Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 4

Due date: November 20, 2015

This exercise sheet deals with the application of spectral theory to study ill-posed operator equations of the form: Given f, find u s.t. $Ku = f$, where K is not continuously invertible, meaning that small errors on the "measurement" f might lead to arbitrary large deviations in the solution u. Such a situation occurs in many important physical problems and engineering applications, such as X-ray tomography, PET imaging or parameter identification for PDEs, where the ill-posedness posses severe difficulties in practical applications.

Problem 4.1. [Generalized inverse]. Let $T \in \mathcal{L}(X, Y)$ with X, Y Hilbert spaces. We are interested in a generalized approximate solution to $Tx = y$:

- $\hat{x} \in X$ is called least squares solution of $Tx = y$ if
$$\|T\hat{x} - y\| = \inf\{\|Tz - y\| \,:\, z \in X\}.$$
- $\hat{x} \in X$ is called best approximate solution of $Tx = y$ if $\hat{x}$ is a least squares solution and
$$\|\hat{x}\| = \inf\{\|z\| \,:\, z \text{ is a least squares solution of } Tx = y\}.$$

We define the (Moore-Penrose) generalized inverse of T as

$$T^\dagger : \mathcal{D}(T^\dagger) \to X \tag{1}$$
$$y \mapsto T^\dagger y = x^\dagger \text{ the best approximate solution of } Tx = y \tag{2}$$

with $\mathcal{D}(T^\dagger) := \mathrm{rg}(T) + \mathrm{rg}(T)^\perp$ the domain of $T^\dagger$.

i) Show that $T^\dagger$ is well and densely defined, linear and has a closed graph.

ii) Deduce that $T^\dagger$ is continuous if and only if T has closed range.

iii) Show that $T^\dagger y \in \ker(T)^\perp$ for $y \in \mathcal{D}(T^\dagger)$.

iv) Show that, with $x^\dagger = T^\dagger y$, $T^*Tx^\dagger = T^*y$ and hence, $x^\dagger = (T^*T)^{-1}T^*y$ if T^*T is invertible.

v) Note that $T(T^\dagger y) = y$ if $y \in \mathrm{rg}(T)$ and $T^\dagger = T^{-1}$ if T is bijective.

Problem 4.2. Let $K \in \mathcal{L}(X, Y)$ be compact, with X, Y Hilbert spaces. We write K according to the lecture as
$$Kx = \sum_{i \in N} \sigma_n (x, v_n) u_n$$
with $N \subset \mathbb{N}$ a countable index set, $(u_n)_n$ and $(v_n)_n$ orthonormal systems and $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ the singular values of K. Show

i) If $\dim(\mathrm{rg}(K)) = \infty$, then $K^\dagger$ is unbounded.

ii) $y \in \mathcal{D}(K^\dagger) \Leftrightarrow \sum_{\substack{n \in N \\ \sigma_n \neq 0}} \frac{|(y, u_n)|^2}{\sigma_n^2} < \infty$.

iii) For $y \in \mathcal{D}(K^\dagger)$, $K^\dagger y = \sum_{\substack{n\in N\\ \sigma_n \neq 0}} \frac{(y,u_n)}{\sigma_n} v_n$.

Problem 4.3. Consider the one-dimensional heat equation: Given $u_0 \in \mathcal{C}([0,\pi])$ (compatible with the boundary conditions), find $u \in \{v \in C([0,\pi]\times[0,1]) \,:\, \partial_x v,\, \partial^2_{xx} v,\, \partial_t v \in C((0,\pi)\times(0,T])\}$ such that

$$\begin{aligned} \partial_t u(x,t) &= \partial^2_{xx} u(x,t),\ x \in (0,\pi), t \in (0,T] \\ u(0,t) &= u(\pi,t) = 0,\ t \in (0,T] \\ u(x,0) &= u_0(x),\ x \in [0,\pi] \end{aligned}$$

By separation of variables we get that the unique solution as above can be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \varphi_n(x)$$

with $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ and $c_n = \int_0^\pi u_0(\tau)\varphi_n(\tau)\mathrm{d}\tau$. We are interested in obtaining the initial temperature v_0 from the final temperature $f(x) = u(x,1)$.

i) Show that this problem can be written as solving $Ku_0 = f$ with $K : L^2([0,\pi]) \to L^2([0,\pi])$ a compact, self-adjoint operator.

ii) Determine the eigenvalues and eigenvectors of K

iii) Show that the problem is severely-ill-posed: Determine when $f \in \mathcal{D}(K^\dagger)$ and show that already an error of about 10^{-8} in (f,φ_5) (the fifth Fourier coefficient of the data) can lead to an error larger than 10^3 in the solution.