START-Projekt Y 305-N18

Karl–Franzens–Universität Graz

A PDE-Constrained Generalized Nash Equilibrium Problem with Pointwise Control and State Constraints

Michael Hintermüller Thomas Surowiec

IFB-Report No. 56 March 2012

A–8010 GRAZ, MOZARTGASSE 14/II, AUSTRIA

Supported by the
Austrian Science Fund (FWF)

Der Wissenschaftsfonds.

A PDE-Constrained Generalized Nash Equilibrium Problem with Pointwise Control and State Constraints

M. Hintermüller* and T. Surowiec†

Abstract

A generalized Nash equilibrium problem (GNEP) is formulated in which, in addition to pointwise constraints on both the control and state variables, the feasible sets are partially governed by the solutions of a linear elliptic partial differential equation. The decisions (optimal controls) of the players arise in their competitors optimization problems via the righthand side of the partial differential equation. The existence of a (pure strategy) Nash equilibrium for the GNEP is demonstrated via a relaxation argument under the presence of a constraint qualification. A numerical method based on a non-linear Gauss-Seidel iteration is presented and numerical results are provided.

1 Introduction

Given the wealth of physical, biological, economic, and financial phenomena that can be modeled by solutions of partial differential equations, it is only natural that they should arise as constraints in many practical optimization problems. Moreover, many real world problems involve the interactions of multiple decision makers, each of whom acts according to their own preferences in a non-cooperative manner. It is for this reason that N-person, non-cooperative games in which the strategy sets are partially governed by the solutions of partial differential equations are a natural subject of study. To the best of our knowledge, despite the relevance of such models, there appears to be no treatment in the literature of problems of the type considered in this paper.

The particular class of equilibrium problems chosen for this paper contain what we believe to be essential components of an optimization problem whose feasible set is partially governed by the solution of a partial differential equation (PDE). Though the class considered here is simple in structure from a finite dimensional perspective, the function space setting introduces additional difficulties, such as the existence of Lagrange multipliers. These aspects, along with the potentially large scale of the discretized models, add to the already nontrivial task of studying generalized Nash equilibrium problems (GNEPs). In this respect, this paper is meant to serve as a foundation on which future investigations of GNEPs in function space may be built.

*Department of Mathematics, Humboldt University of Berlin, Germany and Department of Mathematics and Scientific Computing, Karl-Franzens-University of Graz, Austria (hint@math.hu-berlin.de).

†Department of Mathematics, Humboldt University of Berlin, Germany (surowiec@math.hu-berlin.de).

The paper is structured as follows. In Section 2, we introduce the equilibrium problem and demonstrate the existence of a Nash equilibrium. In Section 3, we develop an algorithm in function space for finding generalized Nash equilibria. This method is then tested and discussed in Section 4, after a suitable discretization.

We will use a standard notation throughout the paper and we refer the reader to [1] for details about Lebesgue and Sobolev spaces, to [12, 27] for regularity theory of solutions of partial differential equations, and for any further notions of functional analysis to [28]. Much of the standard theory of PDE-constrained optimization can be found in [17] and [25].

2 A GNEP in Function Space

Throughout the text, we let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, or 3, be open and bounded. We use " $a.e.\,\Omega$ " to represent the phrase "almost everywhere on Ω". The spaces of all functions u for which $|u|^2$ is Lebesgue integrable will be denoted by $L^2(\Omega)$, whereas $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ represents the Sobolev space of all $L^2(\Omega)$ functions y such that $|\nabla y|^2$ is Lebesgue integrable, where ∇y represents the weak derivative of y, and for which $y|_{\partial\Omega} = 0$ holds. Note that $\nabla y(x) \in \mathbb{R}^d$, in which case $|\nabla y|$ represents the pointwise Euclidean norm on $\mathbb{R}^d$. The dual space of $H_0^1(\Omega)$ will be denoted by $H^{-1}(\Omega)$. Due to the assumed boundedness of Ω, we may define the norm on $H_0^1(\Omega)$ by $||y||_{H_0^1(\Omega)} := ||\nabla y||_{L^2(\Omega)}$. The Sobolev spaces $W_0^{m,p}(\Omega)$, where $m \in \mathbb{Z}$ and $1 \leq p \leq +\infty$, are defined analogously to $H_0^1(\Omega)$ with their respective dual spaces denoted by $W_0^{-m,p'}(\Omega)$, where $p' \in \mathbb{R}_+ \cup \{+\infty\}$ such that $1/p + 1/p' = 1$. For a subset $A \subset \Omega$, we use the symbols $m(A)$ and χ_A to represent the Lebesgue measure and the characteristic function, respectively, and we let $-\Delta = -\text{div} \cdot \nabla$ be the standard Laplacian. The following data assumptions are used throughout:

- The boundary $\partial\Omega \subset R^{d-1}$ is regular enough such that if $f \in L^2(\Omega)$, then the (unique) solution $u : \Omega \to \mathbb{R}$ of the Poisson equation with homogeneous Dirichlet boundary conditions and righthand side f can be continuously embedded into the Sobolev space $W_0^{1,r}(\Omega)$, with $r > d$, if $d > 1$, and $r = d$, if $d = 1$.
- $N \geq 2, N \in \mathbb{N}$.
- $a_i, b_i \in L^2(\Omega)$ with $a_i < b_i$, $a.e.\,\Omega$, for all $i = 1, \ldots, N$.
- $\psi \in W^{1,r}(\Omega)$ with $\psi|_{\partial\Omega} < 0$, $a.e.\,\Omega$, $r > d$, if $d > 1$, $r = d$, otherwise.
- $y_d^i \in L^2(\Omega)$, for all $i = 1, \ldots, N$.
- $\alpha_i > 0$, for all $i = 1, \ldots, N$.
- $B_i \subset \Omega$, $m(B_i \cap B_j) = 0$, for all $i, j = 1, \ldots, N$, $i \neq j$.
- $f \in L^2(\Omega)$.

As a notational convention, we define the product spaces $L^2(\Omega)^N := \Pi_{i=1}^N L^2(\Omega)$, $H_0^1(\Omega)^N := \Pi_{i=1}^N H_0^1(\Omega)$, and, for $u \in L^2(\Omega)^N$, $v \in L^2(\Omega)$, we let (v, u_{-i}) represent the vector field in $L^2(\Omega)^N$ obtained by replacing u_i in u by v.

The choice of boundary $\partial\Omega$ allows us to work with problems for which Ω has a non-smooth boundary and is convex as well as for cases in which $\partial\Omega$ is locally homeomorphic to the graph of a Lipschitz continuous function without the convexity requirement on Ω. In the first case, a well-known result from Kadlec, [18], shows that solutions of the Poisson equation with homogeneous Dirichlet boundary conditions is in $W^{2,2}(\Omega)\cap H_0^1(\Omega)$, whereas a famous result from Nečas, [20], shows that such a solution in the second case, i.e., $\partial\Omega$ Lipschitz and Ω non-convex, is in the fractional Sobolev space $W_0^{m,2}(\Omega)$, with $m \in [1, 3/2)$. In both cases, the Sobolev embedding theorem allows the solution to be embedded into $W_0^{1,r}(\Omega)$ with r as required (see [1] and Theorems 1.4.4.1, 2.2.2.3, 3.2.1.2 in [12]). Furthermore, the choice of r in relation to d allows us again to apply the Sobolev embedding theorem to show that $W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ continuously.

Given these data assumptions, we consider an N-player game in which each player i has a desired state y_d^i and cost of control $\frac{\alpha_i}{2}||\cdot||^2_{L^2(\Omega)}$. Each player i is assigned a subset B_i of Ω on which their control u_i can affect the state of the system via the righthand side of a linear elliptic partial differential equation. The players seek to minimize both the distance of the equilibrium state to their respective desired states in the $L^2(\Omega)$-norm as well as their overall costs. This must all be done in such a way that the control lies pointwise almost everywhere between the prescribed bounds a_i and b_i and such that the equilibrium state satisfies the obstacle condition "$y \geq \psi,\ a.e.\,\Omega$." In other words, each player i seeks to solve the following optimization problem in which the decisions of its competitors, denoted throughout by $u_{-i} \in L^2(\Omega)^{N-1}$, arise as exogenous parameters:

$$\begin{array}{l} \min \frac{1}{2}||y - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \text{ over } (u_i, y) \in L^2(\Omega) \times H_0^1(\Omega) \\ \text{subject to (s.t.)} \\ -\Delta y = \chi_{B_i} u_i + \sum_{\substack{k=1 \\ k\neq i}}^{n} \chi_{B_k} u_k + f, \quad a_i \leq u_i \leq b_i,\ a.e.\,\Omega, \quad y \geq \psi,\ a.e.\,\Omega. \end{array} \tag{1}$$

We refer to a point $(u, y) \in L^2(\Omega)^N \times H_0^1(\Omega)$ such that $(u_i, y) \in L^2(\Omega) \times H_0^1(\Omega)$ is feasible for problem (1) for all $i = 1, \ldots, N$ as a *feasible strategy.* For simplicity, we often use

$$U_i := \left\{ v \in L^2(\Omega) \,|\, a_i \leq v \leq b_i,\ a.e.\,\Omega \right\}.$$

We define solutions (equilibria) for this game in a standard sense.

Definition 2.1 (**Nash Equilibrium**). *A feasible strategy (u, y) is referred to as a Nash equilibrium provided the following condition holds for all $i = 1 \ldots, N$:*

$$\frac{1}{2}||y - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \leq \frac{1}{2}||y' - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i'||^2_{L^2(\Omega)},$$

$$\forall u' \in U_i,\ \forall y' \geq \psi,\ a.e.\,\Omega : -\Delta y' = \chi_{B_i} u_i' + \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k + f. \tag{2}$$

In other words, no player can reduce the value of their objective functional by unilaterally changing their decision.

