Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 5

Due date: December 4, 2015

Remember that the intermediate exam will take place on December 4!

Problem 5.1. Proof Proposition 1.46 of the lecture.

Remember that for the adjoint of an operator to be well-defined, its domain needs to be dense. While for a densely defined continuous operator $A : \mathrm{dom}(A) \subset X \to Y$ we always have $\mathrm{dom}(A^*) = Y^*$ and hence A^{**} is well defined, this is not true for general operators. The following example shows that, even for closed operators with dense range, A^{**} is not necessarily well defined.

Problem 5.2. Set $X = \ell^1$ such that $X^* \cong \ell^\infty$. Define $A : \mathrm{dom}(A) \subset X \to X$ by

$$\mathrm{dom}(A) = \{(u_n)_n \in \ell^1 \mid (nu_n)_n \in \ell^1\}, \quad A(u_n)_n = (nu_n)_n$$

Show that

i) A is densely defined and closed.

ii) Determine $D(A^*)$, A^*, $\overline{D(A^*)}$.

Problem 5.3. Let X, Y be two normed spaces and $A : \mathrm{dom}(A) \subset X \to Y$ be a densely defined operator.

i) Show that $\ker(A^*) = \mathrm{rg}(A)^\perp$ and $\ker(A) \subset \mathrm{rg}(A^*)_\perp$.

ii) Show that, if A is closed, $\ker(A) = \mathrm{rg}(A^*)_\perp$. (Hint: Hahn-Banach)