x,\Omega_1,\sigma_1\right) = \frac{2\sigma_2\left(\sigma_1-\sigma_2\right)}{\sigma_2+\sigma_1}\nabla\phi(x)\cdot\nabla\phi^{*}(x)$$

where $\Omega_1 = \emptyset$. Thus, the discretization of $D_T\left(x,\Omega_1,\sigma_1\right)$ is

$$\boldsymbol{D_T} = 2\boldsymbol{\mathcal{R}}\left[\boldsymbol{\sigma_2^k}\left(\frac{\boldsymbol{\sigma_1^k}-\boldsymbol{\sigma_2^k}}{\boldsymbol{\sigma_2^k}+\boldsymbol{\sigma_1^k}}\right)\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}^{*}\right] \tag{4.2.24}$$

where $\boldsymbol{\sigma_2^k}\left(\frac{\boldsymbol{\sigma_1^k}-\boldsymbol{\sigma_2^k}}{\boldsymbol{\sigma_2^k}+\boldsymbol{\sigma_1^k}}\right)$ is calculated element-wise and $k = 1$. Because $\Omega_1 = \emptyset$ then $\chi_1 = 0$ and so $\sigma_h^k = \sigma_{1,h}^k\chi_1 + \sigma_{2,h}^k\left(1-\chi_1\right) = \sigma_{2,h}^k$. And so the discretized forward and adjoint solutions are solved via

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_2^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\phi} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}\left[f_h\right] \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \boldsymbol{\mathcal{A}}\left[\sigma_2^k\right] & \boldsymbol{p} \\ \boldsymbol{p}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}^* \\ \hat{\boldsymbol{\phi}}\boldsymbol{*} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{B}}\left[\phi_h^\delta - \tilde{V}_h\right] \\ 0 \end{bmatrix}.$$

Observe that σ_2^k is given so the matrices on the left-hand side of both equations are well-defined. The multiplication of the matrix $\boldsymbol{\mathcal{R}}$ in (4.2.24) is necessary because

$$\boldsymbol{\sigma_2^k}\left(\frac{\boldsymbol{\sigma_1^k} - \boldsymbol{\sigma_2^k}}{\boldsymbol{\sigma_2^k} + \boldsymbol{\sigma_1^k}}\right)\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}^* \in \mathbb{R}^{(N+1)^2}$$

and we want to use $\boldsymbol{D_T}$ to get an initial update for $\boldsymbol{\chi_1}$ which is in $\mathbb{R}^{N^2}$. Finally, the initial update $\boldsymbol{\chi_1^k}$ is given by

$$\boldsymbol{\chi_1^k} = \text{bool}\left(\boldsymbol{D_T} < -C\right)$$

where C is a predetermined thresholding constant.

Chapter 5

Numerical Results

In this chapter, we present the results of our numerical simulations. All results have been computed on Compaq 11-PX003 with 2 GB of RAM. Microsoft Windows 7 Professional (64-bit) was the operating system used. All codes were written in MATLAB. For simplicity, we will use the notations of all quantities in their continuous form. We will drop the δ and h in all of our notations. For example, we will use χ_m instead of χ_m^δ or $\chi_{m,h}^\delta$.

In the first section, we consider the case when $M = 2$. We will discuss how we obtain the initial guess for χ_m using topological derivative and then present our result. We will also consider the case when χ_m is a disjoint set. We then proceed with the numerical results on approximating χ_m given different initial estimates. This is necessary because we are working under the assumption that the initial estimate for χ_m is given. After that, we present our numerical results for the case $M \geq 3$ where the initial estimates for χ_m, $m \in \{1, 2, \ldots, M-1\}$ are given. We also consider a special case when χ_m is given and σ_m is unknown. At the end of the chapter, we present an approach in which the number of segments does not need to be given.

As stated in the previous chapter, we will let $\Omega = [0,1]^2$. We discretized Ω using a 70×70 grid, that is, we selected $N = 70$ so that $h = 1/70$. Before we begin our calculations, we first need to define the current patterns f on $\partial\Omega$ that we will use to solve the forward problem. It is worth noting that the recovery of σ given the boundary voltages is independent of the choice of f (see, e.g., [22]). We just need to select f so that it has a zero boundary integral. In all our simulations, we will use

$$f(x, y) = 2xy - x - y.$$

It can be easily shown that $\int_{\partial\Omega} f dS = 0$. Given f and the corresponding boundary voltage $\tilde{V}$, we solved the forward problem using FEM as discussed in the previous chapter.

5.1 Image Reconstruction using Topological Derivative

We now present the results of our numerical simulations for the case $M = 2$. This means that the quantity σ can be expressed as $\sigma_1\chi_1 + \sigma_2(1 - \chi_1)$. Let Ω_1 be the support of χ_1. We first present our numerical results when Ω_1 is a connected set and then proceed when Ω_1 is disjoint. The value of σ_2 is known because the value is assumed to be the same at the boundary. In our calculations, we used $\sigma_2 = 6.7$. This value is based on the

