Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 7

Due date: January 15, 2016

Problem 7.1. Let $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < \infty$. Show

i) If $f \in L^p(\Omega, \mathbb{K})$, and $\eta : \mathbb{R}^n \to \mathbb{R}$ is a mollifier kernel, then $\eta_\epsilon * f \to f$ in $L^p(\Omega, \mathbb{K})$ for $\epsilon \to 0$.

ii) If $f \in L^1_{\text{loc}}(\Omega, \mathbb{K})$ such that $[f] = 0$, then $f = 0$. Deduce that, if a distribution is regular, its representing function is unique.

(Hint: i) Use techniques from the lecture. ii) Exploit that, if f is not zero, it is strictly positive or negative a.e. on a compact set with non-zero measure.)

Problem 7.2. [Integration by parts] Let $\Omega \subset \mathbb{R}^n$ be open, $f, g \in C^k(\Omega, \mathbb{K})$ such that one of them has compact support in Ω. Show that

$$\int_\Omega D^\alpha f(x) g(x) \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha g(x) \, \mathrm{d}x$$

for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = \sum_{i=1}^n \alpha_i \leq k$. Deduce that, if $h \in C^k(\Omega, \mathbb{K})$, $D^\alpha[h] = [D^\alpha h]$ for all $|\alpha| \leq k$.
(Hint: First consider the Fubini theorem to compute $\int_{\mathbb{R}^n} \partial_{x_i} f(x) \, \mathrm{d}x$ for compactly supported f.)

Remember Green's formula: For $u, v \in C^1(\overline{\Omega})$ with Ω a bounded Lipschitz domain, we have

$$\int_\Omega \partial_{x_i} uv + \int_\Omega u \partial_{x_i} v = \int_{\partial\Omega} uv\nu_i$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the a.e. defined unit outward normal vector to $\partial\Omega$. (See for instance [Grisvard, Elliptic Problems in Nonsmooth Domains, Theorem 1.5.3.1] and [Necas, Les Methodes directes en theorie des equations elliptiques, Chapter 3, §1, Theorem 1.1] for a definition of Lipschitz domain and a proof in a more general setting.

Problem 7.3. [Fundamental solution for the Laplace operator] Define $f : \mathbb{R}^3 \to \mathbb{R}$ as $f(x) = \frac{1}{4\pi|x|}$ (with $|x|$ the Euclidean norm) for $x \neq 0$ and 0 else. Show

i) $\Delta f(x) = \sum_{i=1}^3 \partial_{x_i x_i} f(x) = 0$ for all $x \neq 0$.

ii) $f \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $\Delta[f](\varphi) = -\varphi(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$, i.e., $\Delta[f]$ is the negative delta-distribution.

iii) For $g \in C_c^\infty(\mathbb{R}^3)$, define $u(x) = (g * f)(x)$. Show that $u \in C^\infty(\mathbb{R}^m)$ and

$$-\Delta u = g$$

Hint: for ii) you might use Greens formula on $\mathbb{R}^3 \setminus B_\epsilon(0)$ and consider the limit $\epsilon \to 0$.

Problem 7.4. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $(\Omega_i)_{i=1}^n$ a sequence of bounded Lipschitz domains such that $\overline{\Omega} = \bigcup_{i=1}^n \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Let $v \in C(\overline{\Omega})$ be such that $v|_{\overline{\Omega}_i} \in C^1(\overline{\Omega}_i)$ for all i. Show that $D^\alpha[v]$ is a regular distribution for $|\alpha| \leq 1$ and determine its representing function.