As the constraint sets of each player i depend on the decisions of its competitors, this type of problem is often referred to as a generalized Nash equilibrium problem (GNEP). Some alternate names for this problem class are, to name only a few, pseudo-games, social equilibrium problems, and abstract economies. This category of games has been investigated since Debreu [5] and Arrow and Debreau [2] in the 1950s. A significant amount of work over the last two decades in the finite dimensional context has been completed, as can be seen in the recent survey paper by Facchinei and Kanzow [7].

GNEPs are notoriously difficult to solve numerically as they essentially require the solution of a quasi-variational inequality. To see this, recall that since the Laplace operator $-\Delta$ is an isometric isomorphism from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$, and since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we can write y as a function linearly dependent on the righthand side of the PDE in (1). We denote this solution operator by

$$y(u) = y(u_i, u_{-i}) := (-\Delta)^{-1}(\chi_{B_i} u_i + \sum_{\substack{k=1 \\ k \neq i}}^{n} \chi_{B_k} u_k + f).$$

Since $L^2(\Omega)$ is compactly embedded into $H^{-1}(\Omega)$ and $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, y is completely continuous from $L^2(\Omega) \to H_0^1(\Omega)$. This can then by used to rewrite the GNEP as the game in which the component problems are given by

$$\begin{array}{l} \min \frac{1}{2}||y(u_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \text{ over } u_i \in L^2(\Omega) \\ \text{s.t.} \\ \quad a_i \leq u_i \leq b_i, \ a.e.\, \Omega, \quad y(u_i, u_{-i}) \geq \psi, \ a.e.\, \Omega. \end{array} \tag{3}$$

Now let $\Gamma_i : L^2(\Omega)^{N-1} \rightrightarrows L^2(\Omega)$ be the multifunction defined by

$$\Gamma_i(u_{-i}) := \left\{ v \in L^2(\Omega) \,|\, u_i \in U_i, \ y(u_i, u_{-i}) \geq \psi, \ a.e.\, \Omega. \right\}.$$

It is easy to see that Γ_i has closed convex values. Therefore, for any fixed u_{-i}, one can derive the first-order necessary and sufficient optimality condition for the i^{th} problem (in the form of a variational inequality):
Find $u_i \in \Gamma_i(u_{-i})$:

$$(\alpha_i u_i, v - u_i)_{L^2(\Omega)} + (y_{u_i}(u_i, u_{-i})^*(y(u_i, u_{-i}) - y_d^i), v - u_i)_{L^2(\Omega)} \geq 0, \forall v \in \Gamma(u_{-i}) \tag{4}$$

Here, the adjoint operator $y_{u_i}(\cdot, u_{-i})^*$ at u_i is given by $\chi_{B_i}(-\Delta)^{-1}$.

By coupling together each of the variational inequalities (4), one obtains a quasi-variational inequality formulation of the GNEP (3). Then due to convexity, we see that a feasible strategy $u \in L^2(\Omega)^N$ for (3) is a Nash equilibrium if and only if it solves the quasi-variational inequality.

There are two main difficulties that must be surmounted in order not only to demonstrate the existence of a generalized Nash equilibrium for the GNEP (1), but also, for the development of an efficient numerical method. First, the classical existence theory for N-player noncooperative games, based on the application of Kakutani's fixed point theorem, is developed in such a way that the decisions of each opposing player may only perturb their competitors' utility functions and not their strategy sets. Second, the derivation of multiplier-based necessary and sufficient optimality conditions for each nonlinear program that comprises (1) is significantly more difficult than in the finite dimensional setting. The ability to derive KKT-type optimality conditions is essential for the development of a numerical method as we shall see in Section 3. For these two reasons, we define a class of parameter dependent Nash equilibrium problems

(NEPs) by using a smooth, convex penalty function for the pointwise constraint on the state variable y. This leads to the component problems given by

$$\begin{aligned}&\min \tfrac{1}{2}||y(u_i,u_{-i})-y_d^i||^2_{L^2(\Omega)}+\tfrac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)}+\tfrac{\gamma}{2}||(\psi-y(u_i,u_{-i}))_+||^2_{L^2(\Omega)} \text{ over } u_i\in L^2(\Omega)\\&\text{s.t. } a_i\le u_i\le b_i,\ a.e.\,\Omega.\end{aligned}\tag{5}$$

Here, $(\cdot)_+=\max(0,\cdot)$, in the pointwise almost-everywhere sense. We refer to the γ-dependent NEPs by the notation NEP$_\gamma$ and for convenience, we refer to NEP$_\gamma$ by (5), despite the slight abuse of notation. Note that the idea to penalize the shared constraints in finite dimensional GNEPs was first introduced by Fukushima and Pang in [22], see also [8].

Our first result deals with the existence of Nash equilibria for NEP$_\gamma$ (5). We first recall a famous result of Ky Fan/Kakutani [10] as formulated in [26].

Theorem 2.2. *Let S be a compact convex set in a real locally convex topological space X and let $\psi: S\rightrightarrows S$ such that $\psi(x)\subset S$ is convex and compact for all $x\in S$. If $x_n\to_X x$ and $y_n\in\psi(x_n)$ such that $y_n\to_X y$ implies $y\in\psi(x)$, then there exists an $x^*\in S$ such that $x^*\in\psi(x^*)$.*

Theorem 2.3 (Existence of a Nash Equilibrium for NEP$_\gamma$). *For all $\gamma>0$, the associated NEP$_\gamma$ (5) has a Nash equilibrium.*

Proof. We need to adapt (5) to the setting of Theorem 2.2. To begin, we define the Banach spaces X_i for $i=1,\dots,N$ by $X_i:=(L^2(\Omega),\tau_{weak})$, i.e., X_i is $L^2(\Omega)$ endowed with the weak topology τ_{weak}. We then let $X:=\Pi_{i=1}^N X_i$ be the real locally convex topological space required in Theorem 2.2 and set $S_i:=\mathrm{cl}\,\{U_i\}_{X_i}$. Due to the equivalence of weak and strong closure for convex sets in reflexive Banach spaces, $S_i=U_i$. Accordingly, we define $S\subset X$ by $S:=\Pi_{i=1}^N S_i$. The weak compactness of closed convex subsets in reflexive Banach spaces implies that S is convex and compact in X. Using these spaces and subsets, we define the best response functions $\psi_i^\gamma: X\to X_i$, $i=1,\dots,N$:

$$\begin{aligned}\psi_i^\gamma(u):=\Bigg\{v_i\in S_i\,\Bigg|\,&\frac{1}{2}||y(v_i,u_{-i})-y_d^i||^2_{L^2(\Omega)}+\frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)}+\frac{\gamma}{2}||(\psi-y(v_i,u_{-i}))_+||^2_{L^2(\Omega)}=\\&\inf_{w_i\in S_i}\frac{1}{2}||y(w_i,u_{-i})-y_d^i||^2_{L^2(\Omega)}+\frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)}+\frac{\gamma}{2}||(\psi-y(w_i,u_{-i}))_+||^2_{L^2(\Omega)}\Bigg\}\end{aligned}$$

along with the multifunction $\psi^\gamma: S\rightrightarrows S$ given by $\psi^\gamma(u):=\psi_1^\gamma(u)\times\cdots\times\psi_N^\gamma(u),\ u\in S$.

Now let $u^n\to u$ in X and $v^n\in\psi^\gamma(u^n)$ such that $v^n\to v$ in X. By definition this means $u_i^n\to u_i$, $v_i^n\to v_i$ weakly in $L^2(\Omega)$ for each $i=1,\dots,N$. Moreover, $v^n\in\psi^\gamma(u^n)$ implies that for each $i=1,\dots,N$ the following holds

$$\begin{aligned}&\frac{1}{2}||y(v_i^n,u_{-i}^n)-y_d^i||^2_{L^2(\Omega)}+\frac{\alpha_i}{2}||v_i^n||^2_{L^2(\Omega)}+\frac{\gamma}{2}||(\psi-y(v_i^n,u_{-i}^n))_+||^2_{L^2(\Omega)}\le\\&\qquad\frac{1}{2}||y(w_i,u_{-i}^n)-y_d^i||^2_{L^2(\Omega)}+\frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)}+\frac{\gamma}{2}||(\psi-y(w_i,u_{-i}^n))_+||^2_{L^2(\Omega)},\ \forall w_i\in S_i.\end{aligned}$$

Since $y(\cdot,u_{-i}^n)$ is completely continuous from $L^2(\Omega)^N$ to $H_0^1(\Omega)$ and the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is continuous, $y(\cdot,u_{-i}^n)$ is continuous from X to $H_0^1(\Omega)$, and therefore, to $L^2(\Omega)$. In addition, we know that $||\cdot||^2_{L^2(\Omega)}$ is weakly lower semicontinous on $L^2(\Omega)$ and thus, lower semicontinuous on X_i. Since the mapping $(\psi - \cdot)_+$ is convex, the composition with $||\cdot||^2_{L^2(\Omega)}$ is also lower semicontinuous on X. Passing to the limit inferior in the previous inequality, we obtain

$$\frac{1}{2}||y(v_i,u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(v_i,u_{-i}))_+||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(w_i,u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i,u_{-i}))_+||^2_{L^2(\Omega)},\ \forall w_i \in S_i.$$

It follows that $v \in \psi^\gamma(u)$. Then by Theorem 2.2, there exists some $u^* \in S$ such that $u^* \in \psi^\gamma(u^*)$. In other words, there exists a $u^* \in S$ such that for all $i = 1,\dots,N$

$$\frac{1}{2}||y(u_i^*,u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(u_i^*,u_{-i}^*))_+||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(w_i,u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i,u_{-i}^*))_+||^2_{L^2(\Omega)},\ \forall w_i \in S_i,$$

This concludes the proof. □

In order to demonstrate that the GNEP (3) has a Nash equilibrium, we will require the fulfillment of a constraint qualification.

