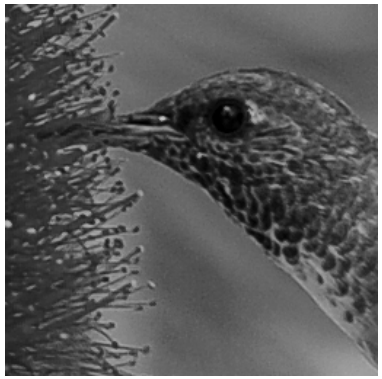
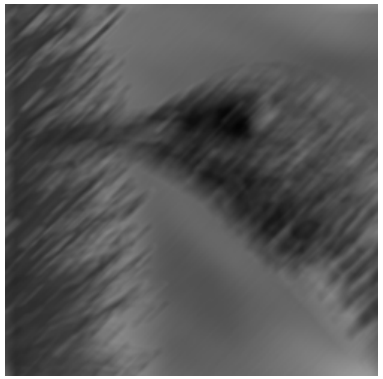
Motivation

Problem setting: With u an unknown, sharp image, let

$$u_0 = Tu \quad \text{be given. Obtain } u!$$



u



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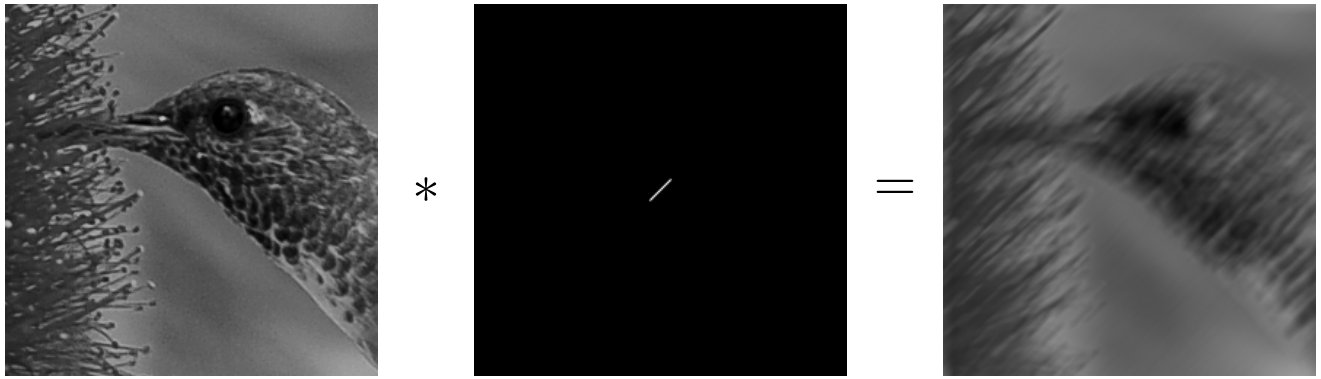

u $\quad * \quad$ k $\quad = \quad$ $u_0 = Tu$

Model: For $\Omega = [-1, 1]^2$, $k \in L^2(\Omega)$, define

$$T : L^2(\Omega) \to L^2(\Omega)$$
$$u \mapsto Tu(x) := \int_\Omega u(x-y)k(y)\,\mathrm{d}y$$

Then, T is well-defined, linear and bounded.

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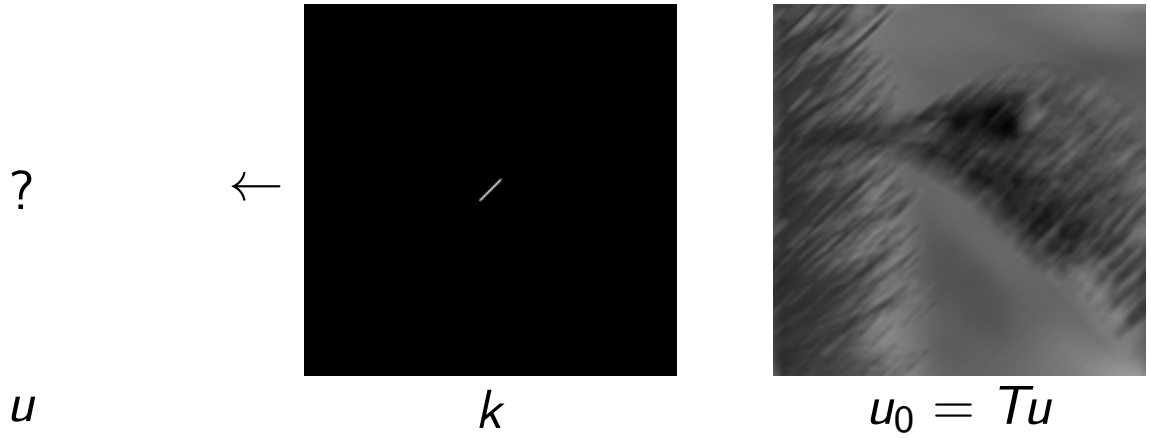

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Inverse Problem: Given u_0, T, obtain u.