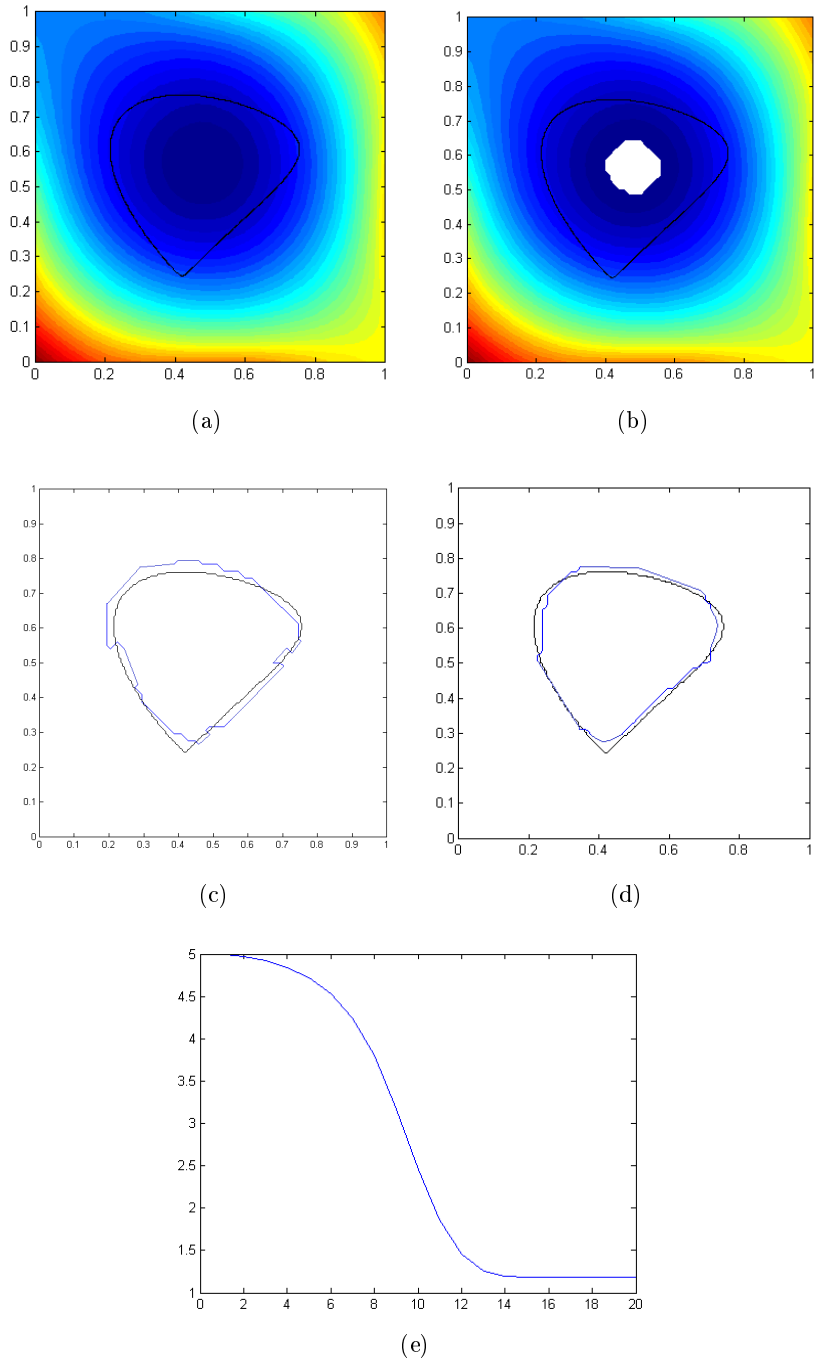
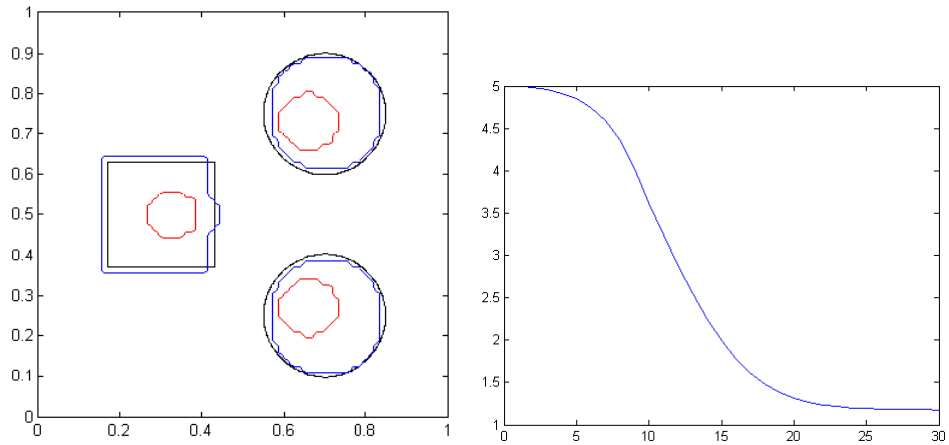


Figure 5.1.1: A reconstruction of a connected segment Ω_1. (A) The contour of the topological derivative $D_T\left(x;\Omega_1;\sigma_1^{\text{init}}\right)$ where $\Omega_1 = \emptyset$ is superimposed to the true solution (black contour). (B) The initial estimate for Ω_1 after thresholding the topological derivative. (C) The approximated solution (blue) compared to the true solution (black) without TV regularization and (D) with TV regularization. (E) The conductivity σ_1 over the iterations.



(a) (b)

Figure 5.1.2: A reconstruction of a disjoint segment Ω_1. (A) The initial estimate of Ω_1 calculated using topological derivative (red); the final estimate of Ω_1 after termination of the algorithm (blue); the actual geometry of Ω_1 (black). (B) Convergence of σ_1 over the iterations.

conductivity of the blood as stated in Table (2.1). Similarly, for the true value of the conductivity at the inner cavity, we used the conductivity of the lungs during expiration, that is, $\sigma_1 = 1$. We set the initial estimate σ_1^{init} for σ_1 to be 5. At the beginning of the iteration, we use the parameters $\alpha = 10^4$, $\epsilon = 10^{-4}$, $\theta = 2$, $\omega = 0.5$, $\nu = 2$ and $\zeta = 0.5$. In our first simulation, we considered the set of all points (x, y) in $[0, 1]^2$ such that

$$z(u, v) = \frac{1}{2}\left(u^4 + v^4 - 16u^2 - 16v^2 + 5u + 5v\right) \leq -39$$

where $u(x) = 5x - 10$ and $v(y) = 7y - \frac{3}{2}$. Unfortunately, this set is a union of two disjoint sets. Hence, we add additional conditions to u and v. We discard the other set by excluding all points $(x, y) \in [0, 1]^2$ such that $u(x) > 0.17$ and $v(y) < 0.1$. We let the remaining set as the true Ω_1. This can be seen in Figure (5.1.1A). The illustrations of our numerical simulations for this particular Ω_1 can be seen in Figure (5.1.1). In Figure (5.1.1A), the contour of the topological derivative $D_T\left(x, \Omega_1, \sigma_1^{\text{init}}\right)$ is shown. The topological derivative is computed by setting $\Omega_1 = \emptyset$. For comparison, the actual Ω_1 is superimposed to the topological derivative. Observe that most negative values of the topological derivative (contours in dark blue) are located inside the actual Ω_1. This justifies our proposition to use the most negative values of the topological derivative to get an initial estimate for χ_m. As stated in the last section of Chapter 3, the initial estimate for χ_m is obtained by setting $\chi_m(x) = 1$ for all $x \in \Omega$ such that $D_T\left(x, \Omega_1, \sigma_1^{\text{init}}\right) < -C$ for some $C > 0$. In this particular example, we chose $C = 10$. The obtained initial update Ω_1^{init} is illustrated in Figure (5.1.1B). Figure (5.1.1C) shows our estimate for Ω_1 compared to the actual Ω_1 without using the smoothing from TV-regularization. Here, we set $\gamma = 0$. Notice that the obtained solution, although a good approximation, lacks smoothness. To give χ_1 smoothness, we implement the algorithm with the help of TV-regularization and we set $\gamma = 10^{-2}$ and $\tau = 10^{-3}$. Finally, the estimate for Ω_1 compared to the actual Ω_1 with the aid of the smoothing from TV-regularization is illustrated in Figure (5.1.1D). The algorithm for this particular example was terminated