Definition 2.4 (**Strict Uniform Feasible Responses**). *We will say that the GNEP satisfies the strict uniform feasible response constraint qualification (SUFR), if there exists an $\varepsilon > 0$, for all $i = 1,\dots,N$:*

$$\forall u_{-i} \in U_{-i},\ \exists u_i \in U_i \ :\ y(u_i,u_{-i}) \geq \psi + \varepsilon,\ a.e.\,\Omega.$$

One could interpret SUFR to require that each player has a feasible response to any strategy by its competitors such that the resulting state fulfills its constraint strictly uniformly. In the first part of the proof of Theorem 2.5, we will see that for the convergence of a sequence of equilibria $\{u^\gamma\}_\gamma$ to a feasible strategy of GNEP it suffices for SUFR to hold with $\varepsilon = 0$. However, for the convergence to a Nash equilibrium, $\varepsilon > 0$ is required.

Theorem 2.5 (**Consistency of the Relaxed Problems**). *If the GNEP (1) satisfies the SUFR, then there exists a sequence of penalty parameters $\gamma \to +\infty$ and an associated sequence of Nash equilibria $\{u^\gamma\}$ for the NEP$_\gamma$'s (5) such that for all $i = 1,\dots,N$, $u_i^\gamma \rightharpoonup_{L^2(\Omega)} u_i^*$ as $\gamma \to +\infty$, where u^* is a Nash equilibrium for the GNEP.*

Proof. Let $U := \Pi_{i=1}^N U_i$ and fix an arbitrary $\gamma > 0$. According to Theorem 2.3, each NEP$_\gamma$ has a Nash equilibrium $u^\gamma \in U$. By definition, $a_i \leq u_i^\gamma \leq b_i,\ a.e.\,\Omega$. Therefore, the sequence/path of equilibria $\{u^\gamma\}_{\gamma>0}$ is uniformly bounded in $L^2(\Omega)^N$. As U is weakly closed and $L^2(\Omega)^N$ a Hilbert space, there exists a subsequence, denoted by γ', and some element $u^* \in U$ such that $u^{\gamma'} \rightharpoonup u^*$ in $L^2(\Omega)^N$.

According to the SUFR, there exists an $\varepsilon > 0$ and a sequence $\{v^{\gamma'}\} \subset U$ such that $y(v_i^{\gamma'}, u_{-i}^{\gamma'}) \geq \psi + \varepsilon,\ a.e.\,\Omega$. As in the previous argument, we can deduce the uniform boundedness of $\{v^{\gamma'}\}_{\gamma'>0}$ in $L^2(\Omega)^N$. Thus, there exists a constant $M \geq 0$, independent of γ', such that

$$\frac{1}{2}||y(u_i^{\gamma'}, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma'}||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(u_i^{\gamma'}, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma'}||^2_{L^2(\Omega)} + \frac{\gamma'}{2}||(\psi - y(u_i^{\gamma'}, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(v_i^{\gamma'}, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i^{\gamma'}||^2_{L^2(\Omega)} \leq M.$$

Using the weak lower semicontinuity of the $L^2(\Omega)$-norm, it follows that

$$\frac{1}{2}||y(u_i^*, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} \leq \liminf_{\gamma' \to +\infty} \left[\frac{1}{2}||y(u_i^{\gamma'}, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma'}||^2_{L^2(\Omega)}\right].$$

Therefore, $\frac{\gamma'}{2}||(\psi - y(u_i^{\gamma'}, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)}$ is bounded as $\gamma' \to +\infty$. But this can only hold if $||(\psi - y(u_i^{\gamma'}, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)} \to 0$. Due to the complete continuity of the solution operator y from $L^2(\Omega)^N$ to $H_0^1(\Omega)$ and the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we also have $||(\psi - y(u_i^{\gamma'}, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)} \to ||(\psi - y(u_i^*, u_{-i}^*))_+||^2_{L^2(\Omega)}$. Thus, $u^* \in U$ such that $y(u_i^*, u_{-i}^*) \geq \psi,\ a.e.\,\Omega$. In other words, there exists a subsequence of equilibria of the NEP$_\gamma$ that converges weakly to a feasible strategy for the GNEP (1). Our next step is to demonstrate that u^* is also a generalized Nash equilibrium.

Fix an arbitrary $i \in \{1, \ldots, N\}$, take $v_i \in U_i$ such that $y(v_i, u_{-i}^*) \geq \psi,\ a.e.\,\Omega$, and $\{u_{-i}^n\} \subset U_{-i}$ such that $u_{-i}^n \rightharpoonup u_{-i}^*$ in $L^2(\Omega)^{N-1}$. Using these functions, we construct a sequence $\{v_i^n\} \subset U_i$ such that $v_i^n \to v_i$ in $L^2(\Omega)$.

According to the SUFR condition, there exists a constant $\varepsilon > 0$ and, for each n, a point $v_i^{n,0} \in U_i$ such that $y(v_i^{n,0}, u_{-i}^n) \geq \psi + \varepsilon,\ a.e.\,\Omega$. Clearly, $\{v_i^{n,0}\}$ is bounded in $L^2(\Omega)$. Since U_i is convex, the points $v_i^n(\lambda) = \lambda v_i^{n,0} + (1-\lambda)v_i \in U_i$ for all $\lambda \in (0,1)$. Due to the linearity of the solution operator y, it holds for each $\lambda \in (0,1)$ that

$$\begin{aligned} y(v_i^n(\lambda), u_{-i}^n) &= y(\lambda v_i^{n,0} + (1-\lambda)v_i, u_{-i}^n) \\ &= \lambda y(v_i^{n,0}, u_{-i}^n) + (1-\lambda)y(v_i, u_{-i}^n) \\ &\geq \lambda(\psi + \varepsilon) + (1-\lambda)y(v_i, u_{-i}^n). \end{aligned}$$

As discussed at the beginning of this section, the assumed regularity of $\partial\Omega$, with $r > d$, $d \in \{2,3\}$, and $r = d$, $d = 1$, yields $y(v_i, \cdot) : L^2(\Omega)^{N-1} \to W_0^{1,r}(\Omega)$, which allows us to embed solutions of the state equation into the space of continuous functions over $\overline{\Omega}$. This renders the solution operator $y(v_i, \cdot)$ continuous from $L^2(\Omega)^{N-1} \to C(\overline{\Omega})$. It follows from the convergence of $y(v_i, u_{-i}^n) \to y(v_i, u_{-i}^*)$ in $C(\overline{\Omega})$ that there exists $n_0 \in \mathbb{N}$ such that $y(v_i, u_{-i}^n) \geq \psi - 1/2^n$ on Ω for all $n \geq n_0$. By defining $\lambda_n := (1/2^n)/(\varepsilon + 1/2^n)$, we obtain a null sequence, whose elements all lie in the interval $(0,1)$, and for which $y(v_i^n(\lambda_n), u_{-i}^n) \geq \psi,\ a.e.\,\Omega$. Then since

$$\begin{aligned} ||v_i^n(\lambda_n) - v_i||_{L^2(\Omega)} &= ||\lambda_n v_i^{n,0} + (1-\lambda_n)v_i - v_i||_{L^2(\Omega)} \\ &= |\lambda_n|||v_i^{n,0} - v_i||_{L^2(\Omega)} \\ &\leq |\lambda_n|(||v_i^{n,0}||^2_{L^2(\Omega)} + ||v_i||_{L^2(\Omega)}), \end{aligned}$$

it follows that $v_i^n(\lambda_n) \to v_i$ in $L^2(\Omega)$. This implies then that any element $v_i \in U_i$ such that $y(v_i, u_{-i}^*) \geq \psi,\ a.e. \,\Omega$ can be obtained by such a sequence $\{v_i^n(\lambda_n)\}_{n=1}^{\infty}$.

Let $\gamma' \to +\infty$ be as in the above and define $X_i := \left\{v_i \in U_i \,\middle|\, y(v_i, u_{-i}^*) \geq \psi,\ a.e.\,\Omega\right\}$. Note that X_i is non-empty due to the SUFR condition. Since for all such γ', $u^{\gamma'}$ is a Nash equilibrium for NEP$_{\gamma'}$, it holds that

$$\frac{1}{2}||y(u_i^{\gamma'}, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma'}||^2_{L^2(\Omega)} + \frac{\gamma'}{2}||(\psi - y(u_i^{\gamma'}, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(v_i, u_{-i}^{\gamma'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)} + \frac{\gamma'}{2}||(\psi - y(v_i, u_{-i}^{\gamma'}))_+||^2_{L^2(\Omega)},\ \forall v_i \in X_i,$$

Now, for any $v_i \in X_i$, we can construct a strongly converging sequence $v_i^{\gamma'}$ analogously to $v_i^n(\lambda_n)$, such that $v_i^{\gamma'} \to_{L^2(\Omega)} v_i$. Upon substitution of this sequence into the previous inequality, passing to the limit inferior over γ' yields the following inequality for all $i = 1, \ldots, N$:

$$\frac{1}{2}||y(u_i^*, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} \leq \frac{1}{2}||y(v_i, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)},\ \forall v_i \in X_i$$

as was to be shown. $\square$

Now that we have shown the existence of a Nash equilibrium for the GNEP (3), we derive first order optimality conditions, which will be needed in the coming sections for the development of an implementable solution method.

