Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 8

Due date: January 29, 2016

Problem 8.1. Let $\Omega \subset \mathbb{R}^n$, $1 < p \leq \infty$ and $p' \in [1, \infty)$ such that $1 = 1/p + 1/p'$. Show that the following are equivalent:

- $u \in W^{1,p}(\Omega)$
- $u \in L^p(\Omega)$ and $\left|\int_\Omega u \partial_i \varphi\right| \leq C \|\varphi\|_{p'}$ for all $\varphi \in C_c^\infty(\Omega)$, $1 \leq i \leq n$

Does this also hold for $p = 1$?

Problem 8.2. Show that for general $\Omega \subset \mathbb{R}^n$, $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$.
(Hint: Consider $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < |x| < 1,\, 0 < y < 1\}$)

Problem 8.3. Show that a self-adjoint extension of a symmetric, densely defined, operator is not necessarily unique.
(Hint: Consider ∂_{xx} on $L^2([0, 2\pi])$ and appropriate boundary conditions.)