after 20 iterations. Despite the ill-possessiveness of the inverse EIT problem, the obtained solution is pretty decent. And note that we obtained this without any *a priori* estimate on Ω_1. Recall that we expressed σ_1 as a function of χ_1. But it is still important to make an analysis on how σ_1 behaves in the course of the algorithm. Because of the term $\int_\Omega \alpha\left(\chi_1+\epsilon\right)\left|\nabla\sigma_1\right|^2 dV$ in the functional, the value of σ_1 tends to be constant in the whole Ω. As stated at the beginning of this section, α is selected be 10^4. This forces σ_1 to be almost constant in the whole Ω. In Figure (5.1.1E), we showed the convergence of the conductivity σ_1 over the iterations. What is shown in this figure is the comparison of the mean of σ_1 in the whole Ω as the algorithm progresses. We can see that after the 15th step, the approximation for the true value of σ_1 does not change that much anymore. The algorithm is terminated after 20 iterations with the mean value of σ_1 in the whole Ω equal to 1.18. This is considerably close to the actual value 1.

Let us now consider the example when Ω_1 is a union of disjoint sets. We will select Ω_1 to be the union of two circles and a square. In this case, it is assumed that σ_1 is constant in all these three inclusions. The two circles are of the same radius 0.15 whose centers are located at $(0.7, 0.25)$ and $(0.7, 0.75)$. The square has length 0.26 with center located at $(0.3, 0.5)$. We illustrate our simulation in Figure (5.1.2). In Figure (5.1.2A), we show the numerical result obtained after 30 iterations. Again, using topological derivative the initial estimate (red) for Ω_1 was computed. The actual geometry of Ω_1 is in black and the estimated geometry is in blue. The algorithm was terminated after 30 iterations. We implemented the algorithm with the same set of parameters used in the previous example. We also incorporated the TV-regularization and used the parameters $\gamma = 10^{-2}$ and $\tau = 10^{-3}$. Observe that despite the fact that Ω_1 is disjoint, the estimate for Ω_1 is relatively accurate. Again, the term $\int_\Omega \alpha\left(\chi_1+\epsilon\right)\left|\nabla\sigma_1\right|^2 dV$ in the functional assures that σ_1 is constant in Ω_1. The penalty for σ_1 to become constant outside Ω_1 is small, so that σ_1 has the tendency to deviate from being constant outside Ω_1. Therefore, because Ω_1 is disjoint, it might take more steps to reach a good approximation of σ_1. The convergence of σ_1 over the iterations is shown in the second figure. It took 25 steps in the iteration before a good approximation is attained as compared to Figure (5.1.1D) in which it took only 15 steps. After the algorithm was terminated, the approximated value of σ_1 is 1.31. Because there are more cavities, it is expected that the recovered conductivity is less accurate compared to the example illustrated in (5.1.1E). Because the problem is ill-posed anyway, the obtained approximation is considerably good. Moreover, the approximated conductivity value was simultaneously recovered with a disjoint geometry Ω_1.

5.2 Image Reconstruction Given an A Priori Estimate

Unfortunately, the use of topological derivative in obtaining an initial update can only be applied when the conductivity is the same in all of the inclusions of Ω. In reality, this is not the case. For example, it can be seen in Table (2.1) that conductivities of healthy tissues show contrast. In this section, we present our simulations when an initial estimate Ω_1^{init} is provided. But first, we will justify that the solution obtained using our proposed method does not depend on the choice of the initial update. We consider the particular case when Ω_1 is given by

$$\Omega_1 = \left\{ (x,y) \in \Omega \middle| \, (x-0.7)^2 + \frac{(y-0.5)^2}{4} \leq 0.15^2 \right\}.$$

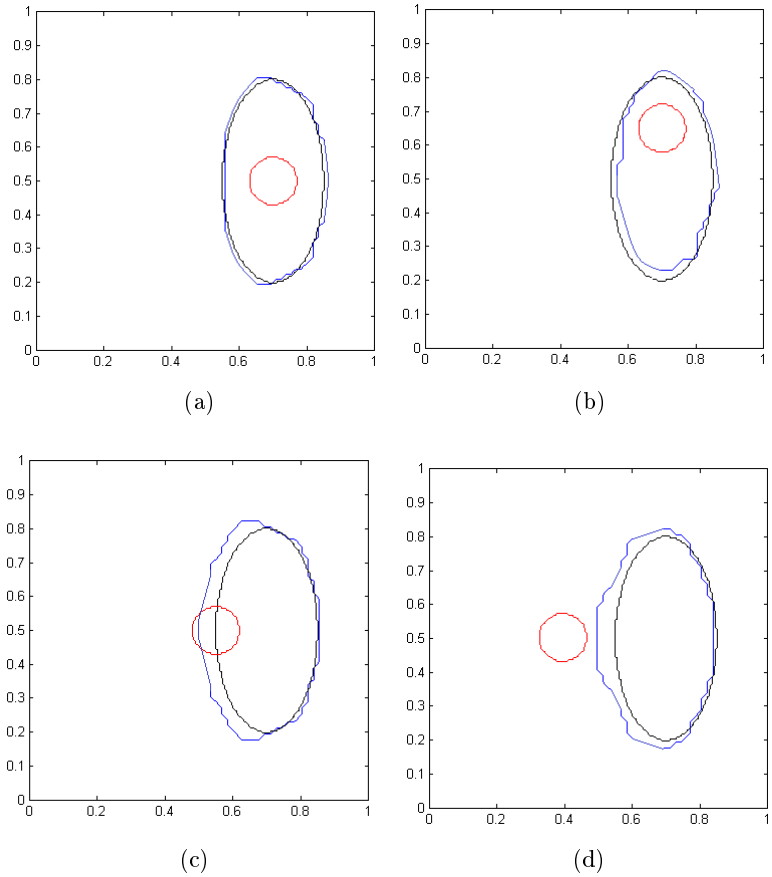


Figure 5.2.1: A reconstruction of Ω_1 using different initial estimate Ω_1^{init}. The actual geometry of Ω_1 is in black. The different initial estimates are in red and the corresponding approximated estimates after termination of the algorithm are in blue.