Proposition 2.6 (Necessary and Sufficient Optimality Conditions NEP$_\gamma$). *For any $\gamma > 0$, a feasible strategy u^γ is a Nash equilibrium for NEP$_\gamma$ (5) if and only if there exists a $y^\gamma \in H_0^1(\Omega)$ and for all $i = 1, \ldots, N$, a $p_i^\gamma \in H_0^1(\Omega)$ such that*

$$u_i^\gamma = \frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - (\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - b_i)_+ + (-(\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - a_i))_+, \tag{6}$$

$$-\Delta y^\gamma = \chi_{B_i} u_i^\gamma + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k^\gamma + f, \tag{7}$$

$$-\Delta p_i^\gamma = y_d^i - y^\gamma + \gamma(\psi - y^\gamma)_+. \tag{8}$$

Proof. By applying the argument used for (4) to the current setting, we can derive first-order necessary and sufficient optimality conditions for a Nash equilibrium u^γ of the form: Find $u^\gamma \in U$ such that for all $i = 1, \ldots, N$

$$(\alpha_i u_i^\gamma, v - u_i^\gamma)_{L^2(\Omega)} + (y_{u_i}(u_i^\gamma, u_{-i}^\gamma)^*(y(u_i^\gamma, u_{-i}^\gamma) - y_d^i - \gamma(\psi - y(u_i^\gamma, u_{-i}^\gamma))_+), v - u_i^\gamma)_{L^2(\Omega)} \geq 0, \forall v \in U_i.$$

By letting $p_i^\gamma = (-\Delta)^{-1}(y(u_i^\gamma, u_{-i}^\gamma) - y_d^i - \gamma(\psi - y(u_i^\gamma, u_{-i}^\gamma))_+)$, we obtain the equivalent coupled system for each $i = 1, \ldots, N$:

$$(\alpha_i u_i^\gamma - \chi_{B_i} p_i^\gamma, v - u_i^\gamma)_{L^2(\Omega)} \geq 0, \forall v \in U_i$$

The nonsmooth equation (6) arises from the equivalence between the variational inequality and the projection of $\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma$ onto U_i, (cf. [19, 25]), whereas (7) and (8) follow from the definitions of p^γ and y^γ. $\square$

The following constraint qualification is based on one developed in [24], see also [16, 21].

Definition 2.7 (**A Uniform Range Space Condition**). *We say that the GNEP satisfies the uniform range space constraint qualification (URS) with respect to the control and state spaces* $L^2(\Omega), W_0^{1,r}(\Omega)$, *respectively, with* $1 \leq r \leq +\infty$, *if the following holds for all* $i = 1, \ldots, N$*: There exists a* $\delta_i > 0$ *and a bounded set*

$$M_i \subset \left\{(v, z) \in L^2(\Omega) \times W_0^{1,r}(\Omega) \,|\, v \in U_i, z \geq \psi, \ a.e.\, \Omega\right\}$$

for all $u_{-i} \in U_{-i}$ *such that*

$$\mathbb{B}_{\delta_i}(0) \subset \left\{ -\Delta y - \chi_{B_i} u_i - \sum_{\substack{k=1 \\ k \neq i}}^{N} \chi_{B_k} u_k - f \,\middle|\, (u_i, y) \in M_i \right\}$$

where $\mathbb{B}_{\delta_i}(0)$ *is the open ball of radius* δ_i *in* $W^{-1,r}(\Omega)$.

Whereas the SUFR constraint qualification is concerned mainly with the regularity of the state constraint, the URS condition is needed to ensure the existence of an adjoint state for the GNEP (3). Together the two conditions will be needed to guarantee the convergence of stationary points which satisfy (6)-(8). Nevertheless, we show in the following lemma that the SUFR condition in fact implies the URS condition. Note that $\mathcal{M}(\overline{\Omega})$ represents the space of all bounded Borel measures. The dual of which is the space $C(\overline{\Omega})$ of all continuous functions.

Lemma 2.8 (**SUFR $\Rightarrow$ URS**). *Under the standing data assumptions, suppose that the GNEP (3) satisfies the SUFR condition. Then the URS condition holds with respect to the control and state spaces* $L^2(\Omega)$ *and* $W_0^{1,r}(\Omega)$, *where* $r > d$, *if* $d > 1$, *and* $r = d$, *if* $d = 1$.

Proof. Based on the data assumptions, there exists a constant $C > 0$ such that for all $y \in W_0^{1,r}(\Omega)$, $||y||_{W_0^{1,r}(\Omega)} \geq C||y||_{C(\overline{\Omega})}$. In addition, we know that the inverse operator $(-\Delta)^{-1}$ is bounded in the operator norm $||\cdot||_{op}$ from $W^{-1,r}(\Omega)$ to $W_0^{1,r}(\Omega)$. Suppose then that $\delta > 0$ with

$$\delta \leq \frac{C\varepsilon}{2||(-\Delta)^{-1}||_{op}},$$

where the positive constant ε is taken from the definition of the SUFR condition.

Fix an arbitrary feasible strategy u for the GNEP (3). By the SUFR condition and regularity assumptions on $\partial\Omega$, there exists a $y \in W_0^{1,r}(\Omega)$ and $u_i^\delta \in U_i$ such that

$$-\Delta y - \chi_{B_i} u_i^\delta - \sum_{\substack{k=1 \\ k \neq i}} \chi_{B_k} u_k - f = 0 \text{ and } y \geq \psi + \varepsilon, \ a.e.\, \Omega.$$

Now let $w_\delta \in W^{-1,r}(\Omega)$ such that $||w_\delta||_{W^{-1,r}(\Omega)} < \delta$ and $y_\delta \in W_0^{1,r}(\Omega)$ such that $-\Delta y_\delta = w_\delta$. Then

$$C||y_\delta||_{C(\overline{\Omega})} \leq ||y_\delta||_{W_0^{1,r}(\Omega)} = ||(-\Delta)^{-1}(w_\delta)||_{W_0^{1,r}(\Omega)} \leq ||(-\Delta)^{-1}||_{op}||w_\delta||_{W^{-1,r}(\Omega)} < \frac{C\varepsilon}{2}.$$

Therefore, $-\varepsilon/2 \leq y_\delta \leq \varepsilon/2$ for all $x \in \overline{\Omega}$, from which it follows that $y+y_\delta \geq \psi+\varepsilon/2 \geq \psi,\ a.e.\,\Omega$. Finally, we observe that

$$-\Delta(y+y_\delta) - \chi_{B_i}u_i^\delta - \sum_{\substack{k=1\\k\neq i}} \chi_{B_k}u_k - f = -\Delta y_\delta = w_\delta$$

and

$$||y_\delta + y||_{W_0^{1,r}(\Omega)} \leq C\varepsilon/2 + ||y||_{W_0^{1,r}(\Omega)} = C\varepsilon/2 + ||(-\Delta)^{-1}(\chi_{B_i}u^\delta + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k}u_k + f)||_{W_0^{1,r}(\Omega)} \leq$$

$$C\varepsilon/2 + C'||(-\Delta)^{-1}||_{op}(Vol(B_i)||u^\delta||_{L^2(\Omega)} + \sum_{\substack{k=1\\k\neq i}}^N Vol(B_k)||u_k||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}),$$

where $C' > 0$ is the constant arising from the (continuous) embedding $W^{-1,r}(\Omega) \hookrightarrow L^2(\Omega)$. In light of the boundedness of the sets U_i, $i = 1, \ldots, N$, the assertion follows. □

Theorem 2.9 (Convergence of Stationary Points). *Suppose the GNEP (3) satisfies the SUFR condition. Then there exist sequences $\gamma_n \to +\infty$, $\{u^n\} \subset L^2(\Omega)^N$, $\{y^n\} \subset W_0^{1,r}(\Omega)$, and $\{p^n\} \subset W_0^{1,s}(\Omega)^N$ along with $u^* \in L^2(\Omega)^N$, $y^* \in W_0^{1,r}(\Omega)$, $p^* \in W_0^{1,s}(\Omega)^N$, and $\lambda^* \in \mathcal{M}(\overline{\Omega})$ where, for all $i = 1, \ldots, N$:*

$$u_i^n \to_{L^2(\Omega)} u_i^*, \quad y^n \to_{W_0^{1,r}(\Omega)} y^*, \quad p_i^n \rightharpoonup_{W_0^{1,s}(\Omega)} p_i^*, \quad \gamma_n(\psi - y^n)_+ \rightharpoonup_{\mathcal{M}(\overline{\Omega})} \lambda^*$$

such that (u_i^n, y^n, p_i^n) satisfies (6)-(8) and

$$u_i^* = \frac{1}{\alpha_i}\chi_{B_i}p_i^* - (\frac{1}{\alpha_i}\chi_{B_i}p^* - b_i)_+ + (-(\frac{1}{\alpha_i}\chi_{B_i}p^* - a_i))_+ \tag{9}$$

$$-\Delta y^* = \chi_{B_i}u_i^* + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k}u_k^* + f \tag{10}$$

$$-\Delta p_i^* = y_d^i - y^* + \lambda^*, \tag{11}$$

$$\langle \lambda^*, \varphi\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \geq 0, \forall \varphi \in C(\overline{\Omega}) : \varphi \geq 0, \quad y^* \geq \psi,\ a.e.\,\Omega,\ \langle \lambda^*, y^* - \psi\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} = 0. \tag{12}$$

Proof. According the Theorem 2.5, there exists a sequence $\gamma_n \to +\infty$ along with a sequence $\{u^n\} \subset L^2(\Omega)^N$ of Nash equilibria for the NEP$_{\gamma_n}$ that converges weakly to a Nash equilibrium u^* for the GNEP (3) in the sense that for each i, $u_i^n \rightharpoonup_{L^2(\Omega)} u_i^*$. It follows then from Proposition 2.6 that there exists a $y^n \in H_0^1(\Omega)$ and, for each $i \in \{1, \ldots, N\}$, a $p_i^n \in H_0^1(\Omega)$ such that the relations (6)-(8) hold at (u_i^n, y^n, p_i^n) with $\gamma = \gamma_n$. Given the assumptions of $\partial\Omega$, the sequences $\{y^n\}$ and $\{p_i^n\}_n$ are contained in $W_0^{1,r}(\Omega)$, for all $i = 1, \ldots, N$. We begin by demonstrating the assertions on the sequences of adjoint states $\{p_i^n\}$.