Inverse Problems

Given X, Y Hilbert spaces,

- $T \in \mathcal{L}(X, Y)$ (Forward model)
- $f \in Y$ (Measured data)

Obtain

$$x \in X \text{ s.t. } Tx = y$$

(Compute the *causal factors* from observations)

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Applications

- (Biomedical) Imaging: CT, PET, MR, Material sciences, geosciences, astronomy
- Modeling with PDEs: Parameter identification, time-reversal
- Machine learning, Image/Video compression, Computer Vision

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Possible solution:: Generalized Inverse

$$T^\dagger y := \hat{x} = \operatorname{argmin}_{x \in X} \|x\|_X \text{ s.t. } \hat{x} \in \operatorname{argmin}_{x \in X} \|Tx - y\|_Y$$

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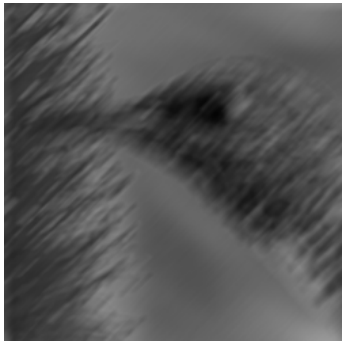
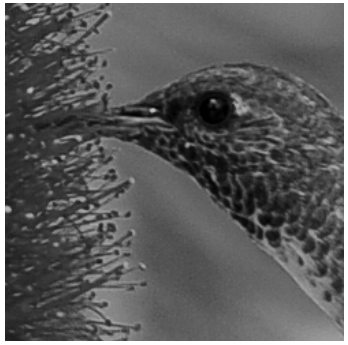
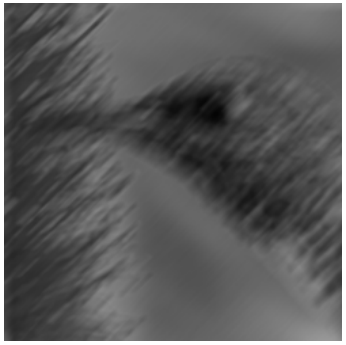


u_0



$T^\dagger(u_0)$

Motivation



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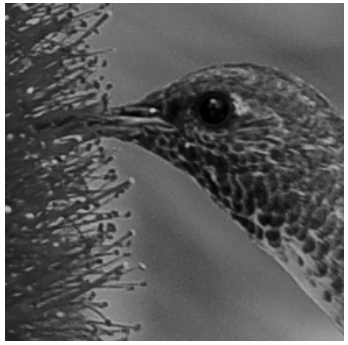

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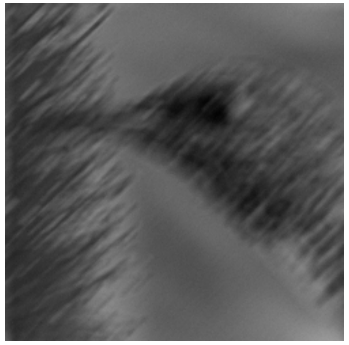
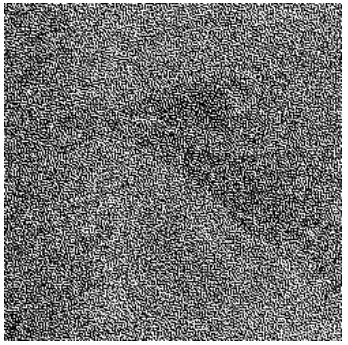