This is shown in black in Figure (5.2.1). We present now the numerical solutions obtained in our simulations if we use different Ω_1^{init}. Just like in the previous section, we are going to use the parameters $\alpha = 10^4$, $\epsilon = 10^{-4}$, $\theta = 2$, $\omega = 0.5$, $\nu = 2$ and $\zeta = 0.5$. Likewise, we incorporate the TV-regularization with parameters $\gamma = 10^{-2}$ and $\tau = 10^{-3}$. We illustrate all our simulations in Figure (5.2.1). To have a good point of comparison, we set Ω_1^{init} to be a circle of radius 0.07 but with different centers. Moreover, we run the algorithm over 30 iterations for each example. We illustrate the first example in Figure (5.2.1A). Here, Ω_1^{init} and the actual Ω_1 share the same center at $(0.7, 0.5)$. This example is similar to the examples discussed in the previous section. This is the expected initial estimate for Ω_1. It can be seen that the approximation after 30 iterations is actually good. In the second example, we moved the center of Ω_1^{init} away from the center of Ω_1 in such a way that Ω_1^{init} is still completely inside Ω_1. It can be seen that a good approximation is also attained. In Figure (5.2.1C), the initial estimate Ω_1^{init} is partially inside Ω_1. Despite this, the approximation, although not as good as the first two examples, is considerably accurate.

What worth noting is the last example shown in Figure (5.2.1D). Here, the initial estimate is outside Ω_1. This is an interesting example because this means that our proposed method has the capability of solving the reconstruction problem even though the initial

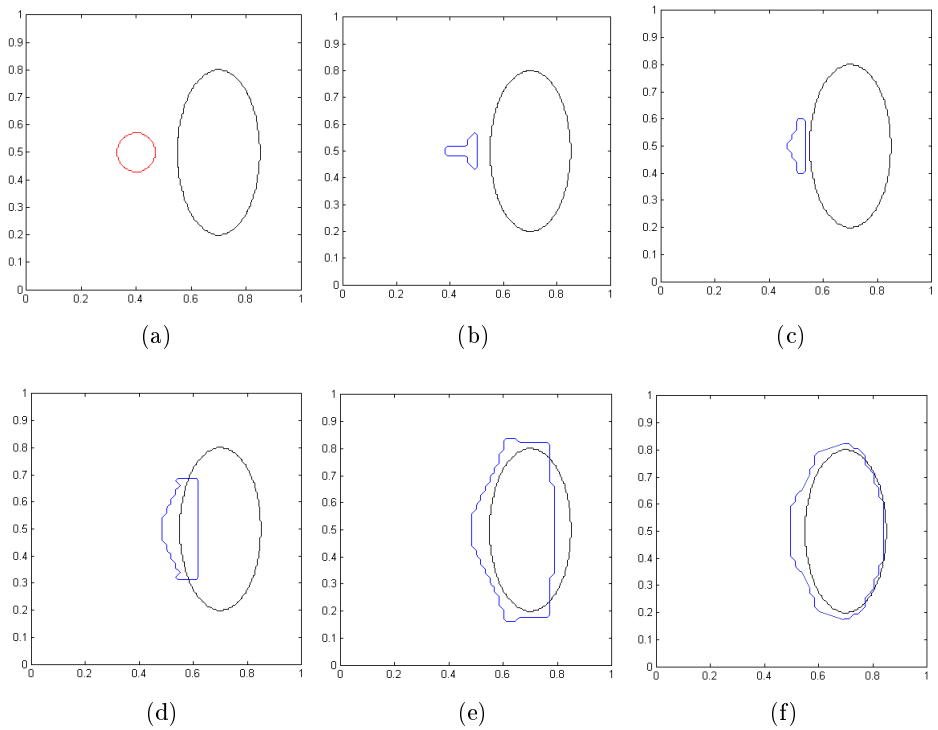
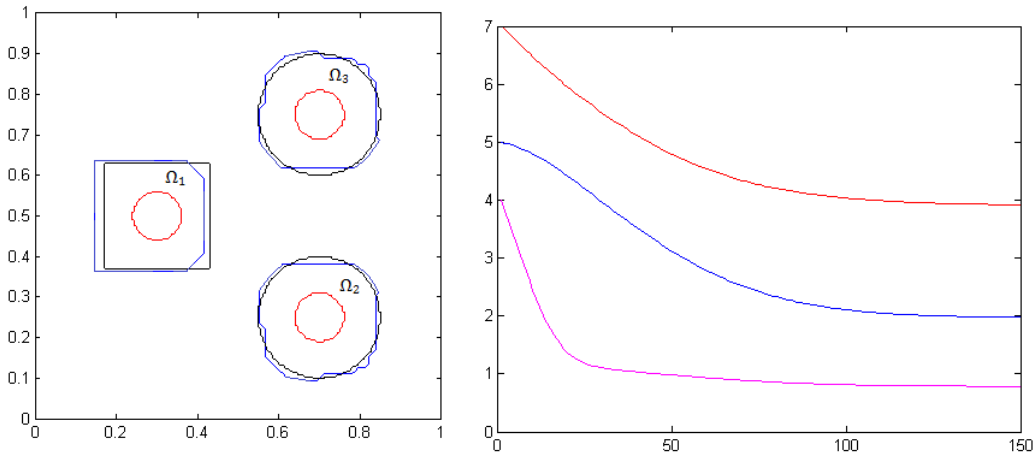


Figure 5.2.2: A reconstruction of Ω_1 with a disjoint initial estimate over iterations. (A) The initial estimate Ω_1^{init} (red) and the actual geometry Ω_1 (black). (B)-(F) The approximated solution (blue) after 2, 4, 9, 20 and 30 steps of the algorithm.

estimate is not *"close"* to the actual solution. Thus, we further analyze this example in Figure (5.2.2). We illustrate in Figure (5.2.2A) the comparison of the initial estimate Ω_1^{init} (red) and the actual geometry Ω_1. Let us denote $\Omega_1^{(k)}$ to be the approximated Ω_1 after the kth step. In Figure (5.2.2B), the comparison between $\Omega_1^{(2)}$ and Ω_1 is depicted. For Ω_1^{init} to be updated into $\Omega_1^{(2)}$, the derivative of the functional J with respect to χ_1 must be highly negative between Ω_1^{init} and Ω_1. This makes the updates $\Omega_1^{(k)}$ *"crawl"* to the actual solution Ω_1. This is depicted in Figures (5.2.2B)-(5.2.2F). Although the solutions obtained in Figures (5.2.1A)-(5.2.1B) are arguably better than the approximated solution shown in Figure (5.2.2F), they do not differ much. The general geometry of Ω_1 is somehow approximated.