Let $(u_i, y) \in M_i$, where M_i is the bounded subset of $L^2(\Omega) \times W_0^{1,r}(\Omega)$ given by the URS condition. Multiplying (8) by $y^n - y$, we obtain

$$\langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} - \gamma_n \int_\Omega (\psi - y^n)_+(y^n - y)dx = \int_\Omega (y_d^i - y^n)(y^n - y)dx$$
$$\Leftrightarrow \langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} - \gamma_n \int_\Omega (\psi - y^n)_+(y^n - \psi + \psi - y)dx = \int_\Omega (y_d^i - y^n)(y^n - y)dx$$
$$\Leftrightarrow \langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} + \gamma_n \int_\Omega (\psi - y^n)_+^2 = \gamma_n \int_\Omega (\psi - y^n)_+(\psi - y)dx + \int_\Omega (y_d^i - y^n)(y^n - y)dx$$

Then since $\psi - y \leq 0,\ a.e.\,\Omega$, it must hold that

$$\langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} \leq \langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} + \gamma_n \int_\omega (\psi - y^n)_+^2 \leq \int_\Omega (y_d^i - y^n)(y^n - y)dx. \tag{13}$$

Clearly, $y^n - y \in W_0^{1,r}(\Omega)$, which in turn implies that $-\Delta(y^n - y) \in W^{-1,r}(\Omega)$. It follows then that $\langle -\Delta p_i^n, y^n - y\rangle_{H^{-1},H_0^1} = \langle p_i^n, -\Delta(y^n - y)\rangle_{W_0^{1,s},W^{-1,r}}$. Since $r > 2$, $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$ continuously. Thus, every $L^2(\Omega)$-function φ defines a bounded linear functional on $W_0^{1,r}(\Omega)$ via $(\varphi, \cdot)_{L^2(\Omega)}$. This allows us to make the following calculation

$$\begin{aligned}
&\langle p_i^n, -\Delta(y^n - y)\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad \langle p_i^n, -\Delta y^n - \chi_{B_i} u_i - \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n - f + \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad \langle p_i^n, \chi_{B_i}(u_i^n - u_i) + \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad (p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)} + \langle p_i^n, \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}}.
\end{aligned} \tag{14}$$

Next, we show that the term $(p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)}$ is bounded. Using (6), we deduce the existence of multipliers $\overline{\lambda}_i^n, \underline{\lambda}_i^n \in L^2(\Omega)$ such that

$$\overline{\lambda}_i^n \geq 0,\ a.e.\,\Omega,\ (\overline{\lambda}_i^n, u_i^n - b_i)_{L^2(\Omega)} = 0, \quad \underline{\lambda}_i^n \geq 0,\ a.e.\,\Omega,\ (\underline{\lambda}_i^n, a_i - u_i)_{L^2(\Omega)} = 0.$$

and $\chi_{B_i} p_i^n = \alpha_i u_i^n + \overline{\lambda}_i^n - \underline{\lambda}_i^n$. But then

$$\begin{aligned}
(p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)} &= (\chi_{B_i} p_i^n, u_i^n - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n + \overline{\lambda}_i^n - \underline{\lambda}_i^n, u_i^n - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)} + (\overline{\lambda}_i^n, u_i^n - b_i + b_i - u_i)_{L^2(\Omega)} \\
&\qquad - (\underline{\lambda}_i^n, u_i^n - a_i + a_i - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)} + (\overline{\lambda}_i^n, b_i - u_i)_{L^2(\Omega)} - (\underline{\lambda}_i^n, a_i - u_i)_{L^2(\Omega)} \\
&\geq (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)}.
\end{aligned} \tag{15}$$

Combining (13)-(15), we obtain the inequality

$$\langle p_i^n, \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} \leq (\alpha_i u_i^n, u_i - u_i^n)_{L^2(\Omega)} + \int_\Omega (y_d^i - y^n)(y^n - y)dx$$
$$\leq |\alpha_i| ||u_i^n||_{L^2(\Omega)} ||u_i - u_i^n||_{L^2(\Omega)} + ||y_d^i - y^n||_{L^2(\Omega)} ||y^n - y||_{L^2(\Omega)}. \quad (16)$$

Since $(u_i, y) \in M_i$ was arbitrarily chosen and M_i is bounded, taking the supremum over both sides of (16) implies there exists a constant $C > 0$ such that

$$\sup_{\substack{\varphi \in W^{-1,r}(\Omega)\\ ||\varphi||_{W^{-1,r}(\Omega)}=1}} \langle p_i^n, \varphi\rangle_{W_0^{1,s},W^{-1,r}} \leq \delta C.$$

It follows that $\{p_i^n\}_n \subset W_0^{1,s}(\Omega)$ is bounded. Given $1 < r < +\infty$, $W_0^{1,s}(\Omega)$ is a reflexive Banach space. Therefore, there exists a subsequence of $\{p_i^n\}_n$, denoted still by n, and an element $p_i^* \in W_0^{1,s}(\Omega)$ such that $p_i^n \rightharpoonup_{W_0^{1,s}(\Omega)} p_i^*$. By the Rellich-Kondrachev theorem, the embedding $W_0^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, in which case, there exists a further subsequence of $\{p_i^n\}_n$, denoted still by n, such that $p_i^n \to_{L^2(\Omega)} p_i^*$. As the $\max(0,\cdot)$-operator is Lipschitz continuous from $L^2(\Omega)$ to $L^2(\Omega)$, the strong convergence of p_i^n to p_i^* implies that $u_i^n \to_{L^2(\Omega)} u_i^*$ for each $i = 1, \dots, N$. Furthermore, we know that the solution operator $y(\cdot)$ of the state equation is (completely) continuous from $L^2(\Omega)$ to $W_0^{1,r}(\Omega)$, from which can can deduce the strong convergence of the sequence $y^n := y(u^n)$ in $W_0^{1,r}(\Omega)$ to y^*. These implications lead to the equations (9) and (10).

Next, we turn our attention to the sequence $\lambda_n := \gamma_n(\psi - y^n)_+$. By the previous arguments, we have from (8) that $\{\lambda_n\}$ is bounded in $W^{-1,s}(\Omega)$. Moreover, the SUFR condition yields the existence of a constant $\varepsilon > 0$ and a (bounded) sequence of controls $\{\tilde{u}_i^n\} \subset U_i$ such that the sequence $\{\tilde{y}^n\}$ defined by $\tilde{y}^n := y(\tilde{u}_i^n, u_{-i}^n)$ satisfies $\tilde{y}^n - \psi \geq \varepsilon$, *a.e.* Ω for all $n \geq 1$. Since $\tilde{y}^n$ must also solve the state equation, it enjoys the increased regularity of y^n, i.e., $\tilde{y}^n \in W_0^{1,r}(\Omega)$. Multiplying (8) by $\tilde{y}^n$ yields the relation

$$\int_\Omega \lambda_n \tilde{y}^n = \langle -\Delta p_i^n, \tilde{y}^n\rangle_{W^{-1,s},W_0^{1,r}} + (y^n - y_d^i, \tilde{y}^n)_{L^2(\Omega)}.$$

Using the continuity of the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$, the boundedness of the sequences $\{\tilde{u}_i^n\}$ and $\{u_{-i}^n\}$, and the definition of the solution operator $y(\cdot)$, we can deduce the existence of constants $C, C' > 0$ such that

$$||\tilde{y}^n||_{L^2(\Omega)} \leq C||\tilde{y}^n||_{W_0^{1,r}} = C||(-\Delta)^{-1}(\chi_{B_i}\tilde{u}_i^n + \sum_{\substack{k=1\\k\neq i}}^{n} \chi_{B_k} u_k^n + f)||_{W_0^{1,r}} \leq$$
$$C||(-\Delta)^{-1}||_{\mathcal{L}(L^2(\Omega),W_0^{1,r})}(\mathrm{Vol}(B_i)||\tilde{u}_i^n||_{L^2(\Omega)} + \sum_{\substack{k=1\\k\neq i}}^{N} \mathrm{Vol}(B_k)||u_k^n||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}) \leq C'.$$

Therefore, there exists a constant $C'' > 0$, independent of n, such that

$$\int_\Omega \lambda_n \tilde{y}^n dx \leq |\int_\Omega \lambda_n \tilde{y}^n dx| = |\langle \lambda^n, \tilde{y}^n\rangle_{W^{-1,s},W_0^{1,r}}| \leq ||\lambda^n||_{W^{-1,s}} ||\tilde{y}^n||_{W_0^{1,r}} \leq C''. \quad (17)$$

Moreover, we have

$$\int_\Omega \lambda_n \tilde{y}^n dx = \int_\Omega \lambda_n(\tilde{y}^n - \psi + \psi)dx = \int_\Omega \lambda_n(\tilde{y}^n - \psi)dx + \int_\Omega \lambda_n \psi dx.$$

By substitution into (17), it follows that

$$\int_\Omega \lambda_n(\tilde{y}^n - \psi)dx \leq C'' - \int_\Omega \lambda_n \psi dx$$

Defining the subset $\mathcal{A}^n := \{x \in \Omega \,|\, \psi - y^n > 0\}$, we then deduce

$$\int_\Omega \lambda_n \psi dx = \int_{\mathcal{A}^n} \lambda_n \psi dx > \int_{\mathcal{A}^n} \lambda_n y^n dx = \int_\Omega \lambda_n y^n dx = \langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}}.$$

Thus,

$$\begin{aligned} 0 \leq \varepsilon \int_\Omega \lambda_n dx \leq \int_\Omega \lambda_n(\tilde{y}^n - \psi)dx \leq C'' - \int_\Omega \lambda_n \psi dx < C'' - \langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}} \\ \leq C'' + |\langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}}| \leq 2C''. \end{aligned}$$

Using the SUFR condition and the pointwise almost-everywhere non-negativity of λ_n, it holds that

$$0 \leq \int_\Omega \lambda_n dx = ||\lambda_n||_{L^1} \leq 2\varepsilon^{-1} C'',$$

from which it follows that the sequence $\{\lambda_n\}$ is bounded in $L^1(\Omega)$. Therefore, there exists a subsequence of λ_n, denoted still by n and an element $\lambda^* \in \mathcal{M}(\overline{\Omega})$ such that $\{\lambda_n\}$ converges in the weak topology $\sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))$ to λ^* (see e.g., Theorem IV.6.2 in [6] or Corollary 2.4.3 in [3]). The limiting adjoint equation (11) thus follows.

