Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 6

Due date: December 18, 2015

Problem 6.1. Let H be a Hilbert space and $T : \text{dom}(T) \subset H \to H$ be a densely defined operator. Show

i) If $S : \text{dom}(S) \subset H \to H$ is such that $T \subset S$, then $S^* \subset T^*$.

ii) If T is essentially self-adjoint, it admits a unique self-adjoint extension.

iii) If T is self-adjoint, it does not admit a non-trivial symmetric extension.

Problem 6.2. Let X, Y be a Banach spaces, $T : \text{dom}(T) \subset X \to Y$ a densely defined operator.

i) Assuming $X = Y$, show that if the resolvent set of T is non-empty, then T is closed.

ii) In case T is a bounded operator defined on all of X, remember that, T is invertible if and only if T^* is invertible.

iii) We say that an operator admits an inverse on its range if it is injective (and hence the inverse mapping on its range exists). Show that, if T admits an inverse on its range and the range of T is dense, then also T^* admits an inverse on its range and

$$(T^*)^{-1} = (T^{-1})^*$$

iv) Give an example where $T : \text{dom}(T) \subset X \to X$ is densely defined, admits an inverse on its range, but T^* is not injective.