Now that we have justified that our proposed method can solve the inverse problem using different initial estimates, we can now present a result when $M > 2$. We go back to the example illustrated in Figure (5.1.2). This means that $M = 4$. In this example, the problem was treated as if σ has the same value in all of the inclusions. We now assume that σ has different values in Ω_1, Ω_2 and Ω_3. We will assign Ω_1 to be the square with center at $(0.3, 0.5)$ with length 0.26, Ω_2 to be the circle with center at $(0.75, 0.25)$ and radius 0.15, and Ω_3 to be the circle with center at $(0.75, 0.75)$ and radius 0.15. We selected Ω_i^{init} , $i = 1, 2, 3$ to be circles with the same radius 0.06 but different locations. This is shown in Figure (5.2.3A). Let σ_1 , σ_2 and σ_3 be the conductivity values at Ω_1, Ω_2 and Ω_3, respectively. We are going to use the same paramaters in the previous examples. We choose the values $\sigma_1 = 2$, $\sigma_2 = 4$ and $\sigma_3 = 1$. For comparison, we selected



(a) (b)

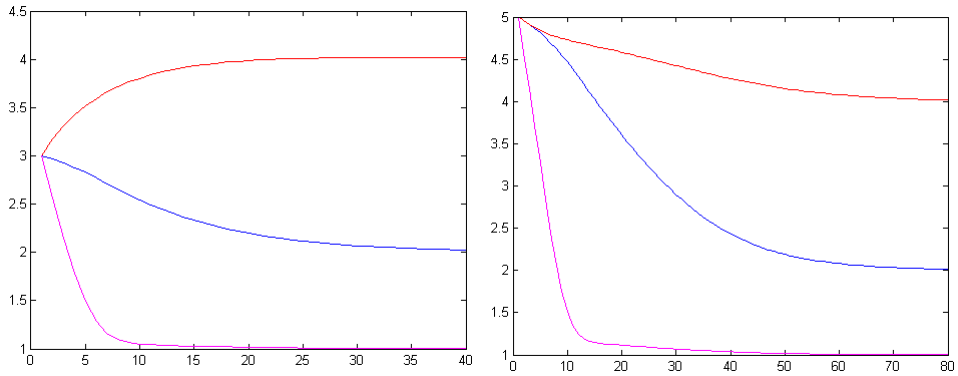
Figure 5.2.3: An image reconstruction with $M = 4$. (A) The comparison of the actual geometry (black), the approximated geometry (blue) and the given initial estimate (red). (B) The convergence of σ_1 (blue), σ_2 (red) and σ_3 (magenta) over the iterations.

the initial conductivity values to be of the same distance from the actual values. That is, $\sigma_1^{\text{init}} = 5$, $\sigma_2^{\text{init}} = 7$ and $\sigma_3^{\text{init}} = 4$. We present the result of our implementation in Figure (5.2.3). The algorithm was terminated after 150 iterations with the same sets of paramaters used in the previous examples. In Figure (5.2.3A), we show the comparison of the actual solution, the approximation solution and the initial estimate. It can be inferred that a good approximation is attained. In Figure (5.2.3B), we illustrate the convergence of conductivity values over iterations. Compared to our examples in Figure (5.1.2), it took more steps for all of the conductivity values to reach the solution. This is expected because the number of segments in this example is more. The conductivity values obtained when after the algorithm stopped for σ_1, σ_2 and σ_3 are 1.98, 3.91 and 0.79, respectively. Compared to the actual values, the obtained solutions can be seen to be good approximations.

5.3 Recovering Conductivity with Known Geometry

The EIT problem is typically an image reconstruction of the body Ω. But what if the geometry of Ω is known and the conductivity values are not? This particular case is studied in [38] using Simulated Annealing, a heuristic optimization technique. In [27], this problem is solved using a level set method. Thus, we present in this section the numerical results for this subproblem. We again consider the example in Figure (5.1.2) and Figure (5.2.3).

We will present two approaches in solving the problem. The first approach is by simply implementing Algorithm (3) for a fixed value of M and by assuming that χ_m is known. Because the geometry of Ω is known, the steps necessary for updating χ_m are skipped. For this approach, we will use the same set of paramaters applied in the previous examples. Because the geometry is known, the TV regularization is not included. Same goes with the second approach to be discussed later. This approach is expected to work because this is just a simplification of the problem when χ_m is unknown. Indeed, as



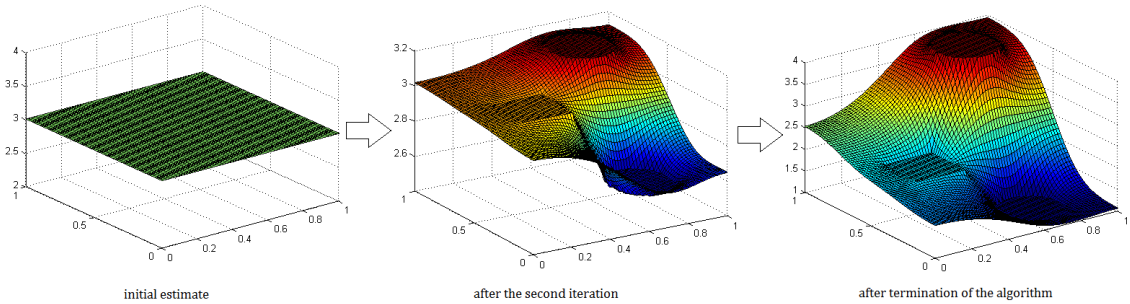
(a) (b)

Figure 5.3.1: Recovery of σ with known geometry by setting $M = 4$. The convergence of σ_1 (blue), σ_2 (red) and σ_3 (magenta) over iterations using (A) $\sigma_1^{\text{init}} = \sigma_2^{\text{init}} = \sigma_3^{\text{init}} = 3$ and (B) $\sigma_1^{\text{init}} = \sigma_2^{\text{init}} = \sigma_3^{\text{init}} = 5$.