It remains to verify the complementarity relations (12). The feasibility of y^* follows from the fact that $y^n \to_{W_0^{1,r}(\Omega)} y^*$ implies $y^n - \psi \to_{L^1(\Omega)} y^* - \psi$. Hence, there exists a subsequence of $\{y^n - \psi\}$ that converges pointwise almost everywhere to $y^* - \psi$, in which case, $y^* \geq \psi, \ a.e. \, \Omega$. Now let $\varphi \in C(\overline{\Omega})$ such that $\varphi \geq 0$ on Ω. Then $0 \leq (\lambda_n, \varphi)_{L^2(\Omega)} = \langle \lambda_n, \varphi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}$. Passing to the limit in n, yields $\langle \lambda^*, \varphi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \geq 0$. Since both $W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W^{2,2}(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous and $0 \geq (\lambda_n, y^n - \psi)_{L^2(\Omega)} = \langle \lambda_n, y^n - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})}$, $\langle \lambda^*, y^* - \psi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \leq 0$ holds. By the feasibility of y^*, the latter holds as an equality. $\square$

This concludes our theoretical study of the GNEP. Before continuing, we note that one could easily extend some of these arguments to include bilateral constraints on the state and/or more general (linear) differential operators than $-\Delta$, provided the solutions are regular enough. One could also consider more control constraints, assuming they remain convex and bounded, however the simple reformulation of the variational inequality used in the previous proposition may no longer be available.

3 The Algorithm

Due to the constructive nature of Theorem 2.5 and 2.9, we can develop an infinite dimensional solution algorithm for the GNEP (3). The algorithm works by approximating a Nash equilibrium for (3) by Nash equilibria obtained for each NEP_γ along some sequence of constants $\{\gamma\}$ with $\gamma \to +\infty$. We describe this outer loop in Algorithm 1. It follows from Theorem 2.9 that the iterates $(u^\gamma, y^\gamma, p^\gamma)$ of Algorithm 1 converge to a point (u^*, y^*, p^*) in $L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ as $\gamma \to +\infty$ where u^* is a Nash equilibrium for (3) and for which there exists a $\lambda^* \in L^2(\Omega)^N$ such that $(u^*, y^*, p^*, \lambda^*)$ satisfies (9)-(11). Note that the data needed for Algorithm 1 also includes the model data, e.g., α_i, y_d^i, f, a_i, b_i, etc.

Algorithm 1

Data: $\gamma_0 > 0$ $N \in \mathbb{N}$.
1: Choose $(u^0, y^0, p^0) \in L^2(\Omega)^N \times H_0^1(\Omega) \times H_0^1(\Omega)^N$ and set $k := 0$.
2: **repeat**.
3: Solve the coupled optimality conditions derived from (6)-(8) with $\gamma = \gamma_0$ to obtain $(u^{k+1}, y^{k+1}, p^{k+1})$ using initial values (u^k, y^k, p^k) for the solution algorithm.
4: Choose $\gamma_{k+1} > \gamma_k$.
5: Set $k := k + 1$.
6: **until** some stopping criterion is fulfilled.

In order to find an equilibrium for each γ_k in Algorithm 1, we propose that one solves the coupled system of necessary and sufficient first-order optimality conditions (6)-(8) derived in Proposition 2.6. The solution of this system can be done in two ways: an all-at-once approach or a non-linear Gauss-Seidel iteration. In this paper, we develop a solver for the "inner loop" in Algorithm 1 based on a non-linear Gauss-Seidel iteration in which the i^{th} optimality system (6)-(8) is solved with u_{-i}^k fixed. Each of these subproblems is solved via a nonsmooth Newton step. In the sequel, we discuss the details of the Newton step.

In the following discussion, fix $i \in \{1, \ldots, N\}$, $\gamma > 0$, and let u^γ be an abitrarily fixed feasible strategy for NEP_γ. Note that for any such u^γ, the i^{th} system (6)-(8) can be reduced to a problem in y^γ and p_i^γ:

$$-\Delta y^\gamma = \chi_{B_i}\left(\frac{1}{\alpha_i} p_i^\gamma - \left(\frac{1}{\alpha_i} p_i^\gamma - b_i\right)_+\right) + \left(-\left(\frac{1}{\alpha_i} p_i^\gamma - a_i\right)\right)_+ + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k^\gamma + f,$$

$$-\Delta p_i^\gamma = y_d^i - y^\gamma + \gamma(\psi - y^\gamma)_+.$$

Using these two nonsmooth equations, we define the mapping $F_i^\gamma : L^2(\Omega)^{N-1} \times H_0^1(\Omega) \times H_0^1(\Omega) \to H^{-1}(\Omega) \times H^{-1}(\Omega)$ by

$$F_i^\gamma(u_{-i}^\gamma, z, q) := \left(\begin{array}{c} -\Delta z - \chi_{B_i}\left(\frac{1}{\alpha_i} q - \left(\frac{1}{\alpha_i} q - b_i\right)_+ + \left(-\left(\frac{1}{\alpha_i} q - a_i\right)\right)_+\right) - \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k^\gamma - f. \\ -\Delta q + z - y_d^i - \gamma(\psi - z)_+ \end{array} \right) \tag{18}$$

Our goal in each subproblem iteration is to find $(\bar{y}^\gamma, \bar{p}_i^\gamma) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$F_i^\gamma(\bar{u}_{-i}^\gamma, \bar{y}^\gamma, \bar{p}_i^\gamma) = 0. \tag{19}$$

Once this is done, we set $\bar{u}_i^\gamma = \frac{1}{\alpha_i}\bar{p}_i^\gamma - (\frac{1}{\alpha_i}\bar{p}_i^\gamma - b_i)_+ + (-(\frac{1}{\alpha_i}\bar{p}_i^\gamma - a_i))_+$. The Gauss-Seidel iteration then updates u^γ by replacing u_i^γ with $\bar{u}_i^\gamma$ and proceeds to the next nonsmooth equation (19). As F_i^γ contains the nonsmooth operators $\max(0,\cdot)$, a generalized derivative concept is needed in order to define a Newton step In the following definition, taken from [4] and [13], let X, Y be Banach spaces, $D \subset X$ an open subset of X, and $F : D \to Y$.

Definition 3.1. *The mapping $F : D \subset X \to Y$ is said to be Newton-differentiable on the open subset $U \subset D$, if there exists a family of mappings $G : U \to \mathcal{L}(X,Y)$ such that*

$$\lim_{h\to 0} \frac{1}{||h||_X} ||F(x+h) - F(x) - G(x+h)h||_Y = 0,$$

for every $x \in U$.

One typically refers to G as the Newton derivative for F on U. A well-known result from [13], shows that

$$G_\delta(y)(x) = \begin{cases} 1 & \text{if} \quad y(x) > 0 \\ 0 & \text{if} \quad y(x) < 0 \\ \delta & \text{if} \quad y(x) = 0 \end{cases} \tag{20}$$

for every $y \in X$ and $\delta \in \mathbb{R}$ is a Newton-derivative of the $\max(0,\cdot)$, provided one has $\max(0,\cdot) : L^p(\Omega) \to L^q(\Omega)$ with $1 \leq q < p \leq \infty$. Suppose now that we wish to solve the equation $F(x) = 0$. If a Newton-derivative of F is available, then a generalized Newton step can be derived. For completeness, we provide the following result, which can be found in [4], as well as [13].

Theorem 3.2. *Suppose that $F(x^*) = 0$ and that F is Newton-differentiable on an open neighborhood U of x^* with Newton derivative G. If $G(x)$ is nonsingular for all $x \in U$ and the set $\left\{||G(x)^{-1}||_{\mathcal{L}(Y,X)} : x \in U\right\}$ is bounded, then the semismooth Netwon iteration*

$$x^{k+1} = x^k - G(x^k)^{-1}F(x^k) \tag{21}$$

converges superlinearly to x^*, provided $||x^0 - x^*||_X$ is sufficiently small.

At this point, we have enough tools to develop a semismooth Newton algorithm for solving (19). Suppose $(z,q) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is the current iterate. We define the following approximations of the active sets:

$$\begin{aligned} &A_i^a := \left\{ x \in \Omega \,\middle|\, a_i(x) - \frac{1}{\alpha_i}q(x) > 0 \right\}, A_i^b := \left\{ x \in \Omega \,\middle|\, \frac{1}{\alpha_i}q(x) - b_i(x) > 0 \right\}, \\ &A_y := \{ x \in \Omega \,|\, \psi(x) - z(x) > 0 \}. \end{aligned}$$

Additionally, we define approximations for the inactive sets by

$$I_i := \Omega \setminus (A_i^a \cup A_i^b), \quad J_i := \Omega \setminus A_y.$$

In the following, we let $(\delta y, \delta p)$ denote the difference between the current iterate (z,q) and previous iterate (y,p) in the Newton step. Then by using G_0 in (20) as the Newton-derivative of the $\max(0,\cdot)$ operator, one can easily show that (21) (applied to (19)) is equivalent to solving the following system in $(\delta y, \delta p)$:

$$-\Delta\delta y - \chi_{B_i\cap I_i}\delta p = \Delta y + \chi_{B_i\cap A_i^a} a + \chi_{B_i\cap A_i^b} b + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k + f,$$

$$-\Delta\delta p - \delta y - \gamma\chi_{A_y}\delta y = \Delta p + y_d^i - y + \gamma(\psi - y)_+$$

and then setting $z = y + \delta y$ and $q = p + \delta p$. This indicates, as was expected from the results in [13], that the nonsmooth Newton step is equivalent to a primal-dual active set strategy. This fact indicates that the Newton iteration may stopped when the previous active sets are equal to the current active sets.