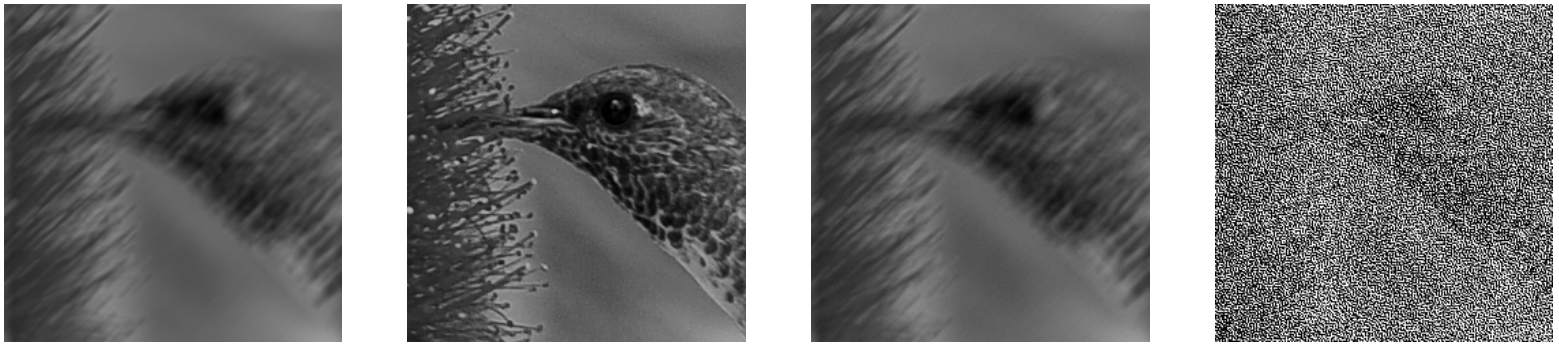
$u_0 + \delta$



$T^\dagger(u_0 + \delta)$

Problem: $T^\dagger$ continuous $\Leftrightarrow$ T has closed range

Motivation



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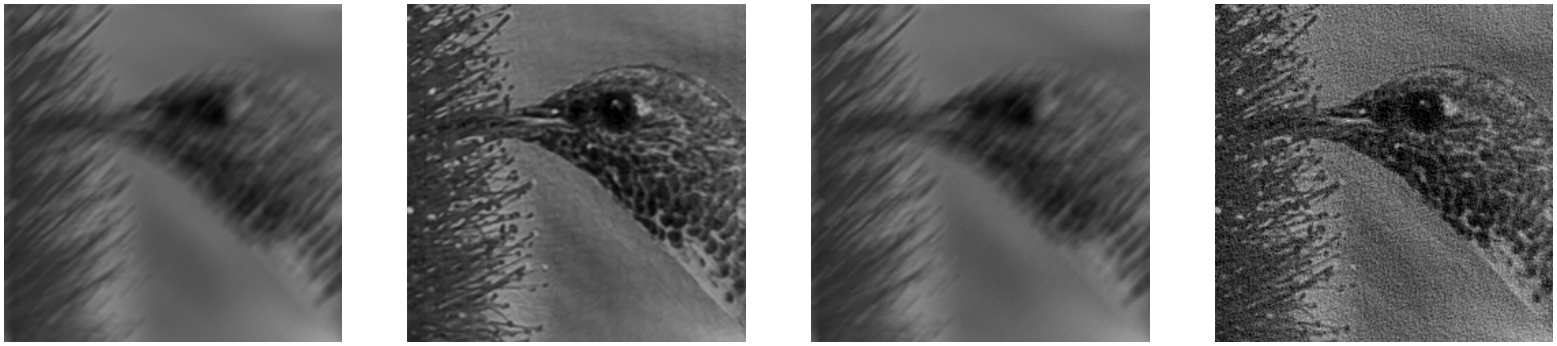

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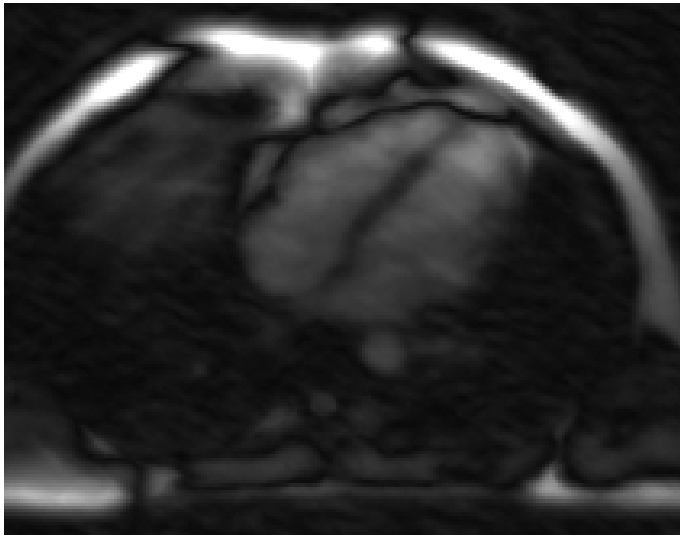
Application: Magnetic resonance imaging