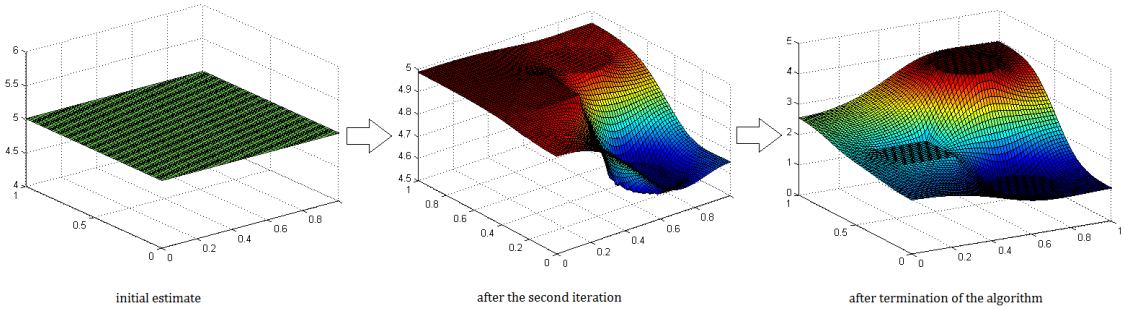
shown in Figure (5.3.1), we can see that the method converges. We used the same values used in Figure (5.2.3) for the actual values for σ_1, σ_2 and σ_3. That is, $\sigma_1 = 2$, $\sigma_2 = 4$ and $\sigma_3 = 1$. In Figure (5.2.3), the method converges when different initial estimates are used. In our current example, we use the same starting values for σ_1, σ_2 and σ_3. In Figures (5.3.1A) and (5.3.1B), we implemented the algorithm using 3 and 5 as the starting values for the conductivity in the inclusions, respectively. In both examples, convergence can be observed.

The second approach is by assuming that $M = 2$ and allowing the conductivity to assume different values in the different inclusions. This will be similar to the example illustrated in Figure (5.1.2) but with the additional condition that the conductivity is not necessarily equal in all of the inclusions. Thus, we wish to estimate σ_1 in the whole domain Ω given χ_1. Contrary to the previous examples, we will not use the same set of parameters. For this approach, we will use $\alpha = 10^{-1}$ instead of $\alpha = 10^4$. By doing this, the penalty for σ_1 to be constant in Ω_1 is not as big. Therefore, σ_1 can assume other constant values at the other inclusions. This is illustrated in Figure (5.3.2). In the first example shown in Figure (5.3.2A), we set $\sigma_1^{\text{init}} = 3$ across Ω. Observe that at the second step, it already deviated away from being constant. The values are still in the neighborhood of 3 but we can see that the values deviate away from 3 towards the actual solution. Both the conductivity values at Ω_1 and Ω_2 are now less than 3 which can be considered good because the conductivity values at Ω_1 and Ω_2 are 2 and 1, respectively. Similarly, the conductivity values at Ω_3 at the second step are now more than 3 which makes sense because the actual conductivity value at Ω_3 is 4. After 30 iterations, the algorithm converged as shown in the last image of Figure (5.3.2A). After the algorithm terminated, the mean values of the approximated σ_1 at Ω_1, Ω_2 and Ω_3 are 1.9995, 1.0001 and 3.9964, respectively. Another example is shown in Figure (5.3.2B). In this example, the initial estimate σ_1^{init} is set to be 5 in the whole domain Ω. It can be seen that for this example, the second approach also worked. The advantage of this approach over the other one is that you do not need to know the number of segments. This is also computationally better. The first approach gets an update for σ_1, σ_2 and σ_3. The second approach only updates one quantity. And what if M is large? It means that for the first approach, you have to implement the algorithm $M - 1$ times to get an

update for each of the conductivities. On the other hand, for the second approach, you only need to implement the algorithm one time regardless of how large M is.



(a)

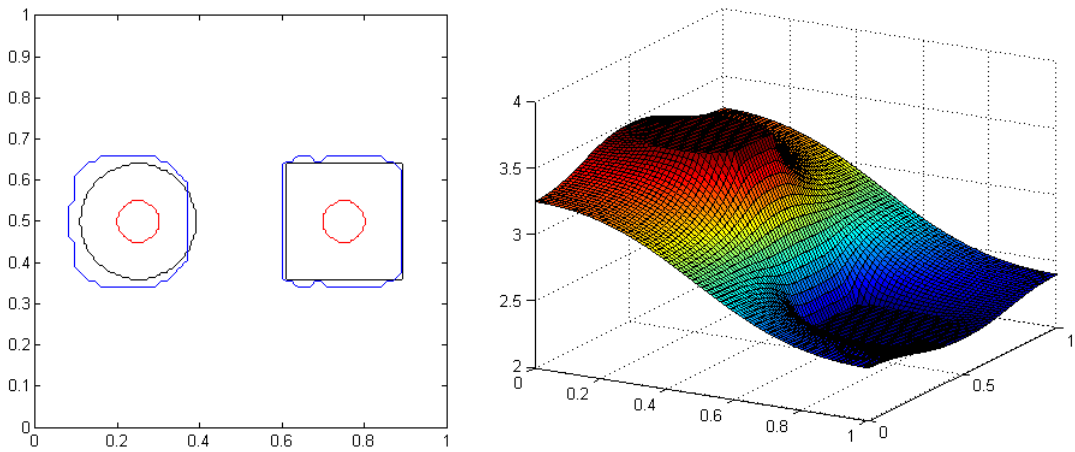


(b)

Figure 5.3.2: Recovery of σ with known geometry by setting $M = 2$. The approximation of σ_1 after the second iteration and the approximation of σ_1 after the termination of the algorithm when (A)$\sigma_1^{\text{init}} = 3$ in Ω and (B) $\sigma_1^{\text{init}} = 5$ in Ω.

5.4 Reconstructing Images of a Disjoint Inclusion

To end this chapter, we will consider the case when $M = 2$ and Ω_1 is a disjoint set. In this case, the conductivity values across Ω_1 are not constant. For example, let Ω_1 be a union of two disjoint sets Ω_1^a and Ω_1^b. Suppose the conductivities at these two sets are not equal, say σ_1^a and σ_1^b. This means that we want σ_1 to have values σ_1^a and σ_1^b at Ω_1^a and Ω_1^b, respectively. Observe that we can solve this problem by using the approach in the example illustrated in Figure (5.2.3). In this section, we present an alternative solution. We will proceed in the same way as how we proceeded in the example illustrated in Figure (5.3.2). We will assume that $M = 2$ and σ_1 is not constant across Ω_1. We will choose α to be not too large so that the penalty for σ_1 to be constant across Ω_1 is not too high. We will still make sure that α is bigger than ϵ to guarantee that σ_1 has more penalty to be constant in Ω_1. We illustrate the result of our simulation in Figure (5.4.1). In this particular example, we let Ω to be a union of a circle and a square . The circle is centered at $(0.25, 0.5)$ with radius 0.14 and the square is centered at $(0.75, 0.5)$ with radius 0.14. The actual conductivities at the circle and the square are 3.5 and 2, respectively. It can be seen in Figure (5.4.1A) the reconstructed geometry of Ω_1 (blue) after 30 iterations. For comparison, the actual geometry (black) and the initial estimate (red) are also shown. The parameters used are the same as the previous examples except for α where we use the value $\alpha = 10^{-2}$. Observe that the image reconstruction is considerably good. For the recovery of the conductivities, we refer to Figure (5.4.1B). After the termination of