According the Theorem 3.2, the semismooth Newton method is merely locally convergent. Though there exist a number of sophisticated globalization schemes [9, 11, 23], see also [14] for a function space treatment of these issues, we chose to implement an Armijo-type line search and thus update our steps by $y_+ = y + \tau\delta y$, and $p_+ = p + \tau\delta p$, where τ is step size. Despite the fact that no descent is guaranteed along such a "path", we observed satisfying results. Future work might include the implementation of a more properly globalized Newton step. To see that each F_i^γ is in fact Newton differentiable, we refer the reader to the proof of Proposition 5.5 in [14], which can be easily adapted to our setting.

These new considerations lead to the inner loop algorithm Algorithm 2.

Algorithm 2

Data: $\gamma > 0$ $N \in \mathbb{N}$.

1: Choose $(u^0, y^0, p^0) \in L^2(\Omega)^N \times H_0^1(\Omega) \times H_0^1(\Omega)$ and set $l := 0$.
2: **repeat**.
  3: **for** $i = 1, \dots, N$.
  4: Using a semismooth Newton step, solve $F_i^\gamma(u_{-i}^l, z, q) = 0$ in (z, q)
    5: Set $v = \frac{1}{\alpha_i} q - (\frac{1}{\alpha_i} q - b_i)_+ + (-(\frac{1}{\alpha_i} q - a_i))_+$.
    6: Set $(u_i^{l+1}, y^{l+1}, p_i^{l+1}) = (v, z, q)$.
    7: Set $u^l = (u_i^{l+1}, u_{-i}^l)$ and $y^l = y^{l+1}$.
  7: **end**.
5: Set $l := l + 1$.
6: **until** some stopping criterion is fulfilled.

The convergence of Algorithm 2 requires the convergence of a non-linear Gauss-Seidel step. Though such iterations may be either slow or prone to cycles, we observed good behaviour in our numerical experiments. A theoretical treatment of this question will be the subject of future work.

4 Numerical Experiments

Throughout this section, we let $N = 4$ and $\Omega = (0, 1) \times (0, 1)$. In order to discretize the problem, we considered a uniform grid with mesh size h, and we discretized the Laplacian $-\Delta$ by finite differences using a standard 5-point stencil. The examples were solved up to $h = 1/512$. Note that a mesh size of $h = 1/512$ corresponds to generalized Nash equilibrium problems with over three million free variables (counting the state, adjoint state, and control). To the best of our knowledge, these are the largest GNEPs solved to date. Using the variable ν defined by,

$$\nu := \frac{\sum_{i=1}^{N} ||u_i^{k+1} - u_i^k||_{L^2(\Omega)}}{\sum_{i=1}^{N} ||u_i^{k+1} - u_i^0||_{L^2(\Omega)}},$$

we set updated γ whenever $\nu \leq 1e-06$ by setting $\gamma^{k+1} = 2\gamma^k$. The nonsmooth Newton iterations stopped when the active sets for iteration k were equal to $k-1$. One topic of future research will be to develop a path-following strategy for the γ-updates as developed in [15], in order to avoid unnecessary/costly iterations. Due to its dependency on the accuracy and convergence properties of the innermost loop, i.e., the nonsmooth Newton solver, the issue of finding a proper starting point for Algorithm 1 is of great importance. In our experiments, we calculated a starting value for the problem by first setting $(u, y, p) = (0, 0, 0)$ and then obtained (u^0, y^0, p^0) by running one loop of the Gauss-Seidel iteration Algorithm 2. Though this can be a rather computationally intesive step, it provided us with a vector of feasible controls u_{-i}^0.

Example 4.1. For this example, we let

$$\widetilde{B}_1 = \left]\frac{1}{8}, \frac{7}{8}\right[\times \left]\frac{1}{8}, \frac{7}{8}\right[, \ \widetilde{B}_2 = \left]\frac{2}{8}, \frac{6}{8}\right[\times \left]\frac{2}{8}, \frac{6}{8}\right[, \ \widetilde{B}_3 = \left]\frac{3}{8}, \frac{5}{8}\right[\times \left]\frac{3}{8}, \frac{5}{8}\right[,$$

and define

$$B_1 = \Omega \setminus \widetilde{B}_1, \ B_2 = \widetilde{B}_1 \setminus \widetilde{B}_2 \ B_3 = \widetilde{B}_2 \setminus \widetilde{B}_3, \ B_4 = \widetilde{B}_3,$$

where the complement is taken in Ω. We used the desired states

$$\begin{aligned}
y_d^1(\mathbf{x}_1, \mathbf{x}_2) &= 1e3 \max(0, 4(0.25 - \max(|\mathbf{x}_1 - 0.25|, |\mathbf{x}_2 - 0.25|))), \\
y_d^2(\mathbf{x}_1, \mathbf{x}_2) &= 1e3 \max(0, 4(0.25 - \max(|\mathbf{x}_1 - 0.75|, |\mathbf{x}_2 - 0.25|))), \\
y_d^3(\mathbf{x}_1, \mathbf{x}_2) &= 1e3 \max(0, 4(0.25 - \max(|\mathbf{x}_1 - 0.25|, |\mathbf{x}_2 - 0.75|))), \\
y_d^4(\mathbf{x}_1, \mathbf{x}_2) &= 1e3 \max(0, 4(0.25 - \max(|\mathbf{x}_1 - 0.75|, |\mathbf{x}_2 - 0.75|))).
\end{aligned}$$

for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega$ and defined the obstacle ψ by

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \cos(2\sqrt{(\mathbf{x}_1^2 - 0.5)^2 + (\mathbf{x}_2^2 - 0.5)^2}) - 0.7.$$

By letting $A := [0.35, 0.65] \times [0.35, 0.65] \subset \mathbb{R}^2$, we defined the fixed righthand side f by

$$f = -\chi_A(\Delta\psi + 5).$$

Note that this choice of ψ is smooth and clearly satisfies $\psi|_{\partial\Omega} < 0$. Therefore, ψ fulfills the requirements used throught the theoretical portions of this paper. Finally, we let $a_i = 0$ and $b_i = 5$, for $i = 1, \ldots, 4$. For simplicity, we chose $\alpha_i = 1$ for all $i = 1, \ldots, 4$. Further experiments with asymmetric costs and bounds yielded similar results.

The behaviour of the algorithm over various mesh sizes can be seen in Tables 1 and 2. Note that the maxima, minimia, and average values in Table 2 are to be understood in the following sense: the symbol Max (Min) represents the maximum (minimum) number of iterations over

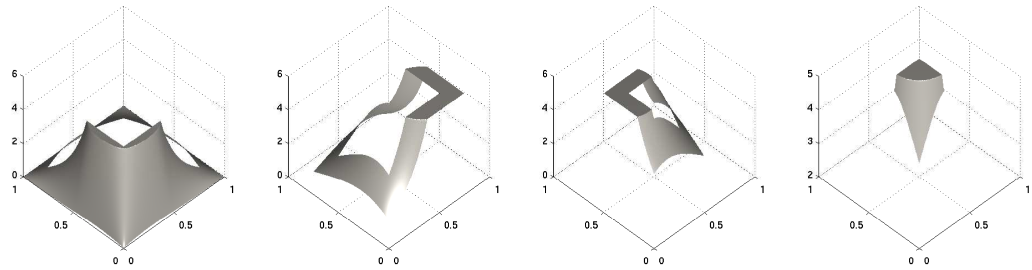

Figure 1: A Nash equilibrium $(u_1^*, u_2^*, u_3^*, u_4^*)$ (l-r) for Exp 4.1.

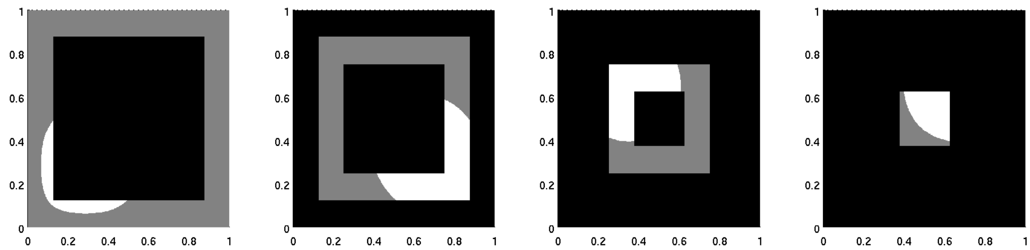

Figure 2: Active sets at equilibrium: $u_i^* = b_i$ (white), $u_i^* = a_i$ (black) for Exp 4.1.

all Gauss-Seidel iterations over all four players, whereas Avg represents the average of the average number of iterations over all players per Gauss-Seidel iteration per γ. We note that both the Gauss-Seidel iterations as well as the nonsmooth Newton steps exhibited a strong mesh independence, see Table 1, We assumed that the discretization error dominated the regularization error and chose therefore, to stop the outer loop (Gauss-Seidel) iterations, once $\gamma^{-1} \leq h^2$. Finally, we note that in Figure 1, the equilibrium controls are plotted such that the portions where $u_i = 0$ have been removed.