- Magnetic resonance imaging is an important clinical tool
- Difficulty: Data acquisition very slow
- Forward model:

$$T \ldots \text{ subsampled Fourier transform}$$

- Ill-posedness: Lack of data



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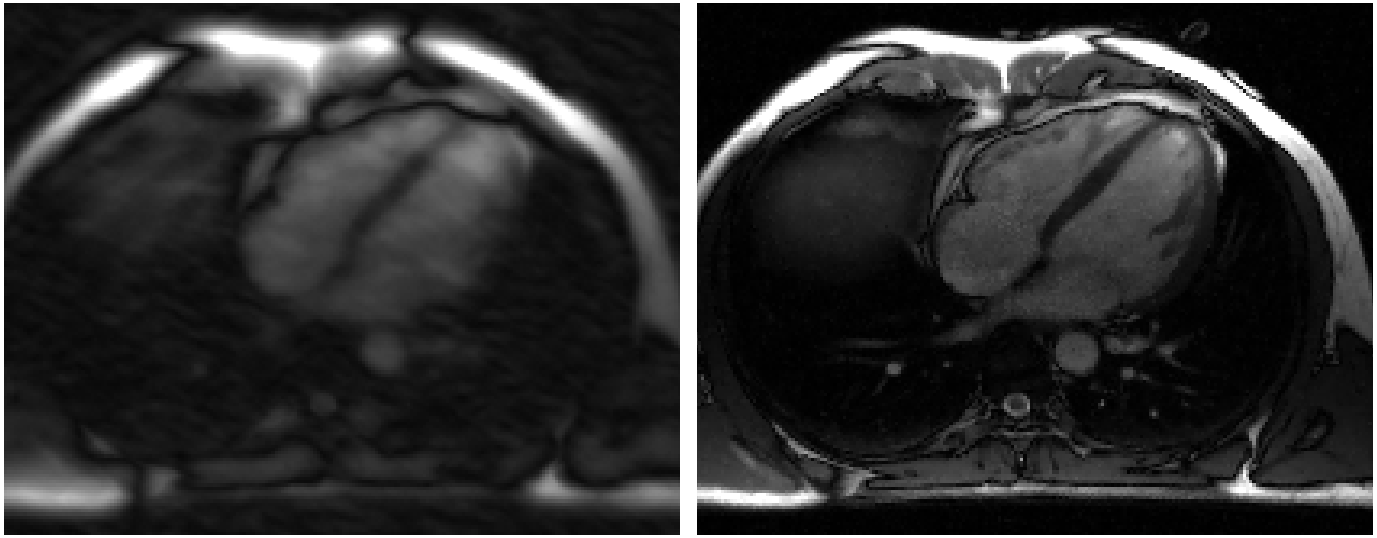

Unregularized Regularized

Application: Electron tomography

- Imaging at nano-scale: material science, semiconductors
- Difficulty: Limited and noisy measurements
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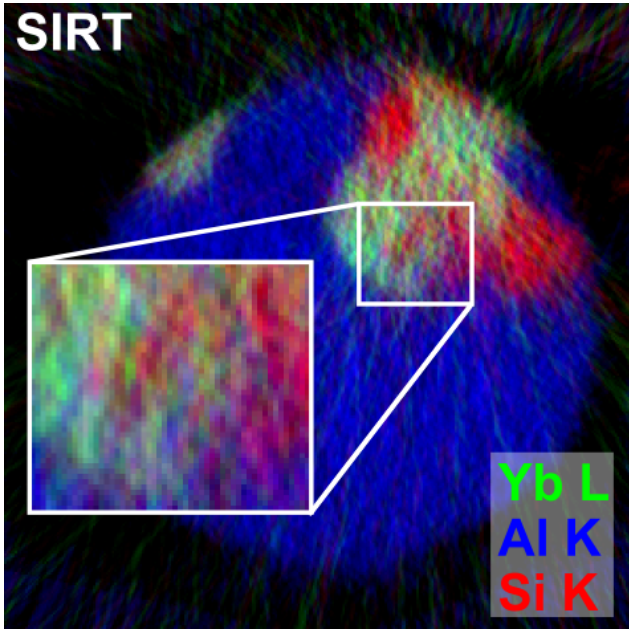