(a) (b)

Figure 5.4.1: A reconstruction of the image of Ω by assuming that $M = 2$ and Ω_1 is a disjoint set. (A) The reconstructed image of Ω_1 (blue) compared with the initial guess (red) and the actual solution (black). (B) The recovered conductivity σ_1 after the termination of the algorithm.

the algorithm, the value of σ_1 is 3.45 at the circle and 2.11 at the square. Compared to the actual values 3.5 and 2, the recovery is pretty good considering the fact that it was done simultaneously with the reconstruction of the image of Ω_1 . The advantage of this approach is that you don't need to know the number of the inclusions of Ω_1. Although the reconstructed image is not as good compared to the other methods discussed in this chapter, this approach is more general and can be used to solve the EIT problem with disjoint inclusions. This approach also needs less computation time because the recovery of the conductivity and the reconstruction of the geometry are done only once. In the method illustrated in Figure (5.2.3), the recovery and reconstruction are done for each of the inclusions which means that the higher the number of inclusions, the longer time it needs to implement the method. On the other hand, the method presented in this section does not depend on the number of inclusions. The computation time is the same regardless of how many inclusions the body Ω has.

Chapter 6

Conclusion

The EIT problem is the image reconstruction of the conductivity distribution of a body Ω given data on the boundary $\partial\Omega$. One of the applications of EIT is the imaging of the internal organs of a human body. Because the conductivities of healthy tissues show great contrast, the image reconstruction can be treated as a segmentation problem. In this work, we were able to find a suitable segmentation method that can be used to solve the EIT problem. By adding regularity terms to the usual least-square distance at the boundary, we were able to formulate a functional that when minimized gives a good reconstruction of the conductivity distribution σ. The geometry of the inclusions Ω_m can be treated as a function in Ω by taking its characteristic function χ_m. Because σ is piecewise constant on these inclusions, it can be expressed as the sum of χ_m multiplied with their corresponding conductivity values σ_m. This way, the formulated functional J was expressed in terms of the σ_m and χ_m. By fixing χ_m and getting the derivative of J with respect to σ_m, an optimality condition was calculated and σ_m was expressed in terms of χ_m. Hence, J became dependent on one quantity only and its gradient can be computed. Given an initial geometry, an update is solved using the method of steepest descent. A multi-phase segmentation approach to the EIT problem was then introduced. By introducing a mollification on χ_m and using the Schauder's Fixed Point theorem, the existence of a fixed point of the proposed method was proved. Using the Finite Element Method, a discretization of the proposed method was introduced. The convergence of the discretization to the continuous case as the step size approaches zero was proved. Although the EIT problem is severely ill-posed, the numerical results obtained were promising. With the help of topological derivative, an initial update for χ_m was computed. Unfortunately, the use of topological derivative can only be implemented if the conductivities are the same in all the inclusions, that is, if σ can be expressed as $\sigma_1\chi_1 + \sigma_2(1-\chi_2)$. For the case when $M \geq 3$, an initial estimate for χ_m is necessary to apply our proposed method. Fortunately, the initial estimate does not necessarily need to be *"too close"* to the actual solution as shown in our simulations. Even if the estimate and the actual solution do not have a common point, a decent approximation can be obtained. When the geometry of the inclusions of Ω are known, the EIT problem becomes a problem of solving the conductivity values. Numerical experiments were also made and our proposed method generates solutions that are independent on the number of inclusions. We admit that when $M \geq 3$, an initial estimate is necessary. Moreover, we considered the continuous case of the EIT problem. However, in real experiments, the Complete Electrode Model (CEM) of EIT is used. Devising a way to get an initial estimate and testing our proposed method for the CEM are subjects of forthcoming

works.

Appendix A

Notations

We present here the notations used in this work. Some well-used notations (e.g., $\mathbb{R}$ for the set of real numbers) are omitted. We use bold symbols for vectors and matrices. Bold characters in lower case are usually used to denote vectors and bold characters in uppercase are used to denote matrices. Let $m, n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be non-empty, bounded and open. We also let u, v to be real-valued functions defined in Ω and $\boldsymbol{u}$ to be a vector-valued function defined in Ω ($\boldsymbol{u} : \Omega \to \mathbb{R}^n$). Furthermore, let f and $\{f_k\}_{k=1}^{\infty}$ to be measurable functions that map Ω to $\mathbb{R}^n$. Suppose x is an element of real-normed space X and $\{x_k\}_{k=1}^{\infty}$ is a sequence in X.

$\lvert a \rvert$	absolute value of the real number a
$\mu(\Omega)$	Lebesgue measure of a set Ω
$\boldsymbol{a} \cdot \boldsymbol{b}$	scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$
$\mathbb{R}^{m \times n}$	the space of real m by n matrices
$\partial\Omega$	boundary of the set Ω
$\bar{\Omega}$	closure of the set Ω
a.e.	abbreviation for almost everywhere, a property is valid in a certain domain except on a set of measure 0
id	the identity function
$\mathrm{supp}(u)$	the support of the function u
$\ker(u)$	the kernel of the function u
∇u	the gradient of the function u
$\nabla \cdot \boldsymbol{u}$	the divergence of the vector-valued function $\boldsymbol{u}$
$u \equiv v$	u is identical with v
$u := v$	use to define u equalling v
$u * v$	convolution of u and v
$\int_{\Sigma} u dS$	integral of u over Σ with respect to $(n-1)$-dimensional surface measure
$\int_{\Omega} u dV$	integral of u over Ω with respect to n-dimensional volume measure

$C(\Omega)$	the space of continuous functions on Ω
$C^m(\Omega)$	the space of $m-$times continuously differentiable functions on Ω
$C^\infty(\Omega)$	the space of infinitely differentiable functions on Ω
$C_0^m(\Omega)$	the space of $m-$times continuously differentiable functions with compact support on Ω
$L^p(\Omega)$	the space of $p-$integrable $(1 \leq p < \infty)$ or essentially bounded $(p = \infty)$ functions on Ω
$W^{m,p}(\Omega)$	the Sobolev space of L^p functions with weak derivatives in L^p up to order m
$H^m(\Omega)$	$W^{m,2}(\Omega)$
Y_1/Y_2	quotient space of Y_1 by Y_2
$\|\cdot\|_Y$	norm on an arbitrary space Y
$Y_1 \hookrightarrow Y_2$	inclusion of Y_1 in Y_2 with continuous injection
W^*	dual of the vector space W
$x_k \to x$ in X	convergence in the strong topology
$x_k \rightharpoonup x$ in X	convergence in the weak topology
$f_k \to f$ a.e.	convergence almost everywhere

Appendix B

Calculus Facts

In this part of the thesis, we present all the classical results repeatedly used in this work. This summary outlines facts relevant to this work. Other trivial results are left out. For the calculus facts presented below, we let $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be non-empty, bounded and open. We assume that Ω has a suffciently smooth boundary $\partial\Omega$. We will also let X and Y to be a Banach spaces. Furthermore, we let H to be a Hilbert space together with an inner product $\langle \cdot, \cdot \rangle$ and equipped with the norm $\|\cdot\|_H$. All of the results shown below will not be proved but a reference for each of them will be given.