Example		Mesh Width h				
4.1		1/32	1/64	1/128	1/256	1/512
γ	2000	0	3	3	3	3
	4000		1	1	1	1
	8000			1	1	1
	16000			1	1	1
	32000				1	1
	64000				1	1
	1.28e+05					1
	2.56e+05					1

Table 1: Total Gauss-Seidel iterations per γ for various mesh widths h

Example 4.1		Mesh Width h				
		1/32	1/64	1/128	1/256	1/512
γ	2000	2/1/1.75	3/1/1.67	3/1/1.67	3/1/2.58	4/3/3.17
	4000		3/1/1.5	3/1/1.5	3/1/2.5	4/3/3.25
	8000			3/1/1.5	3/1/2.5	4/3/3.25
	16000			3/1/1.5	3/1/2.5	4/3/3.25
	32000				3/1/2.5	4/3/3.25
	64000				3/1/2.5	4/3/3.25
	1.28e+05					4/3/3.25
	2.56e+05					4/3/3.25

Table 2: Max/Min/Avg nonsmooth Newton iterations per γ for various mesh widths h

Example 4.2		Mesh Width h				
		1/32	1/64	1/128	1/256	1/512
γ	2000	0	1	2	2	2
	4000		1	1	1	1
	8000			1	1	1
	16000			1	1	1
	32000				1	1
	64000				1	1
	1.28e+05					1
	2.56e+05					1

Table 3: Total Gauss-Seidel iterations per γ for various mesh widths h

Example 4.2. Perhaps one of the major difficulties that can arise when solving GNEPs is a non-trivial biactive set, i.e., when at iteration k, the set $\big\{x \in \Omega \,|\, y^k(x) = \psi(x)\big\} \cap \{x \in \Omega \,|\, \lambda_n(x) = 0\}$ has positive Lebesgue measure. In this example, we consider exactly such a case. Let ψ be as in Example 4.1 and define the subsets

$$B_1 = \left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[, \; B_2 = \left]\frac{1}{2}, 1\right[\times \left]0, \frac{1}{2}\right[, \; B_3 = \left]0, \frac{1}{2}\right[\times \left]\frac{1}{2}, 1\right[, \; B_4 = \left]\frac{1}{2}, 1\right[\times \left]\frac{1}{2}, 1\right[,$$

and

$$\tilde{B}_1 = \left]\frac{1}{4}, \frac{1}{2}\right[\times \left]\frac{1}{4}, \frac{1}{2}\right[, \; \tilde{B}_2 = \left]\frac{1}{2}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{1}{2}\right[, \; \tilde{B}_3 = \left]\frac{1}{4}, \frac{1}{2}\right[\times \left]\frac{1}{2}, \frac{3}{4}\right[, \; \tilde{B}_4 = \left]\frac{1}{2}, \frac{3}{4}\right[\times \left]\frac{1}{2}, \frac{3}{4}\right[.$$

Let $A := \tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3 \cup \tilde{B}_4$ and define the control bounds $a_i = 0$, $i = 1, \ldots, 4$ and

$$b_i := \chi_{B_i} - \frac{1}{2}\chi_{\tilde{B}_i}, (i = 1, \ldots, 4).$$

We define a common desired state, shared by all players, by

$$y_d^i(\mathbf{x}_1, \mathbf{x}_2) = 5e2 \min(\max(0, \psi(\mathbf{x}_1, \mathbf{x}_2)), 0.2), (i = 1, \ldots, 4),$$

and we set $\alpha_i = 1.1e\text{-}3$, for all $i = 1, \ldots, 4$, and $f = -\Delta(\psi)_+ - 1/2\chi_A - \chi_{\Omega \setminus A}$.

The behaviour of the algorithm over various mesh sizes can be seen in Tables 3 and 4. As before the Gauss-Seidel iterations exhibited no mesh dependence, see Table 1. In addition, we see that the nonsmooth Newton steps behave without any noticable mesh dependence. In Figure 3, we provide the equilibrium state; active, biactive, and inactive sets; and the multiplier λ^*. The Nash equilibrium can be seen in Figure 4, as before, we have removed the parts of the graphs at which the function values are zero to facilitate visibility.

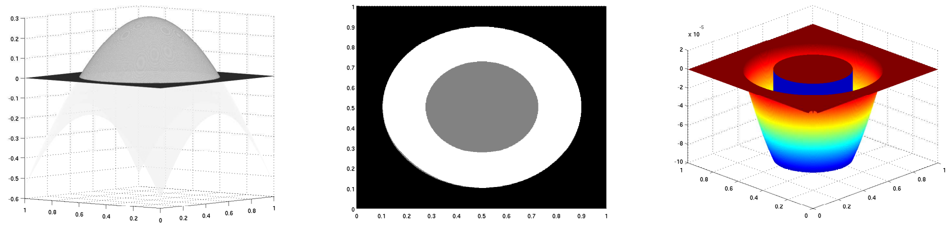

Figure 3: l: y^* (dark) and ψ (light); m: active (white), biactive (grey), and inactive (black) sets; r: λ^*

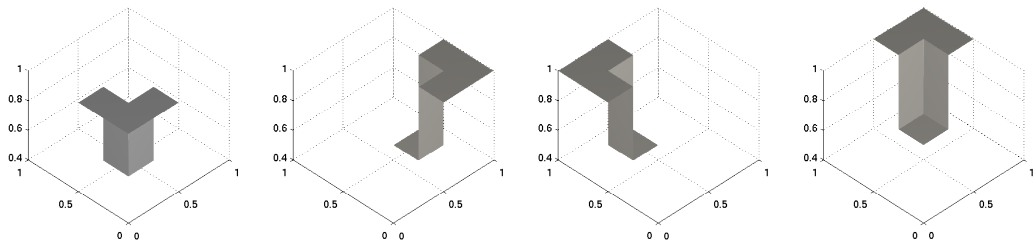

Figure 4: A Nash equilibrium $(u_1^*, u_2^*, u_3^*, u_4^*)$ (l-r) for Exp 4.2

Example 4.2		Mesh Width h				
		1/32	1/64	1/128	1/256	1/512
γ	2000	1/1/1	1/1/1	1/1/1	3/1/1.5	2/1/1.12
	4000		1/1/1	1/1/1	3/1/2	1/1/1
	8000			1/1/1	3/1/2	1/1/1
	16000			1/1/1	3/1/2	1/1/1
	32000				3/1/2	1/1/1
	64000				3/1/2	1/1/1
	1.28e+05					1/1/1
	2.56e+05					1/1/1

Table 4: Max/Min/Avg nonsmooth Newton iterations per γ for various mesh widths h

References

[1] Adams, R. A., and Fournier, J. J.-F. *Sobolev Spaces*, second ed. Elsevier, Amsterdam, 2008.

[2] Arrow, K., and Debreu, G. Existence of an equilibrium for a competitive economy. *Econometrica 22* (1954), 265–290.

[3] Attouch, H., Buttazzo, G., and Michaille, G. *Variational Analysis in Sobolev and BV Spaces.* MPS-SIAM Series on Optimization. SIAM and MPS, Philadelphia, 2006.

[4] Chen, X., Nashed, Z., and Qi, L. Smoothing methods and semismooth methods for nondifferentiable operator equations. *SIAM J. Numer. Anal. 38*, 4 (2000), 1200–1216 (electronic).

[5] Debreu, G. A social equilibrium existence theorem. *Proc. Natl. Acad. Sci.*.

[6] Dunford, N., and Schwartz, J. T. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988.

[7] Facchinei, F., and Kanzow, C. Generalized Nash equilibrium problems. *4OR 5*, 3 (2007), 173–210.

[8] Facchinei, F., and Kanzow, C. Penalty methods for the solution of generalized Nash equilibrium problems. *SIAM J. Optim. 20*, 5 (2010), 2228–2253.

[9] Facchinei, F., and Pang, J.-S. *Finite-dimensional variational inequalities and complementarity problems. Vol. II.* Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[10] Fan, K. Fixed Points and Minimax Theorems in Locally Convex Topological Linear Spaces. *Proceedings of the National Academy of Science 28*, 2 (1952).

[11] Ferris, M. C., and Munson, T. S. Interfaces to PATH 3.0: design, implementation and usage. *Comput. Optim. Appl. 12*, 1-3 (1999), 207–227. Computational optimization—a tribute to Olvi Mangasarian, Part I.

[12] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[13] Hintermüller, M., Ito, K., and Kunisch, K. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim. 13*, 3 (2002).

[14] Hintermüller, M., and Kopacka, I. A smooth penalty approach and a nonlinear multigrid algorithm for elliptic MPECs. *Computational Optimization and Applications* (2009). DOI 10.1007/s10589-009-9307-9.

[15] Hintermüller, M., and Kunisch, K. Path-following methods for a class of constrained minimization problems in function space. *SIAM J. Optim. 17*, 1 (2006), 159–187 (electronic).

[16] HINTERMÜLLER, M., AND KUNISCH, K. PDE-constrained optimization subject to pointwise constraints on the control, the state, and its derivative. *SIAM J. Optim. 20*, 3 (2009), 1133–1156.

[17] HINZE, M., PINNAU, R., ULBRICH, M., AND ULBRICH, S. *Optimization with PDE constraints*, vol. 23 of *Mathematical Modelling: Theory and Applications*. Springer, New York, 2009.

[18] KADLEC, J. The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain. *Czechoslovak Math. J. 14 (89)* (1964), 386–393.

[19] KINDERLEHRER, D., AND STAMPACCHIA, G. *An Introduction to Variational Inequalities and Their Applications*. Academic Press, New York and London, 1980.

[20] NEČAS, J. Sur la coercivité des formes sesquilinéaires, elliptiques. *Rev. Roumaine Math. Pures Appl. 9* (1964), 47–69.

[21] NEITTAANMÄKI, P., SPREKELS, J., AND TIBA, D. *Optimization of Elliptic Systems*. Springer Monographs in Mathematics. Springer, New York, 2006.

[22] PANG, J.-S., AND FUKUSHIMA, M. Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Comput. Manag. Sci. 2*, 1 (2005), 21–56.

[23] RALPH, D. Global convergence of damped Newton's method for nonsmooth equations via the path search. *Math. Oper. Res. 19*, 2 (1994), 352–389.

[24] TIBA, D., AND ZĂLINESCU, C. On the necessity of some constraint qualification conditions in convex programming. *J. Convex Anal. 11*, 1 (2004), 95–110.

[25] TRÖLTZSCH, F. *Optimal control of partial differential equations*, vol. 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels.

[26] VIDYASAGAR, M. V. A New Approach to N-person, Nonzero-Sum, Linear Differential Games. *JOTA 18*, 1 (1976), 477–487.

[27] WLOKA, J. *Partielle Differentialgleichungen*. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982.

[28] YOSIDA, K. *Functional Analysis*. Spinger-Verlag, Berlin, 1980.