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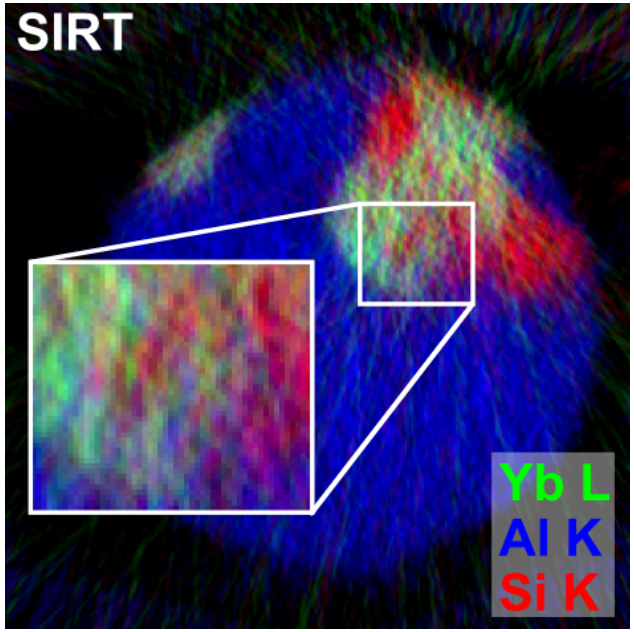

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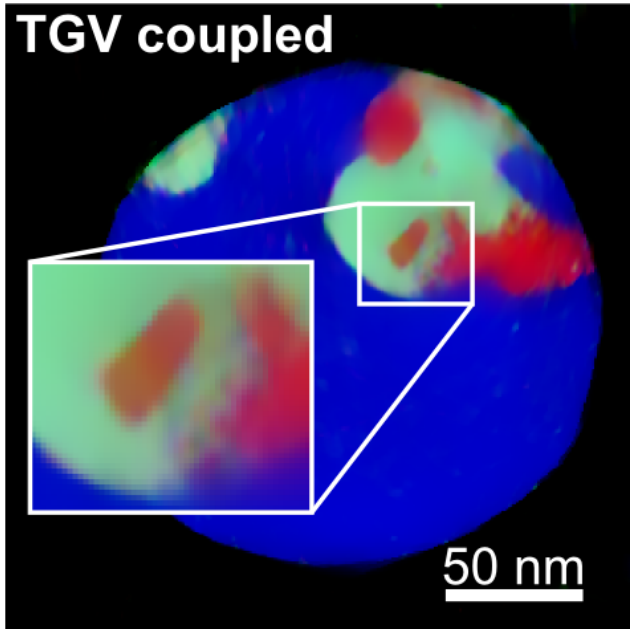

Regularized

Application: JPEG decompression

- JPEG is the most popular image compression standard
- Difficulty: High compression causes artifacts
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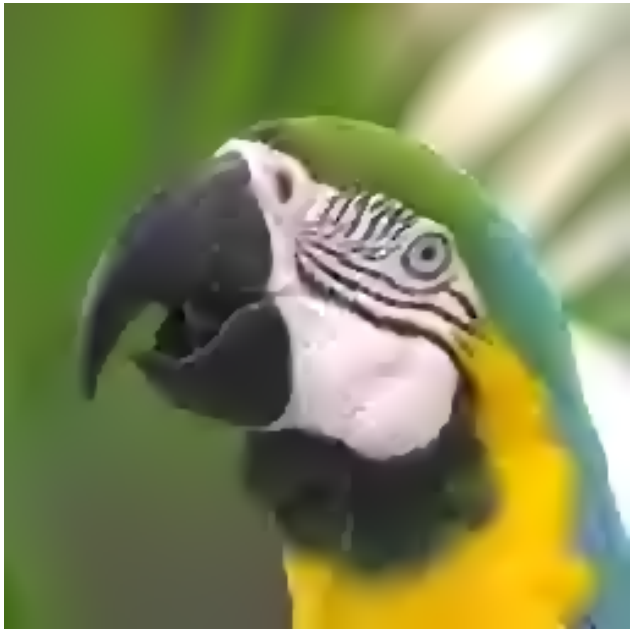

Regularized

Conclusion

- Inverse Problems are highly relevant for diverse applications
- Regularization is a key for good results
- Mathematical Inverse Problems: Combination of theory (functional analysis, measure theory, PDEs,...) and numerical implementation