In this section, we present all inequalities used in this work. The first four inequalities are discussed and proven in detail in [2] while the discussion of the fifth inequality can be found in [4]. The last inequality can be found in [13, 9].

- **Minkowski Inequality.** Suppose $1 \leq p \leq \infty$. For any $f, g \in L^p(\Omega)$, then

  $$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

  This inequality is the *"triangle inequality"* for the space $L^p(\Omega)$. In this work, we took the liberty of using either of these names.

- **Cauchy-Schwarz Inequality.** Let $f, g \in H$ then

  $$|\langle f, g \rangle_H| \leq \|f\|_H \|g\|_H.$$

  In this work, this inequality is repeatedly applied to $L^2(\Omega)$, a Hilbert space.

- **Hölder's Inequality.** Suppose $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then

  $$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

- **Young's Inequality for Convolution.** Suppose $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For any $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then

  $$\|f * g\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

- **Chebyshev's Inequality.** Given a measurable function $f : \Omega \to \mathbb{R}$, then

  $$\mu(\{x \in \Omega : f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \int_\Omega |f|^p \, dV$$

  for any $\epsilon > 0$ and for all $0 < p < \infty$.

- **Poincare-Friedrichs Inequality.** Suppose $1 \leq p \leq \infty$, then there exists $C > 0$ such that
$$\|f\|_{L^p(\Omega)} \leq C \left(\left| \int_{\partial\Omega} f dS \right| + \|\nabla f\|_{L^p(\Omega)} \right)$$
for all $f \in W^{1,p}(\Omega)$. The constant C depends only on Ω.

We also employ the following classical results from real and functional analysis.

- **Lax-Milgram Theorem.** [30] Let $a : H \times H \to \mathbb{R}$ be a bilinear functional, that is, linear in the first and the second arguments. Suppose a is bounded, that is, $\exists C_1 > 0$ such that
$$|a(u,v)| \leq C_1 \|u\|_H \|v\|_H \quad \forall u, v \in H$$
and coercive, that is, $\exists C_2 > 0$ such that
$$|a(u,u)| \geq C_2 \|u\|_H^2 \quad \forall u \in H.$$
Furthermore, let $b : H \to \mathbb{R}$ be bounded linear functional, that is, $\exists C_3 > 0$ such that
$$|b(v)| \leq C_3 \|v\|_H \quad \forall v \in H.$$
Then there exists a unique solution $u^* \in H$ such that
$$a(u^*, v) = b(v) \quad \forall v \in H.$$
Furthermore,
$$\|u^*\|_H \leq \frac{C_3}{C_2}.$$

- **Trace Theorem.** [30] There exists a bounded linear operator $T : H^1(\Omega) \to L^2(\Omega)$ such that

  1. $Tu = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\bar{\Omega})$, and
  2. $$\|Tu\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}$$
  for each $u \in H^1(\Omega)$ with the constant C depending only on Ω.

  T is often referred to as the *trace operator*.

- **General Sobolev Imbedding Theorem.** [30] Let $u \in W^{k,p}(\Omega)$.

  1. If $k < \frac{n}{p}$ then $u \in L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. In addition,
  $$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}$$
  where C depends only on Ω, k, p and n.

2. If $k > \frac{n}{p}$ then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}\left(\bar{\Omega}\right)$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integr} \\ \text{any positive number } <1, & \text{otherwise.} \end{cases}$$

In addition,

$$\|u\|_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}\left(\bar{\Omega}\right)} \leq C \|u\|_{W^{k,p}(\Omega)}$$

where C depends only on Ω, k, p, γ and n.

- **Riesz-Representation Theorem.** [30] Let H^* be the dual of H. For any $u^* \in H^*$, there exists $u \in H$ such that

$$u^*(v) = \langle u, v \rangle_H$$

for all $v \in H$.

- **Directional Derivative.** [33] Let $f : X \to Y$ and $u, \delta u \in X$, then

$$\frac{\delta f}{\delta u}(u; \delta u) = \lim_{\eta \to 0} \frac{f(u + \eta \delta u) - f(u)}{\eta}$$

is called the directional derivative of f in the direction of δu. Furthermore, if the directional derivative exists for all $\delta u \in X$ and $\frac{\delta f}{\delta u}(u; \delta u)$ is a bounded linear operator from X to Y, then f is Gateaux differentiable at u with Gateaux derivative $\frac{\delta f}{\delta u}$.

- **Symmetric Difference of Sets.** [39] Suppose $\mathcal{B}(\Omega)$ is the Borel $\sigma-$Algebra. Let $A_1, A_2 \in \mathcal{B}(\Omega)$, then the *symmetric difference* of A_1 and A_2 is defined to be

$$A_1 \triangle A_2 := (A_1 \backslash A_2) \cup (A_2 \backslash A_1),$$

or equivalently,

$$A_1 \triangle A_2 := (A_1 \cup A_2) \setminus (A_2 \cap A_1).$$

For any $A_1, A_2, A_3 \in \mathcal{B}(\Omega)$, the following holds:

  - $A_1 \triangle A_2 = A_2 \triangle A_1$
  - $(A_1 \triangle A_2) \triangle A_3 = A_1 \triangle (A_2 \triangle A_3)$
  - $A_1 \triangle \emptyset = A$
  - $A_1 \triangle A_1 = \emptyset$
  - $A_1 \cap (A_2 \triangle A_3) = (A_1 \cap A_2) \triangle (A_1 \cap A_3)$.

- **The Schauder's Fixed Point Theorem.** [34] Let K be a convex subset of X and suppose $\Upsilon : K \to X$ is continuous. Suppose $\Upsilon(K)$ is compact in K. Then Υ has a fixed point in K.

- **Weak Sequential Compactness in a Hilbert Space.** [49] A bounded sequence $\{u_n\}_{n=1}^{\infty}$ in H contains a weakly convergent subsequence $\{u_{n_l}\}_{l=1}^{\infty}$. In other words, $\exists u^* \in H$ such that $u_{n_l} \rightharpoonup u^*$ in H as $l \to \infty$.

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